Conformal Embeddings of an Open Riemann Surface into Another —A Counterpart of Univalent Function Theory—

Masakazu SHIBA*

Hiroshima University, Hiroshima 739-8527, Japan

We study conformal embeddings of a noncompact Riemann surface of finite genus into compact Riemann surfaces of the same genus and show some of the close relationships between the classical theory of univalent functions and our results. Some new problems are also discussed. This article partially intends to introduce our results and to invite the function-theorists on plane domains to the topics on Riemann surfaces.

KEYWORDS: univalent functions, Riemann surfaces, continuations of a Riemann surface, span

1. Introduction

The rôle of univalent functions in complex analysis is doubtlessly important. Various problems concerning the coefficients of the seties expansion have been above all deeply studied and many remarkable results are now in our possession. Univalent function theory is still today an active field. The domain of definition of a univalent function is, by the definition, limited to a plane domain—a nonempty connected open subset of the Riemann sphere.

Although it is desirable to study analyticity on general Riemann surfaces, univalent functions make sense only on planar Riemann surfaces. The main object of the present paper shall be hence univalent analytic mapping, i.e. conformal embedding of a Riemann surface into another. This causes however a new difficulty; we cannot designate the image of the given surface under such a mapping at the outset. In other words, we have to consider the following problem at first. What kind of Riemann surfaces do we have to choose as the image? The notion of continuations of a Riemann surface serves the purpose, as the discussion below shows.

The first half of the present paper is devoted to a short summary of our basic results. In order to compare them with the classical theory of univalent functions, we confine ourselves to those topics whose counterpart can be easily found. (For other topics, see for instance, [38], [23], [39], and [40].) As an example of such topics we focus our attention to the span in the second half of the paper.

The span is first defined by M. Schiffer ([30]) for multiply connected plain domains, and has been since then generalized in many different directions (see e.g., [15], [27], [28], [29]). Hamano-Maitani-Yamaguchi [10] considered a certain holomorphic family of finite Riemann surfaces parametrized by the unit disk $\mathbb{D}$ and showed that one of the generalized spans (the harmonic span in [28]) is a subharmonic function on $\mathbb{D}$. Concerning the Schiffer span for plane domains a similar result is obtained by S. Hamano ([9]). The span with which we are concerned in this paper is, on the other hand, of different kind and presently defined only for Riemann surfaces of genus one, but shares many interesting function-theoretic properties with the Schiffer span (cf. [35], [36], [37], and [38]).

We finally state our recent results which shows that our span is a natural object in the theory of functions of several complex variables. The details will appear elsewhere as a joint work with S. Hamano and H. Yamaguchi.

2. Continuations of a Riemann Surface

We start with an open (= noncompact) Riemann surface R of genus g (which may be zero). We assume the finiteness of g but nothing about the ideal boundary of R. As a natural generalization of the univalent functions we consider the conformal embeddings of R into closed (= compact) surfaces and show that they bear much similarity. In case $g \geq 1$, however, some new and less trivial problems arise. For example, we have to comprehend the totality of the possible closed Riemann surfaces $\hat{R}$ into which R can be conformally embedded.

It is worthwhile noting here that we are concerned with an *abstract* open Riemann surface (as a one-dimensional complex analytic manifold) R. Looking for a closed Riemann surface into which R can be conformally embedded is therefore finding a realization of R (resp. its ideal boundary) as a subdomain (resp. its relative boundary) on a closed

Riemann surface.

To discuss the above problem precisely we recall a formal definition. In this paragraph let R be any Riemann surface. If there is an injective holomorphic mapping $\hat{\iota}$ of R into another Riemann surface $\hat{R}$, then the pair $(\hat{R}, \hat{\iota})$ is called a *continuation* (or prolongation, extension) of R. (cf. [29]). Specifically, $(\hat{R}, \hat{\iota})$ is called a *compact continuation* of R if $\hat{R}$ is a closed surface, and a *continuation of the same genus* if the genus of $\hat{R}$ is the same as that of R. In this article we use a shorter term "closing" instead of the long phrase "compact continuation of the same genus."

As pioneering studies on the continuations of a Riemann surface we refer to T. Radó [26] and S. Bochner [3]. The main concern of [3] was in the case $g = \infty$, but even in the case of finite genus the concept of continuations is very useful as we see in the following.

