Bases of matrix units for fiber-commutative coherent configurations

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Introduction

In the theory of association schemes, commutative association schemes are essential and fundamental. An association scheme has a set of adjacency matrices. An association scheme is said to be commutative if their adjacency matrices are pairwise commutative. The adjacency algebra of an association scheme is defined as the algebra over the complex field $\mathbb{C}$ spanned by all adjacency matrices. By the definition of adjacency algebras, it has the set of adjacency matrices as a basis and it is closed with respect to the transpose and the Hadamard product. If association schemes are commutative, then those adjacency algebras are also commutative.

For a commutative association scheme, the primitive idempotents are defined by algebraic properties of the adjacency algebra. The primitive idempotents are in the adjacency algebra and the set of all primitive idempotents is also a basis of the adjacency algebra. Thus the adjacency algebra has two bases, the set of all adjacency matrices and the set of all primitive idempotents. Moreover, adjacency matrices and primitive idempotents are idempotents with respect to the Hadamard product and the matrix multiplication, respectively. This is one of the reasons to study commutative association schemes actively.

On the other hand, the sets of the primitive idempotents of non-commutative association schemes or coherent configurations are not bases of their adjacency algebras, where coherent configurations are defined as one of the generalizations of association schemes. Higman [7] showed that adjacency algebras of coherent configurations are semisimple. This implies that, by the representation theory of algebras called Wedderburn's theorem, each adjacency algebra is isomorphic to a direct sum of full matrix algebras. Thus Higman asserts that each adjacency algebra has a certain second basis which corresponds to a disjoint union of sets of matrix units by its isomorphism. As a fact, since the isomorphism is not determined uniquely, the second basis is not determined uniquely.

In historical backgrounds, Higman [7] wrote a paper for coherent configurations in 1975. In Higman's paper, it is not important whether association schemes are commutative or not. In 1984, Bannai and Ito [2] published a book on commutative association schemes. This book revealed many properties of commutative association schemes. After this book was published, many researchers studied commutative association schemes. On the other hand, there are few researches for non-commutative association schemes or coherent configurations.

In this thesis, we generalize sets of primitive idempotents of commutative association schemes to sets of some matrices of non-commutative association schemes and coherent configurations. However, in general cases, the second bases written in Higman's paper are not determined uniquely. Thus we focus on fiber-commutative coherent configurations and Schurian schemes given by imprimitive permutation groups satisfying the nearly multiplicity-free condition. Their common point is that the adjacency algebras of them have subalgebras which are adjacency algebras of commutative association schemes. By the uniqueness of primitive idempotents of commutative association schemes, we may determine the second bases written in Higman's paper uniquely in some sense and call them *bases of matrix units*. In particular, since fiber-commutative coherent configurations have commutative association schemes on each fiber, Hobart and Williford [10] revealed that, for fiber-commutative coherent configurations, primitive idempotents of commutative association schemes on each fiber can be taken as a part of bases of matrix units. By this fact, we can recognize bases of matrix units for fiber-commutative coherent configurations as a generalization of primitive idempotents of commutative association schemes.

By determining bases of matrix units of fiber-commutative coherent configurations uniquely, we can generalize some theorems and lemmas for primitive idempotents of commutative association schemes to bases of matrix units of fiber-commutative coherent configurations. In this thesis, we generalize Krein conditions, absolute bounds and fusions for commutative association schemes to those for fiber-commutative coherent configurations by using bases of matrix units.

Hobart [9] and, Hobart and Williford [10] essentially showed Krein conditions and absolute bounds for coherent configurations, respectively. However, for fiber-commutative coherent configurations, we can simplify both of them by using bases of matrix units. As an example of Krein conditions for fiber-commutative coherent configurations, we compute Krein conditions for the fiber-commutative coherent configurations given by generalized quadrangles.

In addition, to generalize fusions for commutative association schemes to those for fiber-commutative coherent configurations, we refer to papers by Bannai [1] and Muzychuk [13]. They showed independently an equivalent condition for commutative association schemes to have fusion schemes, which is called the Bannai-Muzychuk criterion. The Bannai-Muzychuk criterion includes conditions for the first eigenmatrices of commutative association schemes. To generalize this criterion, we also generalize the first eigenmatrices of commutative association schemes to those of fiber-commutative coherent configurations. Bases of matrix units enable those generalizations. In this thesis, we prove an equivalent condition for fiber-commutative coherent configurations to have fusion configurations. Since the specialization for commutative association schemes of this equivalence is the same as the Bannai-Muzychuk criterion (see Corollary 3.3.1), this equivalence is a natural generalization of the Bannai-Muzychuk criterion. Moreover, as more applications of this equivalence, we fuse any fiber-commutative coherent configurations to construct the trivial fusion configurations and the fiber-commutative coherent configurations given by the permutation group $\mathbb{Z}_3^4 \rtimes S_6$.

Moreover, we construct bases of matrix units for some non-commutative association schemes. Some association schemes are given by transitive permutation groups and these are called the Schurian schemes. A transitive permutation group satisfies the multiplicity-free condition (see Lemma 4.1.1), if and only if the Schurian scheme given by the permutation group is commutative. For a permutation group, we define the *nearly multiplicity-free* condition, which is a generalization of the multiplicity-free condition. For a transitive permutation group satisfying the nearly multiplicity-free condition, its Schurian scheme is non-commutative. However, we may define *the bases of matrix units* for the adjacency algebra of its Schurian scheme and these bases are determined uniquely. As examples, we construct bases of matrix units for Schurian schemes given by the symmetric groups acting on ordered pairs and the dihedral groups acting on themselves. The number of classes of former Schurian schemes is independent of degrees of the symmetric groups. On the other hand, that of latters depends on degrees of the dihedral groups.

In the sense of the representation theory of algebras, bases of matrix units for fiber-commutative coherent configurations and Schurian schemes given by transitive permutation groups satisfying the nearly multiplicity-free condition have a common concept. Since adjacency algebras of coherent configurations are semisimple and it means that adjacency algebras are isomorphic to direct sums of full matrix algebras. By using these isomorphisms, matrix units in direct sums of full matrix algebras can construct bases of adjacency algebras.

Contents

Chapter 1

Association schemes

1.1 Association schemes

Let X be a finite set and $R_0, R_1, \ldots, R_d \subset X \times X$ be binary relations on X.

Definition 1.1.1. A pair $(X, \{R_i\}_{i=0}^d)$ is called an *association scheme* if

(i) $R_0 = \{(x,x) \mid x \in X\}$,

(ii) $\displaystyle\coprod_{i=0}^{d} R_i = X \times X$,

(iii) for any $i \in \{0,1,\ldots,d\}$, there exists $i' \in \{0,1,\ldots,d\}$ such that $R_{i'} = \{(y,x) \mid (x,y) \in R_i\}$,

(iv) for any $i,j,k \in \{0,1,\ldots,d\}$, the number $p_{i,j}^k = |\{z \in X \mid (x,z) \in R_i, (z,y) \in R_j\}|$ is independent from the choice of $(x,y) \in R_k$.

The numbers $p_{i,j}^k$ is called *intersection numbers.* An association scheme is *commutative* if $p_{i,j}^k = p_{j,i}^k$ for all $i,j,k \in \{0,1,\ldots,d\}$. An association scheme is *commutative* if $i = i'$ for all $i \in \{0,1,\ldots,d\}$.

Let $\mathrm{M}_X(\mathbb{C})$ be the full matrix ring indexed by $X \times X$ over the complex field $\mathbb{C}$. Let $(X, \{R_i\}_{i=0}^d)$ be an association scheme. For each $i \in \{0,1,\ldots,d\}$, the *adjacency matrix with respect to* R_i is a square matrix $A_i \in \mathrm{M}_X(\mathbb{C})$ whose entries are defined as $(A_i)_{x,y} = 1$ if $(x,y) \in R_i$ and $(A_i)_{x,y} = 0$ otherwise.

Let $I, J \in \mathrm{M}_X(\mathbb{C})$ be the identity matrix and the all-ones matrix, respectively. By the definition of adjacency matrices, statements in Definition 1.1.1 (i)–(iv) are rewritten as follows:

(i) $A_0 = I$,

(ii) $\sum_{i=0}^{d} A_i = J$,

(iii) for any $i \in \{0, 1, \ldots, d\}$, there exists $i' \in \{0, 1, \ldots, d\}$ such that $A_{i'} = A_i^T$,

(iv) $A_i A_j = \sum_{i=0}^{d} p_{i,j}^k A_k$.

Moreover, an association scheme is commutative if and only if $A_i A_j = A_j A_i$ for all $i, j \in \{0, 1, \ldots, d\}$. an association scheme is symmetric if and only if A_i are symmetric for all $i \in \{0, 1, \ldots, d\}$.

By the definition of association schemes, row sums of A_i are constant and equal to $p_{i,i'}^0$ for each $i \in \{0, 1, \ldots, d\}$. For each $i \in \{0, 1, \ldots, d\}$, its number is called the *valency with respect to* R_i and denoted by k_i.

Example 1.1.2 (Hamming schemes)**.** Let F be a finite set with its order $|F| = q$ $(q \geq 2)$ and $X = F^n$ for a positive integer n. For $i \in \{0, 1, \ldots, n\}$, R_i is defined by $(x, y) \in R_i$ if and only if $\#\{j \mid x_j \neq y_j\} = i$, where $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$. Then $(X, \{R_i\}_{i=0}^n)$ is a symmetric association scheme called the *Hamming scheme* and denoted by $H(n, q)$.

For a positive integer k, a subset Y of a finite set X is a *k-subset* if $|Y| = k$.

Example 1.1.3 (Johnson schemes)**.** Let V be a finite set with its order $|V| = n$. Let k be a positive integer with $k \leq n/2$ and X be a set of all k-subsets of V. For $i \in \{0, 1, \ldots, n\}$, R_i is defined as $R_i = \{(x, y) \in X \times X \mid |x \cap y| = k - i\}$. Then $(X, \{R_i\}_{i=0}^k)$ is a symmetric association scheme called the *Johnson scheme* and denoted by $J(n, k)$.

Example 1.1.4 (Schurian association schemes)**.** Let G be a transitive permutation group on a finite set X. Then G acts on $X \times X$ by $(x, y)^g = (x^g, y^g)$ for $(x, y) \in X \times X$ and $g \in G$. Let $\Lambda_0, \Lambda_1, \ldots, \Lambda_d$ be all orbits of G on $X \times X$, where $\Lambda_0 = \{(x, x) \mid x \in X\}$. Then $(X, \{\Lambda_i\}_{i=0}^d)$ is an association scheme and called the *Schurian association scheme* or, for brevity, *the Schurian scheme of G on X*.

Let $(X, \{R_i\}_{i=0}^{d})$ be an association scheme and $A_0, A_1, \ldots, A_d$ be its adjacency matrices.

Definition 1.1.5. The *adjacency algebra or Bose-Mesner algebra* $\mathfrak{A}$ is defined as a subalgebra of the full matrix algebra $\mathrm{M}_X(\mathbb{C})$ spanned by $\mathfrak{A} = \langle A_0, A_1, \ldots, A_d \rangle_{\mathbb{C}}$.

By the definition of association schemes, the adjacency algebra $\mathfrak{A}$ satisfies $\dim(\mathfrak{A}) = d + 1$ and has a basis $\{A_0, A_1, \ldots, A_d\}$.

Example 1.1.6 (Thin schemes)**.** For a group G, the Schurian schemes of G on G is called *the thin scheme* of G. The adjacency algebra of the Schurian scheme of G is same as the group ring $\mathbb{C}G$.

1.2 Commutative association schemes

Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$ be a commutative association scheme with $|X| = n$. Since $A_0, A_1, \ldots, A_d$ are commutative each other, $A_0, A_1, \ldots, A_d$ are diagonalized simultaneously by a unitary matrix U. In other words, $A_0, A_1, \ldots, A_d$ have maximal common eigenspaces

$$\mathbb{C}^X = \bigoplus_{i=0}^{r} V_i,$$

where V_i are maximal common eigenspaces. Let E_i be the orthogonal projection from $\mathbb{C}^X$ onto V_i for each $i \in \{0, 1, \ldots, r\}$. Moreover, as a fact, $r = d$ holds. The matrices $E_0, E_1, \ldots, E_d$ are called *primitive idempotents.* By the definition of primitive idempotents, E_i are diagonalized by U and it implies that eigenvalues of E_i are 1 with its multiplicity m_i and 0 with its multiplicity $n - m_i$ for some m_i. In addition, $m_i = \mathrm{rank}(E_i) = \dim(V_i)$ holds. The numbers m_i are called *multiplicities.*

Proposition 1.2.1 ([2, Section 2.3])**.** *For the primitive idempotents $E_0, E_1, \ldots, E_d$, the following hold.*

(i) $\{E_0, E_1, \ldots, E_d\}$ *is a basis of* $\mathfrak{A}$.

