H-Compactness of Elliptic Operators on Weighted Riemannian Manifolds

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In this paper we study the asymptotic behavior of second-order uniformly elliptic operators on weighted Riemannian manifolds. They naturally emerge when studying spectral properties of the Laplace–Beltrami operator on families of manifolds with rapidly oscillating metrics. We appeal to the notion of *H*-convergence introduced by Murat and Tartar. In our main result we establish an *H*-compactness result that applies to elliptic operators with measurable, uniformly elliptic coefficients on weighted Riemannian manifolds. We further discuss the special case of "locally periodic" coefficients and study the asymptotic spectral behavior of compact submanifolds of $\mathbb{R}^n$ with rapidly oscillating geometry.

KEYWORDS: Periodic homogenization, H-convergence, Mosco convergence, Convergence of spectrum

1. Introduction

We study the asymptotic behavior of elliptic operators on families of weighted Riemannian manifolds that might feature fast oscillations. In this introduction we survey the results and the structure of this paper without going into detail. The precise definitions and statements can then be found in Sect. 2.

Convergence of metric measure spaces, in particular, Riemannian manifolds, has attracted an enormous amount of attention. Especially, substantial effort has been devoted to establishing geometric criteria for the convergence of spectral structures, e.g., see [3, 6, 8, 13–15, 17–20, 22, 24].

Our point of view is different. We establish a compactness result that shows that *any* family of (uniformly elliptic) PDEs of the form $-\text{div}_{g_\varepsilon, \mu_\varepsilon}(\mathbb{L}_\varepsilon \nabla_{g_\varepsilon})u = f$ defined on a uniformly bi-Lipschitz diffeomorphic family of weighted Riemannian manifolds $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ admits an *H-convergent* subsequence. The latter notion has been introduced in the context of homogenization of elliptic PDEs on $\mathbb{R}^n$ (in divergence form and of second-order), see [25]. In particular, in our setting it yields the existence of a limiting manifold and a limiting elliptic PDE such that solutions to the elliptic PDE on M_ε converge as $\varepsilon \downarrow 0$ to the solution of the limiting PDE. Our approach in particular allows us to treat Riemannian manifolds which oscillate rapidly on a small length scale $0 < \varepsilon \ll 1$.

This should be compared with the seminal work by Kuwae and Shioya [17], where spectral convergence is established for families of manifolds which are locally bi-Lipschitz diffeomorphic to a reference manifold with a bi-Lipschitz constant converging to 1. In situations where the manifold features rapid oscillations, the family of diffeomorphisms between the manifolds is only uniformly bi-Lipschitz but not locally close to an isometry — and thus the approach in [17] is not applicable. In contrast, as we shall show, it is still possible to establish *H*-convergence, which in the symmetric case (e.g., when considering the Laplace–Beltrami operator on M_ε) implies Mosco-convergence of the associated energy forms, and the convergence of the associated spectrum. Moreover, our approach also applies to non-symmetric PDEs.

For general uniformly bi-Lipschitz diffeomorphic families of manifolds the limiting manifold and PDE depends on the extracted subsequence. However, under geometric conditions for (M_ε), we can uniquely identify the limit by appealing to suitable homogenization formulas (see Sect. 2.2). In the flat case, a natural geometric condition is periodicity of the coefficient field. In the case of PDEs on Riemannian manifolds with a symmetry structure, or for general manifolds that feature periodicity in local coordinates, we obtain similar identification results and homogenization formulas.

The latter might be of interest for applications to diffusion models in biomechanics, which is another motivation of our work. In this context, diffusion and reaction-diffusion processes in biological membranes and through interfaces are studied, e.g., see [1, 10, 28, 30]. One observation made is that "diffusion in biological membranes can appear *anisotropic* even though it is molecularly isotropic in all observed instances," see [30]. We present examples (see

below) where anisotropic diffusion on surfaces emerges on large scales from isotropic diffusion on surfaces with rapidly oscillating geometry.

Examples. Before stating our results in a general form, we illustrate our findings on the level of examples. In the following we present five examples. Four of them consider families of 2-dimensional submanifolds (M_ε) in $\mathbb{R}^3$ given by an explicit formula and depending on a small parameter $\varepsilon > 0$, which is related to the length scale of spatial oscillations of M_ε. In the limit $\varepsilon \downarrow 0$, M_ε Hausdorff-converges (as a subset of $\mathbb{R}^3$) to a reference submanifold $M_0 \subset \mathbb{R}^3$; however, along the limit, the manifold oscillates more and more rapid and the curvature diverges. As a consequence, the spectrum of the associated Laplace–Beltrami operator on M_ε does not converge to the spectrum of the one on M_0 (where M_0 is considered with the metric induced by the ambient Euclidean space). In Lemma 20 below, we show that M_0 can be equipped with an *effective metric* $\hat{g}_0$ (and an effective weight $\hat{\mu}_0$) such that the resulting weighted Riemannian manifold $(M_0, \hat{g}_0, \hat{\mu}_0)$ captures the asymptotic spectral behavior of (M_ε) in the limit $\varepsilon \downarrow 0$, in the sense that the spectrum of the Laplace–Beltrami operator on M_ε converges to the spectrum of the Laplace–Beltrami operator on $(M_0, \hat{g}_0, \hat{\mu}_0)$. In examples (b)–(d) below, it turns out that the limiting manifold $(M_0, \hat{g}_0, \hat{\mu}_0)$ can be realized as a 2-dimensional submanifold $N_0 \subset \mathbb{R}^3$, and thus, the spectral properties of N_0 capture the asymptotic spectral properties of M_ε in the limit $\varepsilon \downarrow 0$. Proofs and further details are presented in Sect. 3.

(a) A one-dimensional example. We start with an elementary, one-dimensional example to clarify the results conceptually. For $\varepsilon = \frac{1}{k}$ with $k \in \mathbb{N}$ we consider the 1-dimensional submanifold $M_\varepsilon \subset \mathbb{R}^2$,

$$M_\varepsilon := \left\{ \begin{pmatrix} x \\ f_\varepsilon(x) \end{pmatrix} ; x \in [0, L] \right\}, \tag{1}$$

where $L \in \mathbb{N}$, $f_\varepsilon(x) := \varepsilon f(\frac{x}{\varepsilon})$ and f denotes a smooth, 1-periodic function with $f(0) = f(1) = 0$ that is not identically 0. By periodicity we note that the density of the Riemannian volume form ρ_ε associated with M_ε weakly-* converges in $L^\infty((0, L))$:

$$\rho_\varepsilon = \sqrt{1 + |f_\varepsilon'|^2} = \sqrt{1 + |f'(\tfrac{\cdot}{\varepsilon})|^2} \overset{*}{\rightharpoonup} \int_0^1 \sqrt{1 + |f'(y)|^2} \, dy =: \rho_0,$$

and $\rho_0 > 1$, since $f \not\equiv 0$. By periodicity (and the conditions on ε and L), the volume of M_ε (which here is just the one-dimensional Hausdorff-measure of M_ε) is independent of ε; more precisely, $\text{vol}_1(M_\varepsilon) = \int_0^L \rho_\varepsilon \, dy = \int_0^L \rho_0 \, dy = L\rho_0$. On the other hand M_ε converges w.r.t. the Hausdorff-distance in $\mathbb{R}^2$ to the submanifold $M_0 := \{ \binom{s}{0} ; s \in [0, L] \}$ with volume $\text{vol}_1(M_0) = L$. The latter is strictly smaller than the volume of M_ε and the loss of volume is due to the emergence of rapid oscillations in the limit $\varepsilon \downarrow 0$. On the other hand, our results (see Lemma 21 and Remark 22) show that the spectrum of the Laplace–Beltrami operator on M_ε converges to the spectrum of the Laplace–Beltrami operator on a weighted Riemannian manifold M_0 whose Riemannian volume form has ρ_0 as the density against the Lebesgue measure. The weighted Riemannian manifold is isometrically isomorphic to a submanifold in $\mathbb{R}^2$, for example, to

$$N_0 := \left\{ \begin{pmatrix} x \\ \sqrt{\rho_0^2 - 1} \, x \end{pmatrix} ; x \in [0, L] \right\}, \tag{2}$$

which is a straight line with the same volume as M_ε, i.e., $\text{vol}_1(N_0) = \rho_0 L$. Note that N_0 is just one (of many) illustrative isometric embeddings of the limit manifold in $\mathbb{R}^2$. The sequence M_ε (for $f(y) = \frac{1}{2\pi} \sin(2\pi y)$ and $L = 2$) and the Hausdorff-limit M_0 are illustrated in Fig. 1.
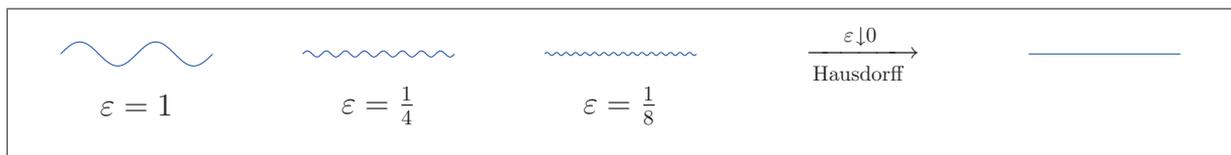

Fig. 1. A one-dimensional example. The three pictures on the left show M_ε defined by (1) with $f(y) = \frac{1}{2\pi} \sin(2\pi y)$ and $L = 2$ for decreasing values of ε. As $\varepsilon \to 0$ these manifolds Hausdorff-converge to the manifold $M_0 = [0, 2] \times \{0\}$, shown on the right. However, the spectrum of the Laplace–Beltrami operator on M_ε converges to the spectrum of the Laplace–Beltrami operator on a submanifold $N_0 \subset \mathbb{R}^2$, see (2). Note that N_0 is (as M_0) a straight line, but its length is $2\rho_0 = \frac{1}{\pi} \int_0^{2\pi} \sqrt{1 + \cos^2(y)} \, dy$ — the length of the oscillating curves M_ε which is strictly larger than 2 — the length of M_0.

(b) A graphical surface with star-shaped corrugations. For $\varepsilon = \frac{1}{k}$ with $k \in \mathbb{N}$, $R > 0$ and a smooth, 2π-periodic function $f : [0, \infty) \to \mathbb{R}$ we introduce the 2-dimensional submanifold of $\mathbb{R}^3$

$$M_\varepsilon := \left\{ \begin{pmatrix} r\sin\theta \\ r\cos\theta \\ \varepsilon f\left(\frac{\theta}{\varepsilon}\right) \end{pmatrix} ; r \in (0,R), \theta \in [0,2\pi) \right\}. \tag{3}$$

In Fig. 2 we present M_ε for some values of ε in the case $f = \sin^2$. As an application of our results we show that the spectrum of the Laplace–Beltrami operator on M_ε converges to the spectrum of the Laplace–Beltrami operator on the submanifold

$$N_0 := \left\{ \begin{pmatrix} \rho_0(r)\sin\theta \\ \rho_0(r)\cos\theta \\ \int_0^r \sqrt{1 - \rho_0'(t)^2}\, \mathrm{d}t \end{pmatrix} ; r \in (0,R), \theta \in [0,2\pi) \right\}, \tag{4}$$

where $\rho_0(r) = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{f'(y)^2 + r^2}\, \mathrm{d}y$, see Fig. 2. For generals values of $\varepsilon > 0$ the manifold M_ε is no longer smooth, but our results can be extended to this case.
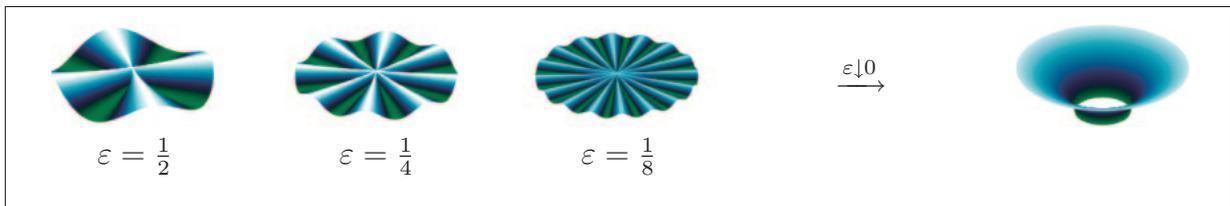

Fig. 2. A family of graphical surfaces with star-shaped corrugations. The three pictures on the left show M_ε defined by (3) with $f = \sin^2$ and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (4). As $\varepsilon \to 0$ the spectrum of the Laplace–Beltrami operator on M_ε converges to the spectrum of the Laplace–Beltrami operator on N_0. The color indicates the height component.

(c) A carambola-shaped sphere in $\mathbb{R}^3$. We can transfer the example above from a graph over $\mathbb{R}^2$ to a sphere with oscillatory perturbation of its radius as depicted in Fig. 3. More precisely, for $\varepsilon = \frac{1}{k}$ with $k \in \mathbb{N}$ and a smooth 2π-periodic function $f \colon [0,\infty) \to \mathbb{R}$ we consider the 2-dimensional submanifold of $\mathbb{R}^3$

$$M_\varepsilon := \left\{ \left(1 + \varepsilon f\left(\tfrac{\theta}{\varepsilon}\right)\right) \begin{pmatrix} \sin\varphi\sin\theta \\ \sin\varphi\cos\theta \\ \cos\varphi \end{pmatrix} ; \varphi \in (0,\pi), \theta \in [0,2\pi) \right\}. \tag{5}$$

In that case a limiting submanifold is given by

$$N_0 := \left\{ \begin{pmatrix} \rho_0(\varphi)\sin\theta \\ \rho_0(\varphi)\cos\theta \\ \int_0^\varphi \sqrt{1 - \rho_0'(t)^2}\, \mathrm{d}t \end{pmatrix} ; \varphi \in (0,\pi), \theta \in [0,2\pi) \right\}, \tag{6}$$

where $\rho_0(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{f'(y)^2 + \sin^2\varphi}\, \mathrm{d}y$. See Fig. 3 for a visualization in the case $f = \sin^2$.
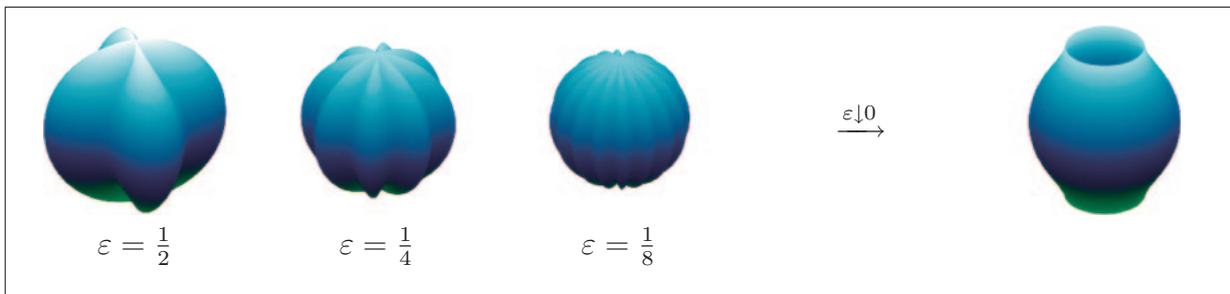

Fig. 3. A family of spheres with radial perturbations oscillating with the longitude. The three pictures on the left show M_ε defined by (5) with $f = \sin^2$ and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (6).

(d) A corrugated, rotationally symmetric submanifold in $\mathbb{R}^3$. In contrast to the previous example we assume a sphere with radial perturbations with the latitude, i.e., for $\varepsilon > 0$ and a smooth π-periodic function $f: [0, \infty) \to \mathbb{R}$ we consider the 2-dimensional submanifold of $\mathbb{R}^3$

$$M_\varepsilon := \left\{ (1 + \varepsilon f(\tfrac{\varphi}{\varepsilon})) \begin{pmatrix} \sin \varphi \sin \theta \\ \sin \varphi \cos \theta \\ \cos \varphi \end{pmatrix} ; \varphi \in (0, \pi), \theta \in [0, 2\pi) \right\}. \tag{7}$$

In that case a limiting submanifold is given by

$$N_0 := \left\{ \begin{pmatrix} \sin \varphi \sin \theta \\ \sin \varphi \cos \theta \\ \int_0^\varphi \sqrt{\frac{\rho_0(t)^2}{\sin^2 t} - \cos^2 t} \, dt \end{pmatrix} ; \varphi \in (0, \pi), \theta \in [0, 2\pi) \right\}, \tag{8}$$

where $\rho_0(\varphi) = \frac{\sin \varphi}{\pi} \int_0^\pi \sqrt{f'(y)^2 + 1} \, dy$. See Fig. 4 for the case $f = \sin^2$.
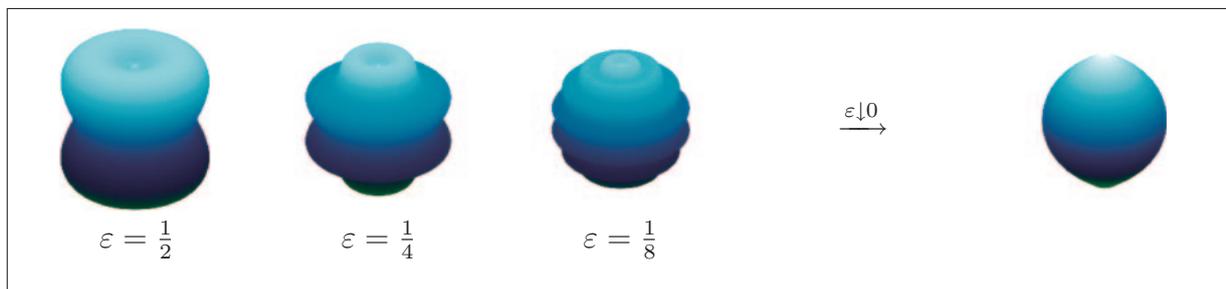

Fig. 4.   A family of spheres with radial perturbations oscillating with the latitude. The three pictures on the left show M_ε defined by (7) with $f = \sin^2$ and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (8).

(e) A locally corrugated graphical surface.   Consider a relatively-compact open set $Y \subset \mathbb{R}^2$ and a set $Z \subset Y$ of isolated points. For every point $z \in Z$ we use a smooth function $\psi_z: [0, \infty) \to [0, 1]$ to define a rotationally symmetric cut-off function $\psi_z(|\cdot - z|)$ such that

$$\begin{cases} \psi_z(0) = 1, \\ \operatorname{supp}\psi_z(|\cdot - z|) \cap \operatorname{supp}\psi_{z'}(|\cdot - z'|) = \emptyset \text{ for all } z' \in Z \setminus \{z\}. \end{cases}$$

Now we consider a smooth T-periodic function $f: [0, \infty) \to \mathbb{R}$ and the set M_ε which is the graph of the function

$$Y \setminus Z \ni x \mapsto \sum_{z \in Z} \varepsilon f\left(\tfrac{|x-z|}{\varepsilon}\right) \psi_z(|x - z|) \in \mathbb{R}^3, \tag{9}$$

which we regard as a two-dimensional submanifold of $\mathbb{R}^3$. In that case a limiting submanifold is given by

$$Y \setminus Z \ni x \mapsto \sum_{z \in Z} \int_0^{|x-z|} \sqrt{\frac{\rho_{0,z}(t)^2}{t^2} - 1} \, dt \in \mathbb{R}^3, \tag{10}$$

where $\rho_{0,z}(r) = \frac{r}{T} \int_0^T \sqrt{f'(y)^2 \psi_z(r)^2 + 1} \, dy$. See Fig. 5 for the case $f = \sin^2$.
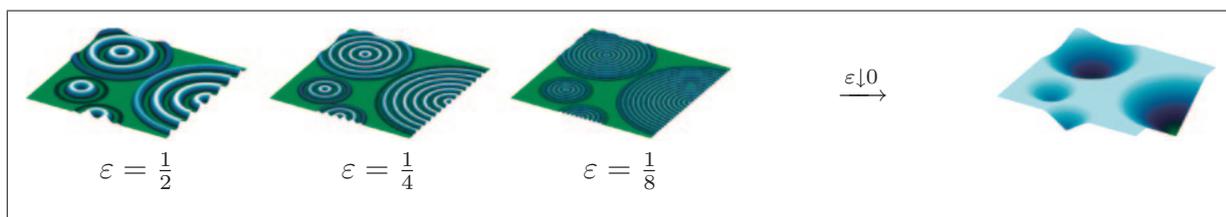
Fig. 5.   A family of locally corrugated graphical surfaces. The three pictures on the left show M_ε defined via (9) with $f = \sin^2$ and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (10).

General setting and the structure of the paper.   Throughout this paper we consider weighted Riemannian manifolds $M = (M, g, \mu)$ with metric g and measure μ. We *always* assume that M is *n-dimensional (with $n \geq 2$), smooth,*

connected, without boundary, and that μ has a smooth positive density against the Riemannian volume associated with g. We refer to the end of the introduction for a summary of standard notation that we use in this paper. The examples discussed above belong to the following general setting:

Definition 1 (Uniformly bi-Lipschitz diffeomorphic families of manifolds). *A family of weighted Riemannian manifolds* $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ *indexed by* $0 < \varepsilon < 1$ *is called* uniformly bi-Lipschitz diffeomorphic, *if there exits a weighted Riemannian manifold* (M_0, g_0, μ_0) *and a constant C such that for all ε there exist diffeomorphisms* $h_\varepsilon \colon M_0 \to M_\varepsilon$ *with*

$$\frac{1}{C}|\xi|_{g_0} \leq |dh_\varepsilon(x)\xi|_{g_\varepsilon} \leq C|\xi|_{g_0} \quad \text{for all } x \in M_0 \text{ and } \xi \in T_x M_0. \tag{11}$$

We call (M_0, g_0, μ_0) *reference manifold.*

In the setting of (11) the Laplace–Beltrami operator on M_ε gives rise to a second-order elliptic operator $\text{div}(\mathbb{L}_\varepsilon \nabla)$ on M_0 with a uniformly elliptic coefficient field $\mathbb{L}_\varepsilon$, i.e.,

$$g_0(\xi, \mathbb{L}_\varepsilon \xi) \geq \tfrac{1}{C^{n+2}}|\xi|_{g_0}^2, \quad g_0(\xi, \mathbb{L}_\varepsilon^{-1}\xi) \geq C^{n+2}|\xi|_{g_0}^2 \quad \text{for every } \xi \in TM_0,$$

see Sect. 2.3 for further details. It is therefore natural to consider homogenization of elliptic operators on the reference manifold with oscillating coefficients and measure. This is done in Sect. 2, where our results are presented.