Any univalent function on a plane domain D defines together with its natural embedding into the Riemann sphere $\hat{\mathbb{C}}$ a closing of D. Although $\hat{\mathbb{C}}$ is only the closed Riemann surface of genus zero, there are infinitely many closings of D. This can be easily seen if D is simply connected, as the Riemann mapping theorem shows.

Among such various closings of a plane domain D there are distinguished ones. Indeed, Koebe's generalized uniformization says D can be conformally mapped onto the so-called extremal horizontal (resp. vertical) slit plane. These mappings yield typical closings of D, and they do not coincide in general. Cf. [5], [8], and [29]. Z. Nehari [25] and M. Mori [24] intended to generalize this classical theorem to open Riemann surfaces of finite genus. However, they were mainly interested in the values of the mapping functions, and hence did not need the notion of closings at all. Their results will be interpreted in the light of "closings."

3. The Embedding Theorem

Let R be a Riemann surface of finite genus g as before. We call a single-valued meromorphic function f on R an S_0-function, if it satisfies any one of the following conditions.

[C] df is a canonical semiexact differential on R in the sense of Kusunoki ([18]).

[D] $\operatorname{Im} df$ is a distinguished harmonic differential on R in the sense of Ahlfors ([1]).

[P] $\operatorname{Im} f$ is a $(Q)L_1$-function on R in the sense of Sario ([29]).

Furthermore, for real $t \in (-1, 1]$, we say that a meromorphic function f is an S_t-function on R if $\operatorname{Im}(e^{-\frac{\pi}{2}it} f)$ is an S_0-function. Cf. [34].

Remark 3.1. The name "S_t-functions" is after their hydrodynamic property. They describe ideal fluid flows on R such that the ideal boundary of R is recognized as impenetrable. They are called *Strömungsfunktionen* in the classical texts (e.g., [16]). Note that S_t-functions and Strömungsfunktionen are not the so-called stream functions. The name "stream function" is usually used to indicate the imaginary part of an S_t-function, so that it is not analytic but harmonic on R. Cf. also [34].

We note that any one of the conditions [C], [D] and [P] implies the existence of a compact set K on R such that the Dirichlet integral

$$\iint_{R \setminus K} |f'(z)|^2 \, dx dy$$

is finite, where $z = x + iy$ stands for a generic local parameter on R. Hence we know that an S_t-function f has only a finite number of poles on R.

The present paper is based on the following embedding theorem. For the proof see [34] together with [41].

Theorem 3.2. *Let $t \in (-1, 1]$. For any nonconstant S_t-function f on R there exist a closed Riemann surface $\hat{R}$ of genus g, a conformal embedding $\hat{\iota} : R \to \hat{R}$, and a meromorphic function $\hat{f}$ on $\hat{R}$ such that*

(1) *the two-dimensional Lebesgue measure of $\hat{R} \setminus \hat{\iota}(R)$ vanishes,*

(2) *each component of $\hat{R} \setminus \hat{\iota}(R)$ consists of a finite number of C^1-arcs which have finitely many points in common,*
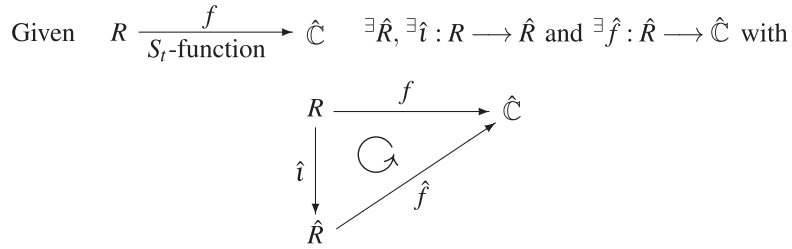
(3) *$f = \hat{f} \circ \hat{\iota}$ on R,*

(4) *$\hat{f}$ is holomorphic on $\hat{R} \setminus \hat{\iota}(R)$, and*

(5) *$\operatorname{Im}(e^{-\frac{\pi}{2}it} f) = \mathrm{const.}$ on each boundary component of $\hat{R} \setminus \hat{\iota}(R)$.*

(See the diagram below.)

According to our terminology $(\hat{R}, \hat{\iota})$ is a closing of R. The closing obtained in this way depends not only on t but also f. In fact, both of $\hat{R}$ and $\hat{\iota}$ are not always uniquely determined, even if t and f are fixed.