(ii) *for any* $i \in \{0, 1, \ldots, d\}, E_i E_j = \delta_{i,j} E_i$,

(iii) $\sum_{i=0}^{d} E_i = I$,

(iv) *for any* $i \in \{0, 1, \ldots, d\}$*, there exists* $\hat{i} \in \{0, 1, \ldots, d\}$ *such that* $E_{\hat{i}} = E_i^T$.

Since $\{A_0, A_1, \ldots, A_d\}$ and $\{E_0, E_1, \ldots, E_d\}$ are bases of $\mathfrak{A}$, A_i are expressed as linear combinations of $E_0, E_1, \ldots, E_d$ and E_i are also expressed as linear combinations of $A_0, A_1, \ldots, A_d$:

$$A_i = \sum_{j=0}^{d} p_i(j) E_j,$$

$$E_i = \frac{1}{n} \sum_{j=0}^{d} q_i(j) A_j.$$

Definition 1.2.2. The *first and second eigenmatrices* are defined as the square matrices P, Q of degree $d+1$ such that

$$P_{ij} = p_j(i),$$
$$Q_{ij} = q_j(i),$$

respectively.

It is trivial that $PQ = QP = nI_{d+1}$.

Let $\circ$ be the Hadamard product or entry-wise product for matrices, i.e. for $A, B \in \mathrm{M}_X(\mathbb{C})$, $A \circ B$ is defined as $(A \circ B)_{i,j} = A_{i,j} B_{i,j}$.

By the definition of adjacency matrices, $A_i \circ A_j = \delta_{i,j} A_i$ and this means that $\mathfrak{A}$ is closed under the Hadamard product.

Definition 1.2.3. Let $E_0, E_1, \ldots, E_d$ be the primitive idempotents of the adjacency algebra $\mathfrak{A}$ of $\mathfrak{X}$. Set

$$E_i \circ E_j = \frac{1}{n} \sum_{k=0}^{d} q_{i,j}^k E_k$$

for some $q_{i,j}^k \in \mathbb{C}$. The coefficients $q_{i,j}^k$ are called the *Krein parameters.*

With respect to the Hadamard product, there is a theorem called the Schur product theorem. It shows that the Hadamard product of two positive definite matrices is also positive semidefinite. Similarly, the following lemma holds.

Lemma 1.2.4 ([2, Lemma 3.9 in Section 2.3])**.** *Let $A, B \in \mathrm{M}_X(\mathbb{C})$ be positive semidefinite Hermitian matrices. Then $A \circ B$ is positive semidefinite.*

Proof. Let $\otimes$ be the Kronecker product. For positive semidefinite Hermitian matrices A, B, $A \otimes B$ is also positive semidefinite. Since, $A \circ B$ is a principal submatrix of $A \otimes B$, $A \circ B$ is also positive semidefinite. □

Theorem 1.2.5 (Krein conditions, [2, Theorem 3.8 in Section 2.3])**.** *Let $(X, \{R_i\}_{i=0}^d)$ be a commutative association scheme. Then Krein parameters $q_{i,j}^k$ are non-negative real numbers for all $i, j, k \in \{0, 1, \ldots, d\}$.*

Proof. For any $i, j \in \{0, 1, \ldots, d\}$, $E_i \circ E_j$ are diagonalized and eigenvalues of $E_i \circ E_j$ are $q_{i,j}^k/n$ with its multiplicity m_k for all $k \in \{0, 1, \ldots, d\}$. Since E_i are positive semidefinite Hermitian matrices, by Lemma1.2.4, $q_{i,j}^k/n$ are non-negative real numbers. □

Theorem 1.2.6 (Absolute bounds, [2, Theorem 4.9 in Chapter II])**.** *Let $(X, \{R_i\}_{i=0}^d)$ be a commutative association scheme, m_i be the multiplicities for $i \in \{0, 1, 2, \ldots, d\}$ and $q_{i,j}^k$ be the Krein parameters for $i, j, k \in \{0, 1, \ldots, d\}$. Then*

$$\sum_{\substack{k \in \{0,1,\ldots,d\} \\ q_{i,j}^k > 0}} m_k = \begin{cases} m_i m_j & \text{if } i \neq j, \\ \frac{1}{2} m_i (m_i + 1) & \text{if } i = j. \end{cases}$$

1.3 Fusion schemes

Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ be a association scheme.

Definition 1.3.1. A *fusion scheme of* $\mathfrak{X}$ is an association scheme $(X, \{S_i\}_{i=0}^{d'})$ such that the same finite set X and binary relations S_i which are disjoint unions of some R_i.

Let $\mathfrak{X}' = (X, \{S_i\}_{i=0}^{d'})$ be a fusion scheme of $\mathfrak{X}$ and $\mathfrak{A}, \mathfrak{A}'$ be the adjacency algebras of $\mathfrak{X}, \mathfrak{X}'$,respectively. Then, by the definition of fusion schemes, $\mathfrak{A}'$ is a subalgebra of $\mathfrak{A}$. Moreover, fusion schemes corresponds to a partition for $\{0, 1, \ldots, d\}$. In other words, for a fusion scheme $\mathfrak{X}'$, there exists a partition $\Delta = \{\delta_0, \delta_1, \ldots, \delta_{d'}\}$ such that $S_i = \coprod_{j \in \delta_i} R_j$.

Theorem 1.3.2 ([1, Lemma 1]). *Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ be a commutative association scheme, $E_0, E_1, \ldots, E_d$ be primitive idempotents of $\mathfrak{X}$ and P be the first eigenmatrix of $\mathfrak{X}$. Then $\mathfrak{X}$ has a fusion scheme if and only if there exist partitions Δ and Γ on $\{0, 1, \ldots, d\}$ such that, for any $\delta \in \Delta, \gamma \in \Gamma$, the submatrix $P_{\gamma,\delta}$ of P indexed by $\gamma \times \delta$ satisfies that row sums of $P_{\gamma,\delta}$ are constant. In this case, the fusion scheme $(X, \{S_\delta\}_{\delta\in\Delta})$ with its primitive idempotents $\{F_\gamma\}_{\gamma\in\Gamma}$ satisfies $S_\delta = \coprod_{j\in\Delta} R_j$ and $F_\gamma = \sum_{j\in\Gamma} E_j$ hold for $\delta \in \Delta, \gamma \in \Gamma$.*

Proof. Let $\mathfrak{A}$ be the adjacency algebra of $\mathfrak{X}$. Suppose that $\mathfrak{X}' = (X, \{S_i\}_{i=0}^{d'})$ is a fusion scheme of $\mathfrak{X}$. Then there exists a partition $\Delta = \{\delta_0, \delta_1, \ldots, \delta_{d'}\}$ such that $S_i = \coprod_{j\in\delta_i} R_j$. Let $F_0, F_1, \ldots, F_{d'}$ be primitive idempotents of $\mathfrak{X}'$. For any $i \in \{0, 1, \ldots, d'\}$, since F_i is a idempotent, there exists a subset $\gamma_i \subset \{0, 1, \ldots, d\}$ such that $F_i = \sum_{j\in\gamma_i} E_i$. Moreover, the identity $F_iF_k = \delta_{i,k}F_i$ implies $\gamma_i \cap \gamma_k = \emptyset$ for $i \neq k$. Thus $\{\gamma_0, \gamma_1, \ldots, \gamma_{d'}\}$ is a partition of $\{0, 1, \ldots, d\}$. Let $\mathfrak{A}' \subset \mathfrak{A}$ be the adjacency algebra of $\mathfrak{X}'$. Then $\{F_0, F_1, \ldots, F_{d'}\}$ is a basis of $\mathfrak{A}'$. Since the adjacency matrix $A'_i = \sum_{j\in\delta_i} A_j$ of S_i is in $\mathfrak{A}'$ and it means that row sums of submatrices $P_{\gamma,\delta}$ of P are constant.

The converse is clear. □

Chapter 2

Coherent configurations

2.1 Coherent algebras

Let X be a finite set with order $|X| = n$ and $\mathrm{M}_X(\mathbb{C})$ be the full matrix ring over $\mathbb{C}$ indexed by $X \times X$. For $i, j \in X$, let $E_{i,j} \in \mathrm{M}_X(\mathbb{C})$ be the matrix whose (i, j)-entry is 1 and all other entries are 0.

Definition 2.1.1. Let $I, J \in \mathrm{M}_X(\mathbb{C})$ be the identity matrix and the all-ones matrix, respectively. A *coherent algebra* $\mathfrak{A}$ is defined as a subalgebra of $\mathrm{M}_X(\mathbb{C})$ such that

(i) $I, J \in \mathfrak{A}$,

(ii) $\mathfrak{A}$ is closed under the transpose,

(iii) $\mathfrak{A}$ is closed under the Hadamard product.

This definition means that a coherent algebra $\mathfrak{A}$ is closed under the ordinary matrix product, the Hadamard product and the transpose and has the identity elements with respect to 2 products.

Lemma 2.1.2 ([7, (3.1)])**.** *Every coherent algebra* $\mathfrak{A}$ *is semisimple.*

Let $\{\varphi_s \mid s \in S\}$ be a set of representatives of all irreducible matrix representations of $\mathfrak{A}$ over $\mathbb{C}$ satisfying $\varphi_s(A)^* = \varphi_s(A^*)$ for any $A \in \mathfrak{A}$, where $*$ denotes the transpose-conjugate.

Theorem 2.1.3 (Wedderburn's Theorem). *Let $\mathfrak{A}$ be a semisimple algebra over $\mathbb{C}$. Then $\mathfrak{A}$ is decomposed into*

$$\mathfrak{A} = \bigoplus_{s \in S} \mathfrak{C}_s,$$

where $\mathfrak{C}_s$ is the simple two-sided ideal corresponding to φ_s and $\mathfrak{C}_s \simeq \mathrm{M}_{e_s}(\mathbb{C})$ as algebras for some positive integers e_s.

By this theorem, $\varphi_s|_{\mathfrak{C}_s}$ is an isomorphism from $\mathfrak{C}_s$ to $\mathrm{M}_{e_s}(\mathbb{C})$.

2.2 Coherent configurations

As a generalization of association schemes, we have coherent configurations. Let $R \subset X \times X$ be a binary relation of $X \times X$. The adjacency matrix A with respect to R is defined as $(A)_{x,y} = 1$ if $(x, y) \in R$ and 0 otherwise.

Definition 2.2.1. For a finite set X, let $R_0, R_1, \ldots, R_d \subset X \times X$ be binary relations of $X \times X$ and $A_0, A_1, \ldots, A_d$ be the adjacency matrices. A *coherent configuration* $(X, \{R_i\}_{i=0}^{d})$ is defined as

(i) there exists a subset $K \subset \{0, 1, \ldots, d\}$ such that $\displaystyle\sum_{i \in F} A_i = I$,

(ii) $\displaystyle\sum_{i=0}^{d} A_i = J$,

(iii) for any $i \in \{0, 1, \ldots, d\}$, there exists $i' \in \{0, 1, \ldots, d\}$ such that $A_{i'} = A_i^T$,

(iv) $A_i A_j = \displaystyle\sum_{i=0}^{d} p_{i,j}^{k} A_k$.

The algebra spanned by $A_0, A_1, \ldots, A_d$ over $\mathbb{C}$ is called the *adjacency algebra*.

It is clear that the difference between the definition of association schemes and the definition of coherent configurations is the first condition.

Higman stated that the definition of coherent configurations and the definition of coherent algebras are equivalent (see [8]). In other words, an adjacency algebra of a coherent configuration is equivalent to a coherent algebra.

Let $\mathfrak{X} = (X, \{R_k\}_{k=0}^d)$ be a coherent configuration. By Definition 2.2.1(i), $I \in \mathrm{M}_n(\mathbb{C})$ is decomposed into $(0,1)$-matrices. This implies that X is decomposed into $X = \coprod_{i \in F} X_i$. The each X_i is called a *fiber.* By Definition 2.2.1(iv), for any $k \in \{0, 1, \ldots, d\}$, there exist $i, j \in F$ such that $R_k \subset X_i \times X_j$. For any $i, j \in F$, we denote $r_{i,j} = \#\{k \in \{0, 1, \ldots, d\} \mid R_k \subset X_i \times X_j\}$. Thus the index set $\{0, 1, \ldots, d\}$ can be rearranged by $\{(i, j, a) \mid i, j \in F, a \in \{1, 2, \ldots, r_{i,j}\}\}$ and $\{R_k\}_{k=0}^d = \{R_{i,j,a} \mid i, j \in F, a \in \{1, 2, \ldots, r_{i,j}\}\}$.

Let $A_{i,j,a}$ be the adjacency matrix with respect to $R_{i,j,a}$, We may always assume that

(i) $F = \{1, 2, \ldots, f\}$,

(ii) for any $i, j \in F, a \in \{1, 2, \ldots, r_{i,j}\}, A_{i,j,a}^T = A_{j,i,a}$,

(iii) for any $i \in F, A_{i,i,1} = I_{X_i}$,

where $I_{X_i} \in \mathrm{M}_X(\mathbb{C})$ is the matrix with 1 on (x, x)-entries for $x \in X_i$ and 0 otherwise.