Our main result, cf. Theorem 5, is a compactness result for *H*-convergence. In the symmetric case (e.g., for the Laplace–Beltrami operator) *H*-convergence implies *Mosco-convergence* of the associated energy forms, cf. Lemma 9, and the *convergence of the spectrum* of the associated second-order elliptic operators $-\text{div}(\mathbb{L}_\varepsilon \nabla)$, cf. Lemma 11. In Sect. 2.2 we address the problem of identifying the limiting PDE and manifold. In particular, we provide a *homogenization formula* for manifolds that feature *periodicity* in local coordinates. In Sect. 2.3 we discuss the application to families of parametrized manifolds that are bi-Lipschitz diffeomorphic. In particular, for such families, we establish spectral convergence (along subsequences) in Lemma 20 and discuss the special case of families of submanifolds of $\mathbb{R}^d$, see Lemma 21. In Sect. 3 we discuss concrete examples as the ones presented above. All proofs of the results in this paper are presented in Sect. 4.

Notation.   For the background of the analysis on manifolds, we refer the readers to [7, 12].

- Let $\Omega \subset M$ open. We write $\omega \Subset \Omega$ if ω is an open set such that the closure $\overline{\omega}$ is compact and $\overline{\omega} \subset \Omega$.
- We use h for a diffeomorphism between manifolds and denote its differential by dh. We use $\mathbb{L}$ for a measurable $(1, 1)$-tensor field on a manifold. We call $\mathbb{L}$ also a *coefficient field* on the manifold.
- We use the notation $(\cdot, \cdot)(x) = g(\cdot, \cdot)(x)$ and $|\xi|(x) = \sqrt{g(\xi, \xi)(x)}$ to denote the inner product and induced norm in $T_x M$ at $x \in M$. We tacitly simply write (ξ, η) and $|\xi|$ instead of $(\xi, \eta)(x)$ and $|\xi|(x)$ if the meaning is clear from the context.
- For a (sufficiently regular) function u and vector field ξ on Ω, the gradient of u is denoted by $\nabla_g u$ and the divergence of ξ is denoted by $\text{div}_{g,\mu} \xi$, i.e., we have $g(\nabla_g u, \xi) = \xi u = du(\xi)$ and $-\int_\Omega g(\text{div}_{g,\mu} \xi, u)\, d\mu = -\int_\Omega g(\xi, \nabla_g u)\, d\mu$ provided either u or ξ are compactly supported. In particular, we write $\triangle_{g,\mu} := \text{div}_{g,\mu} \nabla_g$ to denote the (weighted) Laplace–Beltrami operator. If the meaning is clear from the context, we shall simply write ∇, div, and $\triangle$. In some situations the Riemannian manifold will be parametrized by the parameter ε; in that case, we may us the notation ∇_ε, div_ε and $\triangle_\varepsilon$. If there is no danger of confusion, we may drop the index ε in the notation.
- For $\Omega \subset M$ open we denote by $L^2(\Omega, g, \mu)$ the Hilbert space of square integrable functions and denote by

$$\|u\|_{L^2(\Omega,g,\mu)}^2 := \int_\Omega |u|^2 \, d\mu$$

  the associated norm. We denote by $L^2(T\Omega)$ the space of measurable sections ξ of $T\Omega$ such that $|\xi| \in L^2(\Omega, g, \mu)$.
- We denote by $C_c^\infty(\Omega)$ the space of smooth compactly supported functions, and by $H^1(\Omega, g, \mu)$ the usual Sobolev space on (Ω, g, μ), i.e., the space of functions $u \in L^2(\Omega, g, \mu)$ with distributional first derivatives in $L^2(\Omega, g, \mu)$. Equipped with the norm

$$\|u\|_{H^1(\Omega,g,\mu)}^2 := \int_M |u|^2 + |\nabla u|^2 \, d\mu$$

  (and the usual inner product), $H^1(\Omega, g, \mu)$ is a Hilbert space.
- We denote by $H_0^1(\Omega, g, \mu)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega, g, \mu)$. We denote by $H^{-1}(\Omega, g, \mu)$ the dual space to $H_0^1(\Omega, g, \mu)$ and use the notation $\langle F, u \rangle_{(\Omega,g,\mu)}$ to denote the dual pairing of $F \in H^{-1}(\Omega, g, \mu)$ and $u \in H_0^1(M, g, \mu)$. We tacitly simply write Ω (instead of (Ω, g, μ)), $L^2(\Omega)$, $H^1(\Omega)$, $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^1(\Omega)}$, $\langle \cdot, \cdot \rangle$, if the meaning is clear from the context.

2.   Statement of the Main Results

2.1   H-, Mosco-, and spectral convergence

We are interested in second-order elliptic operators of the form

$$-\mathrm{div}(\mathbb{L}\nabla): H_0^1(\Omega) \to H^{-1}(\Omega), \quad \Omega \subset M \text{ open},$$

where $-\mathrm{div} = -\mathrm{div}_{g,\mu}: L^2(T\Omega) \to H^{-1}(\Omega)$ is the adjoint of $\nabla = \nabla_g: H_0^1(\Omega) \to L^2(T\Omega)$, and $\mathbb{L}$ denotes a uniformly elliptic coefficient field defined on Ω. More precisely, for $0 < \lambda \leq \Lambda$ and $\Omega \subset M$ open, we denote by $\mathcal{M}(\Omega, \lambda, \Lambda)$ the set of all measurable coefficient fields $\mathbb{L}: \Omega \to \mathrm{Lin}(T\Omega)$ that are uniformly elliptic, not necessarily symmetric, and bounded in the sense that for μ-a.e. $x \in \Omega$ and all $\xi \in T_x\Omega$

$$g(\xi, \mathbb{L}(x)\xi) \geq \lambda |\xi|^2, \tag{12}$$

$$g(\xi, (\mathbb{L}(x))^{-1}\xi) \geq \tfrac{1}{\Lambda}|\xi|^2. \tag{13}$$

Moreover, we define

$$m_0(\Omega) := -\inf\left\{ \frac{\int_\Omega g(\nabla u, \nabla u)\,\mathrm{d}\mu}{\int_\Omega u^2\,\mathrm{d}\mu}; \quad u \in H_0^1(\Omega) \right\}.$$

(See Remark 2 below for a discussion of $m_0(\Omega)$). Given a family $(\mathbb{L}_\varepsilon)_{\varepsilon>0} \subset \mathcal{M}(\Omega, \lambda, \Lambda)$ and $f \in H^{-1}(\Omega)$ we study the asymptotic behavior as $\varepsilon \downarrow 0$ of the weak solution $u_\varepsilon \in H_0^1(\Omega)$ to the equation

$$mu_\varepsilon - \mathrm{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) = f \quad \text{in } H^{-1}(\Omega), \tag{14}$$

where m denotes a fixed scalar satisfying $m > \frac{m_0(\Omega)}{\lambda}$.

Remark 2.   • *We briefly comment on the constant $m_0(\Omega)$. First, the quotient $\frac{\int_\Omega g(\nabla u, \nabla u)\,\mathrm{d}\mu}{\int_\Omega u^2\,\mathrm{d}\mu}$ appearing in the definition of $m_0(\Omega)$ is the Rayleigh Quotient. Hence, $m_0(\Omega)$ is just the negative of the infimum of the spectrum of the Dirichlet Laplace–Beltrami operator on the weighted Riemannian manifold. In particular, if the spectrum is a pure point spectrum, then $-m_0(\Omega)$ is given by the lowest Dirichlet eigenvalue.*

• *In the special case that $\Omega \Subset M$ is relatively-compact and connected, Poincaré's inequality (for functions with zero mean) holds:*

$$\forall u \in H^1(\Omega): \quad \int_\Omega \left| u - \tfrac{1}{\mu(\Omega)} \int_\Omega u\,\mathrm{d}\mu \right|^2 \mathrm{d}\mu \leq C_\Omega \int_\Omega |\nabla u|^2\,\mathrm{d}\mu.$$

In this case we have $m_0(\Omega) \leq 0$, and in (14) any $m > 0$ is admissible. Also note that, the condition $m_0(\Omega) < 0$ is equivalent to the validity of Poincare's inequality (for functions with vanishing boundary conditions):

$$\forall u \in H_0^1(\Omega): \quad \int_\Omega |u|^2\,\mathrm{d}\mu \leq C_\Omega' \int_\Omega |\nabla u|^2\,\mathrm{d}\mu, \tag{15}$$

where $C_\Omega' > 0$ denotes a generic constant (only depending on n).

• *It is easy to check that $m > m_0(\Omega)$ if and only if*

$$\inf\left\{ \int_\Omega m|u|^2 + g(\nabla u, \nabla u)\,\mathrm{d}\mu; \quad u \in H_0^1(\Omega) \text{ with } \|u\|_{H_0^1(\Omega)} = 1 \right\} > 0.$$

Similarly, one can check that $m > \frac{m_0(\Omega)}{\lambda}$ implies that the bounded, bilinear form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}, \quad a(u,v) := m \int_\Omega uv\,\mathrm{d}\mu + \int_\Omega g(\mathbb{L}\nabla u, \nabla v)\,\mathrm{d}\mu$$

is coercive. Therefore, by the Lax-Milgram lemma, (14) admits a unique weak solution $u_\varepsilon \in H_0^1(\Omega)$ satisfying

$$\|u_\varepsilon\|_{H^1(\Omega)} \leq C(\Omega, \lambda, m)\|f\|_{H^{-1}(\Omega)}. \tag{16}$$

Remark 3.   *The condition (13) is a boundedness condition for $\mathbb{L}$ and equivalent to*

$$g(\eta, \mathbb{L}(x)\xi) \leq \widehat{\Lambda}|\eta||\xi| \quad \text{for all } \xi, \eta \in T_x\Omega, \tag{17}$$

for μ-a.e. $x \in \Omega$ and a suitable constant $\widehat{\Lambda} > 0$ that is independent of $x \in \Omega$. Note that the constant Λ in (13) is stable under H-convergence (in the sense that $\mathcal{M}(\Omega, \lambda, \Lambda)$ is closed under H-convergence, see Proposition 6). In contrast, the constant in the alternative condition (17) is not stable.

H-compactness.   Our first main result is a compactness result concerning the homogenization limit $\varepsilon \downarrow 0$. It relies on the notion of *H*-convergence which goes back to the seminal work by Murat and Tartar ([25]) where the notion is

introduced in the flat case $M = \mathbb{R}^n$. It is a generalization of the notion of *G*-convergence by Spagnolo and De Giorgi. The definition of *H*-convergence can be phrased in our setting as follows:

Definition 4 (*H*-convergence). *Let $\Omega \subset M$ be open. We say a sequence $(\mathbb{L}_\varepsilon) \subset \mathcal{M}(\Omega, \lambda, \Lambda)$ H-converges in (Ω, g, μ) to $\mathbb{L}_0 \in \mathcal{M}(\Omega, \lambda, \Lambda)$ as $\varepsilon \to 0$, if for any relatively-compact open subset $\omega \Subset \Omega$ with $m_0(\omega) < 0$, and any $f \in H^{-1}(\omega)$, the unique solutions $u_\varepsilon, u_0 \in H_0^1(\omega)$ to*

$$-\mathrm{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) = -\mathrm{div}(\mathbb{L}_0 \nabla u_0) = f \quad in \ H^{-1}(\omega)$$

satisfy

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & weakly \ in \ H^1(\omega), \\ \mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla u_0 & weakly \ in \ L^2(T\omega). \end{cases}$$

In that case we write $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ in (Ω, μ, g) as $\varepsilon \to 0$.

Our main result is the following *H*-compactness statement:

Theorem 5. *Let $\lambda, \Lambda > 0$ and let $(\mathbb{L}_\varepsilon)$ denote a sequence in $\mathcal{M}(M, \lambda, \Lambda)$. Then there exist a subsequence (not relabeled) and $\mathbb{L}_0 \in \mathcal{M}(M, \lambda, \Lambda)$ such that the following holds:*

(a) $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ in (M, g, μ).
(b) *For every $\Omega \subset M$ open, every $m > \frac{m_0(\Omega)}{\lambda}$, and sequences $(f_\varepsilon) \subset L^2(\Omega)$ and $(F_\varepsilon) \subset L^2(T\Omega)$ with*

$$\begin{cases} f_\varepsilon \rightharpoonup f_0 & weakly \ in \ L^2(\Omega), \\ F_\varepsilon \to F_0 & in \ L^2(T\Omega), \end{cases}$$

the solutions $u_\varepsilon, u_0 \in H_0^1(\Omega)$ to

$$\begin{aligned} mu_\varepsilon - \mathrm{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) &= f_\varepsilon + \mathrm{div}\, F_\varepsilon \quad in \ H^{-1}(\Omega), \\ mu_0 - \mathrm{div}(\mathbb{L}_0 \nabla u_0) &= f_0 + \mathrm{div}\, F_0 \quad in \ H^{-1}(\Omega), \end{aligned} \tag{18}$$

satisfy

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & weakly \ in \ H_0^1(\Omega), \\ \mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla u_0 & weakly \ in \ L^2(T\Omega). \end{cases}$$

Additionally we have $u_\varepsilon \to u_0$ strongly in $L^2(\Omega)$, if either $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, or $m \neq 0$ and $f_\varepsilon \to f_0$ strongly in $L^2(\Omega)$.

For the proof see Sect. 4.2. The theorem is an extension of a classical result in [25] where (scalar) elliptic operators of the form $-\mathrm{div}(A_\varepsilon \nabla)$ on $\mathbb{R}^n$ are considered. It has been extended to a large class of elliptic equations on $\mathbb{R}^n$ including e.g., linear elasticity [4] and monotone operators for vector valued fields ([5]). See also [31] for a variant that applies to non-local operators.

In the following we briefly comment on the proof of Theorem 5, which is based on Murat and Tartar's method of oscillating test-functions. In contrast to the classical flat case $M = \mathbb{R}^n$, we require a localization argument, since the tangent spaces $T_x M$ change when x varies in M. We therefore first establish *H*-compactness restricted to sufficiently small balls B (see Proposition 6 below) and then argue by covering M with countably many of such balls.

Proposition 6 (*H*-compactness on small balls). *Let $(\mathbb{L}_\varepsilon) \subset \mathcal{M}(M, \lambda, \Lambda)$ and let $B \Subset M$ denote an open ball with radius smaller than the injectivity radius at its center. Then there exits $\mathbb{L}_0 \in \mathcal{M}(\frac{1}{2}B, \lambda, \Lambda)$ and a (not relabeled) subsequence of $(\mathbb{L}_\varepsilon)$ such that $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ in $\frac{1}{2}B$, which is the open ball with the same center point and half the radius of B.*

To lift Proposition 6 from small balls to the whole manifold we cover M by a countable collection of sufficiently small balls and pass to a diagonal sequence that features *H*-convergence on each of these balls. In order to guarantee that the *H*-limits associated with these balls are identical on the intersections of the balls, we appeal to the following lemma, which in particular establishes the uniqueness and locality property of *H*-convergence:

Lemma 7 (Uniqueness, locality, invariance w.r.t. transposition). *Let $\Omega \subset M$ be open and consider a sequences $(\mathbb{L}_\varepsilon) \subset \mathcal{M}(\Omega, \lambda, \Lambda)$ that H-converges to some $\mathbb{L}_0$ in Ω.*

(a) *Let $(\widetilde{\mathbb{L}}_\varepsilon) \subset \mathcal{M}(\Omega, \lambda, \Lambda)$ denote another sequence that H-converges to some $\widetilde{\mathbb{L}}_0$ in Ω. Suppose that for some open $\omega \Subset \Omega$ we have $\mathbb{L}_\varepsilon = \widetilde{\mathbb{L}}_\varepsilon$ in ω for all ε. Then $\mathbb{L}_0 = \widetilde{\mathbb{L}}_0$ μ-a.e. in ω.*
(b) *Consider the coefficient field $\mathbb{L}_\varepsilon^*$ defined by the identity*

$$g(\mathbb{L}_\varepsilon^* \xi, \eta) = g(\xi, \mathbb{L}_\varepsilon \eta) \quad for \ all \ \xi, \eta \in T\Omega,$$

i.e., the adjoint of $\mathbb{L}_\varepsilon$. Then $(\mathbb{L}_\varepsilon^*)$ H-converges in Ω to $\mathbb{L}_0^*$ (the adjoint of $\mathbb{L}_0$).

Finally, to prove that *H*-convergence on the individual balls yields *H*-convergence on the entire manifold, and in order to treat the varying right-hand sides in part (b) of Theorem 5, we apply the following lemma.

Lemma 8. *Let $\Omega \subset M$ be open and $\mathbb{L}_\varepsilon \xrightarrow{H} \mathbb{L}_0$ in Ω. Let $\omega \Subset \Omega$ with $m_0(\omega) < 0$. Then for every $f_\varepsilon, f_0 \in L^2(\omega)$ and $G_\varepsilon, F_\varepsilon, G_0, F_0 \in L^2(T\omega)$ with*

$$\begin{cases} f_\varepsilon \rightharpoonup f_0 & \text{weakly in } L^2(\omega), \\ G_\varepsilon \to G_0 & \text{in } L^2(T\omega), \\ F_\varepsilon \to F_0 & \text{in } L^2(T\omega), \end{cases}$$

the unique solutions $u_\varepsilon, u_0 \in H_0^1(\omega)$ to

$$-\mathrm{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) = f_\varepsilon - \mathrm{div}(\mathbb{L}_\varepsilon G_\varepsilon) - \mathrm{div}\, F_\varepsilon \quad \text{in } H^{-1}(\omega),$$

$$-\mathrm{div}(\mathbb{L}_0 \nabla u_0) = f_0 - \mathrm{div}(\mathbb{L}_0 G_0) - \mathrm{div}\, F_\varepsilon \quad \text{in } H^{-1}(\omega)$$

satisfy

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H_0^1(\omega), \\ \mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla u_0 & \text{weakly in } L^2(T\omega). \end{cases}$$

Mosco-convergence and convergence of the spectrum. If we restrict to the symmetric case, i.e., $\mathbb{L}_\varepsilon$ satisfies

$$g(\mathbb{L}_\varepsilon \xi, \eta) = g(\xi, \mathbb{L}_\varepsilon \eta) \quad \text{for all } \xi, \eta \in TM,$$

the solutions to (18) can be characterized as the unique minimizers in $H_0^1(\Omega)$ to the strictly convex and coercive functional

$$H_0^1(\Omega) \ni u \mapsto \mathcal{E}_{m,\varepsilon}(u) - \int_M f_\varepsilon u + g(F_\varepsilon, \nabla u)\, d\mu,$$

where

$$\mathcal{E}_{m,\varepsilon}(u) := \tfrac{1}{2}\int_\Omega m|u|^2 + g(\mathbb{L}_\varepsilon \nabla u, \nabla u)\, d\mu.$$

In this symmetric situation we can appeal to variational notions of convergence, in particular Γ-convergence and Mosco-convergence. The latter is extensively used to study the convergence properties of the associated evolution (i.e., the semigroup generated by $-\mathrm{div}(\mathbb{L}_\varepsilon \nabla)$), e.g., see [16, 17, 19, 21, 22]. See a work by Hino ([9]) for a non-symmetric generalization of Mosco-convergence. A simple argument (that we outline for the reader's convenience — together with the definition of Mosco-convergence — in the appendix) shows that *H*-convergence implies Mosco-convergence (resp. Resolvent convergence):

Lemma 9 (*H*-convergence implies Mosco-convergence)**.** *Let $\mathbb{L}_\varepsilon \in \mathcal{M}(M, \lambda, \Lambda)$ be symmetric. Suppose $\mathbb{L}_\varepsilon \xrightarrow{H} \mathbb{L}_0$, then the functional $\mathcal{E}_\varepsilon \colon L^2(M) \to \mathbb{R} \cup \{+\infty\}$,*

$$\mathcal{E}_\varepsilon(u) = \begin{cases} \displaystyle\int_M (\mathbb{L}_\varepsilon \nabla u, \nabla u)\, d\mu & u \in H_0^1(M), \\ \infty & \text{otherwise} \end{cases}$$

Mosco-converges to $\mathcal{E}_0 \colon L^2(M) \to \mathbb{R} \cup \{+\infty\}$,

$$\mathcal{E}_0(u) = \begin{cases} \displaystyle\int_M (\mathbb{L}_0 \nabla u, \nabla u)\, d\mu & u \in H_0^1(M), \\ \infty & \text{otherwise.} \end{cases}$$

Remark 10. *The notion of Mosco-convergence only directly yields strong convergence of (u_ε) in $L^2(M)$ (and weak convergence in $H^1(M)$). The notion of H-convergence is a bit stronger, since it also yields convergence of the fluxes $\mathbb{L}_\varepsilon \nabla u_\varepsilon$. In contrast, Mosco-convergence in conjunction with the Div-Curl Lemma, see Lemma 25 below, only yields convergence of the L^2-projection of $\mathbb{L}_\varepsilon \nabla u_\varepsilon$ onto the orthogonal complement of $\{\nabla \phi : \phi \in H_0^1(M)\} \subset L^2(T\Omega)$.*

