Interdisciplinary Information Sciences Vol. 25, No. 1 (2019) 33–44

ISSN 1340-9050 print/1347-6157 online
DOI 10.4036/iis.2019.A.03

Taylor Coefficients of the Conformal Map for the Exterior of the Reciprocal of the Multibrot Set

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In this paper we investigate normalized conformal mappings of the exterior of the reciprocal of the Multibrot set and analyze the growth of the denominator of the coefficients. Our inequality improves Ewing and Schober's result which was presented in [6]. We use the coefficient formula of [13]. The straightforward adaptation of the proof in this paper slightly improves the main theorem of [12].

KEYWORDS: Mandelbrot set, Multibrot set, conformal mapping, complex dynamics

1. Introduction

Let $\mathbb{Z}$ be the set of all integers, $\mathbb{N}_0$ the set of all non-negative integers, $\mathbb{N}$ the set of positive integers, $\mathbb{C}$ the complex plane, $\mathbb{D}$ the open unit disk, $\overline{\mathbb{D}}$ the closed unit disk and $P : \mathbb{C} \to \mathbb{C}$ a non-linear polynomial. Let p be a prime number and x any non-zero rational number. Then there exists a unique integer v such that $x = p^v r/q$ with integers r and q that are not divisible by p. We define the p-adic valuation $\nu_p : \mathbb{Q} \to \mathbb{Z}$ as follows:

$$\nu_p(x) = \begin{cases} v & \text{for } x \in \mathbb{Q} \setminus \{0\}, \\ +\infty & \text{for } x = 0. \end{cases}$$

For any real number x, we set the floor function $\lfloor x \rfloor := \max\{m \in \mathbb{Z} : m \leq x\}$.

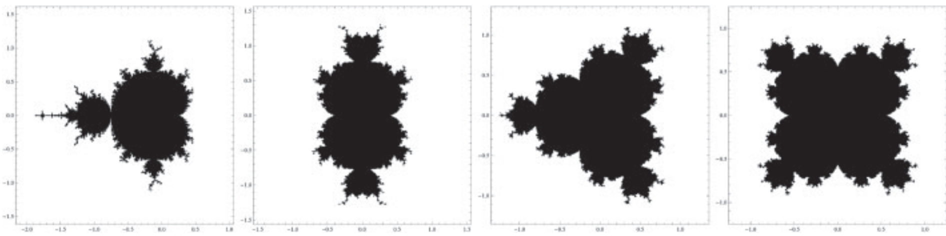

Fig. 1. $\mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5$.

The n-th iteration of the polynomial $P^{\circ n}$ is defined inductively by $P^{\circ(n+1)} = P \circ P^{\circ n}$ with $P^{\circ 0}(z) = z$. The *Multibrot set* $\mathcal{M}_d$ of degree d is the set of all parameters $c \in \mathbb{C}$ for which the *Julia set* of $P_{d,c}(z) := z^d + c$ is connected, where $d \geq 2$ is a fixed integer (see Fig. 1 for $d = 2, 3, 4, 5$). It is known that $\mathcal{M}_d$ is compact and connected. There is an important conjecture which states that *the Mandelbrot set* $\mathcal{M}_2$ is locally connected. This would imply the density conjecture of hyperbolic dynamics in the quadratic family (see [2]). Douady and Hubbard demonstrated connectedness of $\mathcal{M}_2$ by constructing a conformal isomorphism $\Phi : \widehat{\mathbb{C}} \setminus \mathcal{M}_2 \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ that satisfies $\lim_{z\to\infty} \Phi(z)/z = 1$. If the inverse map $\Phi^{-1}(z) =: \Psi(z)$ extends continuously to the unit circle, then $\mathcal{M}_2$ is locally connected, according to Carathéodory's continuity theorem. It should be noted that the convergence of the Laurent series $\Phi^{-1}(z) = z + \sum_{m=0}^{\infty} b_m z^{-m}$ on the unit circle implies that $\mathcal{M}_2$ is locally connected. Jungreis presented a method to compute the coefficients b_m of $\Psi(z)$ in [8]. Several detailed studies of b_m are given in [1, 4, 5, 10]. An analysis of the dynamics of $P_{d,c}(z) := z^d + c$ with an integer $d \geq 2$ is presented in [15]. Constructing the canonical map $\Psi_d(z) = z + \sum_{m=0}^{\infty} b_{d,m} z^{-m}$ of $\widehat{\mathbb{C}} \setminus \mathcal{M}_d$, Yamashita [15] analyzed the coefficients $b_{d,m}$. The function $f_d(z) := 1/\Psi_d(1/z)$ is the normalized conformal map f_d from $\mathbb{D}$ to $\mathcal{R}_d := \mathbb{C} \setminus \{1/z : z \in \mathcal{M}_d\}$ which fixes the origin and satisfies $f_d'(0) > 0$. If f_d has a continuous extension to the boundary, the Multibrot set is locally connected. The conformal map $f_d(z) : \mathbb{D} \to \mathcal{R}_d$ has the Taylor expansion of the form $f_d(z) = z + a_{d,1}z + a_{d,2}z^2 + a_{d,3}z^3 + \cdots$ (see [6, 13]). Ewing and Schober studied the coefficients $a_{2,m}$ in [6]. They

