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An Extremal Problem for Univalent Functions

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For a real constant b, we give sharp estimates of $\log|f(z)/z| + b\arg[f(z)/z]$ for subclasses of normalized univalent functions f on the unit disk.

KEYWORDS: close-to-convex function, variability region, extremal problem

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized so that $f(0) = 0$ and $f'(0) = 1$. The subclass $\mathcal{S}$ of $\mathcal{A}$ consisting of all univalent functions has attracted much interest for many years in the univalent function theory. In the present paper, we are primarily interested in the extremal problem to find the value of

$$\Psi_z(t, \mathcal{F}) = \sup_{f\in\mathcal{F}} \mathrm{Re}\left[e^{it}\log\frac{f(z)}{z}\right]$$

for a subclass $\mathcal{F}$ of $\mathcal{S}$ and $z \in \mathbb{D}$. Here and hereafter, $g(z) = \log[f(z)/z] = \log|f(z)/z| + i\arg[f(z)/z]$ will be understood as the holomorphic branch of logarithm determined by $g(0) = 0$. It is often more convenient to consider the quantities

$$\Phi_z^+(b, \mathcal{F}) = \sup_{f\in\mathcal{F}}\left\{\log\left|\frac{f(z)}{z}\right| + b\arg\frac{f(z)}{z}\right\}$$

and

$$\Phi_z^-(b, \mathcal{F}) = \inf_{f\in\mathcal{F}}\left\{\log\left|\frac{f(z)}{z}\right| + b\arg\frac{f(z)}{z}\right\}.$$

Then, we have the obvious relation

$$\Psi_z(t, \mathcal{F}) = \begin{cases} (\cos t)\Phi_z^+(-\tan t, \mathcal{F}) & \text{if } \cos t > 0, \\ (\cos t)\Phi_z^-(-\tan t, \mathcal{F}) & \text{if } \cos t < 0. \end{cases} \tag{1.1}$$

Therefore, the first problem is essentially equivalent to finding the values of $\Phi_z^\pm(b, \mathcal{F})$ (except for the case when $e^{it} = \pm i$). We also consider the quantities

$$\Psi(t, \mathcal{F}) = \sup_{z\in\mathbb{D}} \Psi_z(t, \mathcal{F})$$

and

$$\Phi^+(b, \mathcal{F}) = \sup_{z\in\mathbb{D}} \Phi_z^+(b, \mathcal{F}) \quad \text{and} \quad \Phi^-(b, \mathcal{F}) = \inf_{z\in\mathbb{D}} \Phi_z^-(b, \mathcal{F}).$$

The above extremal problems will reduce to geometric ones, once we know about the shape of the variability region $W_z(\mathcal{F})$ of $\log[f(z)/z]$ for a subclass $\mathcal{F}$ of $\mathcal{S}$ and a fixed point $z \in \mathbb{D}$ defined by

$$W_z(\mathcal{F}) = \left\{\log\frac{f(z)}{z} : f \in \mathcal{F}\right\}.$$

Indeed, for instance, we have

$$\Psi_z(t, \mathcal{F}) = \sup_{w\in W_z(\mathcal{F})} \mathrm{Re}[e^{it}w] = \sup_{u+iv\in W_z(\mathcal{F})} (u\cos t - v\sin t).$$

We note that $W_z(\mathcal{F}) = W_r(\mathcal{F})$ for $r = |z|$ if $\mathcal{F}$ is rotationally invariant; in other words, if the function $e^{-i\theta}f(e^{i\theta}z)$

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belongs to $\mathcal{F}$ whenever $f \in \mathcal{F}$ and $\theta \in \mathbb{R}$. The union

$$W(\mathcal{F}) = \bigcup_{z \in \mathbb{D}} W_z(\mathcal{F})$$

is called the full variability region of $\log[f(z)/z]$ for $\mathcal{F}$.

In the present paper, we will discuss those regions of variability and the corresponding extremal problems for typical subclasses of $\mathcal{S}$.

2. Univalent Functions

Grunsky [4] gave a description of $W_z(\mathcal{S})$ (see also [3, §10.9]).