Another consequence of *H*-convergence is convergence of the spectrum. In the following we briefly recall some well-known facts regarding the spectral theory for the operator $\mathcal{L}_\varepsilon := -\mathrm{div}(\mathbb{L}_\varepsilon \nabla) : H_0^1(\Omega) \to H^{-1}(\Omega)$. We consider an open, relatively-compact subset $\Omega \Subset M$ and suppose that $m_0(\Omega) < 0$, so that Poincaré's inequality (15) is available and the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ is compact. Moreover, we consider a symmetric, uniformly elliptic coefficient field $\mathbb{L}_\varepsilon \in \mathcal{M}(M, \lambda, \Lambda)$. We call (λ, u) with $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$ and eigenpair of $\mathcal{L}_\varepsilon$, if $\mathcal{L}_\varepsilon u = \lambda u$ in $H^{-1}(\Omega)$. To study the spectrum of $\mathcal{L}_\varepsilon$ we consider the associate resolvent operator $\mathcal{R}_\varepsilon : L^2(\Omega) \to L^2(\Omega)$, $\mathcal{R}_\varepsilon := \mathcal{L}_\varepsilon^{-1}$. It is a compact, self-adjoint operator on $L^2(\Omega)$ and thus the spectral theorem implies that the spectrum of $\mathcal{R}_\varepsilon$ consists only of the real point spectrum with strictly positive eigenvalues. Moreover, the associated (normalized) eigenfunctions form an orthonormal

basis of $L^2(\Omega)$. The spectrum of $\mathscr{R}_\varepsilon$ is in one-to-one relation with the spectrum of $\mathscr{L}_\varepsilon$ in the sense that (λ, u) is an eigenpair of $\mathscr{L}_\varepsilon$ if and only if $(\frac{1}{\lambda}, u)$ is an eigenpair of $\mathscr{R}_\varepsilon$. We thus conclude that: The spectrum of $\mathscr{L}_\varepsilon$ only consists of the real point spectrum, all eigenvalues are strictly positive, and that we can find an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions. The following statement shows that if $\mathbb{L}_\varepsilon$ is H-convergent, then the eigenspaces and eigenvalues converge. The statement is a direct consequence of [11, Lemma 11.3 and Theorem 11.5, see also Theorem 11.6] combined with Theorem 5:

Lemma 11 (*H*-convergence implies spectral convergence). *Let $(\mathbb{L}_\varepsilon)$ be a sequence of symmetric coefficient fields in $\mathcal{M}(M, \lambda, \Lambda)$ and suppose that $\mathbb{L}_\varepsilon \xrightarrow{H} \mathbb{L}_0$. Consider an open, relatively-compact set $\Omega \subset M$ with $m_0(\Omega) < 0$. For $\varepsilon \geq 0$ we consider the operator*

$$-\mathrm{div}(\mathbb{L}_\varepsilon \nabla) : H_0^1(\Omega) \to H^{-1}(\Omega),$$

and let

$$0 < \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \lambda_{\varepsilon,3} \leq \cdots$$

denote the list of increasingly ordered eigenvalues, where eigenvalues are repeated according to their multiplicity. Let $u_{\varepsilon,1}, u_{\varepsilon,2}, u_{\varepsilon,3}, \ldots$ denote a list of associated eigenfunctions (forming an orthonormal basis of $L^2(\Omega)$). Then for all $k \in \mathbb{N}$,

$$\lambda_{\varepsilon,k} \to \lambda_{0,k},$$

and if $s \in \mathbb{N}$ denotes the multiplicity of $\lambda_{0,k}$, i.e.,

$$\lambda_{0,k-1} < \lambda_{0,k} = \cdots = \lambda_{0,k+s-1} < \lambda_{0,k+s} \quad (\text{with the convention } \lambda_{0,0} = 0),$$

then there exists a sequence $\bar{u}_{\varepsilon,k}$ of linear combinations of $u_{\varepsilon,k}, \ldots, u_{\varepsilon,k+s-1}$ such that

$$\bar{u}_{\varepsilon,k} \to u_{0,k} \quad \text{strongly in } L^2(\Omega).$$

2.2 Identification of the limit via local coordinate charts

For a general sequence of coefficient fields $(\mathbb{L}_\varepsilon)$ the H-limit $\mathbb{L}_0$ obtained by Theorem 5 depends on the choice of the subsequence. In contrast, if the coefficient field features a special structure, then the H-limit is unique, the convergence holds for the entire sequence and one might even have a homogenization formula for $\mathbb{L}_0$. In the flat case $M = \mathbb{R}^n$ such results are classical. The simplest (non-trivial) example is periodic homogenization when $\mathbb{L}_\varepsilon(x) = \mathbb{L}(\frac{x}{\varepsilon})$ where $\mathbb{L}$ is periodic, i.e., $\mathbb{L}(\cdot + k) = \mathbb{L}(\cdot)$ a.e. in $\mathbb{R}^n$ for all $k \in \mathbb{Z}^n$; another example is stochastic homogenization, when $\mathbb{L}_\varepsilon(x) = \mathbb{L}(\frac{x}{\varepsilon})$ and $\mathbb{L}$ is sampled from a stationary and ergodic ensemble, see the seminal papers [29] or [26] for a self-contained introduction to periodic and stochastic homogenization. In the flat case these results rely on the fact that we can define an ergodic group action on the manifold M. For general manifolds this is not possible. In this section we first make the simple observation that a coefficient field locally H-converges if and only if the coefficient field expressed in local coordinates H-converges, and secondly, obtain H-convergence and a homogenization formula for *locally* periodic coefficient fields on general manifolds.

For this purpose we fix $(\Omega, \Psi; x^1, x^2, \ldots, x^n)$ a local coordinate chart of M, a relatively-compact set $U \Subset \Psi(\Omega) \subset \mathbb{R}^n$, and set $\omega := \Psi^{-1}(U) \subset \Omega$. We will suppress Ψ when the meaning is clear from the context. In particular, for the representation of a function u on Ω in local coordinates we shall simply write u instead of $u \circ \Psi^{-1}$. We associate to $\mathbb{L} \in \mathcal{M}(\omega, \lambda, \Lambda)$ a density ρ and a coefficient field $A : U \to \mathbb{R}^{n \times n}$ with components

$$A_{ij} := \rho \, g(\mathbb{L}\nabla_g x^i, \nabla_g x^j) \quad \text{for all } i, j = 1, \ldots, n, \quad \rho = \sigma\sqrt{\det g}, \tag{19}$$

where σ is the density of μ against the Riemannian volume measure.

Lemma 12. *Let $\mathbb{L} \in \mathcal{M}(\omega, \lambda, \Lambda)$ and let $A : U \to \mathbb{R}^{n \times n}$ be defined by (19). Then there exist $0 < \lambda' \leq \Lambda' < \infty$ (only depending on Ψ, U, λ, and Λ) such that we have*

$$\forall \xi \in \mathbb{R}^n : \quad A\xi \cdot \xi \geq \lambda'|\xi|^2 \quad and \quad A^{-1}\xi \cdot \xi \geq \frac{1}{\Lambda'}|\xi|^2 \quad a.e. \text{ in } U,$$

where "·" denotes the scalar product in $\mathbb{R}^n$.

Next we express the elliptic equation in local coordinates. For $f \in L^2(\omega)$ and $\xi \in L^2(T\omega)$ let $u \in H_0^1(\omega)$ be the unique solution to

$$-\mathrm{div}_{g,\mu}(\mathbb{L}\nabla_g u) = f - \mathrm{div}_{g,\mu}\xi \quad \text{in } H^{-1}(\omega),$$

that is

$$\int_\omega g(\mathbb{L}\nabla_g u, \nabla_g \varphi)\,\mathrm{d}\mu = \int_\omega f\varphi\,\mathrm{d}\mu + \int_\omega g(\xi, \nabla_g \varphi)\,\mathrm{d}\mu \quad \text{for all } \varphi \in H_0^1(\omega).$$

Let $F \in L^2(TU) \cong L^2(U; \mathbb{R}^n)$ be the vector field on U with the components $F^i = dx^i(\xi)$. Then

$$-\text{div}(A\nabla u) = \rho f - \text{div}(\rho F) \quad \text{in } H^{-1}(U), \tag{20}$$

that is, for any $\psi \in C_c^\infty(U)$

$$\int_U A\nabla u \cdot \nabla \psi \, dx = \int_U \rho f \psi \, dx + \int_U \rho F \cdot \nabla \psi \, dx,$$

where "$\cdot$" stands for the scalar product in $\mathbb{R}^n$.

With help of this transformation we can make the following observation:

Lemma 13. *Let $\mathbb{L}_\varepsilon, \mathbb{L}_0 \in \mathcal{M}(\omega, \lambda, \Lambda)$ and denote by A_ε, A_0 be defined by (19). Then the following assertions are equivalent.*
  (1) *$(\mathbb{L}_\varepsilon)$ H-converges to $\mathbb{L}_0$ on (ω, g, μ).*
  (2) *(A_ε) H-converges to A_0 on U equipped with the standard Euclidean metric and measure.*

On the level of A_ε (which is defined on the "flat" open subset $U \subset \mathbb{R}^n$), we can naturally consider periodic homogenization. In the following we denote by $Y := [0,1)^n$ the reference cell of periodicity and by $H_\#^1(Y)$ the Hilbert-space of Y-periodic functions $\phi \in H^1(Y)$ with zero average, i.e., $\int_Y \phi = 0$. We denote by $\mathcal{M}_{\text{per}}(\lambda, \Lambda)$ the class of Y-periodic coefficient fields $A: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ with ellipticity constants $0 < \lambda \le \Lambda < \infty$, that is

$$A(\cdot, y) \text{ is continuous for a.e. } y \in \mathbb{R}^n, \tag{21}$$

$$A(x, \cdot) \text{ is measurable and } Y\text{-periodic for each } x \in \mathbb{R}^n, \tag{22}$$

$$A(x,y)\xi \cdot \xi \ge \lambda |\xi|^2 \text{ and } A(x,y)^{-1}\xi \cdot \xi \ge \tfrac{1}{\Lambda}|\xi|^2 \text{ for each } x \in \mathbb{R}^n, \text{ a.e. } y \in \mathbb{R}^n$$

$$\text{and all } \xi \in \mathbb{R}^n. \tag{23}$$

It is a classical result (see e.g., [2, Theorem 2.2]) that for $A \in \mathcal{M}_{\text{per}}(\lambda, \Lambda)$ the sequence $A_\varepsilon(x) := A(x, \frac{x}{\varepsilon})$ H-converges to a homogenized coefficient field A_{hom} which is characterized as follows:

$$A_{\text{hom}}(x)e_j = \int_Y A(x,y)(\nabla_y \phi_j(x,y) + e_j)\, dy, \tag{24}$$

where (e_j) is the standard basis in $\mathbb{R}^n$, and $\phi_j(x, \cdot) \in H_\#^1(Y)$ denotes the unique weak solution to

$$\int_Y A(x,y)(\nabla_y \phi_j(x,y) + e_j) \cdot \nabla_y \psi(y)\, dy = 0 \quad \text{for all } \psi \in H_\#^1(Y). \tag{25}$$

For our purpose we require a small variant of this classical result which includes an additional shift in the definition of A_ε:

Lemma 14. *Let $A \in \mathcal{M}_{\text{per}}(\lambda, \Lambda)$ and $r \in \mathbb{R}$. The sequence $A_\varepsilon(x) := A(x, \frac{x+r}{\varepsilon})$ H-converges on $\mathbb{R}^n$ to A_{hom} as defined in (24).*

Since we could not find a suitable reference in the literature we give the argument in the appendix. By appealing to periodic homogenization, we can make the following observation:

Lemma 15 (Homogenization formula). *Let $\mathbb{L}_\varepsilon, \mathbb{L}_0 \in \mathcal{M}(M, \lambda, \Lambda)$ and suppose that $(\mathbb{L}_\varepsilon)$ H-converges to $\mathbb{L}_0$ on M. Fix a local coordinate chart $(\Omega, \Psi; x^1, x^2, \ldots, x^n)$ and let A_ε, A_0 be the coefficient fields on $U \Subset \Psi(\Omega)$ associated with $\mathbb{L}_\varepsilon$ and $\mathbb{L}_0$ defined by (19). Suppose local periodicity in the sense that there exists a $Y := [0,1)^n$-periodic coefficient field $L: \mathbb{R}^n \to \mathbb{R}^{n \times n}$ such that*

$$g\left(\mathbb{L}_\varepsilon(x)\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right) = L_{ij}(x, \tfrac{x}{\varepsilon}) \quad \text{for a.e. } x \in \Omega.$$

Then $\mathbb{L}_0$ on $\omega = \Psi^{-1}(U) \subset \Omega$ in local coordinates takes the form

$$(A_{\text{hom}})_{ij} = \rho g(\mathbb{L}_0 \nabla_g x^i, \nabla_g x^j) \quad \text{a.e. in } U,$$

where $A_{\text{hom}}: U \to \mathbb{R}^{d \times d}$ is defined by (24) with $A(x,y) := \rho(x)L(y)$.

2.3 Asymptotic behavior of the Laplace–Beltrami on parametrized manifolds

In this section we consider weighted Riemannian manifolds $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ that are bi-Lipschitz diffeomorphic to a reference manifold (M_0, g_0, μ_0) in the sense of Definition 1. In particular, below we shall consider the special case of submanifolds of $\mathbb{R}^d$ and study the asymptotic behavior of the associated Laplace–Beltrami operator. In our approach we pull the Laplace–Beltrami operator on M_ε, $\Delta_{g_\varepsilon, \mu_\varepsilon}$, back to the reference manifold M_0 by appealing to the diffeomorphism h_ε from Definition 1. In this way we obtain a family of elliptic operators on M_0 with coefficients $\mathbb{L}_\varepsilon$. By appealing to our result on H-compactness, cf. Theorem 5, we may extract a subsequence along which the elliptic operators H-converge to a limiting operator of the form $\text{div}(\mathbb{L}_0 \nabla)$. In the symmetric case, we may combine this with our results with Lemma 9 and 11 to deduce Mosco-convergence and convergence of the spectrum.

We start with a transformation rule. It invokes the following notation: If (M, g, μ) and (M_0, g_0, μ_0) are Riemannian

manifolds, and $h : M_0 \to M$ a diffeomorphism, then for every function f on M we denote by $\overline{f} := f \circ h$ the pullback of f along h. Moreover, we denote by $(dh^{-1})^* : TM_0 \to TM$ the adjoint of the differential $dh^{-1} : TM \to TM_0$ of h^{-1} given by

$$g((dh^{-1})^*\xi, \eta)(h(x)) = g_0(\xi, dh^{-1}\eta)(x) \quad \text{for all } \xi \in T_x M_0, \ \eta \in T_{h(x)}M.$$

Lemma 16 (Transformation lemma). *Let (M, g, μ) and (M_0, g_0, μ_0) be weighted Riemannian manifolds and assume that there exists a bi-Lipschitz diffeomorphism $h : M_0 \to M$ satisfying (11). Let σ and σ_0 denote the densities of μ and μ_0 w.r.t. the Riemannian volume measures associated with g and g_0, respectively. We use the notation $\overline{f} := f \circ h$ and $\overline{u} := u \circ h$ for the pullback along h. We define a density function ρ and a coefficient field $\mathbb{L}$ on M_0 by the identities*

$$\rho := \frac{\overline{\sigma}}{\sigma_0}\sqrt{\frac{\det \overline{g}}{\det g_0}} \quad \text{and} \quad g_0(\mathbb{L}\xi, \eta) = \rho\,\overline{g}((dh^{-1})^*\xi, (dh^{-1})^*\eta),$$

where $\overline{\sigma} := \sigma \circ h$ and $\overline{g} := g \circ h$ denote the pulled back quantities. Moreover we consider the metric $\hat{g}_0$ and the measure $\hat{\mu}_0$ on M_0 given by

$$d\hat{\mu}_0 := \rho\,d\mu_0 \quad \text{and} \quad \hat{g}_0(\mathbb{L}\xi, \eta) := \rho\,g_0(\xi, \eta),$$

Then the following are equivalent:
(a) *$u \in H^1(M)$ is a solution to*

$$(m - \Delta_{g,\mu})u = f \quad \text{in } H^{-1}(M, g, \mu);$$

(b) *$\overline{u} \in H^1(M_0)$ is a solution to*

$$(m\rho - \mathrm{div}_{g_0,\mu_0}(\mathbb{L}\nabla_{g_0}))\overline{u} = \rho\overline{f} \quad \text{in } H^{-1}(M_0, g_0, \mu_0);$$

(c) *$\overline{u} \in H^1(M_0)$ is a solution to*

$$(m - \Delta_{\hat{g}_0,\hat{\mu}_0})\overline{u} = \overline{f} \quad \text{in } H^{-1}(M_0, \hat{g}_0, \hat{\mu}_0).$$

In the rest of this section, we consider the following setting:

Assumption 17 (Family of uniformly bi-Lipschitz diffeomorphic manifolds). *We denote by $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ a family of weighted Riemannian manifolds that are bi-Lipschitz diffeomorphic to a reference manifold (M_0, g_0, μ_0) in the sense of Definition 1. We assume that $H^1(M_0, g_0, \mu_0)$ is compactly embedded in $L^2(M_0, g_0, \mu_0)$. We denote by σ_ε and σ_0 the densities of μ_ε and μ_0 w.r.t. the Riemannian volume measures associated with g_ε and g_0, respectively. Moreover, we define ρ_ε and $\mathbb{L}_\varepsilon$ by the identities*

$$\rho_\varepsilon := \frac{\overline{\sigma}_\varepsilon}{\sigma_0}\sqrt{\frac{\det \overline{g}_\varepsilon}{\det g_0}} \quad \text{and} \quad g_0(\mathbb{L}_\varepsilon \xi, \eta) = \rho_\varepsilon\,\overline{g}_\varepsilon((dh_\varepsilon^{-1})^*\xi, (dh_\varepsilon^{-1})^*\eta) \tag{26}$$

with $\overline{\sigma}_\varepsilon := \sigma_\varepsilon \circ h_\varepsilon$ and $\overline{g}_\varepsilon := g_\varepsilon \circ h_\varepsilon$.

We introduce the following notion of strong L^2-convergence for functions defined on the variable spaces $L^2(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$:

Definition 18. *In the setting of Assumption 17. Let $f_\varepsilon \in L^2(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ and $f_0 \in L^2(M_0, \hat{g}_0, \hat{\mu}_0)$. We say (f_ε) strongly converges to f_0 in L^2, if*

$$\int_{M_\varepsilon} f_\varepsilon(\psi \circ h_\varepsilon^{-1})\,d\mu_\varepsilon \to \int_{M_0} f_0\psi\,d\hat{\mu}_0 \quad \text{for all } \psi \in C_c^\infty(M_0), \quad \text{and}$$

$$\int_{M_\varepsilon} |f_\varepsilon|^2\,d\mu_\varepsilon \to \int_{M_0} |f_0|^2\,d\hat{\mu}_0. \tag{27}$$

Lemma 19 (*H*-Compactness of bi-Lipschitz diffeomorphic manifolds). *Consider the setting of Assumption 17. Then there exists a subsequence for $\varepsilon \to 0$ (not relabeled) such that the following holds:*
(a) *There exist a density ρ_0 and a uniformly elliptic coefficient field $\mathbb{L}_0$ on M_0 such that (ρ_ε) converges to ρ_0 weak-$*$ in $L^\infty(M_0)$, and $(\mathbb{L}_\varepsilon)$ H-converges to $\mathbb{L}_0$ in (M_0, g_0, μ_0).*
(b) *Define a measure $\hat{\mu}_0$ and a metric $\hat{g}_0$ on M_0 via the identities*

$$d\hat{\mu}_0 := \rho_0\,d\mu_0 \quad \text{and} \quad \hat{g}_0(\mathbb{L}_0\xi, \eta) = \rho_0\,g_0(\xi, \eta).$$

Let $m > m_0(M_0, g_0, \mu_0)$ and let $u_\varepsilon \in H^1(M_\varepsilon)$ and $u_0 \in H^1(M_0)$ denote the unique solutions to

$$(m - \Delta_{g_\varepsilon,\mu_\varepsilon})u_\varepsilon = f_\varepsilon \quad \text{in } H^{-1}(M_\varepsilon, g_\varepsilon, \mu_\varepsilon), \tag{28a}$$

$$(m - \Delta_{\hat{g}_0,\hat{\mu}_0})u_0 = f_0 \quad \text{in } H^{-1}(M_0, \hat{g}_0, \hat{\mu}_0), \tag{28b}$$

and suppose that

$$f_\varepsilon \to f_0 \quad \text{strongly in } L^2 \text{ in the sense of } (27).$$

Then

$$u_\varepsilon \to u_0 \quad \text{strongly in } L^2 \text{ in the sense of (27)}.$$

The coefficient field $\mathbb{L}_\varepsilon$ in Lemma 19 is symmetric and uniformly elliptic (with respect to g_0) by construction. Therefore, similarly to Lemma 11 we may deduce convergence of the spectrum of the Laplace–Beltrami operators. To that end, we additionally suppose that M_0 is compact and $m_0(M_0) < 0$. Thanks to (11), the weighted Riemannian manifolds M_ε satisfy the same properties, and thus the spectrum of $-\Delta_{g_\varepsilon, \mu_\varepsilon}$ consists only of the real point spectrum with strictly positive eigenvalues.

Lemma 20 (Spectral convergence of bi-Lipschitz diffeomorphic manifolds)**.** *Suppose that M_0 is compact and $m_0(M_0) < 0$. Consider the setting of Assumption 17, and let $\overline{g}_0, \overline{\mu}_0$ be defined as Lemma 19 (b). For $\varepsilon \geq 0$ consider the operator*

$$\begin{cases} -\Delta_{g_\varepsilon, \mu_\varepsilon} : H_0^1(M_\varepsilon, g_\varepsilon, \mu_\varepsilon) \to H^{-1}(M_\varepsilon, g_\varepsilon, \mu_\varepsilon) & \text{for } \varepsilon > 0, \\ -\Delta_{\hat{g}_0, \hat{\mu}_0} : H_0^1(M_0, \hat{g}_0, \hat{\mu}_0) \to H^{-1}(M_0, \hat{g}_0, \hat{\mu}_0) & \text{for } \varepsilon = 0, \end{cases}$$

and let

$$0 < \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \lambda_{\varepsilon,3} \leq \cdots,$$

denote the list of increasingly ordered eigenvalues with eigenvalues being repeated according to their multiplicity. Let $u_{\varepsilon,1}, u_{\varepsilon,2}, u_{\varepsilon,3}, \ldots$ denote the associated eigenfunctions. Then for all $k \in \mathbb{N}$,

$$\lambda_{\varepsilon,k} \to \lambda_{0,k},$$

and if $s \in \mathbb{N}$ is the multiplicity of $\lambda_{0,k}$, i.e.,

$$\lambda_{0,k-1} < \lambda_{0,k} = \cdots = \lambda_{0,k+s-1} < \lambda_{0,k+s} \quad \text{(with the convention } \lambda_{0,0} = 0\text{)},$$

then there exists a sequence $(\bar{u}_{\varepsilon,k})_\varepsilon$ of linear combinations of $u_{\varepsilon,k}, \ldots, u_{\varepsilon,k+s-1}$ such that

$$\bar{u}_{\varepsilon,k} \to u_{0,k} \quad \text{strongly in } L^2 \text{ in the sense of (27)}. \tag{29}$$

We finally discuss the special case of submanifolds of $\mathbb{R}^d$. In the following lemma we collect (without proof) some consequences that directly follow from Lemma 16, 19, and 20 applied to the special case.