Received September 2, 2014; Accepted September 29, 2017; J-STAGE Advance published June 7, 2019
2019 Mathematics Subject Classification: Primary 37F45, Secondary 30D05.
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gave some coefficients formulas and estimate for the growth of denominator, and presented that infinitely many coefficients are 0. Also infinitely many non-zero terms are determined. In [13, 14], we investigated properties of the coefficients $a_{d,m}$ of the conformal map $f_d(z) = z + \sum_{m=2}^{\infty} a_{d,m} z^m$ of the exterior of $\mathcal{R}_d := \{1/z : z \in \mathcal{M}_d\}$. Some values of $a_{d,m}$ are presented in Tables 1, 2, 3, 4 for $d = 2, 3, 4, 6$ respectively.

In particular, $a_{d,m}$ is *d-adic rational*, i.e. there exist $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$ such that $a_{d,m} = k/d^n$ (see [13, Lemma 4]). Ewing–Schober [6] studied the coefficients of f_2 and obtained an inequality for the growth of the denominator, which can be presented in the following way.

Theorem 1.1 ([6, Theorem 2]). *For $m \geq 2$, the inequality*

$$-\nu_2(a_{2,m}) \leq 2m - 3 \tag{1.1}$$

holds. Equality holds if $m = 2^k + 1$ for some $k \in \mathbb{N}$.

In this note, we improve this inequality in a general way. In particular, our result is as follows when $d = 2$.

Corollary 1.2 (Corollary of the main theorem). *For all $m \geq 2$, the inequality*

$$-\nu_2(a_{2,m}) \leq \nu_2((2m-2)!) \tag{1.2}$$

holds. Equality holds if and only if m is odd. Furthermore let n be a positive integer so that $2^n \leq m \leq 2^{n+1} - 1$. If m is even and $m > 2^{n+2-\nu_2(m)} - 2n + 3\nu_2(m) - 3$,

$$-\nu_2(a_{2,m}) \leq \nu_2(m) + \nu_2((m-1)!) + \sum_{k=1}^{n+1-\nu_2(m)} 2^k (n - k + 1 - \nu_2(m)) \tag{1.3}$$

holds.

Remark 1.3. We note that the inequality

$$\nu_2((2m-2)!) \leq 2m - 3$$

holds for all m. Further if m even and $m > 2^{n+2-\nu_2(m)} - 2n + 3\nu_2(m) - 3$, then

$$\nu_2(m) + \nu_2((m-1)!) + \sum_{k=1}^{n+1-\nu_2(m)} d^k (n - k + 1 - \nu_2(m)) < \nu_2((2m-2)!)$$

(see the proof of the main theorem for details and Tables 5, 6, 7).

Remark 1.4. The condition m is even and $m > 2^{n+2-\nu_2(m)} - 2n + 3\nu_2(m) - 3$ is not empty. At least if $n \geq 2\nu_2(m)$ and $\nu_2(m) > 1$, then

$$\begin{aligned} & 2^{n+2-\nu_2(m)} - 2n + 3\nu_2(m) - 3 \\ & = 1 - \nu_2(m) + (2\nu_2(m) - n)2 + 2^2 + 2^3 + \cdots + 2^{n+1-\nu_2(m)} \\ & < 1 + 2 + 2^2 + \cdots + 2^{n+1-\nu_2(m)} \\ & \leq 1 + 2 + 2^2 + \cdots + 2^{n-1} \\ & < m \end{aligned}$$

since $2^n \leq m \leq 2^{n+1} - 1$.

2. Preliminaries

We will use the following lemmas.

Lemma 2.1 ([7, pp. 69–72, and pp. 102–114]). *Let $x, y \in \mathbb{R}$ and p be a prime number. The following relations are valid:*

$$\lfloor x \rfloor + m = \lfloor x + m \rfloor \text{ for } m \in \mathbb{Z}. \tag{2.1}$$

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor. \tag{2.2}$$

$$\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor \text{ for } m \in \mathbb{N}. \tag{2.3}$$

$$\nu_p(mn) = \nu_p(m) + \nu_p(n) \text{ for } m, n \in \mathbb{Z}. \tag{2.4}$$

$$\nu_p(m/n) = \nu_p(m) - \nu_p(n) \text{ for } m \in \mathbb{Z},\ n \in \mathbb{Z} \setminus \{0\}. \tag{2.5}$$

$$\nu_p(m+n) \geq \min\{\nu_p(m), \nu_p(n)\} \text{ for } m, n \in \mathbb{Z}. \tag{2.6}$$

$$\nu_p(m!) = \sum_{l=1}^{\infty} \left\lfloor \frac{m}{p^l} \right\rfloor \text{ for } m \in \mathbb{N}_0. \tag{2.7}$$

Further the following Lemmas will be used in the proof.