Theorem A (Grunsky). *For $z \in \mathbb{D}$ with $r = |z|$, the variability region $W_z(\mathcal{S})$ of $\log[f(z)/z]$ for $\mathcal{S}$ is the closed disk*

$$\left| w - \log \frac{1}{1-r^2} \right| \le \log \frac{1+r}{1-r}.$$

In particular, we see that for a fixed $z \in \mathbb{D}$ with $|z| = r$ and a real constant t,

$$\Psi_z(t, \mathcal{S}) = (\cos t) \log \frac{1}{1-r^2} + \log \frac{1+r}{1-r} = (1 - \cos t) \log(1+r) - (1 + \cos t) \log(1-r).$$

For more general extremal problems, the reader may refer to the monograph [1] by Alexsandrov.

In particular, letting $r \to 1^-$ in the above, we obtain $\Psi_z(t, \mathcal{S}) \to +\infty$ if $\cos t > -1$ and $\Psi_z(\pi, \mathcal{S}) = 2\log(1+r) \to 2\log 2 = \log 4$. Hence, in view of (1.1), we obtain $\Phi^+(b, \mathcal{S}) = +\infty$ for $b \in \mathbb{R}$ and

$$\Phi^-(b, \mathcal{S}) = \begin{cases} -\infty & \text{if } b \neq 0, \\ -\log 4 & \text{if } b = 0. \end{cases}$$

More precisely, as a corollary of the Grunsky theorem, we have the following.

Corollary. *The full variability region $W(\mathcal{S})$ of $\log[f(z)/z]$ for $\mathcal{S}$ is the half-plane $\{w : \operatorname{Re} w > -\log 4\}$.*

Proof. Since $\mathcal{S}$ is rotationally invariant, $W_z(\mathcal{S}) = W_r(\mathcal{S})$ for $r = |z|$. For a fixed $\eta \in \mathbb{R}$, the intersection of the disk $W_r(\mathcal{S})$ with the horizontal line $\operatorname{Im} w = \eta$ is the segment with the endpoints

$$-\log(1-r^2) \pm \sqrt{\left(\log \frac{1+r}{1-r}\right)^2 - \eta^2} + \eta i$$

for r so close to 1 that $|\eta| \le \log[(1+r)/(1-r)]$. Since

$$-\log(1-r^2) - \sqrt{\left(\log \frac{1+r}{1-r}\right)^2 - \eta^2} \to -\log 4$$

whereas

$$-\log(1-r^2) + \sqrt{\left(\log \frac{1+r}{1-r}\right)^2 - \eta^2} \to +\infty$$

as $r \to 1^-$, we see that $\{w \in W(\mathcal{S}) : \operatorname{Im} w = \eta\} = \{x + i\eta : x > -\log 4\}$. The proof is now complete. □

3. Starlike Functions and Convex Functions

For the subclasses $\mathcal{S}^*$ and $\mathcal{K}$ of $\mathcal{S}$ consisting of starlike and convex functions respectively, Marx [8, Satz B, C] obtained essentially the following result.

Theorem B (Marx). *For $z \in \mathbb{D}$ with $r = |z|$, the variability region $W_z(\mathcal{S}^*)$ of $\log[f(z)/z]$ for $\mathcal{S}^*$ is given as*

$$\{-2\log(1-\zeta) : |\zeta| \le r\}$$

and $W_z(\mathcal{K})$ of $\log[f(z)/z]$ for $\mathcal{K}$ is

$$\{-\log(1-\zeta) : |\zeta| \le r\} = \frac{1}{2} W_z(\mathcal{S}^*).$$

Note that $W_r(\mathcal{S}^*)$ is nothing but the image of the disk $|\zeta| \le r$ under the mapping $\log[k(z)/z]$, where $k(z)$ is the Koebe function; namely, $k(z) = \frac{z}{(1-z)^2}$. Similarly, $W_r(\mathcal{K})$ is the image of the disk $|\zeta| \le r$ under the mapping $\log[l(z)/z]$, where

$l(z) = z/(1-z)$ is an extremal convex function. It is easy to see that the regions $W_r(\mathcal{S}^*)$ and $W_r(\mathcal{K})$ are convex and symmetric with respect to the real axis.