Lemma 21. *Consider the setting of Assumption 17, and assume that*
- *M_ε are n-dimensional submanifolds of the Euclidean space $\mathbb{R}^d$ with g_ε and μ_ε induced by the standard metric and measure of $\mathbb{R}^d$;*
- *the reference manifold M_0 is a subset of the Euclidean space $\mathbb{R}^n$, i.e., $M_0 \subset \mathbb{R}^n$, $g_0(\xi, \eta) := \xi \cdot \eta$, and $d\mu_0 = dx$.*

Then:
(a) *The formulas in (26) turn into*

$$\rho_\varepsilon = \sqrt{\det(dh_\varepsilon^\mathsf{T} dh_\varepsilon)} \quad \text{and} \quad \mathbb{L}_\varepsilon = \rho_\varepsilon (dh_\varepsilon^\mathsf{T} dh_\varepsilon)^{-1},$$

where dh_ε denotes the Jacobian of h_ε.
(b) *An application of Lemma 19 yields the existence of a density ρ_0 and a coefficient field $\mathbb{L}_0 \in \mathcal{M}(M_0, \frac{1}{C_0}, C_0)$ (with $C_0 > 0$ only depending on n, λ, Λ and the constant C in (11)) such that*

$$\rho_\varepsilon = \sqrt{\det(dh_\varepsilon^\mathsf{T} dh_\varepsilon)} \xrightarrow{*} \rho_0 \quad \text{weakly-* in } L^\infty(M_0),$$

$$\mathbb{L}_\varepsilon = \rho_\varepsilon (dh_\varepsilon^\mathsf{T} dh_\varepsilon)^{-1} \xrightarrow{H} \mathbb{L}_0 \quad \text{on } M_0,$$

for a subsequence (not relabeled), and the limiting Riemannian manifold $(M_0, \hat{g}_0, \hat{\mu}_0)$ is then given by

$$d\hat{\mu}_0 = \rho_0 \, dx \quad \text{and} \quad \hat{g}_0(\xi, \eta) = \rho_0 \mathbb{L}_0^{-1} \xi \cdot \eta.$$

(c) *If additionally M_0 is open and bounded and has a Lipschitz boundary, then the conclusion of Lemma 20 on spectral convergence holds.*

Remark 22 (Realizability of $(M_0, \hat{g}_0, \hat{\mu}_0)$)**.** *If the limiting metric $\hat{g}_0$ is smooth, then it is realizable in $\mathbb{R}^m$ with m large enough, i.e., there exists an isometry $h_0 : (M_0, \hat{g}_0, \hat{\mu}_0) \to \mathbb{R}^m$ such that $N_0 := h_0(M_0)$ is a n-dimensional submanifold of $\mathbb{R}^m$ (with induced metric and measure from $\mathbb{R}^m$). Such an embedding is characterized by the identity*

$$dh_0^\mathsf{T} dh_0 = \rho_0 \mathbb{L}_0^{-1}. \tag{30}$$

Indeed, this follows by the Nash embedding theorem provided the dimension of the ambient space m is large enough. However, in the general case, we cannot necessarily give an explicit definition of the immersion h_0. In the examples that we discuss in Sect. 3 below, we study parametrized, $n = 2$-dimensional submanifolds of $\mathbb{R}^3$ that converge to a

limiting manifold that is realizable as a 2-dimensional submanifold of $\mathbb{R}^3$ and given by an explicit formula.

Note that if one introduces a different reference manifold $\widetilde{M}_0$ with a diffeomorphism $\psi: \widetilde{M}_0 \to M_0$, the same calculations can be done with $\tilde{h}_\varepsilon := h_\varepsilon \circ \psi: \widetilde{M}_0 \to M_\varepsilon$ instead of h_ε, which does not necessarily satisfy the uniform ellipticity conditions, and one ends up with the isometric embedding $\tilde{h}_0 = h_0 \circ \psi: \widetilde{M}_0 \to \mathbb{R}^m$. Thus, in practice, the calculations to identify the limiting manifold can be done with diffeomorphisms which are not uniformly elliptic, as long as there exist uniformly elliptic diffeomorphisms.

3. Examples

In the following we consider two examples of laminate-like coefficient fields. We study each of them by appealing to homogenization in the flat case via local charts. Note that the coefficient fields in the following examples are intrinsic objects that could be considered without using charts, and so the respective *H*-limit, even though it is studied and expressed in local coordinates, is not bound to charts.

3.1 Laminate-like coefficient fields on spherically symmetric manifolds

Let $0 < R \leq \infty$ and $s \in C^\infty([0, R))$ such that $s(r) > 0$ if $r > 0$, $s(0) = 0$, and $s'(0) = 1$. We consider the 2-dimensional spherically symmetric manifold $M = \{(x_1, x_2) = (r, \theta) \in [0, R) \times \mathbb{S}^1\}$ equipped with the Riemannian metric

$$g = dr^2 + s^2(r)\, d\theta^2$$

in the polar coordinates (r, θ) (see e.g., [7]). For example,

- $\mathbb{R}^2$ is a model with $R = \infty$ and $s(r) = r$;
- $\mathbb{S}^2$ without pole is a model with $R = \pi$ and $s(r) = \sin r$;
- $\mathbb{H}^2$ is a model with $R = \infty$ and $s(r) = \sinh r$.

For the sake of simplicity we normalize $\mathbb{S}^1$ to have circumference 1. Consider $\mathbb{L}_\varepsilon \in \mathcal{M}(M, \lambda, \Lambda)$ of the form

$$\mathbb{L}_\varepsilon(r, \theta) = \mathbb{L}_\#\left(r, \theta, \tfrac{\theta}{\varepsilon}\right) \quad \text{a.e. in } M$$

and assume that $M \ni (r, \theta) \mapsto \mathbb{L}_\#(r, \theta, y)$ is continuous for a.e. $y \in \mathbb{R}$ and $y \mapsto \mathbb{L}_\#(r, \theta, y)$ is measurable and 1-periodic for all $(r, \theta) \in M$. Denoting by $\{\phi(t)\}$ the one-parameter group

$$\phi(t): x \mapsto \exp_x\left(t\,\frac{\partial}{\partial\theta}\right), \quad x \in M \setminus \text{pole(s)}, \ t \in \mathbb{R},$$

the coefficient field $\mathbb{L}_\varepsilon$ oscillates (on scale ε) along ϕ, while it is slowly varying in the radius direction. We therefore call $\mathbb{L}_\varepsilon$ a *laminate-like coefficient field* on M, see Fig. 6.

Fig. 6.   Illustrations of the laminate-like structure of the coefficient field on $\mathbb{R}^2$, $\mathbb{S}^2$, and $\mathbb{H}^2$.

We make the following observations:

(a) By Theorem 5 we have $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ for a subsequence and some coefficient field $\mathbb{L}_0$. As we shall see below, the limit $\mathbb{L}_0$ can be expressed by a "homogenization formula" that uniquely determines $\mathbb{L}_0$ in terms of $\mathbb{L}_\#$. Hence, $\mathbb{L}_0$ is independent of the chosen subsequence and we conclude that $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ for all sequences $\varepsilon \downarrow 0$.

(b) Consider the special case

$$\mathbb{L}_\#(r, \theta, y) := \begin{pmatrix} a_\#(y) & 0 \\ 0 & b_\#(y) \end{pmatrix} \tag{31}$$

with $a_\#, b_\# : \mathbb{R} \to (\lambda, \Lambda)$ measurable and 1-periodic. Above, we tacitly expressed $\mathbb{L}_\#$ w.r.t. polar coordinates, i.e., $(\mathbb{L}_\#)_{ij} := (\frac{\partial}{\partial x^i}, \mathbb{L}_\# \frac{\partial}{\partial x^j})$ where $x = (x^1, x^2) = (r, \theta)$. In this case we may represent $\mathbb{L}_0$ with help of the arithmetic and harmonic mean of $a_\#$ and $b_\#$ to express the diffusivity orthogonal to the flow ϕ and aligned to the flow ϕ, respectively:

$$\mathbb{L}_0 = \begin{pmatrix} \int_0^1 a_\# & 0 \\ 0 & (\int_0^1 b_\#^{-1})^{-1} \end{pmatrix}. \tag{32}$$

In order to prove these claims it suffices to identify $\mathbb{L}_0$ locally. Consider an open, bounded set $\omega \Subset M$. We may assume without loss of generality that $\overline{\omega}$ does not intersect the curve $\{(r,\theta) : \theta = 0\}$. Denote the chart of polar coordinates by Ψ and define $U \subset \mathbb{R}^2$ by $U := \Psi(\omega)$. According to (19) we associate to $\mathbb{L}_\varepsilon$ a coefficient field A_ε on U. It can be written in the form $A_\varepsilon(r,\theta) = A_\#(r,\theta,\frac{\theta}{\varepsilon})$ with

$$A_\#(r,\theta,y) = \begin{pmatrix} s(r) & 0 \\ 0 & s^{-1}(r) \end{pmatrix} \mathbb{L}_\#(r,\theta,y),$$

where we identified $\mathbb{L}_\#(r,\theta,y)$ with the corresponding coefficient matrix in polar coordinates. Since $\mathbb{L}_\varepsilon \xrightarrow{H} \mathbb{L}_0$ on ω, we have $A_\varepsilon \xrightarrow{H} A_0$ on U by Lemma 13. On the other hand, since A_ε is a coefficient field of the form $A_\#(r,\theta,\frac{\theta}{\varepsilon})$ with $A_\#$ being continuous in the first two components and periodic in the third component, the periodic homogenization formula (24) applies and we deduce that A_0 only depends on $\mathbb{L}_\#$ and the metric g (but not on the extracted subsequence). Hence, $\mathbb{L}_0$ is uniquely determined by $\mathbb{L}_\#$ and the metric, and thus H-convergence holds for the entire sequence. This proves (a).

Next, we discuss the special case (31) for which we obtain

$$A_\#(r,\theta,y) = \begin{pmatrix} s(r)a_\#\left(\frac{\theta}{\varepsilon}\right) & 0 \\ 0 & s^{-1}(r)b_\#\left(\frac{\theta}{\varepsilon}\right) \end{pmatrix}$$

and

$$A_0(r,\theta) = \begin{pmatrix} s(r) \int_0^1 a_\# & 0 \\ 0 & s^{-1}(r)(\int_0^1 b_\#^{-1})^{-1} \end{pmatrix}.$$

The above identities can be seen by evaluating (24), which in the case of laminates can be done by hand. This proves (b).

Example 1: A graphical surface with star-shaped corrugations. In the spirit of Definition 1 we start with the reference manifold

$$M_0 = \{(r,\theta); r \in (0,R), \theta \in [0,2\pi]\}$$

for some $R > 0$. Note that M_0 does not include the origin. Now we define a family $M_\varepsilon = h_\varepsilon(M_0)$ of 2-dimensional submanifolds of $\mathbb{R}^3$ (with standard metric and measure induced from $\mathbb{R}^3$) using uniform bi-Lipschitz immersions $h_\varepsilon : M_0 \to \mathbb{R}^3$,

$$h_\varepsilon(r,\theta) = \begin{pmatrix} r\sin\theta \\ r\cos\theta \\ \varepsilon f\left(r,\frac{\theta}{\varepsilon}\right) \end{pmatrix},$$

where $f : (0,\infty) \times [0,\infty) \to \mathbb{R}$ is smooth and 2π-periodic in the second argument. In Fig. 2 in the Introduction we choose $f(r,y) = \sin^2(y)$ to present M_ε for some values of ε.

We follow the path described in Lemma 21 and calculate first

$$dh_\varepsilon^\mathsf{T} dh_\varepsilon = \begin{pmatrix} 1 + \left(\varepsilon\partial_1 f\left(r,\frac{\theta}{\varepsilon}\right)\right)^2 & \varepsilon\partial_1 f\left(r,\frac{\theta}{\varepsilon}\right)\partial_2 f\left(r,\frac{\theta}{\varepsilon}\right) \\ \varepsilon\partial_1 f\left(r,\frac{\theta}{\varepsilon}\right)\partial_2 f\left(r,\frac{\theta}{\varepsilon}\right) & r^2 + \left(\partial_2 f\left(r,\frac{\theta}{\varepsilon}\right)\right)^2 \end{pmatrix},$$

to get the density

$$\rho_\varepsilon = \sqrt{\det(dh_\varepsilon^\mathsf{T} dh_\varepsilon)} = \sqrt{r^2 + r^2\left(\varepsilon\partial_1 f\left(r,\frac{\theta}{\varepsilon}\right)\right)^2 + \left(\partial_2 f\left(r,\frac{\theta}{\varepsilon}\right)\right)^2},$$

and the coefficient field

$$\begin{aligned} \mathbb{L}_\varepsilon &= \rho_\varepsilon(dh_\varepsilon^\mathsf{T} dh_\varepsilon)^{-1} \\ &= 1/\rho_\varepsilon \begin{pmatrix} r^2 + \left(\partial_2 f\left(r,\frac{\theta}{\varepsilon}\right)\right)^2 & -\varepsilon\partial_1 f\left(r,\frac{\theta}{\varepsilon}\right)\partial_2 f\left(r,\frac{\theta}{\varepsilon}\right) \\ -\varepsilon\partial_1 f\left(r,\frac{\theta}{\varepsilon}\right)\partial_2 f\left(r,\frac{\theta}{\varepsilon}\right) & 1 + \left(\varepsilon\partial_1 f\left(r,\frac{\theta}{\varepsilon}\right)\right)^2 \end{pmatrix}. \end{aligned}$$

It turns out that $\rho_\varepsilon \xrightarrow{*} \rho_0$ weakly-$*$ in $L^\infty(M_0)$ with

$$\rho_0(r) = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{(\partial_2 f(r,y))^2 + r^2} \, dy,$$

and using (32) we see $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ with

$$\mathbb{L}_0 = \begin{pmatrix} \frac{1}{2\pi} \int_0^{2\pi} \sqrt{(\partial_2 f(r,y))^2 + r^2}\, dy & 0 \\ 0 & \left(\frac{1}{2\pi} \int_0^{2\pi} \sqrt{(\partial_2 f(r,y))^2 + r^2}\, dy\right)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \rho_0(r) & 0 \\ 0 & \frac{1}{\rho_0(r)} \end{pmatrix}.$$

Thus the limiting metric on M_0 is given by

$$\hat{g}_0(\xi, \eta) = \rho_0 \mathbb{L}_0^{-1} \xi \cdot \eta = \begin{pmatrix} 1 & 0 \\ 0 & \rho_0^2 \end{pmatrix} \xi \cdot \eta.$$

In this situation we finally can find a bi-Lipschitz immersion $h_0 \colon M_0 \to \mathbb{R}^3$ such that $dh_0^\top dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(r, \theta) = \begin{pmatrix} \rho_0(r) \sin\theta \\ \rho_0(r) \cos\theta \\ \int_0^r \sqrt{1 - \rho_0'(t)^2}\, dt \end{pmatrix}.$$

That means, by Remark 22, the (rotationally symmetric) submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$ (with the standard measure and metric induced from $\mathbb{R}^3$), which for the case $f(r,y) = \sin^2(y)$ is pictured in Fig. 2, is the spectral limit of (M_ε). Note that the excluded origin in the reference manifold coincides now with a circle of radius $\lim_{r \downarrow 0} \rho_0(r)$, which for $f(r,y) = \sin^2(y)$ is $\frac{\pi}{2}$.

Example 2: Sphere with radial perturbations oscillating with the longitude. Instead of a graph over $\mathbb{R}^2$ as in the example above we can treat a radially perturbed sphere in the same way. We take an analogous underlying reference manifold

$$M_0 = \{(\varphi, \theta); \varphi \in (0, \pi), \theta \in [0, 2\pi]\}$$

and define the family $M_\varepsilon := h_\varepsilon(M_0)$ of 2-dimensional submanifolds of $\mathbb{R}^3$ via bi-Lipschitz immersions $h_\varepsilon \colon M_0 \to M_\varepsilon$,

$$h_\varepsilon(\varphi, \theta) = \left(1 + \varepsilon f\left(\varphi, \tfrac{\theta}{\varepsilon}\right)\right) \begin{pmatrix} \sin\varphi \sin\theta \\ \sin\varphi \cos\theta \\ \cos\varphi \end{pmatrix},$$

where $f \colon (0, \pi) \times [0, \infty) \to \mathbb{R}$ is differentiable and 2π-periodic in the second argument. In Fig. 3 in the Introduction we choose $f(r,y) = \sin^2(y)$ to picture M_ε for some values of ε. As in the previous example we obtain the following formulas for the limiting density

$$\rho_0(\varphi) = \tfrac{1}{2\pi} \int_0^{2\pi} \sqrt{(\partial_2 f(\varphi, y))^2 + \sin^2\varphi}\, dy,$$

and the limiting metric

$$\hat{g}_0(\xi, \eta) = \tfrac{1}{\rho_0} \mathbb{L}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \rho_0^2 \end{pmatrix} \xi \cdot \eta.$$

Again we can find a bi-Lipschitz immersion $h_0 \colon M_0 \to \mathbb{R}^3$ such that $dh_0^\top dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(\varphi, \theta) = \begin{pmatrix} \rho_0(\varphi) \sin\theta \\ \rho_0(\varphi) \cos\theta \\ \int_0^\varphi \sqrt{1 - \rho_0'(t)^2}\, dt \end{pmatrix}.$$

Thus the (rotationally symmetric) submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$, which for the case $f(r,y) = \sin^2(y)$ is pictured in Fig. 3, is the spectral limit of the sequence (M_ε).

3.2 Concentric laminate-like coefficient fields on Voronoi tesselated manifolds

Let (M, g, μ) be a n-dimensional manifold and $Z \subset M$ a countable closed subset. For $z \in Z$ we denote by M_z the associated Voronoi cell, that is

$$M_z := \{x \in M; d(x, z) < d(x, Z \setminus \{z\})\},$$

where $d(\cdot, \cdot)$ is the geodesic distance on M. We assume the Voronoi tessellation to be fine enough to ensure that for μ-a.e. point $x_0 \in M$ there are $z \in Z$ and $\varrho > 0$ such that
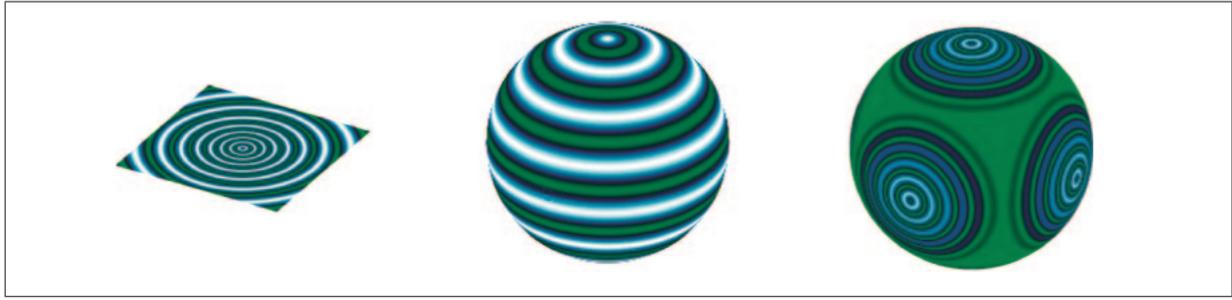
Fig. 7.   Illustration of coefficient fields with laminate-like structure.

for all $x \in B_\varrho(x_0) \subset M_z$ exists exactly one shortest path γ_x from x to z. $\hspace{2cm}$ (33)

We consider a sequence $(\mathbb{L}_\varepsilon)$ in $\mathcal{M}(M, \lambda, \Lambda)$ of rapidly oscillating coefficient fields of the form $\mathbb{L}_\varepsilon(x) = \mathbb{L}(\frac{d(x,Z)}{\varepsilon})$, where $\mathbb{L}(r)$ is 1-periodic in $r \in \mathbb{R}$, see Fig. 7.

By Theorem 5 $(\mathbb{L}_\varepsilon)$ H-converges (up to a subsequence) to some $\mathbb{L}_0 \in \mathcal{M}(M, \lambda, \Lambda)$. We are going to show that $\mathbb{L}_0$ coincides μ-a.e. on M with some constant coefficient field which is uniquely determined by $\mathbb{L}$. In particular the whole sequence $(\mathbb{L}_\varepsilon)$ H-converges to $\mathbb{L}_0$.

In order to prove this, it suffices to identify $\mathbb{L}_0$ locally, i.e., for μ-a.e. $x_0 \in M$. As a first step we construct curvilinear coordinates such that in these coordinates the coefficients locally turn into a laminate up to a small perturbation that vanishes at x_0. In particular we claim that local coordinates $(B_\varrho(x_0), \Psi; x^1, \ldots, x^n)$ exist such that

$$\Psi(x_0) = 0, \tag{34a}$$

$$x^1 = d(\cdot, z) - d(x_0, z), \tag{34b}$$

$$g\left(\tfrac{\partial}{\partial x^1}, \tfrac{\partial}{\partial x^j}\right) = 0 \text{ for } j = 2, \ldots, n, \tag{34c}$$

$$\lim_{x \to x_0} \rho(x) g\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right)(x) = \delta_{ij}. \tag{34d}$$

Indeed, note that by (34b) geodesics through z are mapped to straight lines parallel to the x^1-axis.

Therefore, we fix $x_0 \in M$, $z \in Z$ and $\varrho > 0$ satisfying (33). As in (34b) we set for $x \in B_\varrho(x_0)$

$$x^1(x) := d(x, z) - d(x_0, z).$$

Thanks to (33) x^1 is differentiable and the level set $U_{x_0} := \{x \in B_\rho(x_0); x^1(x) = 0\}$ is a $n-1$-dimensional submanifold of M_z including x_0 and for any point $x \in U_{x_0}$ the tangent space $T_x U_{x_0}$ is orthogonal to $d\gamma_x(0)$, which gives (34c). Assume $\varrho > 0$ to be small enough such that we can choose local normal coordinates $x^2, \ldots, x^n$ of U_{x_0} with $x^j(x_0) = 0$ $(j = 2, \ldots, n)$. By the differentiability of geodesics we can extend these coordinate functions to curvilinear coordinates $x^1, \ldots, x^n$ on $B_\varrho(x_0)$ (with a probably smaller ϱ) in the way that $x^2, \ldots, x^n$ are constant on γ_x for every $x \in B_\varrho(x_0)$. Then we have

$$\lim_{x \to x_0} g\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right)(x) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{35}$$

which yields (34d).