Lemma 2.2 ([13, Theorem 6]). *For $n \in \mathbb{N}$ and fixed $d \in \mathbb{N} \setminus \{1\}$, let $2 \le m \le d^{n+1} - 1$ and r sufficiently large. Then*

$$-ma_{d,m} = \frac{1}{2\pi i} \int_{|w|=r} P_{d,w}^{\circ n}(w)^{m/d^n} \frac{\mathrm{d}w}{w^2}.$$

Lemma 2.3 ([13, Corollary 8]). *Let $n \in \mathbb{N}$ and $2 \le m \le d^{n+1} - 1$. Then*

$$ma_{d,m} = \sum C_{j_1}\left(\frac{m}{d^n}\right) C_{j_2}\left(\frac{m}{d^{n-1}} - dj_1\right) C_{j_3}\left(\frac{m}{d^{n-2}} - d^2 j_1 - dj_2\right)$$
$$\cdots C_{j_n}\left(\frac{m}{d} - d^{n-1} j_1 - d^{n-2} j_2 - \cdots - dj_{n-1}\right),$$

where the sum is over all non-negative indices $j_1, \ldots, j_n$ such that $(d^n - 1)j_1 + (d^{n-1} - 1)j_2 + (d^{n-2} - 1)j_3 + \cdots + (d-1)j_n = m - 1$.

$C_j(a)$ in Lemma 2.3 is the general binomial coefficient, i.e.

$$C_j(a) = \frac{a(a-1)(a-2)\cdots(a-j+1)}{j(j-1)(j-2)\cdots(1)}$$

for any $a \in \mathbb{R}$ and $j \in \mathbb{N}$ with $C_0(a) = 1$.

3. Results

We can now state our main result.

Theorem 3.1. *For fixed $d \in \mathbb{N} \setminus \{1\}$, let $m \in \mathbb{N} \setminus \{1\}$ with $(d-1) \mid (m-1)$. We assume that d is factorized as $d = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$ where $s \in \mathbb{N}, t_1, t_2, \ldots, t_s \in \mathbb{N}$ and $p_1, p_2, \ldots, p_s$ are distinct prime numbers. Let $A = (m-1)/(d-1)$. Then,*

$$-\nu_{p_l}(a_{d,m}) \le \nu_{p_l}(A!) + t_l A \tag{3.1}$$

holds for any $p_l \in \{p_1, p_2, \ldots, p_s\}$. Equality holds if and only if $p_l \nmid m$ and $m = d$. Furthermore let $n \in \mathbb{N}$ so that $d^n \le m \le d^{n+1} - 1$ and set $N = n + 1 - \nu_{p_l}(m)/t_l$. If $p_l \mid m$, then the inequality

$$-\nu_{p_l}(a_{d,m}) \le \nu_{p_l}(m) + \nu_{p_l}(A!) + \sum_{k=1}^{\max\{\lfloor N \rfloor, 0\}} d^k (t_l(n-k+1) - \nu_{p_l}(m)) \tag{3.2}$$

holds when either the two inequalities $n \ge \nu_{p_l}(m)/t_l$ and $m > 1 - \nu_{p_l}(m)/t_l + (2\nu_{p_l}(m)/t_l - n)d + d^2 + d^3 + \cdots + d^{\lfloor N \rfloor} + (N - \lfloor N \rfloor)d^{\lfloor N \rfloor + 1}$ are true, or $n < \nu_{p_l}(m)/t_l$ is satisfied.

Remark 3.2. If $(d-1) \nmid (m-1)$, then $a_{d,m} = 0$ by [13, Corollary 11] and hence $-\nu_{p_l}(a_{d,m}) = -\infty$ for any $p_l \in \{p_1, p_2, \ldots, p_s\}$ (the converse is not true, i.e. there are some zero-coefficients in the case $(d-1) \mid (m-1)$ (see [13, Theorem 10 and Theorem 12]).)

Remark 3.3. If $p_l \mid m$, then the inequality

$$\nu_{p_l}(m) + \nu_{p_l}(A!) + \sum_{k=1}^{\lfloor n+1-\nu_{p_l}(m)/t_l \rfloor} d^k (t_l(n-k+1) - \nu_{p_l}(m)) < \nu_{p_l}(A!) + t_l A$$

holds when the two inequalities $n \ge \nu_{p_l}(m)/t_l$ and $m > 1 - \nu_{p_l}(m)/t_l + (2\nu_{p_l}(m)/t_l - n)d + d^2 + d^3 + \cdots + d^{\lfloor N \rfloor} + (N - \lfloor N \rfloor)d^{\lfloor N \rfloor + 1}$ are true, or $n < \nu_{p_l}(m)/t_l$ is satisfied (see the proof for details).

