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New Invariants for Integral Lattices

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Let Λ be any integral lattice in Euclidean space. It has been shown that for every integer $n > 0$, there is a hypersphere that passes through exactly n points of Λ. Using this result, we introduce new lattice invariants and give some computational results related to two-dimensional Euclidean lattices of class number one.

KEYWORDS: quadratic fields, lattices, lattice invariant

1. Introduction

We consider the following condition on lattices $\Lambda \subset \mathbb{R}^d$.

Definition 1.1 ([1, 5]). If there is a hypersphere in $\mathbb{R}^d$ that passes through exactly n points of Λ for every integer $n > 0$, then Λ is called "universally concyclic."

A lattice generated by $(a, b), (c, d) \in \mathbb{R}^2$, $(ad - bc \neq 0)$, is denoted by $\Lambda[(a, b), (c, d)]$. In [5], Maehara introduced the term "universally concyclic." Then, he and others showed the following results. In [7] and [4], Schinzel, Maehara, and Matsumoto proved that $\mathbb{Z}^2$, that is, $\Lambda[(1, 0), (0, 1)]$, is universally concyclic. Moreover, if $a, b, c, d \in \mathbb{Z}$ are such that $q := ad - bc$ is a prime and $q \equiv 3 \pmod 4$, then $\Lambda[(a, b), (c, d)]$ is universally concyclic. The equilateral triangular lattice $\Lambda[(1, 0), (-1/2, \sqrt{3}/2])]$ and the rectangular lattice $\Lambda[(1, 0), (0, \sqrt{3})]$ are universally concyclic. In [1], it was shown that all integral lattices in $\mathbb{R}^d$ with $d \geq 2$ are universally concyclic.

Remark 1.1. We remark that there exist some nonintegral lattices that are not universally concyclic. Maehara also proved in [5] that if τ is a transcendental number, then $\Lambda[(1, \tau), (0, 1)]$ cannot contain four concyclic points, and hence, it is not universally concyclic. The rectangular lattice $\Lambda[(\alpha, 0), (0, \beta)]$ does not contain five concyclic points if and only if $(\alpha/\beta)^2$ is an irrational number. Hence, some additional integrality conditions are necessary to ensure this property.

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, and let $\mathcal{O}_K$ be its ring of algebraic integers. Let Cl_K be the ideal classes of K. In this paper, we only consider the cases $|\mathrm{Cl}_K| = 1$, namely, d is in the following set: $\{1, 2, 3, 7, 11, 19, 43, 67, 163\}$.

We denote by d_K the discriminant of K:

$$d_K = \begin{cases} -4d & \text{if } -d \equiv 2, 3 \pmod 4 \\ -d & \text{if } -d \equiv 1 \pmod 4. \end{cases}$$

Theorem 1.1 (cf. [9, p. 87]). *Let d be a positive square-free integer, and let $K = \mathbb{Q}(\sqrt{-d})$. Then*

$$\mathcal{O}_K = \begin{cases} \mathbb{Z} + \mathbb{Z}\sqrt{-d} & \text{if } -d \equiv 2, 3 \pmod 4 \\ \mathbb{Z} + \mathbb{Z}\dfrac{-1 + \sqrt{-d}}{2} & \text{if } -d \equiv 1 \pmod 4. \end{cases}$$

Therefore, we consider $\mathcal{O}_K$ to be a lattice in $\mathbb{R}^2$ with the basis

$$\begin{cases} (1, 0), (0, \sqrt{d}) & \text{if } -d \equiv 2, 3 \pmod 4 \\ (1, 0), \left(-\dfrac{1}{2}, \dfrac{\sqrt{d}}{2}\right) & \text{if } -d \equiv 1 \pmod 4, \end{cases}$$

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denoted by $[1, \sqrt{-d}]$, $[1, (-1 + \sqrt{-d})/2]$, respectively. Note that $[1, \sqrt{-1}]$ is the $\mathbb{Z}^2$ lattice.

The main purpose of this paper is to introduce the new lattice invariants (Definition 1.2) and to give some computational results related to two-dimensional Euclidean lattices of class number one (Theorem 1.2).