Given $\quad R \xrightarrow[S_t\text{-function}]{f} \hat{\mathbb{C}} \quad \exists \hat{R}, \exists \hat{\imath} : R \longrightarrow \hat{R}$ and $\exists \hat{f} : \hat{R} \longrightarrow \hat{\mathbb{C}}$ with

$$
\begin{array}{ccc}
R & \xrightarrow{\ f\ } & \hat{\mathbb{C}} \\
{\scriptstyle \hat{\imath}} \downarrow & \circlearrowright \nearrow_{\hat{f}} & \\
\hat{R} & &
\end{array}
$$

Remark 3.3. In condition (2) we do not exclude the case where a component reduces to a point.

Remark 3.4. Conditions (2), (4) and (5) in the above theorem can be described in a more intuitive manner if we use the hydrodynamical terminology. The given function f on R describes an ideal fluid flow, where each ideal boundary component is impenetrable. Meanwhile, condition (4) states that the function $\hat{f}$ constructed in the theorem describes a flow on the closed Riemann surface $\hat{R}$, and condition (2) together with condition (5) means that $\hat{R} \setminus \hat{\imath}(R)$ consists of a set of stream arcs of $\hat{f}$. Therefore, all except for finitely many boundary components are simple C^1-arcs. An exceptional ideal boundary component is realized on $\hat{R}$ as a connected closed set on streamlines with stagnation points on it. It is an interesting problem which ideal boundary component is exceptional. We discuss the problem in another paper (a joint work with H. Yamaguchi).

It turns out that the proof of this theorem can be carried out for multi-valued S_t-functions too, if we change the expression a little. For later use we state the general version in terms of differentials.

Theorem 3.5. *Let $t \in (-1, 1]$ and φ be a nontrivial meromorphic differential on R such that $\mathrm{Im}(e^{-\frac{\pi}{2}it}\varphi)$ satisfies any one of the condition [C], [D] or [P]. Then there exists a closed Riemann surface $\hat{R}$ of genus g, a conformal embedding $\hat{\imath} : R \to \hat{R}$ and a meromorphic differential $\hat{\varphi}$ on $\hat{R}$ such that*
(1) *the two-dimensional Lebesgue measure of $\hat{R} \setminus \hat{\imath}(R)$ vanishes,*
(2) *each component of $\hat{R} \setminus \hat{\imath}(R)$ consists of a finite number of C^1-arcs which have finitely many points in common,*
(3) *$\varphi = \hat{\imath}^*(\hat{\varphi})$ on R, where $\hat{\imath}^*$ stands for the pull-back by $\hat{\imath}$,*
(4) *$\hat{\varphi}$ is holomorphic on $\hat{R} \setminus \hat{\imath}(R)$. and*
(5) *$\mathrm{Im}(e^{-\frac{\pi}{2}it}\hat{\varphi}) = 0$ along each component of $\hat{R} \setminus \hat{\imath}(R)$.*

Remark 3.3 and Remark 3.4 are available for conditions (2), (4), and (5) in Theorem 3.5. To state condition (5) more precisely, let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be the C^1-arcs in condition (2). Then condition (5) above means that $\mathrm{Im}(e^{-\frac{\pi}{2}it}\hat{\varphi})$ vanishes along each γ_n, $1 \le n \le k$.

The closings of R obtained here might be considered only as examples, but this is not the case. Indeed, if the genus g is zero or one, this kind of closings have various remarkable extremal properties and are essentially used to obtain the precise geometric description of the totality of the closings of R, as we see in the next section.

4. Univalent Functions as Closings

4.1 Some elementary observations

We review in the present section an elementary but suggestive example to observe what a univalent function means as a closing of its domain of definition. Let $\mathbb{D}$ be the open unit disk in the z-plane and R its exterior together with point at infinity:

$$
\mathbb{D} := \{|z| < 1\} \quad \text{and} \quad R := \hat{\mathbb{C}} \setminus \bar{\mathbb{D}} = \{1 < |z| \le \infty\}.
$$

The Joukowski transformation

$$
J : z \mapsto w = z + \frac{1}{z}
$$

is a univalent meromorphic function on the Riemann surface R. As is well known J maps the Riemann surface R onto the slit domain in the w-sphere

$$
\Sigma_0 := \hat{\mathbb{C}}_w \setminus s_0,
$$

where s_0 stands for the segment

$$
\{w \in \hat{\mathbb{C}} \mid -2 \le \mathrm{Re}\, w \le 2, \mathrm{Im}\, w = 0\}
$$

on the real axis.