Remark 3.4. The case $p_l \mid m$, $n \ge \nu_{p_l}(m)/t_l$ and $m > 1 - \nu_{p_l}(m)/t_l + (2\nu_{p_l}(m)/t_l - n)d + d^2 + d^3 + \cdots + d^{\lfloor N \rfloor} + (N - \lfloor N \rfloor)d^{\lfloor N \rfloor + 1}$ is not empty. At least if $n \ge 2\nu_{p_l}(m)/t_l$ and $\nu_{p_l}(m)/t_l \in \mathbb{N} \setminus \{1\}$, then

$$\begin{aligned}
&1 - \nu_{p_l}(m)/t_l + (2\nu_{p_l}(m)/t_l - n)d + d^2 + d^3 + \cdots + d^{\lfloor N \rfloor} + (N - \lfloor N \rfloor)d^{\lfloor N \rfloor + 1} \\
&= 1 - \nu_{p_l}(m)/t_l + (2\nu_{p_l}(m)/t_l - n)d + d^2 + d^3 + \cdots + d^{\lfloor N \rfloor} \\
&< 1 + d + d^2 + \cdots + d^{\lfloor N \rfloor} \\
&\le 1 + d + d^2 + \cdots + d^{n-1}
\end{aligned}$$

$$= (d^n - 1)/(d - 1)$$
$$< m$$

because $d^n \leq m \leq d^{n+1} - 1$ and $d = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$.

Using the main theorem, we obtain the following corollary for the growth of the denominator.

Corollary 3.5. *Under the same assumption of the main theorem, $a_{d,m} d^{x(m)} \in \mathbb{Z}$ holds where*

$$x(m) := \max_{1 \leq j \leq s} \left\lceil \frac{\nu_{p_j}(A) + t_j A}{t_j} \right\rceil.$$

The strategy is basically same as in the proof of [12]. However we need a more detailed calculation in the last part.

Proof of Theorem 3.1. We take $d \in \mathbb{N} \setminus \{1\}$, $m \in \mathbb{N} \setminus \{1\}$ with $(d-1)|(m-1)$. Assume that d is factorized as $d = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$ where $s \in \mathbb{N}, t_1, t_2, \ldots, t_s \in \mathbb{N}, p_1, p_2, \ldots, p_s$ are distinct prime numbers and set

$$\begin{aligned} \mathbb{J} = \{&(j_1, j_2, \ldots, j_n) \in (\mathbb{N}_0)^n : \\ &(d^n - 1) j_1 + (d^{n-1} - 1) j_2 + \cdots + (d-1) j_n = m - 1\}. \end{aligned}$$

By Lemma 2.2 if $m = d^n$ for some $n \in \mathbb{N}$, then we have

$$\begin{aligned} d^n a_{d,d^n} &= \frac{1}{2\pi i} \int_{|z|=r} P_{d,w}^{\circ n}(w)^{d^n/d^n} \frac{dw}{w^2} \\ &= \frac{1}{2\pi i} \int_{|z|=r} ((P_{d,w}^{\circ n-1}(w))^d + w) \frac{dw}{w^2} \\ &= 1. \end{aligned}$$

Hence $a_{d,d^n} = 1/d^n$. In particular $a_{d,d} = 1/d$. It means that $-\nu_{p_l}(a_{d,d}) = t_l$ holds when $m = d$.

Take $n \in \mathbb{N}$ such that $d^n \leq m \leq d^{n+1} - 1$. Let $k \in \{1, 2, \ldots, n\}$, $(j_1, j_2, \ldots, j_n) \in \mathbb{J}$ and set

$$\alpha_k = m/d^{n-k+1} - d^{k-1} j_1 - d^{k-2} j_2 - \cdots - d j_{k-1}.$$

Setting $\beta = d^{n-k+1} \alpha_k = m - d^n j_1 - d^{n-1} j_2 - \cdots - d^{n-k+2} j_{k-1}$, we obtain

$$\begin{aligned} C_{j_k}(\alpha_k) &= \frac{\alpha_k(\alpha_k - 1)(\alpha_k - 2) \cdots (\alpha_k - (j_k - 1))}{j_k!} \\ &= \frac{\beta(\beta - d^{n-k+1})(\beta - 2d^{n-k+1}) \cdots (\beta - (j_k - 1)d^{n-k+1})}{d^{j_k(n-k+1)} j_k!}. \end{aligned} \tag{3.3}$$

First we give an estimate for $-\nu_{p_l}(C_{j_k}(\alpha_k))$ for fixed $l \in \{1, 2, \ldots, s\}$ (the calculation is similar to [12]). The denominator of (3.3) satisfies the following equality:

$$\nu_{p_l}(d^{j_k(n-k+1)} j_k!) \overset{(2.4)}{=} \nu_{p_l}(j_k!) + j_k t_l (n - k + 1). \tag{3.4}$$