For each $i \in F$, $(X_i, \{R_{i,i,a}\}_{a=1}^{r_{i,i}})$ is an association scheme.

In particular, if $f = 1$, then the coherent configuration is an association scheme.

Let $\mathfrak{A}$ be the adjacency algebra of $\mathfrak{X}$. Then $\mathfrak{A}$ is decomposed into a direct sum of subspaces:

$$\mathfrak{A} = \bigoplus_{i,j \in F} \mathfrak{A}_{i,j},$$

where $\mathfrak{A}_{i,j}$ is a subspace spanned by $\{A_{i,j,a} \mid a \in \{1, 2, \ldots, r_{i,j}\}\}$. In particular, for each $i \in F$, $\mathfrak{A}_{i,i}$ is a subalgebra and is equivalent to the adjacency algebra of the association scheme $(X_i, \{R_{i,i,a}\}_{a=1}^{r_{i,i}})$. For brevity, we write $\mathfrak{A}_i = \mathfrak{A}_{i,i}$.

Definition 2.2.2. A coherent configuration $(X, \{R_{i,j,a}\}_{i,j,a})$ is called *fiber-commutative*, if $\mathfrak{A}_i$ are commutative for all $i \in F$. A coherent configuration $(X, \{R_{i,j,a}\}_{i,j,a})$ is called *fiber-symmetric*, if $\mathfrak{A}_i$ consists only of symmetric matrices for all $i \in F$.

2.3 Bases of matrix units

Let $\mathfrak{A}$ be the adjacency matrix of a coherent configuration $(X, \{R_{i,j,a}\}_{i,j,a})$.

By Lemma 2.1.2 and Theorem 2.1.3, $\mathfrak{A}$ is decomposed into

$$\mathfrak{A} = \bigoplus_{s \in S} \mathfrak{C}_s,$$

where $\mathfrak{C}_s$ is a simple two-sided ideal and $\mathfrak{C}_s \simeq \mathrm{M}_{e_s}(\mathbb{C})$ for a positive integer e_s. This implies that there exists a basis $\{\varepsilon^s_{i,j} \in \mathfrak{A} \mid i, j \in F_s\}$ of $\mathfrak{C}_s$ satisfying

$$\varepsilon^s_{i,j}\varepsilon^s_{k,l} = \delta_{j,k}\varepsilon^s_{i,l}, \tag{2.1}$$

$$\varepsilon^s_{i,j}{}^* = \varepsilon^s_{j,i}, \tag{2.2}$$

where $|F_s| = e_s$. Note that there is a good reason not to take $F_s = \{1, \ldots, e_s\}$. This will become clear after Lemma 2.4.1. By [10, Theorem 8], we can choose $\varepsilon^s_{i,j}$ in such a way that

$$\varepsilon^s_{i,j} \in \bigcup_{k,l=1}^{f} \mathfrak{A}_{k,l} \quad (i, j \in F_s,\ s \in S). \tag{2.3}$$

Note that, since $\mathfrak{A}_{k,l}\mathfrak{A}_{k',l'} = 0$ if $l \neq k'$, (2.3) implies

$$\varepsilon^s_{i,i} \in \bigcup_{k=1}^{f} \mathfrak{A}_k \quad (i \in F_s,\ s \in S). \tag{2.4}$$

This is also mentioned in the proof of [10, Theorem 8]. Since $\overline{\mathfrak{C}_s}$ is also a simple two-sided ideal, there exists $\hat{s} \in S$ such that $\mathfrak{C}_{\hat{s}} = \overline{\mathfrak{C}_s}$. If $\mathfrak{X}$ is fiber-symmetric, then $s = \hat{s}$ for all $s \in S$ by (2.4). Note that $\{\overline{\varepsilon^s_{i,j}} \mid i, j \in F_s\}$ is a basis of $\mathfrak{C}_{\hat{s}}$ satisfying (2.1). Since $\overline{\mathfrak{A}_{k,l}} = \mathfrak{A}_{k,l}$ for all $k, l \in \{1, \ldots, f\}$,

$$\overline{\varepsilon^s_{i,j}} \in \bigcup_{k,l=1}^{f} \mathfrak{A}_{k,l} \quad (i, j \in F_s,\ s \in S).$$

This implies that we can choose $\{\varepsilon^s_{i,j} \mid i, j \in F_s\}$ and $\{\varepsilon^{\hat{s}}_{i,j} \mid i, j \in F_s\}$ in a manner compatible with complex conjugation.

Definition 2.3.1. For each $s \in S$, a basis $\{\varepsilon^s_{i,j} \mid i, j \in F_s\}$ of $\mathfrak{C}_s$ is called *a basis of matrix units for* $\mathfrak{C}_s$ if (2.1) and (2.3) hold. If $\{\varepsilon^s_{i,j} \mid i, j \in F_s\}$ is a basis of matrix units for $\mathfrak{C}_s$ for each $s \in S$, then their disjoint union is called *bases of matrix units for* $\mathfrak{A}$ provided that $F_s = F_{\hat{s}}$ and

$$\overline{\varepsilon^s_{i,j}} = \varepsilon^{\hat{s}}_{i,j} \quad (i, j \in F_s, s \in S).$$

Note that bases of matrix units are not determined uniquely (see [7]), however we will see later that they are essentially unique for the fiber-commutative case.

Lemma 2.3.2. *The center of* $\mathfrak{A}$ *is contained in* $\bigoplus_{k=1}^{f} \mathfrak{A}_k$.

Proof. This is immediate from (2.4), since $\sum_{i\in F_s} \varepsilon_{i,i}^s$ is the central idempotent corresponding to $\mathfrak{C}_s$. □

Let $J_{k,l}$ be the matrix in $\mathfrak{A}$ with 1 in all entries indexed by $X_k \times X_l$ and 0 otherwise. Without loss of generality, we may assume that $\mathfrak{C}_1 = \mathfrak{A}\varepsilon_1\mathfrak{A}$, where

$$\varepsilon_1 = \sum_{k=1} \frac{1}{|X_k|} J_{k,k}. \tag{2.5}$$

This implies that we may also assume that

$$\varepsilon_{k,l}^1 = \frac{1}{\sqrt{|X_k||X_l|}} J_{k,l} \tag{2.6}$$

for any $k, l \in F_1$, where $F_1 = \{1, \dots, f\}$.

For the reminder of this section, we fix bases of matrix units $\{\varepsilon_{i,j}^s \mid s \in S,\ i, j \in F_s\}$ for $\mathfrak{A}$. Let $\Lambda_s = F_s^2 \times \{s\}$ for each $s \in S$ and $\Lambda = \coprod_{s\in S} \Lambda_s$. Moreover, we denote $\varepsilon_\lambda = \varepsilon_{i,j}^s$ for $\lambda = (i, j, s) \in \Lambda$. Define $n_\lambda = \sqrt{|X_k||X_l|}$, where $\lambda \in \Lambda$ and $\varepsilon_\lambda \in \mathfrak{A}_{k,l}$. Let $\circ$ denote the Hadamard (entry-wise) product of matrices. Since $\mathfrak{A}$ is closed with respect to $\circ$, there exist $q_{\lambda,\mu}^\nu \in \mathbb{C}$ such that

$$n_\lambda\varepsilon_\lambda \circ n_\mu\varepsilon_\mu = \sum_{\nu\in\Lambda} q_{\lambda,\mu}^\nu n_\nu\varepsilon_\nu. \tag{2.7}$$

Definition 2.3.3. The complex numbers $q_{\lambda,\mu}^\nu$ appearing in (2.7) are called *Krein parameters with respect to bases of matrix units* $\{\varepsilon_\lambda \mid \lambda \in \Lambda\}$.

Let $\mathcal{P}_F$ denote the set of all the positive semidefinite hermitian matrices in $\mathrm{M}_F(\mathbb{C})$.

Theorem 2.3.4 (Krein conditions [9, Lemma 1])**.** *For any* $s, t, u \in S$, $B = (b_{i,j}) \in \mathrm{M}_{F_s}(\mathbb{C})$ *and* $C = (c_{i,j}) \in \mathrm{M}_{F_t}(\mathbb{C})$, *let* $\tilde{Q}_{s,t}^u(B, C)$ *denote the matrix in* $\mathrm{M}_{F_u}(\mathbb{C})$ *whose* (m, n)*-entry is*

$$\sum_{i,j\in F_s} \sum_{k,l\in F_t} b_{i,j} c_{k,l} q_{(i,j,s),(k,l,t)}^{(m,n,u)}. \tag{2.8}$$

Then
$$\tilde{Q}^u_{s,t}(B,C) \in \mathcal{P}_{F_u} \quad (B \in \mathcal{P}_{F_s},\ C \in \mathcal{P}_{F_t}). \tag{2.9}$$

Let η be the mapping from Λ to $\{1,\ldots,f\}^2$ defined by $\varepsilon_\lambda \in \mathfrak{A}_{\eta(\lambda)}$ for $\lambda \in \Lambda$, or equivalently,
$$\eta(i,j,s) = (k,l) \text{ if } \varepsilon^s_{i,j} \in \mathfrak{A}_{k,l}. \tag{2.10}$$

Lemma 2.3.5. *For each $s \in S$, define*
$$F'_s = \{k \mid 1 \le k \le f, (k,k) \in \{\eta(i,i,s) \mid i \in F_s\}\}.$$
Then $\eta(\Lambda_s) = F'^2_s$.

Proof. First, we prove $\eta(\Lambda_s) \subset F'^2_s$. For $(i,j,s) \in \Lambda_s$, suppose $\eta(i,j,s) = (k,l)$. Namely, $\varepsilon^s_{i,j} \in \mathfrak{A}_{k,l}$. By (2.1) and (2.4), $\varepsilon^s_{i,i} \in \mathfrak{A}_k$ and $\varepsilon^s_{j,j} \in \mathfrak{A}_l$ hold. Thus $\eta(i,i,s) = (k,k)$ and $\eta(j,j,s) = (l,l)$ and these mean $k,l \in F'_s$.

Conversely, suppose $\eta(i,i,s) = (k,k)$ and $\eta(j,j,s) = (l,l)$, where $i,j \in F_s$. Then $\varepsilon^s_{i,i} \in \mathfrak{A}_k$ and $\varepsilon^s_{j,j} \in \mathfrak{A}_l$. By (2.1), we obtain $\varepsilon^s_{i,j} \in \mathfrak{A}_{k,l}$. Thus $(k,l) = \eta(i,j,s) \in \eta(\Lambda_s)$. □

Lemma 2.3.6. *Let $\lambda,\mu,\nu \in \Lambda$. If $q^\nu_{\lambda,\mu} \neq 0$, then $\eta(\lambda) = \eta(\mu) = \eta(\nu)$.*

Proof. By the definition of η, $\varepsilon_\lambda \in \mathfrak{A}_{\eta(\lambda)}$, $\varepsilon_\mu \in \mathfrak{A}_{\eta(\mu)}$, and $\varepsilon_\nu \in \mathfrak{A}_{\eta(\nu)}$ hold. If $\eta(\lambda) \neq \eta(\mu)$, then $\varepsilon_\lambda \circ \varepsilon_\mu = 0$, and this means $q^\nu_{\lambda,\mu} = 0$ for any $\nu \in \Lambda$. If $\eta(\lambda) = \eta(\mu) \neq \eta(\nu)$, then $\varepsilon_\lambda \circ \varepsilon_\mu \in \mathfrak{A}_{\eta(\lambda)}$ and this means that $q^\nu_{\lambda,\mu} = 0$. □

By Lemma 2.3.6, the expansion (2.7) is simplified to
$$\varepsilon_\lambda \circ \varepsilon_\mu = \frac{\delta_{\eta(\lambda),\eta(\mu)}}{n_\lambda} \sum_{\substack{\nu \in \Lambda \\ \eta(\nu)=\eta(\lambda)}} q^\nu_{\lambda,\mu} \varepsilon_\nu. \tag{2.11}$$

For brevity, we write a basis of matrix units $\{\varepsilon^s_{i,j} \mid i,j \in F_s\}$ as $\{\varepsilon^s_{i,j}\}$ and we define $Z \circ \{\varepsilon^s_{i,j}\} = \{\zeta_{i,j}\varepsilon^s_{i,j} \mid i,j \in F_s\}$ for a matrix $Z = (\zeta_{i,j}) \in \mathrm{M}_{F_s}(\mathbb{C})$.