We introduce the following new lattice invariants $\mathrm{uc}(\Lambda, n)$.

Definition 1.2. Let $\Lambda \subset \mathbb{R}^d$ be an integral lattice. For $n \in \mathbb{N}$, the *universally concyclic number* $\mathrm{uc}(\Lambda, n)$ (or $\mathrm{uc}(n)$ for short) is defined by the square of the minimum value among the radii of the hyperspheres that pass through exactly n points of Λ.

If two lattices Λ_1 and Λ_2 are isomorphic, then $\mathrm{uc}(\Lambda_1, n) = \mathrm{uc}(\Lambda_2, n)$ for all $n \in \mathbb{N}$. Therefore, $\mathrm{uc}(\Lambda, n)$ is an invariant of the lattice Λ. In [3], Maehara proposed the following problem:

Problem 1.1. Determine the $\mathrm{uc}(\mathbb{Z}^2, n)$ for $n = 3, \ldots, 10$.

In this paper, we determine the $\mathrm{uc}(\Lambda, n)$ for some n and Λ whose class number is one.

The following table provides the computational results.

Theorem 1.2. *Let $K = \mathbb{Q}(\sqrt{-d})$ as in Theorem 1.1. Concyclic numbers of two-dimensional Euclidean lattices $\mathcal{O}_K$ of class number one for $n \le 10$ if $1 \le d \le 163$ are determined as indicated in Table 1.*

Table 1.

$-d$	d_K	$\mathcal{O}_K$	uc(3)	uc(4)	uc(5)
-1	-2^2	$[1, \sqrt{-1}]$	$5^2/2 \cdot 3^2$	$1/2$	$5^4/2 \cdot 3^2$
-2	-2^3	$[1, \sqrt{-2}]$	$3^2/2^3$	$3/2^2$	$3^4/2^3$
-3	-3	$[1, (1 + \sqrt{-3})/2]$	$1/3$	$7/2^2$	$7^2 \cdot 13^2/11^2$
-7	-7	$[1, (1 + \sqrt{-7})/2]$	$2^2/7$	$2^3/7$	$2^4/7$
-11	-11	$[1, (1 + \sqrt{-11})/2]$	$3^2/11$	$3 \cdot 5/11$	$3^4/11$
-19	-19	$[1, (1 + \sqrt{-19})/2]$	$5^2/19$	$5 \cdot 7/19$	$5^4/19$
-43	-43	$[1, (1 + \sqrt{-43})/2]$	$11^2/43$	$11 \cdot 13/43$	$11 \cdot 13 \cdot 17 \cdot 23/2^2 \cdot 43$
-67	-67	$[1, (1 + \sqrt{-67})/2]$	$17^2/67$	$17 \cdot 19/67$	$17 \cdot 19 \cdot 23 \cdot 29/2^2 \cdot 67$
-163	-163	$[1, (1 + \sqrt{-163})/2]$	$41^2/163$	$41 \cdot 43/163$	$43^2 \cdot 61^2/3^2 \cdot 163$

uc(6)	uc(7)	uc(8)	uc(9)	uc(10)
$5^2/2^2$	$5^4 \cdot 13 \cdot 17/2 \cdot 11^2$	$5/2$	$5^2 \cdot 13^2/2 \cdot 3^2$	$5^4/2^2$
$3^2/2^2$	$3^6/2^3$	$3^3/2^2$	$3^2 \cdot 11^2/2^3$	$3^4/2^2$
1	$7^2 \cdot 13 \cdot 19 \cdot 43/3 \cdot 11^2$	$7 \cdot 13/2^2$	$7^2/3$	$7^4/2^2$
2^2	$2^6/7$	2^3	$2^8/7$	2^4
$3^2 \cdot 5/11$	$3^6/11$	$3^3 \cdot 5/11$	$3^2 \cdot 5^2/11$	$3^4 \cdot 5/11$
$5^2 \cdot 7/19$	$5 \cdot 7^2 \cdot 11 \cdot 17/3^2 \cdot 19$	$5 \cdot 7 \cdot 11/19$	$5^2 \cdot 7^2/19$	$5^4 \cdot 7/19$
$11^2 \cdot 13/43$	$11 \cdot 13^2 \cdot 17 \cdot 23/3^2 \cdot 43$	$11 \cdot 13 \cdot 17/43$	$11^2 \cdot 13^2/43$	$11^4 \cdot 13/43$
$17^2 \cdot 19/67$	$17 \cdot 19^2 \cdot 23 \cdot 29/3^2 \cdot 67$	$17 \cdot 19 \cdot 23/67$	$17^2 \cdot 19^2/67$	$17^4 \cdot 19/67$
$41^2 \cdot 43/163$	$41 \cdot 43 \cdot 61 \cdot 71 \cdot 83/2^2 \cdot 3^2 \cdot 163$	$41 \cdot 43 \cdot 47/163$	$41^2 \cdot 43^2/163$	$41 \cdot 47 \cdot 53 \cdot 71 \cdot 83/3^2 \cdot 163$