In these coordinates the associated coefficient field at $y \in U := \Psi(B_\varrho(x_0))$ can be written as

$$A_\varepsilon(y) = A\left(y, \tfrac{y_1 + d(x_0, z)}{\varepsilon}\right)$$
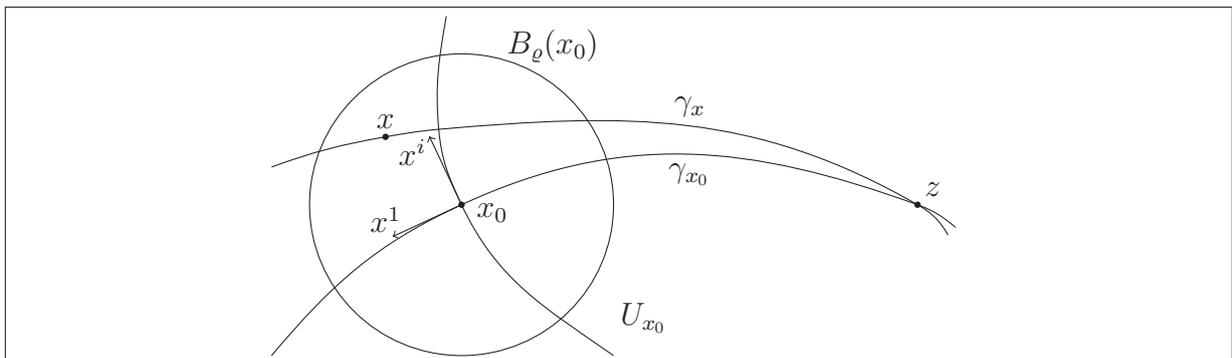
Fig. 8.   Construction of the local coordinates.

for some $A: U \times \mathbb{R}$ continuous in the first, and measurable and 1-periodic in the second argument. This can be seen by considering (19): The coefficient field A_ε on U associated to $\mathbb{L}_\varepsilon$ takes the form

$$(A_\varepsilon)_{ij} = \overline{\rho}\,\overline{g}(\mathbb{L}_\varepsilon \nabla_g x^i, \nabla_g x^j),$$

where $\overline{\rho} := \rho \circ \Psi^{-1}$ and $\overline{g} := g \circ \Psi^{-1}$ denote the representation of the quantities in local coordinates. By the definitions of $\mathbb{L}_\varepsilon$ and x^1 we see that

$$g\big(\mathbb{L}_\varepsilon(x)\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\big) = g\big(\mathbb{L}\big(\tfrac{d(x,Z)}{\varepsilon}\big)\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\big) = g\big(\mathbb{L}\big(\tfrac{x^1(x)+d(x_0,Z)}{\varepsilon}\big)\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\big)$$

is only depending on $x^1(x) = y_1$, and A_ε has the desired form with

$$A_{ij}(y,r) := \overline{\rho}\,\overline{g}(\mathbb{L}(r)\nabla_g x^i, \nabla_g x^j)(y), \tag{36}$$

which is continuous in $y \in U$, and measurable and 1-periodic in $r \in \mathbb{R}$.

For $\varepsilon \to 0$ the homogenized matrix A_{hom} associated with A_ε is given by the homogenization formula (24) for A defined in (36). Therefore A_{hom} continuously depends on $y \in U$. Moreover the matrix $A_{\text{hom}}(0)$ is independent on the initial choice of x_0 and is given by the following weak-$*$ limits in $L^\infty(U)$:

$$\frac{1}{A_{11}\big(0, \tfrac{\cdot}{\varepsilon}\big)} \rightharpoonup \frac{1}{(A_{\text{hom}})_{11}(0)},$$

$$\frac{A_{i1}\big(0, \tfrac{\cdot}{\varepsilon}\big)}{A_{11}\big(0, \tfrac{\cdot}{\varepsilon}\big)} \rightharpoonup \frac{(A_{\text{hom}})_{i1}(0)}{(A_{\text{hom}})_{11}(0)}, \quad i = 2, \dots, n,$$

$$\frac{A_{1j}\big(0, \tfrac{\cdot}{\varepsilon}\big)}{A_{11}\big(0, \tfrac{\cdot}{\varepsilon}\big)} \rightharpoonup \frac{(A_{\text{hom}})_{1j}(0)}{(A_{\text{hom}})_{11}(0)}, \quad j = 2, \dots, n,$$

$$A_{ij}\big(0, \tfrac{\cdot}{\varepsilon}\big) - \frac{A_{i1}\big(0, \tfrac{\cdot}{\varepsilon}\big)A_{1j}\big(0, \tfrac{\cdot}{\varepsilon}\big)}{A_{11}\big(0, \tfrac{\cdot}{\varepsilon}\big)} \rightharpoonup (A_{\text{hom}})_{ij}(0) - \frac{(A_{\text{hom}})_{i1}(0)(A_{\text{hom}})_{1j}(0)}{(A_{\text{hom}})_{11}(0)}, \quad i,j = 2, \dots, n.$$

By Lemma 15, we have

$$(A_{\text{hom}})_{ij} = \overline{\rho}\,\overline{g}(\mathbb{L}_0 \nabla_g x^i, \nabla_g x^j). \quad \text{a.e. in } U.$$

We conclude that $\mathbb{L}_0$ is continuous (μ-a.e.) on $B_\varrho(x_0)$ and thus (using (35)) $g(\mathbb{L}_0(x_0)\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j})(x_0) = (A_{\text{hom}})_{ij}(0)$ for μ-a.e. $x_0 \in M$.

As in the previous example we could consider the special case of a diagonal matrix

$$\mathbb{L}(r)\tfrac{\partial}{\partial x^i} = a_i(r)\tfrac{\partial}{\partial x^i} \quad \text{for } i = 1, \dots, n.$$

Then $\mathbb{L}_0(x_0)$ is a diagonal matrix, too, and we have

$$g\big(\mathbb{L}_0(x_0)\tfrac{\partial}{\partial x^1}, \tfrac{\partial}{\partial x^1}\big)(x_0) = \left(\int_0^1 a_1^{-1}\right)^{-1} \quad \text{and}$$

$$g\big(\mathbb{L}_0(x_0)\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^i}\big)(x_0) = \int_0^1 a_i \quad \text{for } i = 2, \dots, n. \tag{37}$$

Example 3: A radially symmetric corrugated graphical surface. We consider the reference manifold

$$M_0 = \{(r,\theta); r \in (0,R), \theta \in [0, 2\pi)\}$$

for some $R > 0$, and define a family $M_\varepsilon = h_\varepsilon(M_0)$ of 2-dimensional submanifolds of $\mathbb{R}^3$ using uniform bi-Lipschitz immersions $h_\varepsilon: M_0 \to \mathbb{R}^3$,

$$h_\varepsilon(r,\theta) = \begin{pmatrix} r\sin\theta \\ r\cos\theta \\ \varepsilon f\big(r, \tfrac{r}{\varepsilon}\big) \end{pmatrix}, \tag{38}$$

where $f(0,\infty) \times [0,\infty) \to \mathbb{R}$ is differentiable and T-periodic in the second argument. In Fig. 9 we took $f(r,y) = \sin^2(y)$ to illustrate M_ε for some values of ε.

Following Lemma 21 we compute the density

$$\rho_\varepsilon = \sqrt{\det(dh_\varepsilon^\mathsf{T} dh_\varepsilon)} = \sqrt{r^2 + r^2\big(\varepsilon\partial_1 f\big(r, \tfrac{r}{\varepsilon}\big) + \partial_2 f\big(r, \tfrac{r}{\varepsilon}\big)\big)^2},$$

and the coefficient field
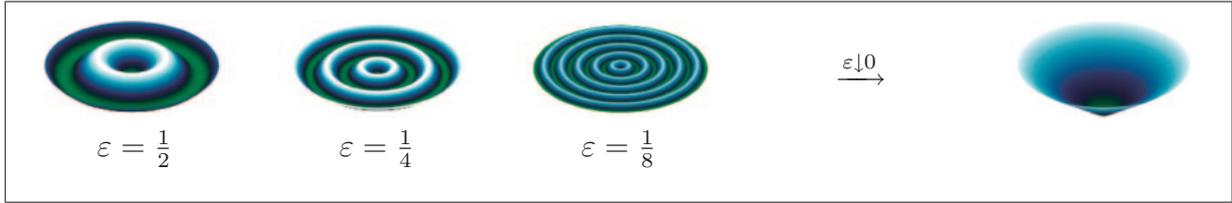
Fig. 9. A family of rotationally symmetric corrugated graphical surfaces. The three pictures on the left show M_ε defined via (38) with $f = \sin^2$ and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (39). As $\varepsilon \to 0$ the spectrum of the Laplace–Beltrami operator on M_ε converges to the spectrum of the Laplace–Beltrami operator on N_0.

$$\mathbb{L}_\varepsilon = \rho_\varepsilon (dh_\varepsilon^\mathsf{T} dh_\varepsilon)^{-1}$$

$$= 1/\rho_\varepsilon \begin{pmatrix} r^2 & 0 \\ 0 & 1 + (\varepsilon \partial_1 f(r, \tfrac{r}{\varepsilon}) + \partial_2 f(r, \tfrac{r}{\varepsilon}))^2 \end{pmatrix}.$$

We find $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ weakly-$*$ in $L^\infty(M_0)$ with

$$\rho_0(r) = \tfrac{r}{T} \int_0^T \sqrt{(\partial_2 f(r, y))^2 + 1}\, dy,$$

and using (37) we see $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ with

$$\mathbb{L}_0 = \begin{pmatrix} \left(\tfrac{1}{rT} \int_0^T \sqrt{(\partial_2 f(r, y))^2 + 1}\, dy \right)^{-1} & 0 \\ 0 & \tfrac{1}{rT} \int_0^T \sqrt{(\partial_2 f(r, y))^2 + 1}\, dy \end{pmatrix}$$

$$= \begin{pmatrix} \tfrac{r^2}{\rho_0(r)} & 0 \\ 0 & \tfrac{\rho_0(r)}{r^2} \end{pmatrix},$$

and get the limiting metric on M_0:

$$\hat{g}_0(\xi, \eta) = \rho_0 \mathbb{L}_0^{-1} \xi \cdot \eta = \begin{pmatrix} \tfrac{\rho_0(r)^2}{r^2} & 0 \\ 0 & r^2 \end{pmatrix} \xi \cdot \eta.$$

We finally find a bi-Lipschitz immersion $h_0 : M_0 \to \mathbb{R}^3$ such that $dh_0^\mathsf{T} dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(r, \theta) = \begin{pmatrix} r \sin\theta \\ r \cos\theta \\ \int_0^r \sqrt{\tfrac{\rho_0(t)^2}{t^2} - 1}\, dt \end{pmatrix}. \tag{39}$$

By Remark 22, the submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$, which for the case $f(r, y) = \sin^2(y)$ is shown in Fig. 9, is the spectral limit of (M_ε).

Example 4: Sphere with radial perturbations oscillating with the latitude. In the same way as in the previous example we can handle the case of a radially perturbed sphere. Again we start with the reference manifold

$$M_0 = \{(\varphi, \theta); \varphi \in (0, \pi), \theta \in [0, 2\pi]\}$$

and define the family $M_\varepsilon := h_\varepsilon(M_0)$ of 2-dimensional submanifolds of $\mathbb{R}^3$ via bi-Lipschitz immersions $h_\varepsilon : \overline{M} \to M_\varepsilon$,

$$h_\varepsilon(\varphi, \theta) = \left(1 + \varepsilon f(\varphi, \tfrac{\varphi}{\varepsilon}) \right) \begin{pmatrix} \sin\varphi \sin\theta \\ \sin\varphi \cos\theta \\ \cos\varphi \end{pmatrix},$$

where $f : (0, \pi) \times [0, \infty) \to \mathbb{R}$ is differentiable and 2π-periodic in the second argument. In Fig. 4 in the Introduction we choose $f(r, y) = \sin^2(y)$ to picture M_ε for some values of ε.

Doing the same calculations as in the example above we end up with the density

$$\rho_0(\varphi) = \tfrac{\sin\varphi}{\pi} \int_0^\pi \sqrt{(\partial_2 f(\varphi, y))^2 + 1}\, dy,$$

and the metric

$$\frac{1}{\rho_0}\mathbb{L}_0 = \begin{pmatrix} \frac{\sin^2 \varphi}{\rho_0(\varphi)^2} & 0 \\ 0 & \frac{1}{\sin^2 \varphi} \end{pmatrix},$$

and again we find a bi-Lipschitz immersion $h_0 \colon M_0 \to \mathbb{R}^3$ such that $dh_0^\top dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(\varphi, \theta) = \begin{pmatrix} \sin \varphi \sin \theta \\ \sin \varphi \cos \theta \\ \int_0^\varphi \sqrt{\frac{\rho_0(t)^2}{\sin^2 t} - \cos^2 t}\, dt \end{pmatrix}.$$

Thus the submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$, which for the case $f(r, y) = \sin^2(y)$ is pictured in Fig. 4, is the spectral limit of the sequence (M_ε).

Example 5: A locally corrugated graphical surface. We finally want to discuss an example with oscillations in several Voronoi cells which can be treated locally.

Let $Y \subset \mathbb{R}^2$ be relatively-compact and open. Consider a set $Z \in Y$ of isolated points. For every point $z \in Z$ we use a smooth function $\psi_z \colon [0, \infty) \to [0, 1]$ to define a rotationally symmetric cut-off function $\psi_z(|\cdot - z|)$ such that

$$\begin{cases} \psi_z(0) = 1, \\ \operatorname{supp}\psi_z(|\cdot - z|) \cap \operatorname{supp}\psi_{z'}(|\cdot - z'|) = \emptyset \text{ for all } z' \in Z \setminus \{z\}. \end{cases}$$

Now we consider a smooth T-periodic function $f \colon [0, \infty) \to \mathbb{R}$ and define M_ε as the graph of the function $h_\varepsilon \colon M_0 := Y \setminus Z \to \mathbb{R}$,

$$h_\varepsilon(x) := \sum_{z \in Z} \varepsilon f\left(\frac{|x - z|}{\varepsilon}\right) \psi_z(|x - z|) \in \mathbb{R}^3,$$

which we regard as a two-dimensional submanifold of $\mathbb{R}^3$. In Fig. 5 in the Introduction we took $f(y) = \sin^2(y)$ to show M_ε for some values of ε.

Doing the same calculations as in the previous examples locally in each Voronoi cell we get a function $h_0 \colon M_0 \to \mathbb{R}$,

$$h_0(x) := x \mapsto \sum_{z \in Z} \int_0^{|x - z|} \sqrt{\frac{\rho_{0,z}(t)^2}{t^2} - 1}\, dt \in \mathbb{R}^3,$$

where $\rho_{0,z}(r) = \frac{r}{T} \int_0^T \sqrt{f'(y)^2 \psi_z(r)^2 + 1}\, dy$, such that the graph of h_0, which is shown in Fig. 5 for $f(y) = \sin^2(y)$, is the spectral limit of (M_ε).

4. Proofs

4.1 Proof of Proposition 6, Lemma 7, and Lemma 8

The argument consists of two parts. In the first part we identify the limiting tensor field $\mathbb{L}_0$. For this purpose, we consider the operators

$$\mathcal{L}_\varepsilon^* \colon H_0^1(B) \to H^{-1}(B), \quad \mathcal{L}_\varepsilon^* u := -\operatorname{div}(\mathbb{L}_\varepsilon^* \nabla), \tag{40}$$

where $\mathbb{L}_\varepsilon^*$ denotes the adjoint of $\mathbb{L}_\varepsilon$ and is defined by the identity $(\mathbb{L}_\varepsilon^* \xi, \eta) = (\xi, \mathbb{L}_\varepsilon \eta)$ for all vector fields ξ, η. Since the operator is uniformly elliptic (with constants independent of ε) we can deduce the existence of a linear isomorphism $\mathcal{L}_0^*$, whose inverse is the limit of $(\mathcal{L}_\varepsilon^*)^{-1}$ in the weak operator topology. Indeed, this follows from the following standard compactness result:

Lemma 23. *Let V be a reflexive separable Banach space and (T_ε) be a sequence of linear operators $T_\varepsilon \colon V \to V'$ that is uniformly bounded and coercive, i.e., there exists $C > 0$ (independent of ε) such that the operator norm of T_ε is bounded by C and*

$$\langle T_\varepsilon v, v \rangle_{V', V} \geq \tfrac{1}{C} \|v\|_V^2 \quad \text{for all } v \in V. \tag{41}$$

Then there exists a linear bounded operator $T_0 \colon V \to V'$ satisfying (41) and for a subsequence (not relabeled) we have $T_\varepsilon^{-1} \rightharpoonup T_0^{-1}$ in the weak operator topology, that is for all $f \in V'$ we have

$$T_\varepsilon^{-1} f \rightharpoonup T_0^{-1} f \quad \text{weakly in } V.$$

(For a proof, e.g., see [25, Proposition 4]). We then show that $\mathcal{L}_0^*$ can in fact be written in divergence form: $\mathcal{L}_0^* = -\operatorname{div}(\mathbb{L}_0^* \nabla)$ with an appropriate $(1, 1)$-tensor field $\mathbb{L}_0^*$. In order to define $\mathbb{L}_0^*$ with help of $\mathcal{L}_0^*$, we introduce auxiliary functions whose gradients span the tangent space. More precisely, we recall the following fact:

Remark 24. *Let $B \Subset M$ denote an open ball with radius smaller than the injectivity radius at its center. Then there*

exist $v_1, \ldots, v_n \in C_c^\infty(B)$ *such that* $T(\frac{1}{2}B)$ *is spanned by the vector fields* $\nabla v_1, \ldots, \nabla v_n$, *i.e.*,

$$\forall y \in \tfrac{1}{2}B: \quad T_y\big(\tfrac{1}{2}B\big) = \mathrm{span}\{\nabla v_1(y), \ldots, \nabla v_n(y)\}. \tag{42}$$

Following ideas of Tartar and Murat, we associate with $v_1, \ldots, v_n$ oscillating test-functions $v_{1,\varepsilon}, \ldots, v_{n,\varepsilon}$ that allow to pass to the limit in products of weakly convergent sequences of the form $(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla v_{i,\varepsilon})$. The argument invokes the following variant of the Div-Curl Lemma for manifolds:

Lemma 25 (Div-Curl Lemma). *Let* $\Omega \subset M$ *be open and let* $(\xi_\varepsilon) \subset L^2(T\Omega)$, $(v_\varepsilon) \subset H^1(\Omega)$ *denote sequences such that*

$$\begin{cases} \xi_\varepsilon \rightharpoonup \xi & \text{weakly in } L^2(T\Omega), \\ \mathrm{div}\,\xi_\varepsilon \to \mathrm{div}\,\xi & \text{in } H^{-1}(\Omega), \end{cases} \quad \text{and} \quad v_\varepsilon \rightharpoonup v \quad \text{weakly in } H^1(\Omega).$$

Then

$$\int_\Omega (\xi_\varepsilon, \nabla v_\varepsilon)\varphi \, \mathrm{d}\mu \to \int_\Omega (\xi, \nabla v)\varphi \, \mathrm{d}\mu \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Moreover, if $v_\varepsilon, v \in H_0^1(\Omega)$, *then*

$$\int_\Omega (\xi_\varepsilon, \nabla v_\varepsilon) \, \mathrm{d}\mu \to \int_\Omega (\xi, \nabla v) \, \mathrm{d}\mu.$$

We present the short proof for the reader's convenience:

Proof of Lemma 25. In the case $v_\varepsilon \in H_0^1(\Omega)$ the statement follows by an integration by parts. In the general case, for $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega (\xi_\varepsilon, \nabla v_\varepsilon)\varphi = \int_\Omega (\xi_\varepsilon, \nabla(v_\varepsilon\varphi)) - \int_\Omega (\xi_\varepsilon, v_\varepsilon \nabla\varphi) = -\langle \mathrm{div}\,\xi_\varepsilon, v_\varepsilon\varphi \rangle - \int_\Omega (\xi_\varepsilon, v_\varepsilon \nabla\varphi). \tag{43}$$

Regarding the first term of the right-hand side of (43),

$$-\langle \mathrm{div}\,\xi_\varepsilon, v_\varepsilon\varphi \rangle \to -\langle \mathrm{div}\,\xi, v\varphi \rangle = \int_\Omega (\xi, v\nabla\varphi) + \int_\Omega (\xi, \varphi\nabla v).$$

For the second term of the right-hand side of (43), since $v_\varepsilon \rightharpoonup v$ in $H^1(\Omega)$, for any relatively compact open set $\Omega' \subset M$, there exists a subsequence of (v_ε) converging to v in $L^2(\Omega')$ by Rellich's theorem; in particular, $v_\varepsilon \nabla\varphi \to v\nabla\varphi$ in $L^2(TM)$ and thus $\int_\Omega (\xi_\varepsilon, v_\varepsilon \nabla\varphi) \to \int_\Omega (\xi, v\nabla\varphi)$. Hence, the right-hand side of (43) converges to $\int_\Omega (\xi, \nabla v)\varphi$. $\qquad\square$

In a second step, we then show that $\mathbb{L}_0$ (the adjoint of $\mathbb{L}_0^*$) is an H-limit of $(\mathbb{L}_\varepsilon)$. To that end we need to consider for (arbitrary but fixed) subdomains $\omega \Subset \Omega$ the localized operators

$$\mathcal{L}_\varepsilon \colon H_0^1(\omega) \to H^{-1}(\omega), \quad \mathcal{L}_\varepsilon u := -\mathrm{div}(\mathbb{L}_\varepsilon \nabla u), \tag{44}$$

and show that $\mathcal{L}_\varepsilon^{-1} \to \mathcal{L}_0^{-1}$ in the weak operator topology.