Usually the (ideal) boundary ∂R of R is recognized as the unit circle $C := \{|z| = 1\}$. It corresponds (in a one-to-one manner) to the doubly traced segment s_0.

We forget $\mathbb{D}$ for a while and consider J only on R. It can be shown that J is an S_0-function on R and therefore by our

Theorem there exists a closing $(\hat{R}, \hat{\iota})$ of R by $w = f(z) = J(z)$. Indeed, we have only to set

$$\hat{R} := \hat{\mathbb{C}}_w, \quad \hat{\iota} := J, \quad \text{and} \quad \hat{f}(w) := w.$$

In standard courses of complex analysis one carefully notes that J is not conformal on the set ∂R, since its derivative vanishes at $z = \pm 1 \in \partial R$. This is true if we use the unit circle as the naïve conformal structure of ∂R. Since R consists of interior points only, we can also use another conformal structure of ∂R. If we conformally saw the both sides of slit in a natural fashion and eliminate the segment s_0 from Σ_0 to obtain the Riemann sphere $\hat{\mathbb{C}}_w$, we have another conformal structure on the ideal boundary, by which J (actually $\hat{J}$) is holomorphic on the ideal boundary too. This is the idea of our Theorem, which is not new but goes back to the Riemann mapping theorem. It concerns only the interior of the domain of definition and pays no attention to the (conformal structure of the) boundary.

In the above observation the univalence of J seems to work substantially. However, our theorem is concerned with a general S_0-function $f : R \to \hat{\mathbb{C}}_w$ too and hence we can construct a closing of R by f. Of course, the situation becomes less trivial. For example, the function

$$h(z) := \{J(z)\}^2 - 2, \quad z \in R$$

is also an S_0-function on R. The function h on R can be continuously extended onto the circle C, by which we understand the set-theoretical boundary of R. (It is of importance to forget the usual conformal structure on C!) The extended function will be denoted by the same letter h. It can be shown that the image of R under h is a two-sheeted covering surface $\hat{R}$ of $\hat{\mathbb{C}}_w$ since h has a single pole of order two at $\infty \in R$. The image of C under h is s_0 as before, but each side of the segment is now traced twice when z moves along C once. By identifying points on C appropriately we obtain a closed Riemann surface of genus zero onto which h is holomorphically extended. To illustrate the nonuniqueness of the identification procedure, we take a real number a with $|a| < 2$ and restart with the two-sheeted covering Riemann surface T_a over $\hat{\mathbb{C}}_w$ which has two branch points over $w = a$ and $w = \infty$. If we remove the set of all points on T_a that lie over the segment s_0, we have a subsurface R_a of T_a. Obviously R_a does not depend on a and actually they are the R. Consequently, there are infinitely many, apparently different, closings of R.

Although T_a $(|a| < 2)$ look like distinct, they are essentially the same; every T_a is, because of its planarity, conformally equivalent to the Riemann sphere $\hat{\mathbb{C}}_w$. The h is not a univalent function on R, but we can regard it as a conformal embedding of R into any closed Riemann surface T_a, $|a| < 2$. The image surface $h(R)$ is uniquely determined, independently of the choice of a. Note that the (covering) Riemann surface T_a is prepared for establishing a one-to-one correspondence between the domain and the image of h. That is, the nonunivalent function h on R can be regarded as a univalent mapping of R into T_a.

Remark 4.1.   The nonuniqueness of the closings can be more clearly understood, if we consider two-sheeted covering surfaces with two fixed branch points other than a and ∞. Such surfaces are of genus one and have different moduli. However, the surface cut along a fixed segment near a as above is always the same.

Remark 4.2.   The cubic of the Joukowski transformation also provides an interesting example. It is one-to-one on the boundary circle C of R but is not univalent in any neighborhood of the boundary. Cf. [17]. In this example also, we have no uniqueness assertion of closings.

4.2   The closings of a plane domain by univalent functions

Here we recall a classical result in the conformal mapping theory. To fix the notation let D be a domain on the Riemann sphere $\hat{\mathbb{C}}_z$ which contains the point at infinity. (We impose no restrictions on the ideal boundary as before.) Let $\mathcal{F} = \mathcal{F}(D, \infty)$ denote the class of functions f which are univalent meromorphic on D and is written as

$$f(z) = z + \frac{\kappa_f}{z} + o(1/z) \text{ about } z = \infty.$$

The theorem below is called "the fundamental theorem in the theory of conformal mapping" or "Koebe's generalized uniformization theorem." Many proofs are known today. Some of the typical proofs can be found, for example, in [5], [8], [14], [19], and [29].