As an application of the Marx theorem, we can solve the corresponding extremal problem. We present it only for starlike functions since we have only to take the half for convex functions.

Theorem 3.1. *For a fixed $z \in \mathbb{D}$ with $|z| = r$ and a real number b,*

$$\Phi_z^+(b, \mathcal{S}^*) = \log(1+b^2) - 2\log(\sqrt{\triangle} - r) + 2b \arctan \frac{br}{\sqrt{\triangle}}$$

and

$$\Phi_z^-(b, \mathcal{S}^*) = \log(1+b^2) - 2\log(\sqrt{\triangle} + r) - 2b \arctan \frac{br}{\sqrt{\triangle}},$$

where $\triangle = 1 + b^2(1-r^2)$.

Proof. By Theorem B, we have for $z \in \mathbb{D}$ with $|z| = r$,

$$\Phi_z^+(b, \mathcal{S}^*) = \sup_{f \in \mathcal{S}^*} \left\{ \log\left|\frac{f(z)}{z}\right| + b \arg \frac{f(z)}{z} \right\} = \sup_{|\xi| \le r} \{-2\log|1-\xi| - 2b\arg(1-\xi)\}.$$

By making use of the maximum principle for harmonic functions,

$$\Phi_z^+(b, \mathcal{S}^*) = \sup_{|\xi| = r} \{-2\log|1-\xi| - 2b\arg(1-\xi)\}.$$

The same argument yields

$$\Phi_z^-(b, \mathcal{S}^*) = \inf_{|\xi| = r} \{-2\log|1-\xi| - 2b\arg(1-\xi)\}.$$

For $\xi = re^{i\theta}$, let

$$\phi(\theta) = -2\log|1 - re^{i\theta}| - 2b\arg(1 - re^{i\theta}) = -\log(1 + r^2 - 2r\cos\theta) + 2b\arctan\frac{r\sin\theta}{1 - r\cos\theta}.$$

We need to find the supremum and infimum of $\phi(\theta)$ over $\mathbb{R}$. Since ϕ is periodic with period 2π, it is enough to find (local) maxima and minima in the interval $[0, 2\pi)$. A simple calculation yields

$$\phi'(\theta) = -2r \cdot \frac{\sin\theta - b\cos\theta + br}{1 - 2r\cos\theta + r^2}.$$

Thus extremal values of $\phi(\theta)$ are attained at the points θ satisfying

$$\sin\theta - b\cos\theta + br = 0.$$

By solving the above equation, we have $\theta = \theta_1, \theta_2$, where

$$\cos\theta_1 = \frac{b^2 r + \sqrt{\triangle}}{1+b^2}, \quad \sin\theta_1 = \frac{b\sqrt{\triangle} - br}{1+b^2}$$

and

$$\cos\theta_2 = \frac{b^2 r - \sqrt{\triangle}}{1+b^2}, \quad \sin\theta_2 = \frac{-b\sqrt{\triangle} - br}{1+b^2}.$$

We note that such θ_1, θ_2 exist uniquely on $[0, 2\pi)$ since

$$\left(\frac{b^2 r \pm \sqrt{\triangle}}{1+b^2}\right)^2 + \left(\frac{\pm b\sqrt{\triangle} - br}{1+b^2}\right)^2 = 1.$$

Thus

$$\Phi_z^+(b, \mathcal{S}^*) = \max\{\phi(\theta_1), \phi(\theta_2)\} = \log(1+b^2) - 2\log(\sqrt{\triangle} - r) + 2b\arctan\frac{br}{\sqrt{\triangle}}$$

and

$$\Phi_z^-(b, \mathcal{S}^*) = \min\{\phi(\theta_1), \phi(\theta_2)\} = \log(1+b^2) - 2\log(\sqrt{\triangle} + r) - 2b\arctan\frac{br}{\sqrt{\triangle}}.$$