Proof of Proposition 6. In the proof we pass to various subsequences and it turns out to be necessary to keep track of them. For a lean notation we denote by $E \subset (0, \infty)$ the set of ε's of the given sequence $(\mathbb{L}_\varepsilon) = (\mathbb{L}_\varepsilon)_{\varepsilon \in E}$. We represent subsequences by means of subsets $E', E'', \ldots \subset E$ that have a cluster point at 0. We follow the convention to write

$$c_\varepsilon \to c_0 \quad (\varepsilon \in E'),$$

if and only if for any sequence $(\varepsilon_j)_{j \in \mathbb{N}} \subset E'$ with $\varepsilon_j \to 0$ we have $c_{\varepsilon_j} \to c_0$.

Step 1. Choice of the subsequence and definition of $\mathbb{L}_0$.

Let $\mathcal{L}_\varepsilon^*$ be defined by (40) and fix $v_1, \ldots, v_n \in C_c^\infty(B)$ according to Remark 24. We claim that there exits a measurable $(1,1)$-tensor field $\mathbb{L}_0 \colon \frac{1}{2}B \to \mathrm{Lin}(T(\frac{1}{2}B))$, a subsequence $E' \subset E$, and functions $(v_{1,\varepsilon}), \ldots, (v_{k,\varepsilon}) \subset H_0^1(B)$ (the so called oscillating test functions) such that for $k = 1, \ldots, n$ and $\varepsilon \in E'$ we have

$$\begin{cases} v_{k,\varepsilon} \rightharpoonup v_k & \text{weakly in } H_0^1(B), \\ v_{k,\varepsilon} \to v_k & \text{in } L^2(B), \\ (\mathcal{L}_\varepsilon^* v_{k,\varepsilon}) & \text{strongly converges in } H^{-1}(B), \\ \mathbb{L}_\varepsilon^* \nabla v_{k,\varepsilon} \rightharpoonup \mathbb{L}_0^* \nabla v_k & \text{weakly in } L^2\big(T(\tfrac{1}{2}B)\big). \end{cases} \tag{45}$$

For the argument note that by uniform ellipticity of $\mathbb{L}_\varepsilon^*$ and the boundedness of B, there exists $C = C(B, \lambda) > 0$ such that

$$\langle \mathcal{L}_\varepsilon^* u, u \rangle = \int_B (\mathbb{L}_\varepsilon^* \nabla u, \nabla u) \geq C\|u\|_{H^1(B)}^2,$$

and thus by Lemma 23 there is $\mathcal{L}_0^* \colon H_0^1(B) \to H^{-1}(B)$ and a subsequence $E'' \subset E$ such that for all $f \in H^{-1}(B)$ and

$\varepsilon \in E''$

$$(\mathcal{L}_\varepsilon^*)^{-1} f \rightharpoonup (\mathcal{L}_0^*)^{-1} f \quad \text{weakly in } H_0^1(B).$$

For $k = 1, \ldots, n$ define

$$v_{k,\varepsilon} := (\mathcal{L}_\varepsilon^*)^{-1} \mathcal{L}_0^* v_k,$$

which by uniform ellipticity of $\mathbb{L}_\varepsilon^*$ and Poincaré's inequality in $H_0^1(B)$ are bounded uniformly in ε. Hence there exits vector fields $\ell_1, \ldots, \ell_n \in L^2(TB)$ and another subsequence $E' \subset E''$ such that we have for $\varepsilon \in E'$

$$\begin{cases} v_{k,\varepsilon} \rightharpoonup v_k & \text{weakly in } H_0^1(B), \\ v_{k,\varepsilon} \to v_k & \text{in } L^2(B), \\ \mathbb{L}_\varepsilon^* \nabla v_{k,\varepsilon} \rightharpoonup \ell_k & \text{weakly in } L^2(TB). \end{cases}$$

Next, we define the tensor field $\mathbb{L}_0^*$ by the identity

$$\forall k \in \{1, \ldots, n\} : \quad \mathbb{L}_0^* \nabla v_k = \ell_k \quad \mu\text{-a.e. in } \tfrac{1}{2}B.$$

Indeed, since $\nabla v_1, \ldots, \nabla v_n$ span $T(\tfrac{1}{2}B)$ the above identity defines $\mathbb{L}_0^*$ uniquely and the last identity in (45) is satisfied by construction. It remains to check the strong convergence of $(\mathcal{L}_\varepsilon^* v_{k,\varepsilon})$. In fact the stronger statement $\mathcal{L}_\varepsilon^* v_{k,\varepsilon} = \mathcal{L}_0^* v_k$ is valid, which is a direct consequence of the definition of $v_{k,\varepsilon}$.

Step 2. H-convergence of $\mathbb{L}_\varepsilon$ to $\mathbb{L}_0$ in $\tfrac{1}{2} B$.

Let the subsequence E', the tensor field $\mathbb{L}_0$, and $(v_{k,\varepsilon})$ be defined as in Step 1. We claim that $(\mathbb{L}_\varepsilon)$ *H*-converges to $\mathbb{L}_0$ in $\tfrac{1}{2}B$ for $\varepsilon \in E'$. To that end let $\omega \Subset \tfrac{1}{2}B$ and let $\mathcal{L}_\varepsilon$ be defined by (44). Arguing as in the previous step, we can find another subsequence $E'' \subset E'$ and a bounded linear, coercive operator $\mathcal{L}_0 \colon H_0^1(\omega) \to H^{-1}(\omega)$ such that

$$\mathcal{L}_\varepsilon^{-1} \rightharpoonup \mathcal{L}_0^{-1} \quad \text{in the weak operator topology for } \varepsilon \in E''. \tag{46}$$

We only need to show that

$$\mathcal{L}_0 u_0 = -\text{div}(\mathbb{L}_0 \nabla u_0), \tag{47}$$

for arbitrary $u_0 \in H_0^1(\omega)$. For the argument set $u_\varepsilon := \mathcal{L}_\varepsilon^{-1} \mathcal{L}_0 u_0$ so that by (46),

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\omega) \text{ and strongly in } L^2(\omega) \text{ for } \varepsilon \in E''. \tag{48}$$

Consider $J_\varepsilon := \mathbb{L}_\varepsilon \nabla u_\varepsilon$. By uniform ellipticity of $\mathbb{L}_\varepsilon$ the sequences (J_ε) is bounded in $L^2(T\omega)$. Hence, there exits $J_0 \in L^2(T\omega)$ and another subsequence $E''' \subset E''$ such that

$$J_\varepsilon = \mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup J_0 \quad \text{weakly in } L^2(T\omega) \text{ for } \varepsilon \in E'''. \tag{49}$$

Combined with the identity $-\text{div} J_\varepsilon = \mathcal{L}_0 u_0$ (which follows from the definition of u_ε) we find that

$$-\text{div} J_0 = \mathcal{L}_0 u_0. \tag{50}$$

Hence, for any test function $\varphi \in C_c^\infty(\omega)$, the convergence properties of $(v_{k,\varepsilon})$ yield

$$\int_\omega (J_\varepsilon, \varphi \nabla v_{k,\varepsilon}) = \int_\omega (J_\varepsilon, \nabla(\varphi v_{k,\varepsilon})) - \int_\omega (J_\varepsilon, v_{k,\varepsilon} \nabla \varphi)$$

$$= \langle \mathcal{L}_0 u_0, \varphi v_{k,\varepsilon} \rangle - \int_\omega (J_\varepsilon, v_{k,\varepsilon} \nabla \varphi)$$

$$\to \langle \mathcal{L}_0 u_0, \varphi v_k \rangle - \int_\omega (J_0, v_k \nabla \varphi)$$

$$= \int_\omega (J_0, \varphi \nabla v_k).$$

On the other hand, since $\mathbb{L}_\varepsilon^* \nabla v_{k,\varepsilon} \rightharpoonup \mathbb{L}_0^* \nabla v_k$ weakly in $L^2(T\tfrac{1}{2}B)$ and $(-\text{div}(\mathbb{L}_\varepsilon^* \nabla v_{k,\varepsilon}))$ strongly converges in $H^{-1}(\tfrac{1}{2}B)$ by (45), the Div-Curl Lemma (Lemma 25) yields

$$\int_\omega (J_\varepsilon, \varphi \nabla v_{k,\varepsilon}) = \int_\omega (\varphi \nabla u_\varepsilon, \mathbb{L}_\varepsilon^* \nabla v_{k,\varepsilon}) \to \int_\omega (\varphi \nabla u_0, \mathbb{L}_0^* \nabla v_k) = \int_\omega (\mathbb{L}_0 \nabla u_0, \varphi \nabla v_k).$$

Hence, by combining the previous two identities we conclude that

$$\int_\omega (\mathbb{L}_0 \nabla u_0, \varphi \nabla v_k) = \int_\omega (J_0, \varphi \nabla v_k).$$

Since $\varphi \in C_c^\infty(\omega)$ is arbitrary and since $\nabla v_1, \ldots, \nabla v_n$ spans $T\omega$, we get $J_0 = \mathbb{L}_0 \nabla u_0$ μ-a.e. in ω. Thus (47) follows from (50). Moreover, since J_0 and $\mathcal{L}_0$ are uniquely determined by $\mathbb{L}_0$, the convergence in (46), (48), and (49) holds for the entire sequence E' (which in particular is independent of ω).

Next we argue that $\mathbb{L}_0 \in \mathcal{M}(\omega, \lambda, \Lambda)$. Indeed, from (48) and (49) and the Div-Curl Lemma (Lemma 25) we learn that for any non-negative $\varphi \in C_c^\infty(\omega)$ we have

$$\int_\omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)\varphi \to \int_\omega (\mathbb{L}_0 \nabla u_0, \nabla u_0)\varphi.$$

By uniform ellipticity of $\mathbb{L}_\varepsilon$ in form of (12), we have $\int_\omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)\rho \geq \lambda \int_\omega |\nabla u_\varepsilon|^2 \rho$, and thus

$$\int_\omega (\mathbb{L}_0 \nabla u_0, \nabla u_0)\varphi \geq \lambda \int_\omega |\nabla u_0|^2 \varphi.$$

Since this is true for all u_0 and φ, we conclude that $\mathbb{L}_0$ satisfies the lower ellipticity condition, cf. (12) μ-a.e. in ω. On the other hand (13) implies

$$\int_\omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)\varphi = \int_\omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \mathbb{L}_\varepsilon^{-1} \mathbb{L}_\varepsilon \nabla u_\varepsilon)\varphi \geq \Lambda \int_\omega |\mathbb{L}_\varepsilon \nabla u_0|^2 \varphi,$$

and thus by the same reasoning as before, we get for μ-a.e. $x \in \omega$ and all $\xi \in T_x\omega$

$$\Lambda |\mathbb{L}_0(x)\xi|^2 \leq (\mathbb{L}_0(x)\xi, \xi).$$

Substituting $\xi = \mathbb{L}_0^{-1}(x)\xi'$ yields the boundedness condition, cf. (13).

Since the above arguments hold for arbitrary $\omega \Subset \frac{1}{2}B$ we deduce that $\mathbb{L}_0 \in \mathcal{M}(\frac{1}{2}B, \lambda, \Lambda)$ and that $(\mathbb{L}_\varepsilon)$ H-converges to $\mathbb{L}_0$ in $\frac{1}{2}B$ for $\varepsilon \in E'$. $\qquad\square$

Next we present the proof of the auxiliary statements Lemma 7 and 8.

Proof of Lemma 7. Step 1: Proof of part (a).
Let $x \in \omega$ and denote by $B \Subset \omega$ an open ball centered at x and with a radius that is smaller than the injectivity radius of Ω at x. Fix $v_1, \ldots, v_n \in C_c^\infty(B)$ according to Remark 24. For $k \in \{1, \ldots, n\}$ set $f \in H^{-1}(B)$ by $f := -\operatorname{div}(\mathbb{L}_0 \nabla v_k)$ and define $v_\varepsilon \in H_0^1(B)$ as the unique solutions to $-\operatorname{div}(\mathbb{L}_\varepsilon \nabla v_\varepsilon) = f$ in $H^{-1}(B)$. By H-convergence of $(\mathbb{L}_\varepsilon)$ and the definition of f we have $v_\varepsilon \rightharpoonup v_k$ weakly in $H_0^1(B)$ and $\mathbb{L}_\varepsilon \nabla v_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla v_k$ weakly in $L^2(B)$. Likewise, by H-convergence of $(\widetilde{\mathbb{L}}_\varepsilon)$ to $\widetilde{\mathbb{L}}_0$ and since $\widetilde{\mathbb{L}}_\varepsilon = \mathbb{L}_\varepsilon$ on B, we find that $\mathbb{L}_\varepsilon \nabla v_\varepsilon \rightharpoonup \widetilde{\mathbb{L}}_0 \nabla v_k$ weakly in $L^2(B)$, and thus $(\widetilde{\mathbb{L}}_0 - \mathbb{L}_0)\nabla v_k = 0$ μ-a.e. in B. Since k was arbitrary, the last identity holds for all $k = 1, \ldots, n$. Hence (42) yields $\mathbb{L}_0 = \widetilde{\mathbb{L}}_0$ μ-a.e. in $\frac{1}{2}B$. Since x is arbitrary, the last identity holds μ-a.e. in ω.

Step 2: Proof of (b).
Let $\omega \Subset \Omega$. We define $\mathcal{L}_\varepsilon$ and $\mathcal{L}_0$ according to (44) and denote the adjoint operators by $\mathcal{L}_\varepsilon^*$, $\mathcal{L}_0^*$, i.e.,

$$\mathcal{L}_\varepsilon^* : H_0^1(\omega) \to H^{-1}(\omega), \quad \mathcal{L}_\varepsilon^* := -\operatorname{div}(\mathbb{L}_\varepsilon^* \nabla),$$
$$\mathcal{L}_0^* : H_0^1(\omega) \to H^{-1}(\omega), \quad \mathcal{L}_0^* := -\operatorname{div}(\mathbb{L}_0^* \nabla).$$

Fix $f \in H^{-1}(\omega)$ and let $u_\varepsilon, u_0 \in H_0^1(\omega)$ be the unique solutions to $\mathcal{L}_\varepsilon^* u_\varepsilon = f$ and $\mathcal{L}_0^* u_0 = f$. It suffice to show that $u_\varepsilon \rightharpoonup u_0$ weakly in $H_0^1(\omega)$ and $\mathbb{L}_\varepsilon^* \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0^* \nabla u_0$ weakly in $L^2(T\omega)$. Since the limiting equation uniquely determines u_0, it suffices to prove the statements up to a subsequence. By a standard energy estimate and the uniform boundedness of $(\mathbb{L}_\varepsilon^*)$ the sequences (u_ε) and $(\mathbb{L}_\varepsilon^* \nabla u_\varepsilon)$ are bounded in $H_0^1(\omega)$ and $L^2(T\omega)$, respectively. Hence, there exits $\tilde{u}_0 \in H_0^1(\omega)$ and $J_0 \in L^2(T\omega)$ such that for a subsequence (not relabeled),

$$\begin{cases} u_\varepsilon \rightharpoonup \tilde{u}_0 & \text{weakly in } H_0^1(\omega), \\ \mathbb{L}_\varepsilon^* \nabla u_\varepsilon \rightharpoonup J_0 & \text{weakly in } L^2(T\omega). \end{cases}$$

In the next two substeps we complete the argument by showing $\tilde{u}_0 = u_0$ and $J_0 = \mathbb{L}_0^* \nabla u_0$.

Substep 2.1. Argument for $\tilde{u}_0 = u_0$: Let $v_0 \in H_0^1(\omega)$ and consider $v_\varepsilon := (\mathcal{L}_\varepsilon)^{-1} \mathcal{L}_0 v_0$. Thanks to $\mathbb{L}_\varepsilon \xrightarrow{H} \mathbb{L}_0$ we have

$$\begin{cases} v_\varepsilon \rightharpoonup v_0 & \text{weakly in } H_0^1(\omega) \text{ and strongly in } L^2(\omega), \\ \mathbb{L}_\varepsilon \nabla v_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla v_0 & \text{weakly in } L^2(T\omega). \end{cases}$$

The Div-Curl Lemma (Lemma 25) thus yields

$$\int_\omega (\mathbb{L}_\varepsilon^* \nabla u_\varepsilon, \nabla v_\varepsilon) = \int_\omega (\nabla u_\varepsilon, \mathbb{L}_\varepsilon \nabla v_\varepsilon) \to \int_\omega (\nabla \tilde{u}_0, \mathbb{L}_0 \nabla v_0) = \int_\omega (\mathbb{L}_0^* \nabla \tilde{u}_0, \nabla v_0)$$
$$= \langle \mathcal{L}_0^* \tilde{u}_0, v_0 \rangle.$$

Since, on the other hand we have $\int_\omega (\mathbb{L}_\varepsilon^* \nabla u_\varepsilon, \nabla v_\varepsilon) = \langle f, v_\varepsilon \rangle \to \langle f, v_0 \rangle$, and since $v_0 \in H_0^1(\omega)$ is arbitrary, we conclude $\mathcal{L}_0^* \tilde{u}_0 = f$ in $H_0^{-1}(\omega)$. Since the kernel of $\mathcal{L}_0^*$ is trivial, we deduce that $\tilde{u}_0 = u_0$.

Substep 2.2: Argument for $J_0 = \mathbb{L}_0^* \nabla u_0$. Let $B \Subset \omega$ be an open ball with radius less than the injectivity radius at its center and fix $v_1, \ldots, v_n \in C_c^\infty(B) \subset C_c^\infty(\omega)$ according to Remark 24. Consider $v_\varepsilon := (\mathcal{L}_\varepsilon)^{-1} \mathcal{L}_0 v_j$ and note that $\mathbb{L}_\varepsilon \xrightarrow{H} \mathbb{L}_0$ yields

$$\begin{cases} v_\varepsilon \rightharpoonup v_j & \text{weakly in } H_0^1(\omega) \text{ and strongly in } L^2(\omega), \\ \mathbb{L}_\varepsilon \nabla v_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla v_j & \text{weakly in } L^2(T\omega). \end{cases}$$

Thus for any $\varphi \in C_c^\infty(\omega)$ the Div-Curl Lemma (Lemma 25) yields

$$\int_\omega (\mathbb{L}_\varepsilon^* \nabla u_\varepsilon, \nabla v_\varepsilon)\varphi \to \int_\omega (J_0, \nabla v_j)\varphi,$$

and thus

$$\int_\omega (\mathbb{L}_\varepsilon^* \nabla u_\varepsilon, \nabla v_\varepsilon)\varphi = \int_\omega (\nabla u_\varepsilon, \mathbb{L}_\varepsilon \nabla v_\varepsilon)\varphi \to \int_\omega (\nabla u_0, \mathbb{L}_0 \nabla v_j)\varphi = \int_\omega (\mathbb{L}_0^* \nabla u_0, \nabla v_j)\varphi.$$

Since $\varphi \in C_c^\infty(\omega)$ is arbitrary because of (42), we get $J_0 = \mathbb{L}_0^* \nabla u_0$. □

Proof of Lemma 8. Let $\mathcal{L}_\varepsilon$ and $\mathcal{L}_0$ be defined by (44) and denote by $\mathcal{L}_\varepsilon^*$ and $\mathcal{L}_0^*$ the adjoint operators. Note that u_0 is uniquely determined by

$$\mathcal{L}_0 u_0 = f_0 - \text{div}(\mathbb{L}_0 G_0) - \text{div} F_0 \quad \text{in } H^{-1}(\omega). \tag{51}$$

We first note that (up to a subsequence) (u_ε) converges weakly in $H_0^1(\omega)$ to some $\tilde{u}_0 \in H_0^1(\omega)$, and $(\mathbb{L}_\varepsilon \nabla u_\varepsilon)$ converges weakly in $L^2(T\omega)$ to some $J_0 \in L^2(T\omega)$. We first claim that $\tilde{u}_0$ solves (51) (which by uniqueness of the solution implies that $\tilde{u}_0 = u_0$). For the argument let $v_0 \in H_0^1(\omega)$ and consider the oscillating test-function $v_\varepsilon := (\mathcal{L}_\varepsilon^*)^{-1} \mathcal{L}_0^* v_0 \in H_0^1(\omega)$. Since $\mathbb{L}_\varepsilon^* \overset{H}{\to} \mathbb{L}_0^*$ by Lemma 7, and $\mathcal{L}_\varepsilon^* v_\varepsilon = \mathcal{L}_0^* v_0$, we deduce that

$$\begin{cases} v_\varepsilon \rightharpoonup v_0 & \text{weakly in } H_0^1(\omega) \text{ and strongly in } L^2(\omega), \\ \mathbb{L}_\varepsilon^* \nabla v_\varepsilon \rightharpoonup \mathbb{L}_0^* \nabla v_0 & \text{weakly in } L^2(T\omega). \end{cases}$$

Thanks to $u_\varepsilon \rightharpoonup \tilde{u}_0$ weakly in $H_0^1(\omega)$ and the Div-Curl Lemma (Lemma 25) we get on the one hand

$$\begin{aligned} \langle \mathcal{L}_\varepsilon u_\varepsilon, v_\varepsilon \rangle &= \int_\omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla v_\varepsilon) = \langle f_\varepsilon, v_\varepsilon \rangle + \int_\omega (G_\varepsilon, \mathbb{L}_\varepsilon^* \nabla v_\varepsilon) + (F_\varepsilon, \nabla v_\varepsilon) \\ &\to \int_\omega f_0 v_0 + \int_\omega (G_0, \mathbb{L}_0^* \nabla v_0) + (F_0, \nabla v_0) \\ &= \int_\omega f_0 v_0 + \int_\omega (\mathbb{L}_0 G_0 + F_0, \nabla v_0), \end{aligned}$$

and on the other hand

$$\begin{aligned} \langle \mathcal{L}_\varepsilon u_\varepsilon, v_\varepsilon \rangle &= \langle \mathcal{L}_\varepsilon^* v_\varepsilon, u_\varepsilon \rangle = \int_\omega (\nabla u_\varepsilon, \mathbb{L}_\varepsilon^* \nabla v_\varepsilon) \to \int_\omega (\nabla \tilde{u}_0, \mathbb{L}_0^* \nabla v_0) = \int_\omega (\mathbb{L}_0 \nabla \tilde{u}_0, \nabla v_0) \\ &= \langle \mathcal{L}_0 \nabla \tilde{u}_0, \nabla v_0 \rangle. \end{aligned}$$

Since $v_0 \in H_0^1(\omega)$ is arbitrary, we conclude that $\tilde{u}_0$ solves (51) and thus $\tilde{u}_0 = u_0$. Moreover, by the argument of Substep 2.1 in the proof of Lemma 7 (b), we deduce that $J_0 = \mathbb{L}_0 \nabla u_0$, which completes the argument. □

4.2 Proof of Theorem 5

The proof is structured as follows: In Step 1 we pass to a subsequence and define the *H*-limit $\mathbb{L}_0$ by appealing to a covering of *M* by balls, Proposition 6, and Lemma 7; (at this point we only have *H*-convergence on balls). In Step 2 we show part (b) of the theorem and recover (a) as a special case.