Theorem 4.3.   *There exists a domain Σ_0^D on $\hat{\mathbb{C}}_w$ containing the point at infinity and a conformal mapping f_0 of D onto Σ_0^D such that*
(1) *$\hat{\mathbb{C}}_w \setminus \Sigma_0^D$ is a compact set of measure zero,*
(2) *each component of $\hat{\mathbb{C}}_w \setminus \Sigma_0^D$ is a horizontal segment (or a point), and*
(3) *f_0 is a unique element in $\mathcal{F}$ that maximizes $\operatorname{Re} \kappa_f$.*

Similarly, there uniquely exists an element f_1 of $\mathcal{F}$ which minimizes $\operatorname{Re} \kappa_f$. It maps D onto a domain Σ_1^D whose complement in $\hat{\mathbb{C}}_w$ is a compact set of measure zero and consists of vertical segments (or points). The functions f_0 and f_1 are known as the extremal horizontal resp. vertical parallel slit mappings. The extremal parallel slit mappings for an arbitrary inclination can be analogously constructed.

These classical results can be more generally reformulated in:

Theorem 4.4.   *Let R be an open Riemann surface of genus zero. Fix a point p on R and a local parameter z about p with $z(p) = 0$. Let $\mathcal{F} = \mathcal{F}(R, p)$ be the class of conformal embedding f of R into $\hat{\mathbb{C}}$ and is written as*

$$f(z) = \frac{1}{z} + \kappa_f z + o(z) \text{ about } p.$$

Then, there exists a complex number κ^* and a nonnegative number ρ_0 such that the circle

$$\{\kappa \in \mathbb{C} \mid \kappa = \kappa^* + \rho_0 e^{\pi i t}, -1 < t \leq 1\}$$

parametrizes the extremal parallel slit mappings of R. Furthermore, the set of coefficients

$$\mathfrak{K} = \mathfrak{K}(R, p) = \{\kappa \in \mathbb{C} \mid \kappa = \kappa_f, f \in \mathcal{F}(R, p)\}$$

is exactly the (possibly degenerated) closed disk

$$\{|\kappa - \kappa^*| \leq \rho_0\}.$$

The disk $\mathfrak{K}$ has many interesting properties (see [5], [8], [14], for example). We mention here, among other things, only the following. M. Schiffer referred to the diameter $\sigma_0 = \sigma_0(R, p) := 2\rho_0$ of $\mathfrak{K}$ as the span of R (more correctly: the span of (R, p)). In a forthcoming paper [9] Hamano studied, along the line of H. Yamaguchi (cf. [10]), the dependence of the span on R from the viewpoint of several complex variables. To state her result more precisely, let $\{R(s)\}_{s \in \mathbb{D}}$ be a family of plane domains such that the boundary of each $R(s)$ consists of a finite number of analytic Jordan curves. If the total space

$$\{(s, p) \mid s \in \mathbb{D}, p \in R(s)\}$$

is a two-dimensional pseudoconvex domain, then the span is a subharmonic function on the parameter domain $\mathbb{D}$.

5.   The Closings of an Open Torus

In this section let R be an open Riemann surface of genus one, unless otherwise stated. Recalling that a closed Riemann surface of genus one is usually called "a torus," we use "an open torus" to mean an open Riemann surface of genus one. In accordance with the classical usage we do not need the term "a closed torus." We impose no particular conditions on the ideal boundary of R; the cardinality of the Kerékjártó-Stoïlow boundary components of R may be uncountable.

To comprehend the family of closings of R quantitatively, it is necessary to consider them in the Torelli space. Namely, we fix once and for all a canonical homology basis $\chi = \{A, B\}$ of R modulo dividing cycles. The pair (R, χ) is referred to as a homologically marked open torus.