We calculated the integer sequences $\mathrm{uc}(\mathbb{Z}^2, 4n)$ and $\mathrm{uc}([1, (1 + \sqrt{-3})/2], 6n)$ for small n, and speculated that they have simple rules. Therefore, we have the following problem:

Problem 1.2. Determine $\mathrm{uc}(\mathbb{Z}^2, 4n)$ and $\mathrm{uc}([1, (1 + \sqrt{-3})/2], 6n)$ for all n.

In this paper, we give a partial answer of Problem 1.2. Namely, we give an exact upperbound of $\mathrm{uc}(\mathbb{Z}^2, 2^{\ell+2})$ and $\mathrm{uc}([1, (1 + \sqrt{-3})/2], 6 \cdot 2^m)$.

Theorem 1.3. *Let ℓ and m be nonnegative integers, let p_i $(i = 1, 2, \ldots)$ be the i-th smallest prime that is congruent to $1 \pmod 4$ (set $p_0 := 1$), and let q_j $(j = 1, 2, \ldots)$ be the j-th smallest prime that is congruent to $1 \pmod 3$ (set $q_0 := 1$).*

(1) *There exists a circle that passes through exactly $2^{\ell+2}$ points (x, y) of $\mathbb{Z}^2$:*

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}\prod_{k=0}^{\ell} p_k.$$

Therefore, we have

$$\mathrm{uc}(\mathbb{Z}^2, 2^{\ell+2}) \le \frac{1}{2}\prod_{k=0}^{\ell} p_k.$$

(2) *The number of the integer solutions of the following equation*

$$\left(x+y\frac{1+\sqrt{-3}}{2}\right)\left(x+y\frac{1-\sqrt{-3}}{2}\right)=x^2+xy+y^2=\prod_{k=0}^{m}q_k$$

is $6\cdot 2^m$. *This means that the circle*

$$|z|=\left(\prod_{k=0}^{m}q_k\right)^{\frac{1}{2}}$$

passes through exactly $6\cdot 2^m$ *points of* $[1,(1+\sqrt{-3})/2]$. *Therefore, we have*

$$\mathrm{uc}([1,(1+\sqrt{-3})/2],6\cdot 2^m)\leq\prod_{k=0}^{m}q_k.$$

In Sect. 2, we give the computational algorithm used in Theorem 1.2. In Sect. 3, we provide the proof of Theorem 1.3. In Sect. 4, we present further problems.

All the computer calculations in this paper were done by Mathematica [8] and C Programming Language [6].

2. Algorithm

In this section, we give the algorithm used to find the square of the minimum value among the radii of the hyperspheres that pass through exactly n points of Λ.

Assume that Λ is one of $\mathcal{O}_K$ in Theorem 1.2. Let ℓ be a positive integer, and let $R\subset\Lambda$ be the set of (x,y) that satisfies $x^2+y^2\leq\ell^2$, $y\geq 0$ and $y\geq-\sqrt{d}x$ if $d=2,3,7,11,19,43,67,163$ (if $d=1$, then let R be the set of (x,y) that satisfies $x^2+y^2\leq\ell^2$, $x\geq 0$ and $y\geq 0$). We shall try to create a hypersphere by taking three vertices on R. Notice that a hypersphere is determined uniquely by taking three vertices over Λ.