Lemma 2.3.7. *Fix $s \in S$. Let $Z = (\zeta_{i,j}) \in \mathrm{M}_{F_s}(\mathbb{C})$. If $Z \circ \{\varepsilon^s_{i,j}\}$ is a basis of matrix units for $\mathfrak{C}_s$, then Z is a positive semidefinite matrix with rank one and $|\zeta_{i,j}| = 1$ for all $i,j \in F_s$.*

Proof. Since $Z \circ \{\varepsilon^s_{i,j}\}$ is a basis of matrix units for $\mathfrak{C}_s$, $Z \circ \{\varepsilon^s_{i,j}\}$ satisfies (2.1). This means that $\zeta^s_{i,j}\zeta^s_{j,k} = \zeta^s_{i,k}$ and $\overline{\zeta^s_{j,i}} = \zeta^s_{i,j}$ for any $i,j,k \in F_s$. Thus $|\zeta_{i,j}| = 1$ holds. Moreover, Since $\zeta^s_{i,j} = \overline{\zeta^s_{1,i}}\zeta^s_{1,j}$ holds, Z is expressed as $Z = \mathbf{z}^*\mathbf{z}$, where $\mathbf{z} = (\zeta_{1,j})_{j \in F_s}$. Thus Z is a positive semidefinite matrix with rank one. □

2.4 Krein conditions for fiber-commutative coherent configurations

In this section, we also use the same notation as the previous section. In other words, $\mathfrak{A}$ is the adjacency algebra of a coherent configuration $\mathfrak{X}$, $\mathfrak{A}$ is decomposed into the direct sum of simple ideals as $\mathfrak{A} = \bigoplus_{s\in S} \mathfrak{C}_s$. Moreover $\{\varepsilon^s_{i,j}\}$ is a basis of matrix units for $\mathfrak{C}_s$, and their union over $s \in S$ is bases of matrix units for $\mathfrak{A}$. In this section, we assume that the coherent configuration $\mathfrak{X}$ is fiber-commutative.

Lemma 2.4.1. *For any $s \in S$ and $k, l \in \{1, \dots, f\}$, $\dim(\mathfrak{C}_s \cap \mathfrak{A}_{k,l}) \le 1$. In other words, the number of pairs (i, j) satisfying $\varepsilon^s_{i,j} \in \mathfrak{A}_{k,l}$ is at most* 1.

Proof. By (2.3), for each $i, j \in F_s$, there exist k, l such that $\varepsilon^s_{i,j} \in \mathfrak{A}_{k,l}$. Thus it suffices to show $\#\{(i, j, s) \in \Lambda_s \mid \varepsilon^s_{i,j} \in \mathfrak{A}_{k,l}\} \le 1$. Suppose $\varepsilon^s_{i,j}, \varepsilon^s_{i'j'} \in \mathfrak{A}_{k,l}$ and $i \ne i'$. By (2.4), we have $\varepsilon^s_{i,i}, \varepsilon^s_{i',i'} \in \mathfrak{A}_k$. Thus $\varepsilon^s_{i,i'} = \varepsilon_{i,i}\varepsilon_{i,i'}\varepsilon_{i',i'} \in \mathfrak{A}_k$ holds. Since $\mathfrak{A}_k$ is commutative, $\varepsilon^s_{i,i'} = \varepsilon^s_{i,i}\varepsilon^s_{i,i'} = \varepsilon^s_{i,i'}\varepsilon^s_{i,i} = 0$, and this is a contradiction. Therefore, we obtain $i = i'$ and similarly, $j = j'$. □

Note that Lemma 2.4.1 is stated implicitly in the proof of [10, Corollary 10] and [3, Proposition 2.1], independently. Since $\eta|_{\Lambda_s} : F_s^2 \times \{s\} \to \{1, \dots, f\}^2$ is injective by Lemma 2.4.1, the set F_s can be taken to be the subset F'_s of $\{1, \dots, f\}$ defined in Lemma 2.3.5.

Definition 2.4.2. For $s \in S$, we define the *support* for $\mathfrak{C}_s$ to be the subset

$$F_s = \{i \in \{1, \dots, f\} \mid \dim(\mathfrak{C}_s \cap \mathfrak{A}_{i,i}) = 1\}.$$

By the definition of F_s, we can take η as $\eta(i, j, a) = (i, j)$ for $i, j \in F_s$. Indeed, by (2.4), we may suppose $\varepsilon^s_{i,i} \in \mathfrak{A}_{i,i}$ for all $i \in F_s$. Then by $\varepsilon^s_{i,j} = \varepsilon^s_{i,i}\varepsilon^s_{i,j}\varepsilon^s_{j,j} \in \mathfrak{A}_{i,j}$, we have $\eta(i, j, s) = (i, j)$.

For brevity, we write $F_{s,t,u} = F_s \cap F_t \cap F_u$. Note that $F_1 = \{1, \dots, f\}$ holds by (2.6).

Thus, for fibewr-commutative coherent configurations, the definition of bases of matrix units can be rewritten as follows.

Definition 2.4.3. Let $\mathfrak{A}$ be the adjacency algebra of a fiber-commutative coherent configuration. *Bases of matrix units for* $\mathfrak{A}$ are defined as matrices $\{\varepsilon^s_{i,j} \mid s \in S, i, j \in F_s\}$ satisfying

(i) for any $s, t \in S$, $i, j \in F_s, k, l \in F_t$, $\varepsilon^s_{i,j}\varepsilon^t_{k,l} = \delta_{s,t}\delta_{j,k}\varepsilon^s_{i,l}$,

(ii) for any $s \in S$, $i, j \in F_s$, ${\varepsilon^s_{i,j}}^* = \varepsilon^s_{j,i}$,

(iii) for any $s \in S$, $i, j \in F_s, \varepsilon^s_{i,j} \in \mathfrak{A}_{i,j}$,

(iv) for any $s \in S$, there exist $\hat{s} \in S$ such that, $F_{\hat{s}} = F_s$ and, for any $i, j \in F_s$, $\overline{\varepsilon^s_{i,j}} = \varepsilon^{\hat{s}}_{i,j}$.

By Lemma 2.3.6, (2.11) can be written as follows: for $(i, j) \in F_s^2 \cap F_t^2$,

$$\varepsilon^s_{i,j} \circ \varepsilon^t_{i,j} = \frac{1}{\sqrt{|X_i||X_j|}} \sum_{\substack{u\in S\\ F_u\ni i,j}} q^{(i,j,u)}_{(i,j,s),(i,j,t)}\varepsilon^u_{i,j}. \tag{2.12}$$

Definition 2.4.4. For $s, t, u \in S$, let $Q^u_{s,t} \in \mathrm{M}_{F_{s,t,u}}(\mathbb{C})$ be the matrix with (i, j)-entry

$$(Q^u_{s,t})_{i,j} = q^{(i,j,u)}_{(i,j,s),(i,j,t)}.$$

The matrix $Q^u_{s,t}$ is called *the matrix of Krein parameters* with respect to the bases of matrix units $\{\varepsilon^s_{i,j}\}, \{\varepsilon^t_{i,j}\}, \{\varepsilon^s_{i,j}\}$ for $\mathfrak{C}_s, \mathfrak{C}_t, \mathfrak{C}_u$.

Note that, by (2.12), $Q^u_{s,t} = Q^u_{t,s}$ holds for any $s, t, u \in S$. Moreover, the matrix $Q^u_{s,t}$ is hermitian by (2.2) and (2.12).

Proposition 2.4.5. *For any $s, t \in S$, we have $Q^t_{1,s} = \delta_{s,t}J_{F_{1,s,t}}$.*

Proof. Immediate from (2.6), (2.12) and Definition 2.4.4. □

Proposition 2.4.6. *For any $s, t \in S$,*

$$Q^1_{s,t} = \delta_{\hat{s},t}\mathrm{tr}(\varepsilon^t_{j,j})J_{F_{s,t,1}}.$$

In particular, $\mathrm{tr}(\varepsilon^t_{j,j})$ is independent of $j \in F_t$.

Proof. By (2.12),

$$\begin{aligned}\left(\varepsilon^s_{i,j} \circ \varepsilon^t_{i,j}\right) \sum_{k,l\in F_1} \varepsilon^1_{k,l} &= \frac{1}{\sqrt{|X_i||X_j|}} \sum_{k,l\in F_1} \sum_{\substack{u\in S\\ F_u\ni i,j}} (Q^u_{s,t})_{i,j}\varepsilon^u_{i,j}\varepsilon^1_{k,l}\\ &= \frac{1}{\sqrt{|X_i||X_j|}}(Q^1_{s,t})_{i,j} \sum_{l\in F_1} \varepsilon^1_{i,l}.\end{aligned}$$

We compute the trace of each side of this identity. By (2.6), the trace of the right-hand side is $(Q^1_{s,t})_{i,j}/\sqrt{|X_i||X_j|}$. On the other hand, the trace of the left-hand side is

$$\begin{aligned}
\operatorname{tr}\left((\varepsilon^s_{i,j}\circ\varepsilon^t_{i,j})\sum_{k,l\in F_1}\varepsilon^1_{k,l}\right) &= \sum_{x,y\in X}\left(\sum_{k,l\in F_1}\varepsilon^s_{i,j}\circ\varepsilon^t_{i,j}\circ\varepsilon^1_{k,l}\right)_{x,y} \\
&= \frac{1}{\sqrt{|X_i||X_j|}}\sum_{x,y\in X}(\varepsilon^s_{i,j}\circ\varepsilon^t_{i,j})_{x,y} \quad \text{(by (2.6))} \\
&= \frac{1}{\sqrt{|X_i||X_j|}}\operatorname{tr}({\varepsilon^s_{i,j}}^T\varepsilon^t_{i,j}) \\
&= \frac{1}{\sqrt{|X_i||X_j|}}\operatorname{tr}(\overline{\varepsilon^s_{j,i}}\varepsilon^t_{i,j}) \quad (\text{by } {\varepsilon^s_{i,j}}^* = \varepsilon^s_{j,i}) \\
&= \frac{1}{\sqrt{|X_i||X_j|}}\operatorname{tr}(\varepsilon^{\hat{s}}_{j,i}\varepsilon^t_{i,j}) \\
&= \frac{1}{\sqrt{|X_i||X_j|}}\delta_{\hat{s},t}\operatorname{tr}(\varepsilon^t_{j,j}).
\end{aligned}$$

By the properties of the trace, $\operatorname{tr}({\varepsilon^s_{i,j}}^T\varepsilon^t_{i,j}) = \operatorname{tr}(\varepsilon^t_{i,j}{\varepsilon^s_{i,j}}^T)$ and this implies $\operatorname{tr}(\varepsilon^t_{j,j}) = \operatorname{tr}(\varepsilon^t_{i,i})$. Thus we obtain $(Q^1_{s,t})_{i,j} = \delta_{\hat{s},t}\operatorname{tr}(\varepsilon^t_{i,i}) = \delta_{\hat{s},t}\operatorname{tr}(\varepsilon^t_{j,j})$, and the result follows. □

Proposition 2.4.7. *For $s,t,u\in S$, let $\mathbf{z}_s\in\mathbb{C}^{F_s}, \mathbf{z}_t\in\mathbb{C}^{F_t}, \mathbf{z}_u\in\mathbb{C}^{F_u}$ be vectors whose entries consist of complex numbers with absolute value 1. Define $\mathbf{z}\in\mathbb{C}^{F_{s,t,u}}$ by*

$$(\mathbf{z})_k = \frac{(\mathbf{z}_s)_k(\mathbf{z}_t)_k}{(\mathbf{z}_u)_k} \quad (k\in F_{s,t,u}).$$

Then $\mathbf{z}^*\mathbf{z}\circ Q^u_{s,t}$ is the matrix of Krein parameters with respect to $\mathbf{z}_s^*\mathbf{z}_s\circ\{\varepsilon^s_{i,j}\}, \mathbf{z}_t^*\mathbf{z}_t\circ\{\varepsilon^t_{i,j}\}, \mathbf{z}_u^*\mathbf{z}_u\circ\{\varepsilon^u_{i,j}\}$.

Proof. By (2.12) and Definition 2.4.4, we have

$$\begin{aligned}
&(\mathbf{z}_s^*\mathbf{z}_s)_{i,j}\varepsilon_{i,j}^s \circ (\mathbf{z}_t^*\mathbf{z}_t)_{i,j}\varepsilon_{i,j}^t \\
&= (\mathbf{z}_s^*\mathbf{z}_s)_{i,j}(\mathbf{z}_t^*\mathbf{z}_t)_{i,j}(\varepsilon_{i,j}^s \circ \varepsilon_{i,j}^t) \\
&= \frac{(\mathbf{z}_s^*\mathbf{z}_s)_{i,j}(\mathbf{z}_t^*\mathbf{z}_t)_{i,j}}{\sqrt{|X_i||X_j|}} \sum_{\substack{u\in S\\ F_u\ni i,j}} (Q_{s,t}^u)_{i,j}\varepsilon_{i,j}^u \\
&= \frac{1}{\sqrt{|X_i||X_j|}} \sum_{\substack{u\in S\\ F_u\ni i,j}} \frac{(\mathbf{z}_s^*\mathbf{z}_s)_{i,j}(\mathbf{z}_t^*\mathbf{z}_t)_{i,j}}{(\mathbf{z}_u^*\mathbf{z}_u)_{i,j}}(Q_{s,t}^u)_{i,j}(\mathbf{z}_u^*\mathbf{z}_u)_{i,j}\varepsilon_{i,j}^u.
\end{aligned}$$

Thus the result follows. □

In particular, if $Q_{s,t}^u$ is positive semidefinite, then $Z \circ Q_{s,t}^u$ is also positive semidefinite. Thus the positive semidefiniteness of $Q_{s,t}^u$ is independent of the choice of bases of matrix units.