Now we use χ to refine the notion of closings. A closing of (R, χ) is, by definition, is a torus $\hat{R}$ together with a canonical homology basis $\hat{\chi} = \{\hat{A}, \hat{B}\}$ and a conformal injection $\hat{\iota} : R \to \hat{R}$ such that $\hat{\iota}(A)$ and $\hat{\iota}(B)$ are homologous to $\hat{A}$ and $\hat{B}$ respectively. A closing of (R, χ) may be thus written as $((\hat{R}, \hat{\chi}), \hat{\iota})$, but we prefer a simpler notation $[\hat{R}, \hat{\chi}, \hat{\iota}]$. Furthermore, two such triples $[\hat{R}, \hat{\chi}, \hat{\iota}]$ and $[\tilde{R}, \tilde{\chi}, \tilde{\iota}]$ shall be identified if there exists a conformal bijection $j : \tilde{R} \to \hat{R}$ with $j \circ \tilde{\iota} = \hat{\iota}$ (cf. the diagram below).

$$(R, \chi) \quad \xrightarrow{\tilde{\iota}} \quad (\tilde{R}, \tilde{\chi}) \qquad\qquad [\tilde{R}, \tilde{\chi}, \tilde{\iota}] \sim [\hat{R}, \hat{\chi}, \hat{\iota}]$$
$$\curvearrowright \qquad \downarrow j \qquad\qquad\qquad \Updownarrow \text{def.}$$
$$\xrightarrow{\hat{\iota}} \quad (\hat{R}, \hat{\chi}) \qquad {}^{\exists} j : (\tilde{R}, \tilde{\chi}) \to (\hat{R}, \hat{\chi}), \text{ conf.surj. with } j \circ \tilde{\iota} = \hat{\iota}$$

Hereafter "a closing" means an equivalence class under the relation $\sim$, but we continue to use the same notation $[\hat{R}, \hat{\chi}, \hat{\iota}]$.

The marked torus $(\hat{R}, \hat{\chi})$ carries a unique holomorphic differential $\hat{\phi}$ whose $\hat{A}$-period is normalized to be 1. The $\hat{B}$-period

$$\tau(\hat{R}, \hat{\chi}) := \int_{\hat{B}} \hat{\phi}$$

of $\hat{\phi}$ is known as the modulus of the marked torus $(\hat{R}, \hat{\chi})$. The length and area on $(\hat{R}, \hat{\chi})$ are measured by the flat metric $|\hat{\phi}|$ that is induced by $\hat{\phi}$. We may and do assume that the cycles $\hat{A}$ and $\hat{B}$ are both geodesic curves. The marked torus $(\hat{R}, \hat{\chi})$ is realized as the quotient space

$$\mathbb{C}/\Pi, \quad \Pi := \{m + n\tau \mid m, n \in \mathbb{Z}\}.$$

The modulus of a closing $[\hat{R}, \hat{\chi}, \hat{\iota}]$ of (R, χ) is defined to be $\tau(\hat{R}, \hat{\chi})$ and denoted by $\tau[\hat{R}, \hat{\chi}, \hat{\iota}]$.

Denote by $\mathcal{C} = \mathcal{C}(R, \chi)$ the family of closings $[\hat{R}, \hat{\chi}, \hat{\iota}]$ of (R, χ) and consider the set

$$\mathfrak{M} = \mathfrak{M}(R, \chi) := \{\tau \in \mathbb{C} \mid \tau = \tau[\hat{R}, \hat{\chi}, \hat{\iota}], [\hat{R}, \hat{\chi}, \hat{\iota}] \in \mathcal{C}(R, \chi)\}.$$

We simply call $\mathfrak{M}(R, \chi)$ the moduli set of the closings of the marked open torus (R, χ). It is obvious that $\mathfrak{M}(R, \chi)$ is included in the upper halfplane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}$, since

$$\text{Im } \tau(\hat{R}, \hat{\chi}) > 0$$

for all $\tau(\hat{R}, \hat{\chi})$.

In [35] and [36] we have proved the following theorem.

Theorem 5.1. *The moduli set $\mathfrak{M}(R, \chi)$ of the closings of (R, χ) is a closed disk in $\mathbb{H}$:*

$$\mathfrak{M}(R, \chi) = \{\tau \in \mathbb{H} \mid |\tau - \tau^*| \le \rho_1\}$$

where $\tau^* \in \mathbb{H}$ and ρ_1 is a real number with $0 \le \rho_1 < \text{Im } \tau^*$. The disk reduces to a single point if and only if R is of class O_{AD}. ($R \in O_{AD}$ means that R carries no nonconstant analytic functions with a finite Dirichlet integral. Cf. [1] or [28].)