First, we shall explain how to plot the three vertices on R. Let (x_i,y_i) be the i-th vertex $(i=1,2,3)$. Set $(x_1,y_1)=(0,0)$, and let (x_2,y_2) vary such that it plots every vertex $(x,y)\in\Lambda$ such that $y/x<\sqrt{d}$ on R. Then, we let (x_3,y_3) vary such that it plots every vertex $(x,y)\in\Lambda$, except for $(x,y)\in\Lambda$ such that $y=0$ on R. This algorithm will provide every hypersphere passing through $(0,0)$ that can be generated by any $(x,y)\in\Lambda$ on R.

Next, we shall explain how to obtain the coordinates for the center and the square of the radius of a hypersphere. Let (x_0,y_0) be the center of a hypersphere, and let D be the square of the radius of the hypersphere. Then,

$$x_0=-(y_2y_3^2+(-y_2^2-x_2^2)y_3+x_3^2y_2)/(2x_2y_3-2x_3y_2),$$
$$y_0=(x_2y_3^2-x_3y_2^2+x_2x_3^2-x_2^2x_3)/(2x_2y_3-2x_3y_2),$$
$$D=\left(\frac{\sqrt{y_2^2+x_2^2}\sqrt{\alpha}}{2x_2y_3-2x_3y_2}\right)^2,$$

where $\alpha=y_3^4-2y_2y_3^3+y_2^2y_3^2+2x_3^2y_3^2-2x_2x_3y_3^2+x_2^2y_3^2-2x_3^2y_2y_3+x_3^2y_2^2+x_3^4-2x_2x_3^3+x_2^2x_3^2$.

Next, we explain how to enumerate the number of lattice points $(x,y)\in\Lambda$ such that $(x-x_0)^2+(y-y_0)^2=D$. Let $x_4\in\Lambda$ move from $[x_0-\sqrt{D}]$ to $[x_0+\sqrt{D}]+1$, where [ ] is the Gauss symbol. For the equation $(x_4-x_0)^2+(y_4-y_0)^2=D$, solve for y_4: $y_4=y_0\pm\sqrt{-x_4^2+2x_0x_4+D-x_0^2}$. Set $c_p=0$. If $x_4\equiv 0\,(\mathrm{mod}\,1)$ and $y_4\equiv 0\,(\mathrm{mod}\,\sqrt{d})$, or if $x_4\equiv 1/2\,(\mathrm{mod}\,1)$ and $y_4\equiv\sqrt{d}/2\,(\mathrm{mod}\,\sqrt{d})$, then $c_p=c_p+1$ (in the case of $d=3,7,11,19,43,67,163$). If $x_4\equiv 0\,(\mathrm{mod}\,1)$ and $y_4\equiv 0\,(\mathrm{mod}\,\sqrt{d})$, then $c_p=c_p+1$ (in the case of $d=1,2$). It is seen that c_p denotes the number of lattice points $(x,y)\in\Lambda$ such that $(x-x_0)^2+(y-y_0)^2=D$ after moving x_4 from $[x_0-\sqrt{D}]$ to $[x_0+\sqrt{D}]+1$. Therefore, we can obtain the hypersphere that passes through exactly c_p points.

Using the above method, since we can find the hyperspheres that pass through exactly c_p points for any $n\in\mathbb{N}$, we can obtain the square of the minimum value of the radius by selecting the smallest radius of any of the hyperspheres that pass through exactly n points of Λ.