Theorem 2.4.8. *For any $s, t, u \in S$, the condition* (2.9) *holds if and only if the matrix of Krein parameters $Q_{s,t}^u$ is positive semidefinite.*

Proof. To prove this equivalence, we simplify (2.9). Let $B = (b_{i,j}) \in \mathcal{P}_{F_s}, C = (c_{i,j}) \in \mathcal{P}_{F_t}$. By Lemma 2.3.6, if $(m,n) \notin F_s^2$ or $(m,n) \notin F_t^2$, then the (m,n)-entry (2.8) of $\tilde{Q}_{s,t}^u(B,C)$ is 0. If $(m,n) \in F_{s,t,u}^2$, then (2.8) is

$$b_{m,n}c_{m,n}(Q_{s,t}^u)_{m,n} = (B' \circ C' \circ Q_{s,t}^u)_{m,n},$$

where $B', C' \in \mathrm{M}_{F_{s,t,u}}(\mathbb{C})$ are the principal submatrices of B, C indexed by $F_{s,t,u}$. Thus $\tilde{Q}_{s,t}^u(B,C)$ has $B' \circ C' \circ Q_{s,t}^u$ as a principal submatrix and all other entries are 0. This implies that $\tilde{Q}_{s,t}^u(B,C) \in \mathcal{P}_{F_u}$ if and only if $B' \circ C' \circ Q \in \mathcal{P}_{F_{s,t,u}}$. In particular, taking B and C to be the all-ones matrices, (2.9) implies $Q_{s,t}^u \in \mathcal{P}_{F_{s,t,u}}$.

Conversely, if $Q_{s,t}^u \in \mathcal{P}_{F_{s,t,u}}$, then $B' \circ C' \circ Q_{s,t}^u \in \mathcal{P}_{F_{s,t,u}}$ for any $B \in \mathcal{P}_{F_s}, C \in \mathcal{P}_{F_t}$ by [2, Lemma 3.9], and (2.9) holds. □

Hobart [9] applied the Krein condition of the coherent configuration defined by a quasi-symmetric design by setting B and C to be all-ones matrices. She commented that there are no choices of B, C which lead to other consequences. Indeed, since the coherent configuration defined by a quasi-symmetric design is fiber-commutative, considering the case $B = C = J$ is sufficient by Theorem 2.4.8.

2.5 Generalized quadrangles

Definition 2.5.1. Let P, L be finite sets and $I \subset P \times L$ be an incidence relation. An incidence structure (P, L, I) is called *a generalized quadrangle with parameters* (s, t) if

(i) for any $l \in L$, $\#\{p \in P \mid (p, l) \in I\} = s + 1$,

(ii) for any $p \in P$, $\#\{l \in L \mid (p, l) \in I\} = t + 1$,

(iii) for any $p \in P$ and $l \in L$ with $(p, l) \notin I$, there exist unique $q \in P$ and unique $m \in L$ such that $(p, m), (q, m), (q, l) \in I$.

Elements of P and L are called *points* and *lines*, respectively.

Let (P, L, I) be a generalized quadrangle with parameters (s, t). For $p, q \in P$, if there exists $l \in L$ such that $(p, l), (q, l) \in I$, then we write $p \sim q$ and say that p and q are *collinear.* Similarly, for $l, m \in L$, if there exists $p \in P$ such that $(p, l), (p, m) \in I$, then we write $l \sim m$ and say that l and m are *concurrent.*

In this section, we apply Theorem 2.4.8 to generalized quadrangles and obtain the following inequalities: If $s, t > 1$, then $s \leq t^2$ and $t \leq s^2$ hold. These inequalities are established in [5, 6], as a consequence of the Krein condition for the strongly regular graph defined by a generalized quadrangle. We also show that no other consequences can be obtained from Theorem 2.4.8 by computing all matrices of Krein parameters.

First, we construct a coherent configuration from a generalized quadrangle. Let $X_1 = P$ and $X_2 = L$ be fibers. Adjacency relations on $X = X_1 \sqcup X_2$

are defined as

$$\begin{aligned}
R_{1,1,1} &= \{(p,p) \mid p \in P\},\\
R_{1,1,2} &= \{(p,q) \in P^2 \mid p \sim q,\ p \neq q\},\\
R_{1,1,3} &= \{(p,q) \in P^2 \mid p \not\sim q\},\\
R_{1,2,1} &= \{(p,l) \in P \times L \mid (p,l) \in I\},\\
R_{1,2,2} &= \{(p,l) \in P \times L \mid (p,l) \notin I\},\\
R_{2,1,1} &= \{(l,p) \in L \times P \mid (p,l) \in I\},\\
R_{2,1,2} &= \{(l,p) \in L \times P \mid (p,l) \notin I\},\\
R_{2,2,1} &= \{(l,l) \mid l \in L\},\\
R_{2,2,2} &= \{(l,m) \in L^2 \mid l \sim m,\ l \neq m\},\\
R_{2,2,3} &= \{(l,m) \in L^2 \mid l \not\sim m\}.
\end{aligned}$$

Then $\mathfrak{X} = (X, \{R_I\}_{I\in\mathcal{I}})$ is a coherent configuration, where $\mathcal{I} = \{(i,j,k) \mid 1 \leq i,j \leq 2,\ 1 \leq k \leq r_{i,j}\}$ and $r_{1,1} = r_{2,2} = 3$, $r_{1,2} = r_{2,1} = 2$. Let $A_{i,j,k}$ be the adjacency matrix of the relation $R_{i,j,k}$, and let $\mathfrak{A}$ be the adjacency algebra of $\mathfrak{X}$. Then $\mathfrak{A}$ is decomposed as

$$\mathfrak{A} = \mathfrak{C}_1 \oplus \mathfrak{C}_2 \oplus \mathfrak{C}_3 \oplus \mathfrak{C}_4,$$

where $\mathfrak{C}_1, \mathfrak{C}_2 \simeq \mathrm{M}_2(\mathbb{C})$ and $\mathfrak{C}_3, \mathfrak{C}_4 \simeq \mathbb{C}$. Moreover, $F_1 = F_2 = \{1,2\}$, $F_3 = \{1\}$, $F_4 = \{2\}$.

For each $\mathfrak{C}_i$, a basis of matrix units can be expressed as follows: For $\mathfrak{C}_1$,

$$\begin{aligned}
\varepsilon^1_{1,1} &= \frac{1}{(st+1)(s+1)}(A_{1,1,1} + A_{1,1,2} + A_{1,1,3}),\\
\varepsilon^1_{2,2} &= \frac{1}{(st+1)(t+1)}(A_{2,2,1} + A_{2,2,2} + A_{2,2,3}),\\
\varepsilon^1_{1,2} &= \frac{1}{(st+1)\sqrt{(s+1)(t+1)}}(A_{1,2,1} + A_{1,2,2}),\\
\varepsilon^1_{2,1} &= \frac{1}{(st+1)\sqrt{(s+1)(t+1)}}(A_{2,1,1} + A_{2,1,2}).
\end{aligned}$$

For $\mathfrak{C}_2$,

$$\begin{aligned}
\varepsilon^2_{1,1} &= \frac{1}{(st+1)(s+t)}(st(t+1)A_{1,1,1} + t(s-1)A_{1,1,2} - (t+1)A_{1,1,3}),\\
\varepsilon^2_{2,2} &= \frac{1}{(st+1)(s+t)}(st(s+1)A_{2,2,1} + s(t-1)A_{2,2,2} - (s+1)A_{2,2,3}),\\
\varepsilon^2_{1,2} &= \frac{1}{(st+1)\sqrt{(s+t)}}(stA_{1,2,1} - A_{1,2,2}),\\
\varepsilon^2_{2,1} &= \frac{1}{(st+1)\sqrt{(s+t)}}(stA_{2,1,1} - A_{2,1,2}).
\end{aligned}$$

For $\mathfrak{C}_3$,

$$\varepsilon^3_{1,1} = \frac{1}{(s+t)(s+1)}(s^2A_{1,1,1} - sA_{1,1,2} + A_{1,1,3}).$$

For $\mathfrak{C}_4$,

$$\varepsilon^4_{2,2} = \frac{1}{(s+t)(t+1)}(t^2A_{2,2,1} - tA_{2,2,2} + A_{2,2,3}).$$

For these bases of matrix units, the matrices of Krein parameters $Q^3_{3,3}$ and $Q^4_{4,4}$ are the 1×1 matrices given by

$$\begin{aligned}
Q^3_{3,3} &= \frac{(st+1)(s-1)(s^2-t)}{(s+t)^2},\\
Q^4_{4,4} &= \frac{(st+1)(t-1)(t^2-s)}{(s+t)^2}.
\end{aligned}$$

By Theorem 2.4.8, both $Q^3_{3,3}$ and $Q^4_{4,4}$ are positive semidefinite, so $s^2 \geq t$ and $t^2 \geq s$ hold, provided $s, t > 1$. The consequences of Theorem 2.4.8 for all other matrices of Krein parameters are trivial. Indeed, the other matrices of Krein parameters are given as follows (we omit those matrices determined by Proposition 2.4.5, and those determined to be zero by Proposition 2.4.6):

$$\begin{aligned}
Q^1_{2,2} &= \frac{st(s+1)(t+1)}{(s+t)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\\
Q^2_{2,2} &= \frac{1}{(s+t)^2} \begin{bmatrix} \sigma(s,t) & \tau(s,t) \\ \tau(s,t) & \sigma(t,s) \end{bmatrix},
\end{aligned}$$

where

$$\sigma(s,t) = (s+1)(t^2(st+2s-1)+s(st-2t-1)),$$
$$\tau(s,t) = (s+t)^{3/2}(st-1)\sqrt{(s+1)(t+1)},$$

and

$$Q^3_{2,2} = \frac{t(st+1)(s+1)(t+1)}{(s+t)^2}, \quad Q^4_{2,2} = \frac{s(st+1)(s+1)(t+1)}{(s+t)^2},$$
$$Q^2_{2,3} = \frac{s(st+1)^2}{(s+t)^2}, \quad Q^3_{2,3} = \frac{t(t+1)(s+1)^2(s-1)}{(s+t)^2},$$
$$Q^2_{2,4} = \frac{t(st+1)^2}{(s+t)^2}, \quad Q^4_{2,4} = \frac{s(s+1)(t+1)^2(t-1)}{(s+t)^2},$$
$$Q^1_{3,3} = \frac{s^2(st+1)}{(s+t)}, \quad Q^2_{3,3} = \frac{s(st+1)(s+1)(s-1)}{(s+t)^2},$$
$$Q^1_{4,4} = \frac{t^2(st+1)}{(s+t)}, \quad Q^2_{4,4} = \frac{t(st+1)(t+1)(t-1)}{(s+t)^2}.$$

2.6 Absolute bounds for commutative coherent configurations

Let $\mathfrak{A}$ be the adjacency algebra of a coherent configuration $\mathfrak{X} = (X, \{R_I\}_{I\in\mathcal{I}})$, and let $\{\varphi_s \mid s \in S\}$ be a set of representatives of all irreducible matrix representations of $\mathfrak{A}$ over $\mathbb{C}$ satisfying $\varphi_s(A)^* = \varphi_s(A^*)$ for any $A \in \mathfrak{A}$. Denote by h_s the multiplicity of φ_s in the standard module $\mathbb{C}^X$. In this section, we assume that $\varphi_s(\varepsilon^s_{i,j}) = E_{i,j}$ for a basis of matrix units $\{\varepsilon^s_{i,j}\}$ for $\mathfrak{C}_s$, where $E_{i,j}$ is $e_s \times e_s$ matrix with (i,j)-entry 1 and all other entries 0. The following bound is known as the absolute bound.

Lemma 2.6.1 ([10, Theorem 5])**.** *For any $s,t \in S$, we have*

$$\sum_{u\in S} h_u \operatorname{rank}\left(\sum_{\lambda\in\Lambda_s}\sum_{\mu\in\Lambda_t}\sum_{\nu\in\Lambda_u} q^\nu_{\lambda,\mu}\varphi_u(\varepsilon_\nu)\right) \le \begin{cases} h_s h_t & \text{if } s \neq t,\\ \binom{h_s+1}{2} & \text{if } s = t.\end{cases}$$

For fiber-commutative coherent configurations, we can simplify this inequality.