To each boundary point

$$\tau_t := \tau^* + \rho_1 \, e^{\pi(t-1)i/2}, \quad -1 < t \le 1$$

of $\mathfrak{M}(R, \chi)$ there corresponds a unique closing $[\hat{R}_t, \hat{\chi}_t, \hat{\iota}_t]$ of (R, χ). The set $\hat{R}_t \setminus \hat{\iota}_t(R)$ has a vanishing area and consists of straight arcs which are parallel to the geodesic on $(\hat{R}, \hat{\chi})$ with inclination $\pi t/2$.

On the other hand, for any interior point τ of the moduli disk there are uncountably many closings $[\hat{R}, \hat{\chi}, \hat{\iota}]$ of (R, χ) with $\tau = \tau[\hat{R}, \hat{\chi}, \hat{\iota}]$; while the marked torus $(\hat{R}, \hat{\chi})$ is unique, there exist uncountably many different embeddings $\hat{\iota} : (R, \chi) \to (\hat{R}, \hat{\chi})$.

Remark 5.2.   The classical contributions by H. Grötzsch ([6], [7]) and M. Heins ([11]) should be here appreciated. Their results can be also viewed as the study of the set of closings of an open torus. They are concerned with the cross ratio of four branch points of the two-sheeted covering of $\hat{\mathbb{C}}$ and the algebraic moduli respectively — instead of the (transcendental) moduli of the closings in our study.

Remark 5.3.   Some of the topics above can be similarly discussed also for $g > 1$. In fact, we can define the set $C(R, \chi)$ analogously for any homologically marked open Riemann surface (R, χ) of finite genus. The moduli set $\mathfrak{M}(R, \chi)$ shall be now replaced with the set $\mathfrak{R}(R, \chi)$ of Riemann period matrices (cf. [43], for instance). We have observed the diagonal of the Riemann period matrices in [36] and [37] to show the relationship between the degeneracy of the Riemann surface (R, χ) and the uniqueness of its closing. This yields a new generalized span for higher genera, which we do not discuss in the present paper, however. The set $\mathfrak{R}(R, \chi)$ is studied also in [31], [32], and [33].

Remark 5.4.   We note that a torus is alternatively represented by an algebraic equation. Based on this fact [12] studied the conformal welding of a slit torus and a slit sphere along the slits. A special case of such welding is the Rankine ovoid in hydrodynamics, which is discussed in [13]. Cf. [44], too.

Remark 5.5.   Even in the case $g = 0$ our observation afford insight into the slit mappings. One of such examples can be found in [40], where another proof of a theorem of Maitani ([20]) is given.

6.   The Span of a Marked Open Torus

The preceding theorem gives a natural generalization of the various classical results. For instance, the particular cases $t = 0$ and $t = 1$ correspond to Koebe's extremal horizontal and vertical slit mappings, respectively. Among other things are the similarity to the results due to de Possel, Grunsky, and Schiffer (cf. [4], [8], [30]). Namely, the moduli disk $\mathfrak{M}(R, \chi)$ plays a rôle similar to the coefficients disk $\mathfrak{K}(R, p)$ in Sect. 4. To see the theorem we first recall that each $[\hat{R}, \hat{\chi}, \hat{\iota}] \in C(R, \chi)$ carries a canonical metric induced by the holomorphic differential $\hat{\varphi}$ on $\hat{R}$ whose $\hat{A}$-period is normalized to be one, $\hat{\chi} = \{\hat{A}, \hat{B}\}$. For example, the total area of a closing $[\hat{R}, \hat{\chi}, \hat{\iota}] \in C(R, \chi)$ is given by

$$\alpha[\hat{R}, \hat{\chi}, \hat{\iota}] := \iint_{\hat{R}} |\hat{\varphi}|^2,$$

which is known to be equal to $\text{Im } \tau[\hat{R}, \hat{\chi}, \hat{\iota}]$.

We now define the complementary area of the closing $[\hat{R}, \hat{\chi}, \hat{\iota}] \in C(R, \chi)$ as

$$\alpha_1[\hat{R}, \hat{\chi}, \hat{\iota}] := \iint_{\hat{R} \setminus \hat{\iota}(R)} |\hat{\varphi}|^2 = \iint_{\hat{R}} |\hat{\varphi}|^2 - \iint_{R} |\iota^*(\varphi)|^2,$$

and furthermore we set for $\tau \in \mathfrak{M}(R, \chi)$

$$A_1(\tau) := \sup\{\alpha_1[\hat{R}, \hat{\chi}, \hat{\iota}] \mid \tau[\hat{R}, \hat{\chi}, \hat{\iota}] = \tau\}.$$