Furthermore we have

$$\begin{aligned} &\nu_{p_l}(\beta - (l-1)d^{n-k+1}) \\ &\quad = \nu_{p_l}(m - d^n j_1 - d^{n-1} j_2 - \cdots - d^{n-k+2} j_{k-1} - (l-1)d^{n-k+1}) \\ &\quad = \nu_{p_l}(m - d^{n-k+1}(d^{k-1} j_1 - d^{k-2} j_2 - \cdots - d j_{k-1} - (l-1))) \\ &\quad \overset{(2.6)}{\geq} \min\{\nu_{p_l}(m), t_l(n-k+1)\} \end{aligned}$$

for all $l \in \{1, 2, \ldots, j_k\}$. Hence the numerator of (3.3) satisfies the following inequality:

$$\begin{aligned} &\nu_{p_l}(\beta(\beta - d^{n-k+1})(\beta - 2d^{n-k+1}) \cdots (\beta - (j_k - 1)d^{n-k+1})) \\ &\quad \overset{(2.4)}{\geq} j_k \min\{\nu_{p_l}(m), t_l(n-k+1)\}. \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$\begin{aligned} -\nu_{p_l}(C_{j_k}(\alpha_k)) &\overset{(2.5)}{\leq} \nu_{p_l}(j_k!) + j_k t_l(n-k+1) - j_k \min\{\nu_{p_l}(m), t_l(n-k+1)\} \\ &= \nu_{p_l}(j_k!) + j_k \max\{t_l(n-k+1) - \nu_{p_l}(m), 0\}. \end{aligned} \tag{3.6}$$

Due to Lemma 2.3, we have

$$-\nu_{p_l}(a_{d,m}) \le \nu_{p_l}(m) + \max_{(j_1,j_2,\dots,j_n)\in\mathbb{J}} \left\{ \sum_{k=1}^{n} \nu_{p_l}(j_k!) \right\} + \max_{(j_1,j_2,\dots,j_n)\in\mathbb{J}} \left\{ \sum_{k=1}^{n} j_k \max\{t_l(n-k+1) - \nu_{p_l}(m), 0\} \right\}.$$

Considering the equation

$$(d^n - 1)j_1 + (d^{n-1} - 1)j_2 + \cdots + (d-1)j_n = m - 1 \tag{3.7}$$

for all $(j_1, j_2, \dots, j_n) \in \mathbb{J}$ and (2.2), we obtain

$$\begin{aligned}\sum_{k=1}^{n} \nu_{p_l}(j_k!) &\overset{(2.7)}{=} \sum_{k=1}^{n} \sum_{h=1}^{\infty} \left\lfloor \frac{j_k}{p_l^h} \right\rfloor \\ &\overset{(2.2)}{\le} \sum_{h=1}^{\infty} \left\lfloor \frac{j_1 + \cdots + j_n}{p_l^h} \right\rfloor \\ &\overset{(3.7)}{\le} \sum_{h=1}^{\infty} \left\lfloor \frac{m-1}{(d-1)p_l^h} \right\rfloor \\ &\overset{(2.7)}{=} \nu_{p_l}(A!)\end{aligned}$$

for all $1 \le k \le n$. We immediately see that equality holds when $j_1 = j_2 = \cdots = j_{n-1} = 0, j_n = (m-1)/(d-1) = A$. Conversely if equality holds, then actually $j_1 = j_2 = \cdots = j_{n-1} = 0, j_n = A$, because $(j_1, j_2, \dots, j_n) \in \mathbb{J}$ satisfies

$$(d^{n-1} + d^{n-2} + \cdots + 1)j_1 + \cdots + (d+1)j_{n-1} + j_n = \frac{m-1}{d-1} = A$$

and hence

$$j_1 + \cdots + j_n = A$$

is valid only when $j_1 = j_2 = \cdots = j_{n-1} = 0, j_n = A$.

If $p_l \nmid m$,

$$\begin{aligned}&\sum_{k=1}^{n} j_k \max\{t_l(n-k+1) - \nu_{p_l}(m), 0\} \\ &= \sum_{k=1}^{n} j_k t_l(n-k+1) \\ &\le \sum_{k=1}^{n} j_k t_l \frac{d^{n-k+1} - 1}{d-1} \\ &\overset{(3.7)}{=} t_l A\end{aligned}$$

holds for all $(j_1, j_2, \dots, j_n) \in \mathbb{J}$, because $\nu_{p_l}(m) = 0$ for $p_l \nmid m$. Equality holds if and only if $j_1 = j_2 = \cdots = j_{n-1} = 0, j_n = A$. We note that if $j_1 = j_2 = \cdots = j_{n-1} = 0, j_n = A$, then

$$C_0(m/d^n) \cdot C_0(m/d^{n-1}) \cdots C_A(m/d) = C_A(m/d) \neq 0.$$

Considering the elements of the sum in the formula in Lemma 2.3, we have $a_{d,m} \neq 0$ and precisely $-\nu_{p_l}(a_{d,m}) = \nu_{p_l}(A!) + t_l A$ if $p_l \nmid m$.