Theorem 2.6.2. *Let $Q^u_{s,t}$ $(s,t,u \in S)$ be the matrices of Krein parameters for $\mathfrak{X}$. For any $s,t \in S$, we have*

$$\sum_{u\in S} h_u \operatorname{rank}(Q^u_{s,t}) \le \begin{cases} h_s h_t & \text{if } s \neq t,\\ \binom{h_s+1}{2} & \text{if } s = t.\end{cases}$$

Proof. By (2.11), for any $u \in S$, we have

$$\sum_{\lambda \in \Lambda_s} \sum_{\mu \in \Lambda_t} \sum_{\nu \in \Lambda_u} q^{\nu}_{\lambda,\mu} \varphi_u(\varepsilon_\nu) = \sum_{i,j \in F_u} q^{(i,j,u)}_{(i,j,s),(i,j,t)} \varphi_u(\varepsilon^u_{i,j})$$
$$= \sum_{i,j \in F_u} (Q^u_{s,t})_{i,j} E_{i,j},$$

and the rank of this matrix is $\mathrm{rank}(Q^u_{s,t})$. By Lemma 2.6.1, the result follows. □

2.7 Eigenmatrices

Let $\mathfrak{X} = (X, \{R_{i,j,a}\}_{i,j,a})$ be a fiber-commutative coherent configuration, $\mathfrak{A}$ be its adjacency algebra and $\{\varepsilon^s_{i,j} \mid s \in S, i, j \in F_s\}$ be the bases of matrix units for $\mathfrak{A}$.

For $i, j \in F$, since the subspace $\mathfrak{A}_{i,j} \subset \mathfrak{A}$ has two bases $\{A_{i,j,a} \mid a \in \{1, 2, \ldots, r_{i,j}\}\}$ and $\{\varepsilon^s_{i,j} \mid s \in S_{i,j}\}$, we have

$$A_{i,j,a} = \sum_{s \in S_{i,j}} p_{i,j,a}(s) \varepsilon^s_{i,j},$$
$$\varepsilon^s_{i,j} = \frac{1}{\sqrt{|X_i||X_j|}} \sum_{i=1}^{r_{i,j}} q_{i,j,s}(a) A_{i,j,a}.$$

Definition 2.7.1. For $i, j \in F$, *the first and second matrices* $P_{i,j}, Q_{i,j}$ *for* $\mathfrak{A}_{i,j}$ is defined as $(P_{i,j})_{s,a} = p_{i,j,a}(s)$ and $(Q_{i,j})_{a,s} = q_{i,j,s}(a)$, respectively. Moreover, *the first and second matrices* P, Q *for* $\mathfrak{A}$ is formally defined as $(P)_{i,j} = P_{i,j}$ and $(Q)_{i,j} = Q_{i,j}$, respectively.

For $i \in F$, since $\{\varepsilon^s_{i,i} \mid s \in S_{i,i}\}$ is the set of all primitive idempotents of the commutative association scheme $(X_i, \{R_{i,i,a}\}_a)$, $P_{i,i}, Q_{i,i}$ are same as eigenmatrices of $(X_i, \{R_{i,i,a}\}_a)$.

Chapter 3

Fusions in fiber-commutative coherent configurations

3.1 Subalgebras having their own bases of matrix units

Let $\mathfrak{A}$ be the adjacency algebra of a fiber-commutative coherent configuration $\mathfrak{X}$ and $\{\varepsilon^s_{i,j} \mid s \in S,\ i,j \in F_s\}$ be the bases of matrix units for $\mathfrak{A}$. In this section, we consider a subalgebra of $\mathfrak{A}$ and reveal some equivalent conditions that the subalgebra has matrices satisfying the conditions in the definition of bases of matrix units.

Let S' be an index set. For $\sigma \in S'$, let $F_\sigma \subset F$ be a subset. Suppose that a set of matrices $\mathcal{E} = \{{\varepsilon'}^\sigma_{i,j} \mid \sigma \in S', i,j \in F_\sigma\}$ is given such that, for any $\sigma \in S', i,j \in F_\sigma$, ${\varepsilon'}^\sigma_{i,j} \in \mathfrak{A}_{i,j}$ holds. We consider when the span of $\mathcal{E}$ is a subalgebra having $\mathcal{E}$ as its bases of matrix units.

Let $\mathfrak{A}'$ be a subspace of $\mathfrak{A}$ defined by $\mathfrak{A}' = \langle \varepsilon \mid \varepsilon \in \mathcal{E} \rangle_{\mathbb{C}}$ and $\Lambda' = \{(i,j,\sigma) \mid \sigma \in S', i,j \in F_\sigma\} = \coprod_{\sigma \in S'} F_\sigma \times F_\sigma \times \{\sigma\}$. Then Λ' is decomposed into

$$\Lambda' = \coprod_{i,j \in F} \{i\} \times \{j\} \times S'_{i,j}, \tag{3.1}$$

where

$$S'_{i,j} = \{\sigma \in S' \mid F_\sigma \ni i,j\}. \tag{3.2}$$

For brevity, we write $S'_i = S'_{i,i}$. and it is clear that $S'_{i,j} = S'_i \cap S'_j$. By (3.1), the subspace $\mathfrak{A}'$ is decomposed into

$$\mathfrak{A}' = \bigoplus_{i,j \in F} \mathfrak{A}'_{i,j},$$

where $\mathfrak{A}'_{i,j} = \langle {\varepsilon'}^{\sigma}_{i,j} \mid \sigma \in S'_{i,j} \rangle_{\mathbb{C}} \subset \mathfrak{A}'$.

Lemma 3.1.1. *Fix $\sigma \in S'$. The following are equivalent.*

(i) For any $i, j, k, l \in F_\sigma$, ${\varepsilon'}^{\sigma}_{i,j} {\varepsilon'}^{\sigma}_{k,l} = \delta_{j,k} {\varepsilon'}^{\sigma}_{i,l}$ and ${{\varepsilon'}^{\sigma}_{i,j}}^* = {\varepsilon'}^{\sigma}_{j,i}$,

(ii) there exists a subset $T_\sigma \subset S$ such that, for any $i, j \in F_\sigma$, and $s \in T_\sigma$, there exist $c^s_{i,j,\sigma} \in \mathbb{C}$ such that

$$
{\varepsilon'}^{\sigma}_{i,j} = \sum_{s \in T_\sigma} c^s_{i,j,\sigma} \varepsilon^s_{i,j}, \;\; (c^s_{i,i,\sigma} = 1, |c^s_{i,j,\sigma}| = 1, \overline{c^s_{i,j,\sigma}} = c^s_{j,i,\sigma}) \tag{3.3}
$$

Proof. Suppose that (i) holds. By ${\varepsilon'}^{\sigma}_{i,i} \in \mathfrak{A}'_{i,i} \subset \mathfrak{A}_{i,i}$, ${\varepsilon'}^{\sigma}_{i,i}$ can be expressed as

$$
{\varepsilon'}^{\sigma}_{i,i} = \sum_{s \in S_i} a_s \varepsilon^s_{i,i}.
$$

Since $\varepsilon^s_{i,i}$ are elements of bases of matrix units,

$$
{{\varepsilon'}^{\sigma}_{i,i}}^2 = \left(\sum_{s \in S_i} a_s \varepsilon^s_{i,i} \right)^2 = \sum_{s \in S_i} a_s^2 \varepsilon^s_{i,i}.
$$

Moreover, by (i), ${\varepsilon'}^{\sigma}_{i,i}$ is an idempotent. It means $a_s \in \{0, 1\}$. Thus there exists $T_\sigma(i) \subset S_i$ such that ${\varepsilon'}^{\sigma}_{i,i} = \sum_{s \in T_\sigma(i)} \varepsilon^s_{i,i}$. For $i \neq j$, ${\varepsilon'}^{\sigma}_{i,j} \in \mathfrak{A}_{i,j}$ can be expressed as

$$
{\varepsilon'}^{\sigma}_{i,j} = \sum_{s \in S_{i,j}} c^s_{i,j,\sigma} \varepsilon^s_{i,j}.
$$

Then, by (i),

$$
{\varepsilon'}^{\sigma}_{j,i} = {{\varepsilon'}^{\sigma}_{i,j}}^* = \sum_{s \in S_{i,j}} \overline{c^s_{i,j,\sigma}} {\varepsilon^s_{i,j}}^* = \sum_{s \in S_{i,j}} \overline{c^s_{i,j,\sigma}} \varepsilon^s_{j,i}.
$$

Thus

$$\begin{aligned}\sum_{s\in T_\sigma(i)} \varepsilon^s_{i,i} &= {\varepsilon'}^\sigma_{i,i} \\ &= {\varepsilon'}^\sigma_{i,j}{\varepsilon'}^\sigma_{j,i} \\ &= \left(\sum_{s\in S_{i,j}} c^s_{i,j,\sigma}\varepsilon^s_{i,j}\right)\left(\sum_{t\in S_{i,j}} \overline{c^t_{i,j,\sigma}}\varepsilon^t_{j,i}\right) \\ &= \sum_{s\in S_{i,j}} |c^s_{i,j,\sigma}|^2\varepsilon^s_{i,i}\end{aligned}$$

and this means that $T_\sigma(i) \subset S_{i,j}$ and $|c^s_{i,j,\sigma}| = 1$ if $s \in T_\sigma(i)$ and 0 otherwise. Thus we obtain ${\varepsilon'}^\sigma_{i,j} = \sum_{s\in T_\sigma(i)} c^s_{i,j,\sigma}\varepsilon^s_{i,j}$, where $|c^s_{i,j,\sigma}| = 1$. By this identity, ${\varepsilon'}^\sigma_{j,i} = {{\varepsilon'}^\sigma_{i,j}}^* = \sum_{s\in T_\sigma(i)} \overline{c^s_{i,j,\sigma}}\varepsilon^s_{j,i}$ follows. This implies that $c^s_{j,i,\sigma} = \overline{c^s_{i,j,\sigma}}$ and $T_\sigma(i)$ is determined independently of the choice of $i \in F_\sigma$.

By direct computation, the converse follows. □

Let

$$\tilde{\Lambda} = \bigcup_{\sigma\in S'} F_\sigma \times F_\sigma \times T_\sigma \tag{3.4}$$

be a subset of Λ.

Lemma 3.1.2. *For all $\sigma \in S'$, suppose that one of the equivalent conditions of* Lemma 3.1.1 *holds. Then the following are equivalent.*

(i) *For any $\sigma, \tau \in S'$ satisfying $\sigma \neq \tau$, ${\varepsilon'}^\sigma_{i,i}{\varepsilon'}^\tau_{i,i} = 0$ holds for any $i \in F_\sigma \cap F_\tau$,*

(ii) *the union $\bigcup_{\sigma\in S'} F_\sigma \times T_\sigma$ is disjoint.*

In particular, (i) *holds if and only if the union* (3.4) *is disjoint.*

Proof. Suppose (i). If $T_\sigma \cap T_\tau \neq \emptyset$, then, by the expression (3.3) in Lemma 3.1.1, for any $s \in T_\sigma \cap T_\tau$ and any $i \in F_\sigma \cap F_\tau$, $\varepsilon^s_{i,i}$ appears in ${\varepsilon'}^\sigma_{i,i}{\varepsilon'}^\tau_{i,i}$. This means ${\varepsilon'}^\sigma_{i,i}{\varepsilon'}^\tau_{i,i} \neq 0$ and this is a contradiction. Thus, for any $\sigma, \tau \in S'$ satisfying $\sigma \neq \tau$, if $F_\sigma \cap F_\tau \neq \emptyset$, then $T_\sigma \cap T_\tau = \emptyset$. Since, for any $\sigma, \tau \in S'$, $(F_\sigma \times T_\sigma) \cap (F_\tau \times T_\tau) = \emptyset$ if and only if $F_\sigma \cap F_\tau = \emptyset$ or $T_\sigma \cap T_\tau = \emptyset$ holds, (ii) follows.

The converse is clear by the expression (3.3) in Lemma 3.1.1. □

If Lemma 3.1.2 (ii) holds, then for any $(i,j,s) \in \Lambda$, $\#\{\sigma \in S' \mid F_\sigma \ni i,j, T_\sigma \ni s\} \leq 1$. In other words, For any $(i,j,s) \in \Lambda$, a unique $(i,j,\sigma) \in \Lambda'$ with $s \in T_\sigma$ is determined if it exists.

If all matrices in the set $\mathcal{E}$ satisfy ones of the equivalent conditions of Lemma 3.1.1 and Lemma 3.1.2, then the expression (3.3) is rewritten as

$$\varepsilon'^{\sigma}_{i,j} = \sum_{s \in T_\sigma} c^s_{i,j} \varepsilon^s_{i,j} \quad (c^s_{i,j} = \overline{c^s_{j,i}}, |c^s_{i,j}| = 1, c^s_{i,i} = 1) \tag{3.5}$$

for all $s \in S', i,j \in F_\sigma$. By Lemma 3.1.2 (ii), for any $(i,j,s) \in \Lambda$, $c^s_{i,j}$ appears only in a uniquely determined $\varepsilon'^{\sigma}_{i,j}$ with $T_\sigma \ni s$. Thus $c^s_{i,j}$ depends only on $(i,j,\sigma) \in \Lambda'$ with $T_\sigma \ni s$. Note that this expression is exacty same as (3.3) and it means that the indices for $c^s_{i,j}$ suffice in the expression (3.3).