The proof is now completed. □

We observe that

$$\Phi_z^+(b, \mathcal{S}^*) + \Phi_z^-(b, \mathcal{S}^*) = -2\log(1 - r^2),$$

which is independent of the parameter b. Since

$$\frac{\partial}{\partial b}\Phi_z^+(b, \mathcal{S}^*) = -\frac{\partial}{\partial b}\Phi_z^-(b, \mathcal{S}^*) = 2\arctan\frac{br}{\sqrt{1 + b^2 - b^2 r^2}},$$

we can see that $\Phi_z^+(b, \mathcal{S}^*)$ is increasing in $b > 0$ and decreasing in $b < 0$. In particular,

$$\Phi_z^+(b, \mathcal{S}^*) \geq \Phi_z^+(0, \mathcal{S}^*) = -2\log(1 - r)$$

and

$$\Phi_z^-(b, \mathcal{S}^*) \leq \Phi_z^-(0, \mathcal{S}^*) = -2\log(1 + r).$$

Letting $r \to 1^-$, we obtain the following corollary.

Corollary 3.2. *For a real number b,*

$$\Phi^+(b, \mathcal{S}^*) = +\infty \quad \textit{and} \quad \Phi^-(b, \mathcal{S}^*) = \log(1 + b^2) - \log 4 - 2b\arctan b.$$

Theorem 3.1 and Corollary 3.2 assure the following result.

Corollary 3.3. *For a fixed $z \in \mathbb{D}$ with $|z| = r$ and a real number b,*

$$\Phi_z^+(b, \mathcal{K}) = \frac{1}{2}\Phi_z^+(b, \mathcal{S}^*), \quad \Phi_z^-(b, \mathcal{K}) = \frac{1}{2}\Phi_z^-(b, \mathcal{S}^*)$$

and

$$\Phi^+(b, \mathcal{K}) = +\infty, \quad \Phi^-(b, \mathcal{K}) = \frac{1}{2}\log(1 + b^2) - \log 2 - b\arctan b.$$

4. Close-to-convex Functions

Biernacki [2] determined the variability region $W_z(\mathcal{C})$ of $\log[f(z)/z]$ for the class $\mathcal{C}$ of linearly accessible functions (now known as *close-to-convex* functions). That is, $W_z(\mathcal{C}) = \{-\log[2u^2/(u + v)] : |u - 1| \leq |z|, |v - 1| \leq |z|\}$. He also showed that $W(\mathcal{C}) \subset \{w : |\operatorname{Im} w| < 3\pi/2\}$. Since that is somewhat implicit, Kato and the authors [5] offered another expression for it; that is, $W_z(\mathcal{C}) = h(\overline{\mathbb{D}}_r)$ for $r = |z|$, where $\overline{\mathbb{D}}_r = \{z : |z| \leq r\}$ and $h(z) = \log(1 + ze^{2i\phi}) - 3\log(1 + z)$, $\phi = \arg(1 + z/3)$. See Fig. 1 for the pictures of the regions $W_r(\mathcal{S})$, $W_r(\mathcal{S}^*)$ and $W_r(\mathcal{C})$. It is, however, still difficult to compute $\Phi_z^\pm(b, \mathcal{C})$ for $z \in \mathbb{D}$. Thus, our main concern in the present paper will be determination of the quantity $\Phi^\pm(b, \mathcal{C})$ because we have a relatively simple expression of $W(\mathcal{C})$.

Lemma 4.1 (Theorem 1.4 in [5]). *The full variability region $W(\mathcal{C})$ for close-to-convex functions is the unbounded Jordan domain whose boundary is the Jordan arc $-\gamma((-2\pi, 2\pi))$. Here,*

$$\gamma(t) = \begin{cases} \log(1 + 3e^{it}) & \text{if } |t| < \pi \\ \log(1 - e^{it}) + \dfrac{t}{|t|}\pi i & \text{if } \pi \leq |t| < 2\pi. \end{cases}$$

Note that the region $W(\mathcal{C})$ is contained in the parallel strip $\{w : |\operatorname{Im} w| < 3\pi/2\}$ as was already shown by Biernacki [2]. By making use of the above lemma, we now describe $\Phi^\pm(b, \mathcal{C})$.