3. Proof of Theorem 1.3

First, we claim that the circle $(2x-1)^2+(2y-1)^2=2\prod_{k=0}^{\ell}p_k$ passes through exactly $2^{\ell+2}$ points of $\mathbb{Z}^2$. By Fermat's $4n+1$ Theorem, for all p_j, there exists $a_j,b_j\in\mathbb{Z}$ such that $p_j=a_j^2+b_j^2$. Therefore, $p_j=(a_j+ib_j)(a_j-ib_j)$. Notice that a_j+ib_j and a_j-ib_j are irreducible elements over $\mathbb{Z}[i]$. Since $2=1^2+1^2$, $2=(1+i)(1-i)$. Hence $\omega\overline{\omega}=2\prod_{k=0}^{\ell}p_k=(1+i)(1-i)\prod_{k=0}^{\ell}p_k=(1+i)(1-i)\prod_{k=0}^{\ell}(a_k+ib_k)(a_k-ib_k)$, where $\omega\in\mathbb{Z}[i]$. We consider the number of possible outcomes for ω. We can express ω as follows:

$\omega=u(1+i)^{\epsilon_0}(1-i)^{1-\epsilon_0}(a_1+ib_1)^{\epsilon_1}(a_1-ib_1)^{1-\epsilon_1}\cdots(a_\ell+ib_\ell)^{\epsilon_\ell}(a_\ell-ib_\ell)^{1-\epsilon_\ell}$, where $u=\pm1,\pm i$, and $\epsilon_n=0,1$ $(n=0,1,\ldots,\ell)$.

It is easily seen that the choice of $(1+i)$ or $(1-i)$ does not depend on the number of possible outcomes of ω, since the absolute value of the real part and the imaginary part of $(1+i)$ and $(1-i)$ is the same.

Consequently, the number of possible outcomes of ω is $4 \cdot 2^{\ell+1}/2 = 2^{\ell+2}$ over $\mathbb{Z}[i]$. From this, the number of $(X, Y) \in \mathbb{Z}^2$ such that $X^2 + Y^2 = 2\prod_{k=0}^{\ell} p_k$ is $2^{\ell+2}$.

Next, we claim that they all correspond to the lattice point $(x, y) \in \mathbb{Z}^2$ such that $(2x-1)^2 + (2y-1)^2 = 2\prod_{k=0}^{\ell} p_k$. Since $X^2, Y^2 \equiv 0, 1 \pmod 4$ and $2\prod_{k=0}^{\ell} p_k \equiv 2 \pmod 4$, $X^2 + Y^2 = \prod_{k=0}^{\ell} p_k$ implies that $X^2 \equiv 1$ and $Y^2 \equiv 1 \pmod 4$. Moreover, it implies that $X \equiv 1$ and $Y \equiv 1 \pmod 2$. Therefore, the number of lattice points $(x, y) \in \mathbb{Z}^2$ such that $(2x-1)^2 + (2y-1)^2 = 2\prod_{k=0}^{l} p_k$ is equivalent to the number of $(X, Y) \in \mathbb{Z}^2$ such that $X^2 + Y^2 = 2\prod_{k=0}^{l} p_k$, $2x - 1 \equiv -1 \equiv 1 (\equiv X) \pmod 2$ and $2y - 1 \equiv -1 \equiv 1 (\equiv Y) \pmod 2$.

Thus, the number of lattice points $(x, y) \in \mathbb{Z}^2$ such that $(2x-1)^2 + (2y-1)^2 = 2\prod_{k=0}^{\ell} p_k$ is just $2^{\ell+2}$.

Next, we claim that the number of the integer solutions of the following equation

$$\left(x + y\frac{1+\sqrt{-3}}{2}\right)\left(x + y\frac{1-\sqrt{-3}}{2}\right) = x^2 + xy + y^2 = \prod_{k=0}^{m} q_k$$

is $6 \cdot 2^m$.

The proof is similar to the first part. Set $\zeta = 1 + \sqrt{-3}/2$. Then, for all q_i, there exists $a_i, b_i \in \mathbb{Z}$ such that

$$\begin{aligned} q_i = a_i^2 + a_i b_i + b_i^2 &= \left(a_i + \frac{b_i}{2} + \frac{\sqrt{-3}b_i}{2}\right)\left(a_i + \frac{b_i}{2} - \frac{\sqrt{-3}b_i}{2}\right) \\ &= (a_i + b_i\zeta)(a_i + b_i\overline{\zeta}). \end{aligned}$$