Step 1. Choice of the subsequence and definition of $\mathbb{L}_0$.
Let (B_j) denote a countable covering of *M* by open balls with $4B_j \Subset M$ such that the radius of B_j is smaller than a quarter of the injectivity radius of *M* at the center of B_j. For every $j \in \mathbb{N}$ Proposition 6 provides a subsequence of $(\mathbb{L}_\varepsilon)$ *H*-converging to some $\mathbb{L}_{j,0} \in \mathcal{M}(2B_j, \lambda, \Lambda)$ in $2B_j$. Thus (by a diagonal subsequence argument) we can choose a subsequence $E' \subset E$ such that $(\mathbb{L}_\varepsilon)$ *H*-converges to $\mathbb{L}_{j,0}$ in $2B_j$ for all $j \in \mathbb{N}$. By Lemma 7 (a) we have $\mathbb{L}_{j,0} = \mathbb{L}_{k,0}$ μ-a.e. in $B_j \cap B_k$, and thus we can choose a coefficient field $\mathbb{L}_0 \in \mathcal{M}(M, \lambda, \Lambda)$ with $\mathbb{L}_0(x) = \mathbb{L}_{j,0}(x)$ for μ-a.e. $x \in B_j$, $j \in \mathbb{N}$.

Step 2. Proof of (b).
Fix $\Omega \subset M$ open, $m > \frac{m_0(\Omega)}{\lambda}$, and take sequences $(f_\varepsilon) \subset L^2(\Omega)$ and $(F_\varepsilon) \subset L^2(T\Omega)$ with $f_\varepsilon \rightharpoonup f_0$ weakly in $L^2(\Omega)$ and $F_\varepsilon \to F_0$ in $L^2(T\Omega)$. Let $u_\varepsilon \in H_0^1(\Omega)$ be the solution to

$$m u_\varepsilon - \text{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) = f_\varepsilon - \text{div} F_\varepsilon \quad \text{in } H^{-1}(\Omega).$$

We extract a subsequence $E'' \subset E'$ such that

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{in } H_0^1(\Omega), \\ \mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup J_0 & \text{in } L^2(T\Omega) \end{cases} \tag{52}$$

for some $u_0 \in H^1(\Omega)$ and $J_0 \in L^2(T\Omega)$. We now claim that u_0 is the (unique) solution in $H_0^1(\Omega)$ to

$$mu_0 - \text{div}(\mathbb{L}_0 \nabla u_0) = f_0 - \text{div } F_0 \quad \text{in } H^{-1}(\Omega) \tag{53}$$

and that $J_0 = \mathbb{L}_0 \nabla u_0$. For the argument we use the covering (B_j) of M described in Step 1. Let $\varphi_j \in C_c^\infty(M)$ denote a partition of unity subordinate to (B_j), in the sense that $\text{supp}\,\varphi_j \Subset B_j$ and $\sum_{j=1}^\infty \varphi_j = 1$. Then for every $\varphi \in H_0^1(\Omega)$ and every $j \in \mathbb{N}$

$$\begin{aligned} \int_\Omega (\mathbb{L}_\varepsilon \nabla(\varphi_j u_\varepsilon), \nabla\varphi) &= \int_\Omega (u_\varepsilon \mathbb{L}_\varepsilon \nabla\varphi_j, \nabla\varphi) + \int_\Omega (\varphi_j \mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla\varphi) \\ &= \int_\Omega (u_\varepsilon \mathbb{L}_\varepsilon \nabla\varphi_j, \nabla\varphi) + \int_\Omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla(\varphi_j\varphi)) - \int_\Omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \varphi\nabla\varphi_j) \\ &= \int_\Omega (u_\varepsilon \mathbb{L}_\varepsilon \nabla\varphi_j, \nabla\varphi) + \int_\Omega (f_\varepsilon - mu_\varepsilon)\varphi_j\varphi + (F_\varepsilon, \nabla(\varphi_j\varphi)) \\ &\quad - \int_\Omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \varphi\nabla\varphi_j) \\ &= \int_\Omega (\mathbb{L}_\varepsilon(u_\varepsilon \nabla\varphi_j), \nabla\varphi) + \int_\Omega (\varphi_j F_\varepsilon, \nabla\varphi) \\ &\quad + \int_\Omega ((f_\varepsilon - mu_\varepsilon)\varphi_j + ((F_\varepsilon - \mathbb{L}_\varepsilon \nabla u_\varepsilon), \nabla\varphi_j))\varphi \\ &= \int_\Omega (\mathbb{L}_\varepsilon G_{j,\varepsilon}, \nabla\varphi) + \int_\Omega (F_{j,\varepsilon}, \nabla\varphi) + \int_\Omega g_{j,\varepsilon}\varphi, \end{aligned} \tag{54}$$

where

$$g_{j,\varepsilon} := (f_\varepsilon - mu_\varepsilon)\varphi_j + ((F_\varepsilon - \mathbb{L}_\varepsilon \nabla u_\varepsilon), \nabla\varphi_j), \quad G_{j,\varepsilon} := u_\varepsilon \nabla\varphi_j, \quad F_{j,\varepsilon} := \varphi_j F_\varepsilon.$$

Moreover set $v_{j,\varepsilon} := \varphi_j u_\varepsilon$ and note that $v_{j,\varepsilon} \in H_0^1(B_j)$. Since (54) holds in particular for all $\varphi \in H_0^1(B_j)$, we infer that $v_{j,\varepsilon}$ is the unique solution in $H_0^1(B_j)$ to

$$-\text{div}(\mathbb{L}_\varepsilon \nabla v_{j,\varepsilon}) = g_{j,\varepsilon} - \text{div}(\mathbb{L}_\varepsilon G_{j,\varepsilon}) - \text{div } F_{j,\varepsilon} \quad \text{in } H^{-1}(B_j).$$

By Step 1 we have $\mathbb{L}_\varepsilon \xrightarrow{H} \mathbb{L}_0$ on $2B_j$. Furthermore, from (52), the compact embedding of $H_0^1(B_j) \subset L^2(B_j)$ (which yields $u_\varepsilon \to u_0$ strongly in $L^2(B_j)$), and the convergence properties of (f_ε) and (F_ε), we deduce that

$$\begin{cases} v_{j,\varepsilon} \rightharpoonup v_{j,0} := \varphi_j u_0 & \text{weakly in } H^1(B_j), \\ g_{j,\varepsilon} \rightharpoonup g_{j,0} := (f_0 - mu_0)\varphi_j + ((F_0 - J_0), \nabla\varphi_j) & \text{weakly in } L^2(B_j), \\ G_{j,\varepsilon} \to G_{j,0} := u_0 \nabla\varphi_j & \text{strongly in } L^2(TB_j), \\ F_{j,\varepsilon} \to F_{j,0} := \varphi_j F_0 & \text{strongly in } L^2(TB_j). \end{cases} \tag{55}$$

Hence, Lemma 8 implies that $v_{j,0} \in H_0^1(B_j)$ is the weak solution to

$$-\text{div}(\mathbb{L}_0 \nabla v_{j,0}) = g_{j,0} - \text{div}(\mathbb{L}_0 G_{j,0}) - \text{div } F_{j,0} \quad \text{in } H^{-1}(B_j),$$

and

$$\mathbb{L}_\varepsilon \nabla v_{j,\varepsilon} \rightharpoonup \mathbb{L}_0 \nabla v_{j,0} \quad \text{weakly in } L^2(TB_j). \tag{56}$$

Since $\sum_{j=1}^\infty \varphi_j = 1$ we deduce that $\sum_{j=1}^\infty \nabla\varphi_j = 0$, and thus

$$\sum_{j=1}^\infty v_{j,0} = u_0, \quad \sum_{j=1}^\infty F_{j,0} = F_0, \quad \sum_{j=1}^\infty G_{j,0} = 0, \quad \sum_{j=1}^\infty g_{j,0} = (f_0 - mu_0).$$

In particular, summation of (56) yields $\mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup J_0 = \mathbb{L}_0 \nabla u_0$ weakly in $L^2(T\Omega)$. Moreover, for any test function $\varphi \in C_c^\infty(\Omega)$ we have on the one hand

$$\int_\Omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla\varphi) = \sum_{j=1}^\infty \int_\Omega (\mathbb{L}_\varepsilon \nabla v_{j,\varepsilon}, \nabla\varphi) \to \sum_{j=1}^\infty \int_\Omega (\mathbb{L}_0 \nabla v_{j,0}, \nabla\varphi) = \int_\Omega (\mathbb{L}_0 \nabla u_0, \nabla\varphi),$$

and on the other hand, by summation of (54), and by (55),

$$\int_\Omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla\varphi) = \sum_{j=1}^\infty \int_\Omega (\mathbb{L}_\varepsilon dv_{j,\varepsilon}, \nabla\varphi)$$

$$= \sum_{j=1}^\infty \int_{B_j} (\mathbb{L}_\varepsilon G_{j,\varepsilon}, \nabla\varphi) + (F_{j,\varepsilon}, \nabla\varphi) + g_{j,\varepsilon}\varphi$$

$$\to \sum_{j=1}^d \int_{B_j} (\mathbb{L}_0 G_{j,0} + F_{j,0}, \nabla\varphi) + g_{0,j}\varphi$$

$$= \int_\Omega (F_0, \nabla\varphi) + (f_0 - mu_0)\varphi.$$

The combination of the previous two identities yields (53). Since the latter admits a unique solution, we deduce that the convergence holds for the entire subsequence E'. Finally we note that if $H_0^1(\Omega)$ is compactly contained in $L^2(\Omega)$, then we even have $u_\varepsilon \to u_0$ strongly in $L^2(\Omega)$. The same conclusion is true if $m \neq 0$ and $f_\varepsilon \to f_0$ strongly in $L^2(\Omega)$. To see this, first note that by $\mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla u_0$ and Lemma 25 we have

$$\int_\Omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon) \to \int_\Omega (\mathbb{L}_0 \nabla u_0, \nabla u_0).$$

Thus, since we may pass to the limit in products of weakly and strongly convergent sequences,

$$m \int_\Omega u_\varepsilon^2 = m \int_\Omega u_\varepsilon^2 + \int_\Omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon) - \int_\Omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)$$

$$= \int_\Omega f_\varepsilon u_\varepsilon + \int_\Omega (F_\varepsilon, \nabla u_\varepsilon) - \int_\Omega (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)$$

$$\to \int_\Omega f_0 u_0 + \int_\Omega (F_0, \nabla u_0) - \int_\Omega (\mathbb{L}_0 \nabla u_0, \nabla u_0) = m \int_\Omega u_0^2.$$

Since $m \neq 0$, this implies $\|u_\varepsilon\|_{L^2(\Omega)} \to \|u_0\|_{L^2(\Omega)}$, which combined with the weak convergence $u_\varepsilon \rightharpoonup u_0$ in $L^2(\Omega)$ yields the claimed strong convergence $u_\varepsilon \to u_0$ in $L^2(\Omega)$. This completes the argument for part (b).

Step 3. Proof of part (a).
Since $m_0(\omega) < 0$, we can take $m = 0$ in part (b) and *H*-convergence immediately follows. $\square$

4.3 Proofs of Lemma 12, 13, and 15

Proof of Lemma 12. Let $\bar{\xi} = (\bar{\xi}^1, \ldots, \bar{\xi}^n), \bar{\eta} = (\bar{\eta}^1, \ldots, \bar{\eta}^n) \in \mathbb{R}^n$ and $\xi, \eta \in T_x M$ such that

$$\begin{cases} \bar{\xi}^i = g(\xi, \frac{\partial}{\partial x^i}) \\ \bar{\eta}^i = g(\eta, \frac{\partial}{\partial x^i}) \end{cases} \quad \text{for } i = 1, \ldots, n.$$

We identify $x \in \Psi^{-1}(U)$ and the corresponding point in U. Since the metric $g(\cdot, \cdot)(x)$ continuously depends on x, since Ψ is a diffeomorphism, and because $U \Subset \Psi(\Omega)$, there exists a constant $C > 0$ such that

$$\frac{1}{C}|\bar{\xi}|^2 \leq \sum_{i,j=1}^n g^{ij}(x)\bar{\xi}^i\bar{\xi}^j = g(\xi, \xi)(x) \leq C|\bar{\xi}|^2 \quad \text{and} \quad \frac{1}{C} \leq \rho(x) \leq C,$$

for all $x \in \Psi^{-1}(U)$, where (g^{ij}) denotes the inverse of the matrix representation (g_{ij}) of g in local coordinates, i.e., $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. Then the uniform ellipticity of $\mathbb{L}$ yields

$$A(x)\bar{\xi} \cdot \bar{\xi} = \rho(x)g(\mathbb{L}\xi, \xi)(x) \geq \lambda\rho(x)g(\xi, \xi)(x) \geq \frac{1}{C'}|\bar{\xi}|^2$$

and

$$A(x)\bar{\xi} \cdot \bar{\eta} = \rho(x)g(\mathbb{L}\xi, \eta)(x) \leq \Lambda\rho(x)|\xi(x)|_g|\eta(x)|_g \leq C'|\bar{\xi}||\bar{\eta}|$$

for some $C' > 0$. Thus the statement follows. $\square$

Proof of Lemma 13. We prove only (2) $\Rightarrow$ (1) as the opposite implication can be proved in the same way. Let $f \in L^2(\omega)$ and $\xi \in L^2(T\omega)$. Let $u_\varepsilon \in H_0^1(\omega)$ with $\varepsilon > 0$ be the solution of

$$-\text{div}_{g,\mu}(\mathbb{L}_\varepsilon \nabla_g u_\varepsilon) = f - \text{div}_{g,\mu} \xi \quad \text{in } H^{-1}(\omega).$$

By (20), u_ε is the solution to

$$-\text{div}(A_\varepsilon \nabla u_\varepsilon) = \rho f - \text{div}(\rho F) \quad \text{in } H^{-1}(U).$$

Since (A_ε) *H*-converges to A_0,

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H_0^1(U), \\ A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0 & \text{weakly in } L^2(U; \mathbb{R}^n), \end{cases} \tag{57}$$

where

$$-\text{div}(A_0 \nabla u_0) = \rho f - \text{div}(\rho F) \quad \text{in } H^{-1}(U). \tag{58}$$

By (57)

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\omega, g). \tag{59}$$

For any $\eta \in L^2(T\omega)$ and $\overline{\eta} = (\overline{\eta}^1, \ldots, \overline{\eta}^n) \in L^2(U; \mathbb{R}^n)$ with $\overline{\eta}^i := g(\eta, \frac{\partial}{\partial x^i})$ for $i = 1, \ldots, n$ we have

$$\int_\omega g(\mathbb{L}_\varepsilon \nabla_g u_\varepsilon, \eta) \, d\mu = \int_U A_\varepsilon(x) \nabla u_\varepsilon \cdot \overline{\eta} \, dx \rightarrow \int_U A_0(x) \nabla u_0 \cdot \overline{\eta} \, dx \quad (\text{as } \varepsilon \rightarrow 0)$$

$$= \int_\omega g(\mathbb{L}_0 \nabla_g u_0, \eta) \, d\mu.$$

Hence,

$$\mathbb{L}_\varepsilon \nabla_g u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla_g u_0 \quad \text{weakly in } L^2(T\omega). \tag{60}$$

Since (58) is equivalent to

$$-\text{div}_{g,\mu}(\mathbb{L}_0 \nabla_g u_0) = f - \text{div}_{g,\mu} \xi \quad \text{in } H^{-1}(\omega),$$

together with (59) and (60) we arrive at the conclusion. □

Proof of Lemma 15. The proof is a direct consequence of Lemma 13 and the well-known fact from periodic homogenization that $A_\varepsilon(x) = A(x, \frac{x}{\varepsilon})$ H-converges to A_{hom}, e.g., see [2, Theorem 2.2]. □

4.4 Proofs of Lemma 16, 19, and 20

Proof of Lemma 16. Step 1. Argument for (a) ⇔ (b).

Since $h: M_0 \rightarrow M$ is a diffeomorphism, the integral transformation formula yields for any function $f \in L^1(M, g, \mu)$

$$\int_M f \, d\mu = \int_{M_0} (f \circ h) \rho \, d\mu_0.$$

To show the equivalence of statement (a) and (b) it only remains to show

$$\overline{g}(\nabla_{\overline{g}} u, \nabla_{\overline{g}} \varphi) \rho = g_0(\mathbb{L} \nabla_{g_0} \overline{u}, \nabla_{g_0} \overline{\varphi})$$

for any test function $\varphi \in C_c^\infty(M)$. To that end we first claim $\nabla_{\overline{g}} u = (dh^{-1})^* \nabla_{g_0} \overline{u}$ (and that the same holds for φ). Indeed, using the definition of the gradient and the adjoint, we have

$$\overline{g}(\nabla_{\overline{g}} u, \xi) = du(\xi) = d(u \circ h)(dh^{-1}\xi) = g_0(\nabla_{g_0} \overline{u}, dh^{-1}\xi) = \overline{g}((dh^{-1})^* \nabla_{g_0} \overline{u}, \xi).$$

Together with the definition of $\mathbb{L}$ we conclude

$$\overline{g}(\nabla_{\overline{g}} u, \nabla_{\overline{g}} \varphi) \rho = \overline{g}((dh^{-1})^* \nabla_{g_0} \overline{u}, (dh^{-1})^* \nabla_{g_0} \overline{\varphi}) \rho = g_0(\mathbb{L} \nabla_{g_0} \overline{u}, \nabla_{g_0} \overline{\varphi}).$$

Step 2. Argument for (b) ⇔ (c).

By the definition of $\hat{\mu}_0$ it suffices to show

$$g_0(\mathbb{L} \nabla_{g_0} \overline{u}, \nabla_{g_0} \overline{\varphi}) = \hat{g}_0(\nabla_{\hat{g}_0} \overline{u}, \nabla_{\hat{g}_0} \overline{\varphi}) \rho.$$

We first observe $\mathbb{L} \nabla_{g_0} \overline{u} = \rho \nabla_{\hat{g}_0} \overline{u}$, which can be seen by the following direct computation, using the definition of $\hat{g}_0$ and of the gradient:

$$\hat{g}_0(\mathbb{L} \nabla_{g_0} \overline{u}, \xi) = \rho \, g_0(\nabla_{g_0} \overline{u}, \xi) = \rho \, d\overline{u}(\xi) = \rho \, \hat{g}_0(\nabla_{\hat{g}_0} \overline{u}, \xi).$$

Again with the definition of the gradient we finally get

$$g_0(\mathbb{L} \nabla_{g_0} \overline{u}, \nabla_{g_0} \overline{\varphi}) = \rho \, g_0(\nabla_{\hat{g}_0} \overline{u}, \nabla_{g_0} \overline{\varphi}) = \rho \, d\overline{\varphi}(\nabla_{\hat{g}_0} \overline{u}) = \rho \, \hat{g}_0(\nabla_{\hat{g}_0} \overline{u}, \nabla_{\hat{g}_0} \overline{\varphi}).$$

□

Proof of Lemma 19. By construction, there exists a constant $C_0 > 0$ (only depending on the constant C of Definition 1 and the dimension n) such that $\mathbb{L}_\varepsilon \in \mathcal{M}(M_0, \frac{1}{C_0}, C_0)$ and $\frac{1}{C_0} \leq \rho_\varepsilon \leq C_0$ a.e. in M_0. Therefore, by weak-∗ compactness in $L^\infty(M_0)$ and by Theorem 5 there exist a subsequence, a density $\rho_0 \in L^\infty(M_0)$ satisfying $\frac{1}{C_0} \leq \rho_0 \leq C_0$, and a coefficient field $\mathbb{L}_0 \in \mathcal{M}(M_0, \frac{1}{C_0}, C_0)$ such that $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ weak-∗ in $L^\infty(M_0)$ and $\mathbb{L}_\varepsilon \overset{H}{\rightarrow} \mathbb{L}_0$ in (M_0, g_0, μ_0) along a subsequence that we do not relabel. This proves statement (a).