Theorem 4.2. *Let b be a real number. Then, $\Phi^+(b, \mathcal{C}) = +\infty$ and*

$$\Phi^-(b, \mathcal{C}) = \begin{cases} \log\dfrac{\sqrt{1 + b^2}}{3 + \sqrt{1 - 8b^2}} - b\arctan\dfrac{3b}{\sqrt{1 - 8b^2}} & \text{if } |b| \leq b_0, \\ \dfrac{1}{2}\log(1 + b^2) - \log 2 - |b|(\arctan|b| + \pi) & \text{if } |b| \geq b_0. \end{cases}$$

Here, $b_0 = 0.24001\ldots$ is the unique solution to the equation

$$b\left(\arctan b - \arctan\frac{3b}{\sqrt{1 - 8b^2}} + \pi\right) = \log(3 + \sqrt{1 - 8b^2}) - \log 2$$

in $0 < b < 1/2\sqrt{2}$.

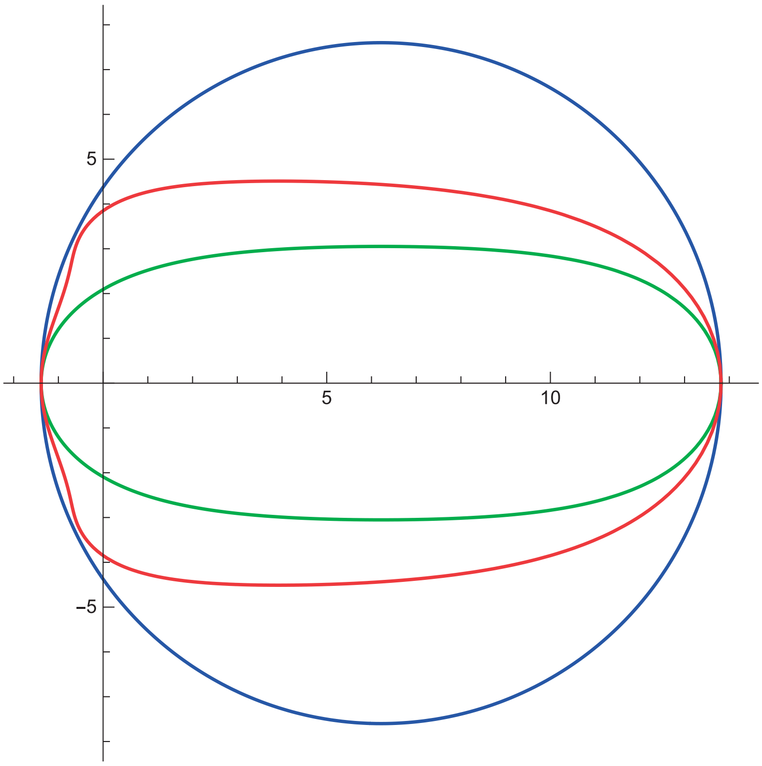

Fig. 1. The regions $W_r(\mathcal{S})$ (bounded by the blue line), $W_r(\mathcal{S}^*)$ (red) and $W_r(\mathcal{C})$ (green) for $r = 0.999$.

Proof. Since $\mathcal{S}^* \subset \mathcal{C}$, Corollary 3.2 yields $\Phi^+(b, \mathcal{C}) \geq \Phi^+(b, \mathcal{S}^*) = +\infty$ for every b.

We next consider $\Phi^-(b, \mathcal{C})$. For brevity, we put $\Phi(b) = \Phi^-(b, \mathcal{C})$ throughout the proof.

In order to prove the theorem, we translate our problem into a geometric one concerning the curve γ given in Lemma 4.1. First of all, we note that γ is symmetric in the sense that $\gamma(-t) = \overline{\gamma(t)}$. Therefore, it is enough to consider the case $0 \leq t \leq 2\pi$ unless otherwise stated. We now study the regularity of the curve $\gamma(t)$ at $t = \pi$. A direct computation shows that the left and right tangent vectors $\gamma'(\pi^-) = 3i/2$ and $\gamma'(\pi^+) = i/2$ have the same direction. Therefore, by a re-parametrization, we see that the boundary of $W(\mathcal{C})$ is of class C^1. We remark, however, that it is not of class C^2. Indeed, this can be confirmed by observing that $\exp(\gamma([0, \pi]))$ and $\exp(\gamma([\pi, 2\pi]))$ are (half-)circles with different radii.