Furthermore we consider the remainder case of the above, i.e., the case $p_l \mid m$. If $nt_l > \nu_{p_l}(m)$, there exists $(j_1, j_2, \dots, j_n) \in \mathbb{J}$ such that

$$\begin{aligned}&\max_{(j_1,j_2,\dots,j_n)\in\mathbb{J}} \left\{ \sum_{k=1}^{n} j_k \max\{t_l(n-k+1) - \nu_{p_l}(m), 0\} \right\} \\ &= \sum_{k=1}^{\lfloor n+1-\nu_{p_l}(m)/t_l \rfloor} j_k(t_l(n-k+1) - \nu_{p_l}(m)) \\ &= \sum_{k=1}^{\lfloor n+1-\nu_{p_l}(m)/t_l \rfloor} j_k t_l(n-k+1) - \sum_{k=1}^{\lfloor n+1-\nu_{p_l}(m)/t_l \rfloor} j_k \nu_{p_l}(m).\end{aligned}$$

We see that

$$\sum_{k=1}^{n'} j_k t_l(n-k+1) \le \sum_{k=1}^{n} j_k t_l(n-k+1)$$
$$\le \max_{(j_1,j_2,\dots,j_n)\in\mathbb{J}} \left\{ \sum_{k=1}^{n} j_k t_l \frac{d^{n-k+1}-1}{d-1} \right\}$$
$$= t_l A.$$

Equality holds if and only if $j_1 = j_2 = \cdots = j_{n-1} = 0, j_n = A$. On the other hand,

$$\sum_{k=1}^{n'} j_k \nu_{p_l}(m) \ge \nu_{p_l}(m).$$

In this case equality holds if and only if $j_n = \cdots = j_{n-1} = 0, j_n = 1$. Hence,

$$\max_{(j_1,j_2,\dots,j_n)\in\mathbb{J}} \left\{ \sum_{k=1}^{n} j_k \max\{t_l(n-k+1) - \nu_{p_l}(m), 0\} \right\}$$
$$< t_l A - \nu_{p_l}(m).$$

Therefore, if $p_l \mid m$ and $nt_l > \nu_{p_l}(m)$,

$$-\nu_{p_l}(a_{d,m}) < \nu_{p_l}(m) + \nu_{p_l}(A!) + t_l A - \nu_{p_l}(m)$$
$$= \nu_{p_l}(A!) + t_l A.$$

Further if $m > 1 - \nu_{p_l}(m)/t_l + (2\nu_{p_l}(m)/t_l - n)d + d^2 + d^3 + \cdots + d^{\lfloor N \rfloor} + (N - \lfloor N \rfloor)d^{\lfloor N \rfloor + 1}$, or $n < \nu_{p_l}(m)/t_l$,

$$\max_{(j_1,j_2,\dots,j_n)\in\mathbb{J}} \left\{ \sum_{k=1}^{n} j_k \max\{t_l(n-k+1) - \nu_{p_l}(m), 0\} \right\}$$
$$= \sum_{k=1}^{\lfloor n+1-\nu_{p_l}(m)/t_l \rfloor} j_k(t_l(n-k+1) - \nu_{p_l}(m))$$
$$\le \sum_{k=1}^{\lfloor n+1-\nu_{p_l}(m)/t_l \rfloor} d^k(t_l(n-k+1) - \nu_{p_l}(m))$$
$$< t_l A - \nu_{p_l}(m)$$

and hence

$$-\nu_{p_l}(a_{d,m}) \le \nu_{p_l}(m) + \nu_{p_l}((m-1)!) + \sum_{k=1}^{\lfloor n+1-\nu_{p_l}(m)/t_l \rfloor} d^k(t_l(n-k+1) - \nu_{p_l}(m)).$$

On the other hand if $nt_l \le \nu_{p_l}(m)$,

$$\max_{(j_1,j_2,\dots,j_n)\in\mathbb{J}} \left\{ \sum_{k=1}^{n} j_k \max\{t_l(n-k+1) - \nu_{p_l}(m), 0\} \right\} = 0$$

and hence $-\nu_{p_l}(a_{d,m}) \le \nu_{p_l}(m) + \nu_{p_l}(A!) < t_l A + \nu_{p_l}(A!)$. □

Remark 3.6. The straightforward adaptation of this proof slightly improves the main result of [12] as follows:

Theorem 3.7 (Improved version of the main theorem of [12]). *For fixed $d \in \mathbb{N} \setminus \{1\}$, let $m \in \mathbb{N}_0$ with $(d-1) \mid (m+1)$. We assume that d is factorized as $d = p_1^{t_1} p_2^{t_2} \dots p_s^{t_s}$ where $s \in \mathbb{N}, t_1, t_2, \dots, t_s \in \mathbb{N}$ and $p_1, p_2, \dots, p_s$ are distinct prime numbers. Let $B = (m+1)/(d-1)$. Then,*

$$-\nu_{p_l}(b_{d,m}) \le \nu_{p_l}(B!) + t_l B$$

holds for any $p_l \in \{p_1, p_2, \dots, p_s\}$. Equality holds if and only if $p_l \nmid m$ and $m = d-2$. Furthermore let $n \in \mathbb{N}$ so that $d^n - 2 \le m \le d^{n+1} - 3$ and set $N = n + 1 - \nu_{p_l}(m)/t_l$. If $p_l \mid m$, the inequality

$$-\nu_{p_l}(a_{d,m}) \le \nu_{p_l}(m) + \nu_{p_l}(B!) + \sum_{k=1}^{\max\{\lfloor N \rfloor, 0\}} d^k(t_l(n-k+1) - \nu_{p_l}(m))$$

holds when either the two inequalities $n \ge \nu_{p_l}(m)/t_l$ and $m > -1 - \nu_{p_l}(m)/t_l + (2\nu_{p_l}(m)/t_l - n)d + d^2 + d^3 + \cdots + d^{\lfloor N \rfloor} + (N - \lfloor N \rfloor)d^{\lfloor N \rfloor + 1}$ are true, or $n < \nu_{p_l}(m)/t_l$ is satisfied.

In particular, the case $d = 2$ is obtained as follows:

Corollary 3.8. *For all $m \geq 0$, the inequality*

$$-\nu_2(b_{2,m}) \leq \nu_2((2m+2)!)$$

holds. Equality holds if and only if m is odd. Furthermore let $n \in \mathbb{N}$ so that $2^n - 2 \leq m \leq 2^{n+1} - 3$. If m is even and $m > 2^{n+2-\nu_2(m)} - 2n + 3\nu_2(m) - 5$, then

$$-\nu_2(b_{2,m}) \leq \nu_2(m) + \nu_2((m+1)!) + \sum_{k=1}^{n+1-\nu_2(m)} 2^k(n-k+1-\nu_2(m)).$$

4. Tables

We show some tables of coefficients $a_{d,m}$ (Tables 1, 2, 3 and 4) and our estimates (Tables 5, 6 and 7). We see that Theorem 3.7 give a better estimate of the main theorem of [12]. However, our estimates are not sharp as Tables 5, 6 and 7.

Table 1. The values of $a_{2,m}$.

m	Numerator	Denominator	$-\nu_2(a_{2,m})$
2	1	2	1
3	1	8	3
4	1	4	2
5	15	128	7
6	0	—	$-\infty$
7	81	1024	10
8	1	8	3
9	1499	32768	15
10	1	32	5
11	16551	262144	18
12	0	—	$-\infty$
13	−19557	4194304	22
14	7	256	8
15	1026129	33554432	25
16	1	16	4
17	78558483	2147483648	31
18	7	512	9
19	496067595	17179869184	34
20	0	—	$-\infty$
21	−506111055	274877906944	38
22	135	4096	12
23	66414150615	2199023255552	41
24	0	—	$-\infty$
25	402782136143	70368744177664	46
26	683	65536	16
27	−7661205650557	562949953421312	49
28	0	—	$-\infty$
29	159606082621811	9007199254740992	53
30	159	16384	14

Table 2. The values of $a_{3,m}$.

m	Numerator	Denominator	$-\nu_3(a_{3,m})$
3	1	3	1
5	1	9	2
7	2	81	4
9	1	9	2
11	52	729	6
13	155	6561	8
15	0	—	$-\infty$
17	2657	59049	10
19	29533	1594323	13
21	0	—	$-\infty$
23	−69655	14348907	15
25	2969930	129140163	17
27	1	27	3
29	23095973	1162261467	19
31	56696777	10460353203	21
33	10	729	6
35	2343898963	94143178827	23
37	24995524274	2541865828329	26
39	0	—	$-\infty$
41	115000492832	22876792454961	28
43	3201040250650	205891132094649	30
45	0	—	$-\infty$
47	−6747874422283	1853020188851841	32
49	27156979500091	16677181699666569	34

Table 3. The values of $a_{4,m}$.

m	Numerator	Denominator	$-\nu_2(a_{4,m})$
4	1	4	2
7	3	32	5
10	1	32	5
13	15	2048	11
16	1	16	4
19	2995	65536	16
22	93	4096	12
25	59451	8388608	23
28	0	—	$-\infty$
31	7405653	268435456	28
34	17127	1048576	20
37	102177851	17179869184	34
43	−1017988077	549755813888	39
46	2092125	134217728	27
49	716781072211	140737488355328	47
52	0	—	$-\infty$
55	−8057836991135	4503599627370496	52
58	−107583317	68719476736	36
61	2910453741726705	288230376151711744	58
64	1	64	6
67	91893393031048069	9223372036854775808	63
70	37808167947	8796093022208	43
73	1318087272305007215	1180591620717411303424	70
76	231	32768	15
79	444913124772728735913	37778931862957161709568	75

Table 4. The values of $a_{6,m}$.