Notice that $a_i + b_i\zeta$ and $a_i + b_i\overline{\zeta}$ are irreducible elements over $\mathbb{Z}[\zeta]$, and $\tau\overline{\tau} = \prod_{k=0}^{m} q_k = \prod_{k=0}^{m}(a_k + b_k\zeta)(a_k + b_k\overline{\zeta})$, where $\tau \in \mathbb{Z}[\zeta]$. We consider the number of possible outcomes for τ. We can express τ as follows: $\tau = u(a_1 + b_1\zeta)^{\mu_1}(a_1 + b_1\overline{\zeta})^{1-\mu_1} \cdots (a_m + b_m\zeta)^{\mu_m}(a_m + b_m\overline{\zeta})^{1-\mu_m}$, where $u = \pm 1, \pm\zeta, \pm\overline{\zeta}$, $\mu_n = 0, 1$ $(n = 1, \ldots, m)$.

As a consequence, the number of possible outcomes of τ is $6 \cdot 2^m$ over $\mathbb{Z}[\zeta]$. From this, the number of $x + y\zeta \in \mathbb{Z}[\zeta]$ such that $x^2 + xy + y^2 = \prod_{k=0}^{m} q_k$ is $6 \cdot 2^m$. Now, since it can be seen that $\mathbb{Z}[\zeta]$ is equivalent to $[1, (1+\sqrt{-3})/2]$, the circle

$$|z| = \left(\prod_{k=0}^{m} q_k\right)^{\frac{1}{2}}$$

passes through exactly $6 \cdot 2^m$ points of $[1, (1+\sqrt{-3})/2]$.

Remark 3.1. We remark that the conditions in Theorem 1.3 "the i-th smallest prime" and "the j-th smallest prime" do not use in the proof of Theorem 1.3. For example, the number of solutions (points of $\mathbb{Z}^2$) is determined by the number of primes appearing in the product

$$\prod_{k=0}^{\ell} p_k.$$

On the other hand, we need these conditions in order to answer Problem 1.3.

4. Further Problems

(1) Find a law in the table of Theorem 1.2.
(2) For $n = \{3, \ldots, 10\}$, determine the $\mathrm{uc}(\Lambda, n)$ for $\Lambda = \mathbb{Z}^3$ and $\mathbb{Z}^4$.
(3) Let

$$\begin{cases} \mathbf{e_1} = \dfrac{1}{\sqrt{12}}(1, 0, 0, 0) \\ \mathbf{e_2} = \dfrac{7}{\sqrt{12}}(0, 1, 0, 0) \\ \mathbf{e_3} = \dfrac{13}{\sqrt{12}}(0, 0, 1, 0) \\ \mathbf{e_4} = \dfrac{19}{\sqrt{12}}(0, 0, 0, 1). \end{cases}$$

Then, we define the two lattices, $L_1 := \langle \mathbf{u_i} \mid i = 1, \ldots, 4 \rangle$ and $L_2 := \langle \mathbf{v_i} \mid i = 1, \ldots, 4 \rangle$, where

$$\begin{cases} \mathbf{u}_1 = 3\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 \\ \mathbf{u}_2 = \mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4 \\ \mathbf{u}_3 = \mathbf{e}_1 - \mathbf{e}_2 + 3\mathbf{e}_3 + \mathbf{e}_4 \\ \mathbf{u}_4 = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 + 3\mathbf{e}_4, \end{cases} \begin{cases} \mathbf{v}_1 = -3\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 \\ \mathbf{v}_2 = \mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4 \\ \mathbf{v}_3 = \mathbf{e}_1 - \mathbf{e}_2 - 3\mathbf{e}_3 + \mathbf{e}_4 \\ \mathbf{v}_4 = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 - 3\mathbf{e}_4. \end{cases}$$

In [2], it was shown that the theta series of L_1 and L_2 are the same, namely, the number of lattice vectors of norm m are the same for all m. However, these two lattices are nonisomorphic, and the proof of this fact is not easy [2].

Therefore, we have the following problem: Determine the $\mathrm{uc}(L_1, n)$ and $\mathrm{uc}(L_2, n)$ for some n, and show that L_1 and L_2 are nonisomorphic.

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