We next study convexity of the curve γ. In [5], we already saw that the curve γ is not convex. More precisely, we compute

$$\frac{d}{dt} \arg \gamma'(t) = \begin{cases} \dfrac{1 + 3\cos t}{|1 + 3e^{it}|^2} & \text{if } 0 \leq t < \pi, \\ \dfrac{1 - \cos t}{|1 - e^{it}|^2} & \text{if } \pi < t < 2\pi. \end{cases}$$

Therefore, the curve γ is convex in $0 < t < \arccos(-1/3)$ and $\pi < t < 2\pi$ and concave in $\arccos(-1/3) < t < \pi$. It is important in the sequel to find the exact form of the convex hull $\widehat{\Omega}$ of Ω, where Ω is an unbounded Jordan domain bounded by the curve γ with $0 \in \Omega$. The newly added boundary $\partial\widehat{\Omega} - \partial\Omega$ consists of the line segment joining the two points of tangency of a common tangent line to γ on two parts $0 < t < \arccos(-1/3)$ and $\pi < t < 2\pi$, and its reflection in the real axis.

We should thus find the common tangent line. Let $\gamma(u)$ and $\gamma(v)$ be the points of tangency of the common tangent line, where $0 < u < \arccos(-1/3)$ and $\pi < v < 2\pi$. Necessary conditions are described by

$$\arg \gamma'(u) = \arg \gamma'(v) = \arg[\gamma(v) - \gamma(u)]. \tag{4.1}$$

Since $\gamma'(v) = e^{iv/2}/(2\sin(v/2))$, we have

$$\frac{v}{2} = \arg \gamma'(v) = \arg \gamma'(u) = u + \frac{\pi}{2} - \alpha, \tag{4.2}$$

where

$$\alpha = \arg(1 + 3e^{iu}) = \arctan \frac{3 \sin u}{1 + 3 \cos u}.$$

Simple computations give us

$$\mathrm{Re}[\gamma(v) - \gamma(u)] = \log\left|\frac{1 - e^{iv}}{1 + 3e^{iu}}\right| = \frac{1}{2} \log \frac{1 - \cos v}{5 + 3 \cos u}$$

and

$$\mathrm{Im}[\gamma(v) - \gamma(u)] = \arg(1 - e^{iv}) + \pi - \arg(1 + 3e^{iu}) = \frac{v + \pi}{2} - \alpha.$$

Hence, the second equation in (4.1) yields the relation

$$\tan \frac{v}{2} = \frac{v + \pi - 2\alpha}{\log(1 - \cos v) - \log(5 + 3 \cos u)}. \tag{4.3}$$

In view of (4.2), we have

$$\tan \frac{v}{2} = \cot(\alpha - u) = \frac{1 + \tan \alpha \tan u}{\tan \alpha - \tan u} = -\frac{3 + \cos u}{\sin u} \tag{4.4}$$

and

$$1 - \cos v = 2 \sin^2 \frac{v}{2} = 2 \cos^2(\alpha - u) = \frac{(3 + \cos u)^2}{5 + 3 \cos u}.$$

Substituting these and (4.2) into (4.3), we obtain

$$-\frac{3 + \cos u}{\sin u} = \frac{u + \pi - 2\alpha}{\log(3 + \cos u) - \log(5 + 3 \cos u)}.$$

It is easy to see that $(3 + \cos u)/\sin u \geq 2\sqrt{2}$, since $0 < u < \arccos(-1/3)$.
We summarize the above observations as follows. The slope of the tangent line to γ increases from $-\infty$ to $-(3 + \cos u)/\sin u = \tan(v/2)$ $(\leq -2\sqrt{2})$ as t moves from 0 to u. The tangent line to γ at $t = u$ is tangent, at the same time, to γ at $t = v$. The part $\gamma((u, v))$ is thus contained in the interior of $\widehat{\Omega}$. The slope of the tangent line to γ increases from $\tan(v/2)$ to 0 as t moves from v to 2π.