m	Numerator	Denominator	$-\nu_2(a_{6,m})$
6	1	6	1
11	5	72	3
16	5	162	1
21	5	384	7
26	7	1458	1
31	8645	6718464	10
36	1	36	2
41	44166115	1934917632	15
46	96545	6377292	2
51	20051	2359296	18
56	224695	57395628	2
61	682050153785	541653102231552	22
66	0	—	$-\infty$
71	510189065505655	38999023360671744	25
76	412426453	41841412812	2
81	120083275	19327352832	31
86	2394396445	753145430616	3
91	49144739612524327415	436689224139720213135 36	34
96	0	—	$-\infty$
101	−2801171227435232984071	628832482761197106914918 4	38
106	52774878534565	6588516227028768	5
111	2597391412505	534362651099136	41
116	19211930633005	7412080755407364	2
121	113778177813131530199081381 5	117355233262825648880889731481 6	46

Table 5. The comparison of of the estimates for $d = 2$.

m	$-\nu_2(a_{2,m})$	Theorem A (1.1)	Corollary 1.1 (1.2)	Corollary 1.1 (1.3)
2	1	1	1	—
3	3	3	3	—
4	2	5	4	3
5	7	7	7	—
6	$-\infty$	9	8	6
7	10	11	10	—
8	3	13	11	7
9	15	15	15	—
10	5	17	16	—
11	18	19	18	—
12	$-\infty$	21	19	12
13	22	23	22	—
14	8	25	23	19
15	25	27	25	—
16	4	29	26	15
17	31	31	31	—
18	9	33	32	—
19	34	35	34	—
20	$-\infty$	37	35	26
21	38	39	38	—
22	12	41	39	—
23	41	43	41	—
24	$-\infty$	45	42	24
25	46	47	46	—
26	16	49	47	45
27	49	51	49	—
28	$-\infty$	53	50	33
29	53	55	53	—
30	14	57	54	48

Table 6. The comparison of of the estimates for $d = 4$ and $p = 2$.

m	$-\nu_2(a_{4,m})$	Theorem 3.1 (3.1)	Theorem 3.1 (3.2)
4	2	2	—
7	5	5	—
10	5	7	6
13	11	11	—
16	4	13	7
19	16	16	—
22	12	18	—
25	23	23	—
28	$-\infty$	25	17
31	28	28	—
34	20	30	—
37	34	34	—
40	$-\infty$	36	17
43	39	39	—
46	27	41	40
49	47	47	—
52	$-\infty$	49	25
55	52	52	—
58	36	54	45
61	58	58	—
64	6	60	24
67	63	63	—
70	43	65	—
73	70	70	—
76	15	72	—
79	75	75	—
82	51	77	—
85	81	81	—
88	0	83	56
91	86	86	—
94	58	88	—
97	95	95	—
100	$-\infty$	97	81
103	100	100	—
106	68	102	—
109	106	106	—
112	$-\infty$	108	46
115	111	111	—
118	75	113	—
121	118	118	—
124	25	120	88
127	123	123	—
130	83	125	—
133	129	129	—
136	19	131	72
139	134	134	—
142	90	136	—
145	142	142	—
148	$-\infty$	144	96
151	147	147	—
154	99	149	—
157	153	153	—
160	$-\infty$	155	58
163	158	158	—
166	106	160	—
169	165	165	—
172	32	167	103
175	170	170	—

Table 7. The comparison of of the estimates for $d = 6$ and $p = 2$.

m	$-\nu_2(a_{6,m})$	Theorem 3.1 (3.1)	Theorem 3.1 (3.2)
6	1	1	—
11	3	3	—
16	1	4	—
21	7	7	—
26	1	8	4
31	10	10	—
36	2	11	6
41	15	15	—
46	2	16	14
51	18	18	—
56	2	19	11
61	22	22	—
66	$-\infty$	23	17
71	25	25	—
76	2	26	13
81	31	31	—
86	3	32	22
91	34	34	—
96	$-\infty$	35	21
101	38	38	—
106	5	39	25
111	41	41	—
116	2	42	21
121	46	46	—
126	$-\infty$	47	29
131	49	49	—
136	1	50	26
141	53	53	—
146	8	54	32
151	56	56	—
156	$-\infty$	57	28
161	63	63	—
166	7	64	38
171	66	66	—
176	$-\infty$	67	36
181	70	70	—
186	$-\infty$	71	41
191	73	73	—
196	2	74	37
201	78	78	—
206	9	79	45
211	81	81	—
216	3	82	42
221	85	85	—
226	11	86	—
231	88	88	—
236	3	89	50
241	94	94	—
246	12	95	—
256	3	98	55
261	101	101	—
266	12	102	98
271	104	104	—
276	$-\infty$	105	58
281	109	109	—
286	16	110	102
291	112	112	—

Acknowledgement

The author is grateful to Professor Toshiyuki Sugawa for his helpful suggestions and constant encouragement. He further expresses his deep gratitude to the referees for their careful reading of this manuscript and for their helpful comments.

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