Next, we prove statement (b). Set $\overline{u}_\varepsilon := u_\varepsilon \circ h_\varepsilon$ and $\overline{f}_\varepsilon := f \circ h_\varepsilon$. By Lemma 16 (b), (28a) is equivalent to

$$(\overline{m} - \mathrm{div}_{g_0,\mu_0}(\mathbb{L}_\varepsilon \nabla_{g_0}))\overline{u}_\varepsilon = \rho_\varepsilon \overline{f}_\varepsilon - (\rho_\varepsilon m - \overline{m})\overline{u}_\varepsilon \quad \text{in } H^{-1}(M_0, g_0, \mu_0), \tag{61}$$

where $\overline{m}$ denotes a (sufficiently large) dummy constant that we introduce in order to be able to apply Theorem 5. By a standard energy estimate, $(\overline{u}_\varepsilon)$ is bounded in $H^1(M_0, g_0, \mu_0)$ and thanks to the compact embedding of $H^1(M_0, g_0, \mu_0)$ in $L^2(M_0, g_0, \mu_0)$ in Assumption 17. Thus there exists $\overline{u}_0 \in H_0^1(M_0, g_0, \mu_0)$ such that $\overline{u}_\varepsilon \to \overline{u}_0$ strongly in $L^2(M_0, g_0, \mu_0)$ (for a further subsequence). Moreover, since $f_\varepsilon \to f_0$ strongly in L^2 in the sense of (27), $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ weak-$*$ in $L^\infty(M_0)$, and since $\frac{1}{C_0} \le \rho_\varepsilon \le C_0$, we deduce that $\rho_\varepsilon f_\varepsilon \rightharpoonup \rho_0 f_0$ weakly in $L^2(M_0, g_0, \mu_0)$, and thus we get for the right-hand side in (61),

Since $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ we conclude with Theorem 5 that $\overline{u}_0$ is a solution to

$$(\overline{m} - \mathrm{div}_{g_0,\mu_0}(\mathbb{L}_0 \nabla_{g_0}))\overline{u}_0 = \rho_0 f_0 - (\rho_0 m - \overline{m})\overline{u}_0 \quad \text{in } H^{-1}(M_0, g_0, \mu_0). \tag{62}$$

Since this PDE admits a unique solution, we conclude that $\overline{u}_\varepsilon \rightharpoonup \overline{u}_0$ weakly in $H^1(M_0, g_0, \mu_0)$, and thus strongly in $L^2(M_0, g_0, \mu_0)$, for the entire sequence. By appealing to the equivalence of (b) and (c) in Lemma 16, we deduce from (62) that $u_0 := \overline{u}_0$ satisfies (28b). It remains to argue that $u_\varepsilon \to u_0$ in the sense of (27). To that end let $\psi \in C_c^\infty(M_0)$. Then, since $\overline{u}_\varepsilon \to u_0$ strongly and $\rho_\varepsilon \rightharpoonup \rho_0$ weakly in $L^2(M_0, g_0, \mu_0)$,

$$\int_{M_\varepsilon} u_\varepsilon(\psi \circ h_\varepsilon^{-1}) \, \mathrm{d}\mu_\varepsilon = \int_{M_0} \overline{u}_\varepsilon \psi \rho_\varepsilon \, \mathrm{d}\mu_0 \to \int_{M_0} u_0 \psi \rho_0 \, \mathrm{d}\mu_0 = \int_{M_0} u_0 \psi \, \mathrm{d}\hat{\mu}_0.$$

Moreover, since $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ in $L^\infty(M_0)$ we have $\overline{u}_\varepsilon \rho_\varepsilon \rightharpoonup u_0 \rho_0$ weakly in $L^2(M_0, g_0, \mu_0)$, and thus

$$\int_{M_\varepsilon} |u_\varepsilon|^2 \, \mathrm{d}\mu_\varepsilon = \int_{M_0} \overline{u}_\varepsilon \, \overline{u}_\varepsilon \rho_\varepsilon \, \mathrm{d}\mu_0 \to \int_{M_0} u_0 \, u_0 \rho_0 \, \mathrm{d}\mu_0 = \int_{M_0} |u_0|^2 \, d\hat{\mu}_0.$$

$\square$

Proof of Lemma 20. The argument is similar to the proof of Lemma 11, which itself is based on [11, Lemma 11.3 and Theorem 11.5]. We only need to treat small changes that come from rewriting the eigenvalue problem on M_ε as a PDE on the reference manifold M_0. For the sake of brevity we only prove that eigenpairs of the Laplace–Beltrami operator on M_ε converge (up to a subsequence) to an eigenpair of the Laplace–Beltrami operator on $(M_0, \hat{g}_0, \hat{\mu}_0)$. The conclusion of the statements of the theorem then follow by appealing to [11, Lemma 11.3 and Theorem 11.5].

We first note that for all $k \in \mathbb{N}$ the sequence $(\lambda_{\varepsilon,k})$ is bounded from above: For the first eigenvalue, (11) implies

$$\lambda_{\varepsilon,1} = \inf\left\{ \int_{M_\varepsilon} g_\varepsilon(\nabla_{g_\varepsilon} u, \nabla_{g_\varepsilon} u) \, \mathrm{d}\mu_\varepsilon ; u \in H_0^1(M_\varepsilon), \|u\|_{L^2(M_\varepsilon)} = 1 \right\}$$

$$= \inf\left\{ \int_{M_0} g_0(\mathbb{L}_\varepsilon \nabla_{g_0}(u \circ h_\varepsilon), \nabla_{g_0}(u \circ h_\varepsilon)) \, \mathrm{d}\mu_0 ; u \in H_0^1(M_\varepsilon), \|u\|_{L^2(M_\varepsilon)} = 1 \right\}$$

$$\le C_0 \inf\left\{ \int_{M_0} g_0(\nabla_{g_0} v, \nabla_{g_0} v) \, \mathrm{d}\mu_0 ; v \in H_0^1(M_0), \|v\|_{L^2(M_0)} = 1 \right\}$$

$$< \infty$$

for some constant $C_0 > 0$ only depending on the constant C in Definition 1 and the dimension n. The analogue statement for the other eigenvalues can be obtained by the Rayleigh-Ritz method with a similar argument. Likewise the sequence of the first eigenvalues $(\lambda_{1,\varepsilon})$ is bounded from below by a positive constant. Indeed, for every eigenpair $(\lambda_\varepsilon, u_\varepsilon)$ we deduce with Lemma 16, (11), and assumption $m_0(M_0) < 0$ that there exists constants $C_0, \overline{C}_0 > 0$ (only depending on the constant C in Definition 1 and the dimension n) such that

$$\lambda_{\varepsilon,1} = \lambda_\varepsilon \|u_{\varepsilon,1}\|_{L^2(M_\varepsilon)}^2 = \int_{M_\varepsilon} g_\varepsilon(\nabla_\varepsilon u_\varepsilon, \nabla_\varepsilon u_\varepsilon) \, \mathrm{d}\mu_\varepsilon$$

$$= \int_{M_0} g_0(\mathbb{L}_\varepsilon \nabla_{g_0}(u_\varepsilon \circ h_\varepsilon), \nabla_{g_0}(u_\varepsilon \circ h_\varepsilon)) \, \mathrm{d}\mu_0 \ge \frac{1}{C_0} \int_{M_0} g_0(\nabla_{g_0}(u_\varepsilon \circ h_\varepsilon), \nabla_{g_0}(u_\varepsilon \circ h_\varepsilon)) \, \mathrm{d}\mu_0$$

$$\ge \frac{1}{C_0} \|u_\varepsilon\|_{L^2(M_0)}^2 \inf\left\{ \int_{M_0} g_0(\nabla_{g_0} v, \nabla_{g_0} v) \, \mathrm{d}\mu_0 ; v \in H_0^1(M_0), \|v\|_{L^2(M_0)}^2 = 1 \right\}$$

$$\ge \frac{1}{C_0} \|u_\varepsilon\|_{L^2(M_0)}^2 \inf\left\{ \int_{M_0} g_0(\nabla_{g_0} v, \nabla_{g_0} v) \, \mathrm{d}\mu_0 ; v \in H_0^1(M_0), \|v\|_{L^2(M_0)}^2 = 1 \right\}$$

$$\ge \overline{C}_0 > 0,$$

where in the last step we in particular used that $m_0(M_0) < 0$. Now, we fix $k \in \mathbb{N}$ and let $(\lambda_{\varepsilon,k}, u_{\varepsilon,k})$ be an eigenpair, i.e.,

$$-\Delta_{g_\varepsilon,\mu_\varepsilon} u_{\varepsilon,k} = \lambda_{\varepsilon,k} u_{\varepsilon,k} \quad \text{in } H^{-1}(M_\varepsilon, g_\varepsilon, \mu_\varepsilon). \tag{63}$$

By passing to a subsequence we may assume that $\lambda_{\varepsilon,k} \to \overline{\lambda}$ as $\varepsilon \to 0$ for some $\overline{\lambda}$. Moreover, w.l.o.g. we may assume that $u_{\varepsilon,k}$ is normalized in the sense that $\int_{M_\varepsilon} |u_{\varepsilon,k}|^2 \, \mathrm{d}\mu_\varepsilon = 1$. Testing (63) with $u_{\varepsilon,k}$ then shows that $\|u_{\varepsilon,k}\|_{H^1(M_\varepsilon)}$ is bounded by a constant independent of ε. We conclude that $\overline{u}_{\varepsilon,k} := u_{\varepsilon,k} \circ h_\varepsilon$ is bounded in $H^1(M_0, g_0, \mu_0)$ and we thus may pass to a further subsequence with $\overline{u}_{\varepsilon,k} \rightharpoonup \overline{u}$ weakly in $H^1(M_0, g_0, \mu_0)$ and strongly in $L^2(M_0, g_0, \mu_0)$, thanks to the compact

embedding of $H^1(M_0, g_0, \mu_0)$ in $L^2(M_0, g_0, \mu_0)$ in Assumption 17. Note that this implies also that $u_{\varepsilon,k} \to \overline{u}$ strongly in L^2 in the sense of (27). We conclude that the right-hand side of (63) is strongly convergent to $\overline{\lambda}\overline{u}$. Thus, by appealing to Lemma 19 (b) we conclude that

$$-\Delta_{\hat{g}_0, \hat{\mu}_0}\overline{u} = \overline{\lambda}\overline{u} \quad \text{in } H^{-1}(M_0, \hat{g}_0, \hat{\mu}_0).$$

Since $\|\overline{u}\|_{L^2(M_0, \hat{g}_0, \hat{\mu}_0)} = 1$ by construction, we conclude that $(\overline{\lambda}, \overline{u})$ is an eigenpair of the Laplace–Beltrami operator on $(M_0, \hat{g}_0, \hat{\mu}_0)$. $\qquad \square$

Appendix: Proofs of Auxiliary Results

A.1   Proof of Lemma 14

We refer to [27] for a similar result in a nonlinear, variational setting.
Step 1. Continuity of $\nabla\phi_i$ in the first argument.
Consider a sequence (x_j) in $\mathbb{R}^n$ converging to some $x_0 \in \mathbb{R}^n$. For simplicity we set

$$\phi_i^j := \phi_i(x_j, \cdot) \quad \text{and} \quad A^j := A(x_j, \cdot)$$

as well as

$$\phi_i^0 := \phi_i(x_0, \cdot) \quad \text{and} \quad A^0 := A(x_0, \cdot).$$

First we note that the continuity of A in the first argument gives $A^j \to A^0$ a.e. on Y and by uniform ellipticity we have $|A^j| \leq \Lambda$ a.e. on Y. Thus we can conclude

$$\int_Y |A^j - A^0|^p \to 0 \tag{A·1}$$

for $1 < p < \infty$.

Now we claim the convergence of $\nabla\phi_i^j$. By (25) we have

$$-\nabla \cdot A^j(\nabla\phi_i^j - \nabla\phi_i^0) = \nabla \cdot ((A^j - A^0)(\nabla\phi_i^0 + e_i)).$$

The uniform ellipticity of A^j allows to estimate

$$\int_Y |\nabla\phi_i^j - \nabla\phi_i^0|^2 \leq \frac{1}{\lambda}\int_Y |(A^j - A^0)(\nabla\phi_i^0 + e_i)|^2.$$

By Meyer's estimate there is $2 < q < \infty$ and $C > 0$ such that $\int_Y |\nabla\phi_i^0|^q \leq C\int_Y |A^0 e_i|^q$ and thus, for $p = \frac{q}{q-2}$ we have

$$\|\nabla\phi_i^j - \nabla\phi_i^0\|_{L^2(Y)} \leq \frac{1}{\sqrt{\lambda}}\|A^j - A^0\|_{L^p(Y)}(\|\nabla\phi_i^0\|_{L^q(Y)} + 1)$$

and (A·1) implies $\|\nabla\phi_i^j - \nabla\phi_i^0\|_{L^2(Y)} \to 0$.
Step 2. H-convergence to A_{hom}.
Fix $r \in \mathbb{R}$. By Theorem 5 there exists a subsequence (not relabeled) s.t. (A_ε) H-converges to some uniformly elliptic coefficient field A_0 on $\mathbb{R}^n$. Let $B \subset \mathbb{R}^n$ denote an arbitrary ball and let $u_\varepsilon \in H^1(B)$ denote the unique weak solution to

$$\begin{cases} -\nabla \cdot A_\varepsilon \nabla u_\varepsilon = 0 & \text{in } B, \\ u_\varepsilon = x_i & \text{on } \partial B. \end{cases}$$

Then $A_\varepsilon \xrightarrow{H} A_0$ implies that $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(B)$, where u_0 is the unique weak solution to

$$\begin{cases} -\nabla \cdot A_0 \nabla u_0 = 0 & \text{in } B, \\ u_0 = x_i & \text{on } \partial B. \end{cases}$$

For $k \in \mathbb{N}$ let $\eta_k \in C_c^\infty(B)$ be a cut-off function with $\eta_k = 1$ in $B_k := \{x \in B : \text{dist}(x, \partial B) > \frac{1}{k}\}$ and consider

$$v_{\varepsilon,k} := x_i + \varepsilon\phi_i\left(x, \frac{x+r}{\varepsilon}\right)\eta_k(x).$$

Then $(v_{\varepsilon,k})$ converges as $\varepsilon \to 0$ to $v_0(x) := x_i$ weakly in $H^1(B)$ and strongly in $L^2(B)$, and a direct computation shows that

$$\nabla v_{\varepsilon,k}(x) = \left(e_i + \nabla\phi_i\left(x, \frac{x+r}{\varepsilon}\right)\right) + (\eta_k - 1)\nabla\phi_i\left(x, \frac{x+r}{\varepsilon}\right) + \varepsilon\phi_i\left(x, \frac{x+r}{\varepsilon}\right)\nabla\eta_k(x),$$

and thus for $w_{\varepsilon,k} := u_\varepsilon - v_{\varepsilon,k} \in H_0^1(B)$ we have (by appealing to the equation for u_ε and for ϕ_i)

$$\int_B A_\varepsilon \nabla w_{\varepsilon,k} \cdot \nabla w_{\varepsilon,k} = -\int_B A_\varepsilon \nabla v_{\varepsilon,k} \cdot \nabla w_{\varepsilon,k}$$

$$= -\int_B A\left(x, \frac{x+r}{\varepsilon}\right)\left(e_i + \nabla\phi_i\left(x, \frac{x+r}{\varepsilon}\right)\right) \cdot \nabla w_{\varepsilon,k}\, dx$$

$$- \int_B A_\varepsilon \big((\eta_k - 1)\nabla\phi_i\big(x, \tfrac{x+r}{\varepsilon}\big) + \varepsilon\phi_i\big(x, \tfrac{x+r}{\varepsilon}\big)\nabla\eta_k(x) \big) \cdot \nabla w_{\varepsilon,k} \, dx$$

$$\leq C(\Lambda) \int_{S_k} \big(|\nabla\phi_i\big(x, \tfrac{x+r}{\varepsilon}\big)| + \varepsilon|\phi_i\big(x, \tfrac{x+r}{\varepsilon}\big)| \|\nabla\eta_k\|_{L^\infty(S_k)} \big) |\nabla w_{\varepsilon,k}| \, dx$$

for some constant $C(\Lambda) > 0$, where $S_k := B \setminus B_k$. The left-hand side is bounded from below by $\lambda \int_B |\nabla w_{\varepsilon,k}|^2$, and thus (by appealing to the Cauchy-Schwarz inequality), we deduce that

$$\int_B |\nabla w_{\varepsilon,k}|^2 \leq C(\lambda, \Lambda) \int_{S_k} |\nabla\phi_i\big(x, \tfrac{x+r}{\varepsilon}\big)|^2 + (\varepsilon|\phi_i\big(x, \tfrac{x+r}{\varepsilon}\big)| \|\nabla\eta_k\|_{L^\infty(S_k)})^2 \, dx.$$

Since $(|\nabla\phi_i(\cdot, \tfrac{\cdot+r}{\varepsilon})|^2)$ is equi-integrable and $|S_k| \to 0$ for $k \to \infty$, we conclude that

$$\limsup_{k\to\infty} \limsup_{\varepsilon\to 0} \int_B |\nabla w_{\varepsilon,k}|^2 = 0,$$

and thus there exists a diagonal sequence (k_ε) (with $k_\varepsilon \to \infty$ as $\varepsilon \to 0$) such that $w_\varepsilon := w_{k_\varepsilon,\varepsilon}$ satisfies $\nabla w_\varepsilon \to 0$ strongly in $L^2(B)$. Hence, with $v_\varepsilon := v_{\varepsilon,k_\varepsilon}$, we conclude that $\nabla u_\varepsilon - \nabla v_\varepsilon \to 0$ in $L^2(B)$. On the other hand, since $v_\varepsilon \to v_0$ strongly in $L^2(B)$, we conclude that $\nabla u_0 = \nabla v_0 = e_i$. Moreover, the *H*-convergence of (A_ε) to A_0 implies $A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0 = A_0 e_i$ weakly in $L^2(B)$, and thus (using $\nabla u_\varepsilon - \nabla v_\varepsilon \to 0$) we have $A_\varepsilon \nabla v_\varepsilon \rightharpoonup A_0 e_i$ weakly in $L^2(B)$.

On the other hand for any $\varphi \in C_c^\infty(B)$ and $\varepsilon > 0$ small enough, we have $\varphi(x)\nabla v_\varepsilon(x) = \varphi(x)(e_i + \nabla\phi_i(x, \tfrac{x+r}{\varepsilon}))$, and thus by periodicity

$$\int \varphi A_\varepsilon \nabla v_\varepsilon = \int \varphi(x) A\big(x, \tfrac{x+r}{\varepsilon}\big)\big(e_i + \nabla\phi_i\big(x, \tfrac{x+r}{\varepsilon}\big)\big) \, dx$$

$$= \int \varphi(x) A\big(x, \tfrac{x}{\varepsilon} + r_\varepsilon\big)\big(e_i + \nabla\phi_i\big(x, \tfrac{x}{\varepsilon} + r_\varepsilon\big)\big) \, dx,$$

where $r_\varepsilon \in Y$ is defined by the identity $\tfrac{r}{\varepsilon} = k + r_\varepsilon$ for some $k \in \mathbb{Z}^d$. We write that expression in the following way:

$$\int \varphi(x) A\big(x, \tfrac{x}{\varepsilon} + r_\varepsilon\big)\big(e_i + \nabla\phi_i\big(x, \tfrac{x}{\varepsilon} + r_\varepsilon\big)\big) \, dx$$

$$= \int \varphi(x - r_\varepsilon) A\big(x - r_\varepsilon, \tfrac{x}{\varepsilon}\big)\big(e_i + \nabla\phi_i\big(x - r_\varepsilon, \tfrac{x}{\varepsilon}\big)\big) \, dx$$

$$= \sum_{z\in\mathbb{Z}^n} \varepsilon^n \int_Y \varphi(\varepsilon z + \varepsilon y - r_\varepsilon) A(\varepsilon z + \varepsilon y - r_\varepsilon, y)(e_i + \nabla\phi_i(\varepsilon z + \varepsilon y - r_\varepsilon, y)) \, dy.$$

Since (r_ε) is a bounded sequence in $Y \subset \mathbb{R}^n$ we may pass to a subsequence (not relabeled) such that $r_\varepsilon \to r_0$ in Y for some $r_0 \in \overline{Y}$. This implies that $\varphi(\cdot + \varepsilon y - r_\varepsilon) \to \varphi(\cdot - r_0)$ strongly in $L^2(U)$ for any $U \subset \mathbb{R}^n$ open and bounded and every $y \in Y$. On the other hand by Step 1 we we have $A^j \nabla\phi_i^j \to A^0 \phi_i^0$ in $L^1(Y)$ and thus we get

$$\sum_{z\in\mathbb{Z}^n} \varepsilon^n \int_Y \varphi(\varepsilon z + \varepsilon y - r_\varepsilon) A(\varepsilon z + \varepsilon y - r_\varepsilon, y)(e_i + \nabla\phi_i(\varepsilon z + \varepsilon y - r_\varepsilon, y)) \, dy$$

$$\to \int_{\mathbb{R}^n} \varphi(x - r_0) \int_Y A(x - r_0, y)(e_i + \nabla\phi_i(x - r_0, y)) \, dy \, dx$$

$$= \int_{\mathbb{R}^n} \varphi(x - r_0) A_{\mathrm{hom}}(x - r_0) e_i \, dx$$

$$= \int_{\mathbb{R}^n} \varphi(x) A_{\mathrm{hom}}(x) e_i \, dx,$$

and we conclude that $\int \varphi(A_0 - A_{\mathrm{hom}})e_i = 0$ for all $\varphi \in C_c^\infty(B)$, which gives $A_0 = A_{\mathrm{hom}}$ a.e. in B. Since B is an arbitrary ball, we conclude that $A_0 = A_{\mathrm{hom}}$ a.e. in $\mathbb{R}^n$. By uniqueness, we conclude that (A_ε) *H*-convergence to A_{hom} for the entire sequence.

A.2 Proof of Lemma 9

We first recall the definition of Mosco-convergence:

Definition 26 (Mosco-convergence). *We say that $(\mathcal{E}_\varepsilon)$ Mosco-converges to $\mathcal{E}_0$ as $\varepsilon \to 0$ if the following two conditions are satisfied.*

(i) *If $u_\varepsilon \rightharpoonup u$ weakly in $L^2(M)$, then*

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u).$$

(ii) *For any $v \in L^2(M)$ there exists $(v_\varepsilon) \subset L^2(M)$ with $v_\varepsilon \rightharpoonup v$ weakly in $L^2(M)$ such that*

$$\limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(v_\varepsilon) \le \mathcal{E}_0(v).$$

For the proof of Lemma 9 we recall that Mosco-convergence is equivalent to resolvent convergence of the operator associated with the Dirichlet form $\mathcal{E}_\varepsilon$. More precisely, for $\varepsilon \ge 0$ consider $\mathcal{L}_\varepsilon : H_0^1(M) \to H^{-1}(M)$, $\mathcal{L}_\varepsilon u := -\mathrm{div}_{g,\mu}(\mathbb{L}_\varepsilon \nabla_g u)$ and denote for $\lambda > 0$ by $G_\varepsilon^\lambda := (\lambda + \mathcal{L}_\varepsilon)^{-1} : L^2(M) \to H_0^1(M)$ the associated resolvent.

Lemma 27 (Theorem 2.4.1 [23]). *The following two conditions are equivalent.*
  (i) *$(\mathcal{E}_\varepsilon)$ Mosco-converges to $\mathcal{E}_0$.*
  (ii) *For any $\lambda > 0$, (G_ε^λ) converges to G_0^λ in the strong operator topology of $L^2(M)$.*

Proof of Lemma 9. We apply Lemma 27. Let $\lambda > 0$, $f_\varepsilon \to f_0$ in $L^2(M)$, and $u_\varepsilon := G_\varepsilon^\lambda f_\varepsilon$. Since $(\mathbb{L}_\varepsilon)$ H-converges to $\mathbb{L}_0$ in M, Theorem 5 implies that $u_\varepsilon \to u_0 := G_0^\lambda f_0$ strongly in $L^2(M)$. $\qquad\square$

Acknowledgments

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