By Lemma 4.1, Ω is the domain $\{-w : w \in W(\mathcal{C})\}$. Then,

$$\Phi(b) = \inf_{w \in W(\mathcal{C})} (\mathrm{Re}\, w + b\, \mathrm{Im}\, w) = -\sup_{X+iY \in \Omega} (X + bY) = -\max_{X+iY \in \overline{\Omega}} (X + bY)$$

Here, we recall that Ω is contained in the region $X < \log 4, |Y| < 3\pi/2$. Hence, the supremum was able to be replaced by the maximum above by taking the points over the closure of Ω.

Since Ω is symmetric in the real axis, we have $\Phi(-b) = \Phi(b)$. Hence, we may assume that $b \geq 0$ in the proof of Theorem 4.2. When $b = 0$, obviously $\Phi(0) = -\gamma(0) = -\log 4$, which agrees with the assertion of the theorem. In the sequel, we thus assume that $b > 0$. For a given $b > 0$, let $Z_0 = X_0 + iY_0$ be a point in $\overline{\Omega}$ at which $X + bY$ takes its maximum over all $X + iY \in \overline{\Omega}$. It is obvious that $Z_0 \in \partial\Omega$ with $Y_0 > 0$ and that the line $X + bY = X_0 + bY_0$ $(= -\Phi(b))$ is tangent to the curve γ at Z_0. Since Z_0 is a support point for the functional $X + bY$ over $\overline{\Omega}$, $Z_0 = \gamma(t)$ for some t with $0 < t \leq u$ or $v \leq t < 2\pi$, where u and v are as above.

When $t \leq u$ $(< \pi)$, we have $X_0 = \log|1 + 3e^{it}| = \frac{1}{2} \log 2(5 + 3 \cos t)$, $Y_0 = \arg(1 + 3e^{it})$. Also, since the slope of the line $X + bY = -\Phi(b)$ is $-1/b$, we have the relation

$$-\frac{1}{b} = \frac{\mathrm{Im}\, \gamma'(t)}{\mathrm{Re}\, \gamma'(t)} = \frac{3 + \cos t}{-\sin t},$$

which yields

$$b = \frac{\sin t}{3 + \cos t}.$$

Thus we have obtained the following equation of $\cos t$

$$\cos^2 t + b^2(3 + \cos t)^2 = 1.$$

Since in this case $0 < b \leq 2\sqrt{2}$ and $\cos t > -1/3$, we have

$$\cos t = \frac{-3b^2 + \sqrt{1 - 8b^2}}{1 + b^2}$$

and

$$\tan Y_0 = \frac{3\sin t}{1+3\cos t} = \frac{3b(3+\cos t)}{1+3\cos t} = \frac{3b}{\sqrt{1-8b^2}}.$$

Hence,

$$\Phi(b) = -(X_0 + bY_0) = \log\frac{\sqrt{1+b^2}}{3+\sqrt{1-8b^2}} - b\arctan\frac{3b}{\sqrt{1-8b^2}} =: p(b)$$

as is stated in the theorem.

When $t \geq v$ ($> \pi$), we have $X_0 = \log|1-e^{it}| = \log[2\sin(t/2)]$ and $Y_0 = \arg(e^{it}-1) = (t+\pi)/2$. Similarly, we have the relation $-\frac{1}{b} = (1-\cos t)/\sin t = \tan(t/2)$, which is equivalent to $b = -\cot\frac{t}{2} = \tan\frac{t-\pi}{2}$. Therefore,

$$\Phi(b) = -(X_0 + bY_0) = -\log\frac{2}{\sqrt{1+b^2}} - b(\arctan b + \pi) =: q(b).$$

Let $b_0 = \sin u/(3+\cos u) = -\cot(v/2)$. Then b_0 must satisfy the relation $p(b_0) = q(b_0)$. Indeed, b_0 is a unique solution to the equation $p(b) = q(b)$ in $0 < b < 1/2\sqrt{2}$, since

$$q'(b) - p'(b) = -\left(\pi + \arctan b - \arctan\frac{3b}{\sqrt{1-8b^2}}\right) < 0,$$

$q(0) - p(0) = \log 2 > 0$ and

$$\lim_{b\to(1/2\sqrt{2})^-}(q(b)-p(b)) = \log\frac{3}{2} - \frac{\sqrt{2}}{8}\left(2\arctan\frac{1}{2\sqrt{2}} + \pi\right) = -0.270\cdots < 0.$$

The proof of Theorem 4.2 has been completed. □

In view of the relation (1.1), we obtain the following.