Lemma 3.1.3. *Let $\overline{\mathcal{E}} = \{\overline{\varepsilon} \mid \varepsilon \in \mathcal{E}\}$. Suppose that ones of the equivalent conditions of* Lemma 3.1.1 *and* Lemma 3.1.2 *hold, and any element in $\mathcal{E}$ is expressed as* (3.5)*. Then $\mathcal{E} = \overline{\mathcal{E}}$ if and only if, for any $\sigma \in S'$, there exists $\hat{\sigma} \in S'$ such that $F_\sigma = F_{\hat{\sigma}}$, $T_{\hat{\sigma}} = \{\hat{s} \mid s \in T_\sigma\}$ holds and $c^{\hat{s}}_{i,j} = \overline{c^s_{i,j}}$ hold for all $s \in T_\sigma, i,j \in F_\sigma$.*

Proof. If $\mathcal{E} = \overline{\mathcal{E}}$, then, for any $\sigma \in S'$, there exists $\hat{\sigma} \in S'$ such that $\overline{\varepsilon'^{\sigma}_{i,j}} = \varepsilon'^{\hat{\sigma}}_{i,j}$ for any $i,j \in F_\sigma$. Then

$$\varepsilon'^{\hat{\sigma}}_{i,j} = \overline{\varepsilon'^{\sigma}_{i,j}} = \sum_{s \in T_\sigma} \overline{c^s_{i,j} \varepsilon^s_{i,j}}. = \sum_{s \in T_\sigma} \overline{c^s_{i,j}} \varepsilon^{\hat{s}}_{i,j}.$$

It means that $T_{\hat{\sigma}} = \{\hat{s} \mid s \in T_\sigma\}$ and $c^{\hat{s}}_{i,j} = \overline{c^s_{i,j}}$. By direct computation, the converse is clear by (3.5). ☐

For a fiber X_i of the fiber-commutative coherent configuration $\mathfrak{X}$, let I_{X_i} be the diagonal matrix with $(I_{X_i})_{x,x} = 1$ if $x \in X_i$ and 0 otherwise.

Lemma 3.1.4. *Suppose that ones of the equivalent conditions of* Lemma 3.1.1 *and* Lemma 3.1.2*. Then $I_{X_i} \in \mathfrak{A}'_{i,i}$ for all $i \in F$ if and only if $\{(i,s) \mid (i,i,s) \in \Lambda\} = \coprod_{\sigma \in S'} F_\sigma \times T_\sigma$.*

Proof. For any $i \in F$, by the definition of fiber-commutative coherent configurations,

$$I_{X_i} = \sum_{s \in S_i} \varepsilon^s_{i,i}.$$

On the other hand, since $\{\varepsilon'^{\sigma}_{i,j} \mid \sigma \in S'_{i,j}\}$ is a basis of $\mathfrak{A}'_{i,j}$ and I_{X_i} is an idempotent in $\mathfrak{A}_{i,i}$, by (3.5) and Lemma 3.1.2 (ii), there exists $\tilde{S} \subset S'_i$ such that

$$I_{X_i} = \sum_{\sigma\in\tilde{S}} \varepsilon'^{\sigma}_{i,i} = \sum_{\sigma\in\tilde{S}} \sum_{s\in T_\sigma} \varepsilon^{s}_{i,i}.$$

Thus $S_i = \coprod_{\sigma\in\tilde{S}} T_\sigma$ holds. By $\coprod_{\sigma\in\tilde{S}} T_\sigma \subset \coprod_{\sigma\in S'_i} T_\sigma \subset S_i$, we obtain $S_i = \coprod_{\sigma\in S'_i} T_\sigma$ for any $i \in F$ and it means that

$$\{(i,s) \mid (i,i,s) \in \tilde{\Lambda}\} = \coprod_{i\in F} \{i\} \times S_i = \{(i,s) \mid (i,i,s) \in \Lambda\}.$$

By Lemma 3.1.2 (ii), for any $(i,i,s) \in \tilde{\Lambda}$, there exists a unique $\sigma \in S$ such that $i \in F_\sigma$ and $s \in T_\sigma$. Thus $\coprod_{\sigma\in S'} F_\sigma \times T_\sigma = \{(i,s) \mid (i,i,s) \in \tilde{\Lambda}\} = \{(i,s) \mid (i,i,s) \in \Lambda\}$.

The converse is clear by (3.5) and Lemma 3.1.2 (ii). □

3.2 Fusions in fiber-commutative coherent configurations

Let $\mathfrak{X} = (X, \{R_{i,j,a}\}_{i,j,a})$ be a fiber-commutative coherent configuration with fibers $\coprod_{i\in F} X_i$, $\mathfrak{A} = \langle A_{i,j,a} \mid i,j \in F, a \in \{1,2,\ldots,r_{i,j}\}\rangle_{\mathbb{C}}$ be the adjacency algebra of $\mathfrak{X}$ and $\{\varepsilon^{s}_{i,j} \mid (i,j,s) \in \Lambda\}$ be the bases of matrix units for the adjacency algebra $\mathfrak{A}$, where $\Lambda = \coprod_{i,j\in F}\{i\} \times \{j\} \times S_{i,j}$. Moreover, let $P = (P_{i,j})$ be the first eigenmatrix of $\mathfrak{X}$.

In this section, we give an equivalent condition for a subalgebra of $\mathfrak{A}$ to be the adjacency algebra of a coherent configuration with the same fibers as those of $\mathfrak{X}$.

Definition 3.2.1. A fiber-commutative coherent configuration $\mathfrak{X}' = (X, \{R'_{i,j,a}\}_{i,j,a})$ is a *fusion configuration with the same fibers as those of* $\mathfrak{X}$ if $\mathfrak{X}$ and $\mathfrak{X}'$ have the same fibers and the adjacency algebra $\mathfrak{A}'$ of $\mathfrak{X}'$ is a subalgebra of $\mathfrak{A}$.

Definition 3.2.2. A family of partitions $\Delta = \{\Delta_{i,j} \mid i,j \in F\}$ is called *admissible* if

(i) for any $i,j \in F$, $\coprod_{\delta\in\Delta_{i,j}} \delta = \{1,2,\ldots,r_{i,j}\}$.

(ii) for any $i \in F, \{1\} \in \Delta_{i,i}$,

(iii) for any $\delta \in \Delta_{i,j}$, $\{a \in \{1, 2, \dots, r_{j,i}\} \mid A^T_{j,i,a} = A_{i,j,b} \text{ for some } b \in \delta\} \in \Delta_{j,i}$.

Lemma 3.2.3. *If $\mathfrak{X}'$ is a fusion configuration with the same fibers as those of $\mathfrak{X}$, then there exists a uniquely determined admissible family of partitions $\Delta = \{\Delta_{i,j} \mid i, j \in F\}$ such that $\mathfrak{X}' = (X, \{R'_{i,j,\delta}\}_{i,j,\delta})$, where $R'_{i,j,\delta} = \coprod_{a\in\delta} R_{i,j,a}$ for $\delta \in \Delta_{i,j}$. Conversely, if $\Delta = \{\Delta_{i,j} \mid i, j \in F\}$ is an admissible family of partitions, then the set $\{A'_{i,j,\delta} \mid i, j \in F, \delta \in \Delta_{i,j}\}$ satisfies conditions of* Definition 2.2.1 (i), (ii), (iii), *where $A'_{i,j,\delta} = \sum_{a\in\delta} A_{i,j,a}$.*

Proof. Let $\{A'_{i,j,b} \mid i, j \in F, b \in \{1, 2, \dots, r'_{i,j}\}\}$ be the set of adjacency matrices of $\mathfrak{X}'$ and $\mathfrak{A}'$ be the adjacency algebra of $\mathfrak{X}'$. Then $\mathfrak{A}'$ is a subalgebra of $\mathfrak{A}$ and it implies that, for any $i, j \in F$ and $b \in \{1, 2, \dots, r'_{i,j}\}$, there exists a subset $\delta_{i,j,b} \subset \{1, 2, \dots, r_{i,j}\}$ such that

$$A'_{i,j,b} = \sum_{a\in\delta_{i,j,b}} A_{i,j,a}.$$

By Definition 2.2.1 (i), (ii), (iii), for all $i, j \in F$, $\Delta_{i,j} = \{\delta_{i,j,b} \mid b \in \{1, 2, \dots, r'_{i,j}\}\}$ satisfy Definition 3.2.2 (i), (ii), (iii) and $\Delta = \{\Delta_{i,j} \mid i, j \in F\}$ is admissible.

The converse is clear by Definition 3.2.2. □

The following theorem essentially reveals the condition in Definition 2.2.1 (iv).

Theorem 3.2.4. *Let $G = (V, E)$ be a bipartite graph with $V = F \cup S$ and edge set $E = \{(i, s) \mid (i, i, s) \in \Lambda\}$. Then $\mathfrak{X}' = (X, \{R'_{i,j,\delta}\}_{i,j,\delta})$ is a fusion configuration with the same fibers as those of $\mathfrak{X}$, where $R'_{i,j,\delta} = \coprod_{a\in\delta} R_{i,j,a}$ for $\delta \in \Delta_{i,j}$, if and only if $\Delta = \{\Delta_{i,j} \mid i, j \in F\}$ is admissible and there exist*

(I) *diagonal matrices $C_{i,j}$ indexed by $S_{i,j} \times S_{i,j}$ with $(C_{i,j})_{s,s} = c^s_{i,j}$ for $i, j \in F$,*

(II) *an index set S' and subsets $T_\sigma \subset S, F_\sigma \subset F$ for $\sigma \in S'$ which gives a partition $\{F_\sigma \times T_\sigma \mid \sigma \in S'\}$ of E into complete bipartite edge-subgraphs; $E = \coprod_{\sigma\in S'} F_\sigma \times T_\sigma$,*

such that, for any $i, j \in F$,

(i) $|\Delta_{i,j}| = |S'_{i,j}|$,

(ii) *for any* $s \in \coprod_{\sigma\in S'_{i,j}} T_\sigma, c^s_{i,i} = 1, |c^s_{i,j}| = 1, c^s_{i,j} = \overline{c^s_{j,i}} = \overline{c^{\hat{s}}_{i,j}}$,

(iii) *for any* $\sigma \in S'_{i,j}, \delta \in \Delta_{i,j}$, *row sums of the submatrix of* $\overline{C_{i,j}}P_{i,j}$ *indexed by* $T_\sigma \times \delta$ *is a constant* $p'_{i,j,\delta}(\sigma)$ *and row sums indexed by* $O_{i,j} \times \delta$ *is* 0,

where $S'_{i,j}$ *is defined by* (3.2) *and* $O_{i,j} = S_{i,j}\backslash\left(\coprod_{\sigma\in S'_{i,j}} T_\sigma\right)$. *Moreover, if* $\mathfrak{X}'$ *is a fusion configuration with the same fibers as those of* $\mathfrak{X}$, *then the first eigenmatrix* $P' = (P'_{i,j})$ *of* $\mathfrak{X}'$ *with respect to bases of matrix units* $\{\varepsilon'^{\sigma}_{i,j} \mid \sigma \in S', i,j \in F_\sigma\}$ *is given by* $(P'_{i,j})_{\sigma,\delta} = p'_{i,j,\delta}(\sigma)$ *for* $\sigma \in S'_{i,j}, \delta \in \Delta_{i,j}$, *where*

$$\varepsilon'^{\sigma}_{i,j} = \sum_{s\in T_\sigma} c^s_{i,j}\varepsilon^s_{i,j} \tag{3.6}$$

for $\sigma \in S', i,j \in F_\sigma$.

Proof. For $i,j \in F, \delta \in \Delta_{i,j}$, let

$$A'_{i,j,\delta} = \sum_{a\in\delta} A_{i,j,a}$$

be the adjacency matrix with respect to $R'_{i,j,\delta}$ of $\mathfrak{X}$.