Then we have

Theorem 6.1. *The area function $A(\tau)$ vanishes on the boundary circle $\partial\mathfrak{M}(R, \chi)$. The maximum complementary area of the closings of (R, χ) is attained at the center τ^* of $\mathfrak{M}(R, \chi)$ and is equal to $\rho_1/2$.*

Remark 6.2. In fact, we can give the function A in a closed form:

$$A_1(\tau) = \frac{\rho_1^2 - r^2}{2\rho_1},$$

where $r := |\tau - \tau^*|$. It can be also shown that the range of $\alpha_1[\hat{R}, \hat{\chi}, \hat{\iota}]$ for $[\hat{R}, \hat{\chi}, \hat{\iota}] \in \mathcal{C}(R, \chi)$ with $\tau = \tau[\hat{R}, \hat{\chi}, \hat{\iota}]$ fixed is exactly the closed interval $[0, A_1(\tau)]$. This result for $g = 1$ suggests a counterpart for $g = 0$. See, for the details, [38].

While the above theorem has a well known prototype in the classical theory of univalent functions, the theorem below shows another feature of the moduli disk. Before stating the theorem we recall that $\mathbb{H}$ carries a hyperbolic metric and any (euclidean) disk in $\mathbb{H}$ can be also regarded as a hyperbolic disk.

Theorem 6.3. *The area ratio*

$$\alpha_1/\alpha = \left(\iint_{\hat{R} \setminus \hat{\iota}(R)} |\hat{\varphi}|^2 \right) \bigg/ \left(\iint_{\hat{R}} |\hat{\varphi}|^2 \right)$$

is maximized at the hyperbolic center of the (hyperbolic) moduli disk. The maximum area ratio is equal to $\tanh(\rho_1/2)$.

Remark 6.4. The area ratio above does not depend on the choice of χ.

On the analogy of the Schiffer span we call

$$\sigma_1 = \sigma_1(R, \chi) := \operatorname{Im}(\tau_1 - \tau_0) = 2\rho_1$$

the span of the marked open torus (R, χ). It is generally not easy to know exactly the span of an open torus. But if it has only one ideal boundary component, we have estimates of its span. See, e.g., [2], [13], [21], and [42]. Cf. also [22].

We are now interested in the problem how the set of closings changes when (R, χ) varies. In case of $g = 1$, this problem can be formulated in terms of the span. Corresponding to the results of Hamano ([9] for $g = 0$, we can discuss the subharmonicity of $\sigma_1(R, \chi)$ when the open torus (R, χ) varies holomorphically. To be more precise, we confine ourselves to those open tori which are the interior of compact bordered Riemann surfaces of genus one. Let

$$\mathbb{D} \ni s \to (R(s), \chi(s))$$

be a smooth family of marked open tori $(R(s), \chi(s))$, where $R(s)$ is the interior of a compact bordered Riemann surface of genus one and $\chi(s)$ a canonical homology basis of $R(s)$ modulo dividing cycles. We consider the span $\sigma_1(R(s), \chi(s))$ as the function of $s \in \mathbb{D}$:

$$\sigma_1(s) := \sigma_1(R(s), \chi(s)).$$

One of our recent results then reads:

Theorem 6.5. *If*

$$\{(s, p) \mid s \in \mathbb{D}, p \in R(s)\}$$

is a two-dimensional pseudo-convex domain, then the span $\sigma_1(s)$ is a subharmonic function on $\mathbb{D}$. Furthermore, if the span is a harmonic function on $\mathbb{D}$ in particular, it is constant on $\mathbb{D}$.

The proof is based on a variational formula for the Laplacian of $\rho_1(s)$, $s \in \mathbb{D}$. The details will appear elsewhere as a joint work with S. Hamano and H. Yamaguchi.

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Note added in proof—The present article was, in accordance with the conference theme, mainly concerned with planar Riemann surfaces and open tori, although a number of results had been known for surfaces of higher genera either (See [31], [32], [33], [36], and [37]). Further generalizations obtained since 2014 will appear elsewhere.

The preannounced paper referred to in Introduction and Section 6 is: Hamano, S., Shiba, M., and Yamaguchi, H., "Hyperbolic span and pseudoconvexity," *Kyoto J. Math.*, **57**: 165–183 (2017), where the hyperbolic span of an open torus introduced in [38] played an important rôle.