Corollary 4.3. *For a real constant t with $\cos t < 0$,*

$$\Psi(t,\mathcal{C}) = \begin{cases} -(\cos t)\log[-(3+\sqrt{1-8\tan^2 t})\cos t] - (\sin t)\arctan\dfrac{3\tan t}{\sqrt{1-8\tan^2 t}} & \textit{if } |\tan t| \leq b_0, \\ -(\cos t)\log[-2\cos t] - |\sin t|(|\pi - t| + \pi) & \textit{if } |\tan t| \geq b_0, \end{cases}$$

where b_0 is given in Theorem 4.2.

5. Application to Power Deformations

As an application of the main theorem, we consider power deformations of a univalent function. Let $c = a + bi$ be a complex number. The power deformation of a function $f \in \mathcal{S}$ with exponent c is defined by

$$f_c(z) = z\left(\frac{f(z)}{z}\right)^c = z\exp(c\log[f(z)/z]).$$

See [6] and [7] for details about the power deformation. We now have

$$|f_c(z)| = |z|\exp(a\log|f(z)/z| - b\arg[f(z)/z]).$$

Therefore, as a corollary of Theorem 4.2, we obtain the following.

Theorem 5.1. *Let $c = a + bi$ be a complex number. If $a > 0$,*

$$\inf_{\substack{f\in\mathcal{C}\\ z\in\mathbb{D}}}\log\left|\frac{f_c(z)}{z}\right| = \begin{cases} a\log\dfrac{\sqrt{a^2+b^2}}{3a+\sqrt{a^2-8b^2}} - b\arctan\dfrac{3b}{\sqrt{a^2-8b^2}} & \textit{if } \left|\dfrac{b}{a}\right| \leq b_0, \\ \dfrac{a}{2}\log(a^2+b^2) - a\log 2a - |b|\left(\arctan\left|\dfrac{b}{a}\right| + \pi\right) & \textit{if } \left|\dfrac{b}{a}\right| \geq b_0, \end{cases}$$

and, if $a < 0$,

$$\sup_{\substack{f\in\mathcal{C}\\ z\in\mathbb{D}}}\log\left|\frac{f_c(z)}{z}\right| = \begin{cases} a\log\dfrac{\sqrt{a^2+b^2}}{-3a+\sqrt{a^2-8b^2}} + b\arctan\dfrac{3b}{\sqrt{a^2-8b^2}} & \textit{if } \left|\dfrac{b}{a}\right| \leq b_0, \\ \dfrac{a}{2}\log(a^2+b^2) - a\log(-2a) - |b|\left(\arctan\left|\dfrac{b}{a}\right| + \pi\right) & \textit{if } \left|\dfrac{b}{a}\right| \geq b_0, \end{cases}$$

where b_0 is given in Theorem 4.2.

REFERENCES

[1] Aleksandrov, I. A., Parametric Extensions in the Theory of Univalent Functions (Russian), Izdat. "Nauka," Moscow (1976).
[2] Biernacki, M., "Sur la représentation conforme des domaines linéairement accessibles," *Prace Mat.-Fiz.*, **44**: 293–314 (1936).
[3] Duren, P. L., Univalent Functions, Springer-Verlag (1983).
[4] Grunsky, H., "Neue Abschätzungen zur konformen Abbildung ein- und mehrfach zusammenhängender Bereiche," *Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin*, **1**: 95–140 (1932).
[5] Kato, T., Sugawa, T., and Wang, L.-M., "Variability regions of close-to-covnex functions," *Anal. Polon. Math.*, **111**: 89–105 (2014).
[6] Kim, Y. C., and Sugawa, T., "On power deformations of univalent functions," *Monatsh. Math.*, **167**: 231–240 (2012).
[7] Kim, Y. C., and Sugawa, T., "On univalence of the power deformation $z(f(z)/z)^c$," *Chinese Ann. Math. Ser. B*, **33**: 823–830 (2012).
[8] Marx, A., "Untersuchungen über schlichte Abbildungen," *Math. Ann.*, **107**: 40–67 (1932).