If $\mathfrak{X}'$ is a fusion configuration with the same fibers as those of $\mathfrak{X}$, then, by Lemma 3.2.3, Δ is admissible. Moreover, the adjacency algebra $\mathfrak{A}'$ of $\mathfrak{X}'$ is decomposed into a direct sum of simple two-sided ideals: $\mathfrak{X}' = \bigoplus_{\sigma\in S'} \mathfrak{C}'_\sigma$. Then $\mathfrak{C}'_\sigma$ gives $F_\sigma = \{i \in F \mid \dim(\mathfrak{A}'_{i,i} \cap \mathfrak{C}'_\sigma) = 1\}$ for each $\sigma \in S'$ and $\mathfrak{A}'$ has bases of matrix units $\{\varepsilon'^{\sigma}_{i,j} \mid \sigma \in S', i,j \in F_\sigma\}$. Let $S'_{i,j} = \{\sigma \in S' \mid \dim(\mathfrak{A}'_{i,j} \cap \mathfrak{C}'_\sigma) = 1\}$. Then, for any $i,j \in F$, $|\Delta_{i,j}| = \dim(\mathfrak{A}'_{i,j}) = |S'_{i,j}|$ holds. By Lemma 3.1.1, for each $\sigma \in S'$, there exists $T_\sigma \subset S$ such that (3.3) holds. By Lemma 3.1.2, $\bigcup_{\sigma\in S'} F_\sigma \times T_\sigma$ is disjoint. By Lemma 3.1.4, $E = \coprod_{\sigma\in S'} F_\sigma \times T_\sigma$ and it means that $\{F_\sigma \times T_\sigma \mid \sigma \in S'\}$ is a partition of E. By (3.5) and Lemma 3.1.3, there exist $c^s_{i,j}$ for $i,j \in F, s \in S_{i,j}$, and $c^s_{i,j}$ satisfy (ii) in the latter conditions for all $i,j \in F, s \in \coprod_{\sigma\in S'_{i,j}} T_\sigma$. Thus, for any $i,j \in F$, we may define a diagonal matrix $C_{i,j}$ indexed by $S_{i,j} \times S_{i,j}$ whose (s,s)-entry is $c^s_{i,j}$ if $s \in \coprod_{\sigma\in S'_{i,j}} T_\sigma$ and 0 otherwise. Since $\{\varepsilon'^{\sigma}_{i,j} \mid \sigma \in S', i,j \in F_\sigma\}$ are bases of matrix units of $\mathfrak{A}'$, $A'_{i,j,\delta}$ is expressed as

$$A'_{i,j,\delta} = \sum_{\sigma\in S'_{i,j}} p'_{i,j,\delta}(\sigma)\varepsilon'^{\sigma}_{i,j} \tag{3.7}$$

for $i,j \in F, \delta \in \Delta_{i,j}$. By $A'_{i,j,\delta} = \sum_{a\in\delta} A_{i,j,a}$ and (3.5), (iii) in the latter conditions holds. In addition, by (3.7), $P' = (P'_{i,j})_{i,j}$ with $P'_{i,j} = (p'_{i,j,\delta}(\sigma))_{\sigma,\delta}$ is the first eigenmatrix of $\mathfrak{X}'$ with respect to $\{{\varepsilon'}^{\sigma}_{i,j} \mid \sigma \in S', i,j \in F_\sigma\}$, where ${\varepsilon'}^{\sigma}_{i,j}$ satisfy (3.6).

Conversely, suppose that Δ is admissible and that the latter conditions are satisfied. For $i,j \in F, \delta \in \Delta_{i,j}, \sigma \in S'$, let ${\varepsilon'}^{\sigma}_{i,j}$ be expressed as (3.6). Since Δ is admissible, the set $\{A'_{i,j,\delta} \mid i,j \in F, \delta \in \Delta_{i,j}\}$ satisfies Definition 2.2.1 (i), (ii), (iii). By (ii), (iii) in the latter conditions,

$$\begin{aligned}
A'_{i,j,\delta} &= \sum_{a\in\delta} A_{i,j,a} \\
&= \sum_{a\in\delta} \sum_{s\in S_{i,j}} p_{i,j,a}(s)\varepsilon^s_{i,j} \\
&= \sum_{s\in S_{i,j}} \left(\sum_{a\in\delta} \overline{c^s_{i,j}} p_{i,j,a}(s) \right) c^s_{i,j}\varepsilon^s_{i,j} \\
&= \sum_{\sigma\in S'_{i,j}} \sum_{s\in T_\sigma} p'_{i,j,\delta}(\sigma) c^s_{i,j}\varepsilon^s_{i,j} \\
&= \sum_{\sigma\in S'_{i,j}} p'_{i,j,\delta}(\sigma){\varepsilon'}^{\sigma}_{i,j}.
\end{aligned}$$

This implies that

$$\langle A'_{i,j,\delta} \mid \delta \in \Delta_{i,j}\rangle_{\mathbb{C}} \subset \langle {\varepsilon'}^{\sigma}_{i,j} \mid \sigma \in S'_{i,j}\rangle_{\mathbb{C}}$$

as a subspace of $\mathfrak{A}_{i,j}$. By the condition (i), the dimensions of these subspaces coincide and it implies these subspaces coincide. By Lemma 3.1.1 and Lemma 3.1.2, $\mathfrak{A}' = \bigoplus_{i,j\in F} \langle A'_{i,j,\delta} \mid \delta \in \Delta_{i,j}\rangle_{\mathbb{C}}$ is closed with respect to the matrix multiplication and it means that $\mathfrak{X}' = (X, \{R'_{i,j,\delta}\}_{i,j,\delta})$ is a fusion configuration with the same fibers as those of $\mathfrak{X}$, where $R'_{i,j,\delta} = \coprod_{a\in\delta} R_{i,j,a}$ for $\delta \in \Delta_{i,j}$. □

3.3 Applications

In this section, we apply Theorem 3.2.4 to commutative association schemes, fiber-commutative coherent configurations and the fiber-commutative coherent configuration given by $\mathbb{Z}_3^4 \rtimes S_6$.

3.3.1 Commutative association schemes

Commutative association schemes are defined as fiber-commutative coherent configurations with $|F| = 1$.

We assume $F = \{1\}$. For brevity, we omit indices given by F. Let $\mathfrak{X} = (X, \{R_a\}_{a=1}^r)$ be a commutative association scheme and $\{\varepsilon^s \mid s \in S\}$ be its primitive idempotents. Since $\mathfrak{X}$ has only one fiber, any fusion configuration with the same fiber as those of $\mathfrak{X}$ has also a fiber X and is called *a fusion schemes.* Then an admissible partition Δ for $\mathfrak{X}$ satisfies

(i) $\{1\} \in \Delta$,

(ii) for any $\delta \in \Delta, \{a \mid A_a^T = A_b \text{ for some } b \in \delta\} \in \Delta$.

Let S' be an index set with $|S'| = |\Delta|$ and, for $\sigma \in S'$, let $T_\sigma \subset S$. Note that, by $|F| = 1$, $F = F_\sigma$ hold for all $\sigma \in S'$. Then the bipartite graph $G = (V, E)$ defined by $\mathfrak{X}$ has vertex set $V = F \cup S = \{1\} \cup S$ and edge set $E = \{(1, s) \mid s \in S\}$. Thus the partition $\{\{1\} \times T_\sigma \mid \sigma \in S'\}$ of E can be identified with the partition $\{T_\sigma \mid \sigma \in S'\}$ of S. Moreover the first eigenmatrix P is decomposed into submatrices indexed by $T_\sigma \times \delta$ for $\sigma \in S', \delta \in \Delta$. In this case, it is clear that $c^s = 1$ for all $s \in S$ and $O = \emptyset$.

Thus Theorem 3.2.4 for commutative association schemes is specialized as follows.

Corollary 3.3.1 (Bannai-Muzychuk criterion, [1, Lemma 1] and [13])**.** *Let $\mathfrak{X} = (X, \{R_i\}_{i=1}^r)$ be a commutative association schemes, $\{\varepsilon^s \mid s \in S\}$ be its primitive idempotents and P be its first eigenmatix. Then $\mathfrak{X}$ has a fusion scheme $\mathfrak{X} = (X, \{R'_\delta\}_{\delta\in\Delta})$ given by a partition Δ, where $R' = \coprod_{a\in\delta} R_a$ for $\delta \in \Delta$, if and only if Δ is admissible and there exists a partition $\{T_\sigma \mid \sigma \in S'\}$ of S such that $|S'| = |\Delta|$ and for any $\sigma \in S', \delta \in \Delta$, row sums of the submatrix indexed by $T_\sigma \times \delta$ of P is constant.*

3.3.2 Trivial fusion configurations with the same fibers

Any fiber-commutative coherent configuration has a trivial fusion configuration with the same fibers. Let $\mathfrak{X} = (X, \{R_{i,j,a}\}_{i,j,a})$ be a fiber-commutative coherent configuration and $\mathfrak{A} = \bigoplus_{i,j\in F} \mathfrak{A}_{i,j}$ be its adjacency algebra. Define an admissible partition $\Delta = \{\Delta_{i,j} \mid i, j \in F\}$ as follows:

(i) for $i \in F, \Delta_{i,i} = \{\{1\}, \{2\}, \ldots, \{r_{i,i}\}\}$,

(ii) for $i, j \in F$ $(i \neq j)$, $\Delta_{i,j} = \{\{1, 2, \ldots, r_{i,j}\}\}$.

By the definition of Δ, it is trivial that Δ gives a subalgebra $\mathfrak{A}' = \bigoplus_{i,j\in F} \mathfrak{A}'_{i,j}$ such that $\mathfrak{A}'_{i,i} = \mathfrak{A}_{i,i}$ and $\mathfrak{A}'_{i,j} = \langle \sum_{a=1}^{r_{i,j}} A_{i,j,a} \rangle_{\mathbb{C}}$ for $i \neq j$ and $\mathfrak{A}'$ is the adjacency algebra of a fusion configuration $\mathfrak{X}'$ with the same fibers as those of $\mathfrak{X}$. By Theorem 3.2.4, the edge set E of the bipartite graph G is decomposed into

$$E = (F \times \{s_0\}) \sqcup \coprod_{\substack{(i,s)\in E \\ s\neq s_0}} \{(i, s)\}$$

where $\mathfrak{C}_{s_0} = \langle \sum_{a=1}^{r_{i,j}} A_{i,j,a} \mid i, j \in F \rangle_{\mathbb{C}}$ is the simple two-sided ideal of $\mathfrak{A}$ corresponding to $s_0 \in S$. In other words, Both $\mathfrak{A}$ and $\mathfrak{A}'$ have $\mathfrak{C}_{s_0}$ as a simple two-sided ideal. In this case, for any $i, j \in F$ $(i \neq j)$, $|S'_{i,i}| = |S_{i,i}|$ and $O_{i,j} = S_{i,j} \backslash \{s_0\}$ hold.

3.3.3 The fiber-commutative coherent configuration given by $\mathbb{Z}_3^4 \rtimes S_6$

There is a unique primitive permutation group $\mathcal{G}$ of degree 81 of the form $\mathcal{G} \simeq \mathbb{Z}_3^4 \rtimes S_6$, where $\mathbb{Z}_3$ is the cyclic group and S_6 is the symmetric group on 6 letters.

Since $\mathcal{G}$ has nontrivial outer automorphisms, we fix an outer automorphism x. Let $\mathcal{G}' = \{(g, g^x) \mid g \in \mathcal{G}\}$ be a permutation group of degree 162. Then the set of all orbits of $\mathcal{G}$ gives a fiber-commutative coherent configuration $\mathfrak{X} = (X, \{R_{i,j,a}\}_{i,j,a})$ with $F = \{1, 2\}$ and $r_{1,1} = r_{2,2} = 4, r_{1,2} = r_{2,1} = 3$. Moreover the adjacency algebra $\mathfrak{A}$ can be decomposed into

$$\mathfrak{A} = \bigoplus_{i=0}^{4} \mathfrak{C}_{s_i},$$

where $\mathfrak{C}_{s_i} \simeq \mathrm{M}_2(\mathbb{C})$ for $i = 0, 1, 2$ and $\mathfrak{C}_{s_i} \simeq \mathbb{C}$ for $i = 3, 4$. By this decomposition, we may write $F_{s_0} = F_{s_1} = F_{s_2} = \{1, 2\}, F_{s_3} = \{1\}, F_{s_4} = \{2\}$. The first eigenmatrix $P = (P_{i,j})$ can be written as

$$P_{1,1} = \begin{array}{c} \\ s_0 \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 30 & 20 & 30 \\ 1 & -6 & 2 & 3 \\ 1 & 3 & -7 & 3 \\ 1 & 3 & 2 & -6 \end{pmatrix} \end{array}, P_{2,2} = \begin{array}{c} \\ s_0 \\ s_1 \\ s_2 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 30 & 20 & 30 \\ 1 & -6 & 2 & 3 \\ 1 & 3 & -7 & 3 \\ 1 & 3 & 2 & -6 \end{pmatrix} \end{array}$$

$$P_{1,2} = P_{2,1} = \begin{array}{c} \\ s_0 \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 15 & 60 & 6 \\ 3 & -6 & 3 \\ -6 & 3 & 3 \end{pmatrix} \end{array}.$$

Note that $A_{i,i,a}$ are symmetric for all $i \in \{1,2\}, a \in \{1,2,3,4\}$ and $A_{1,2,a}^T = A_{2,1,a}$ hold for all $a \in \{1,2,3\}$.

For $\mathfrak{X}$, let $S' = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ and, we define $\Delta = \{\Delta_{i,j} \mid i,j \in \{1,2\}\}$ and T_σ, F_σ for $\sigma \in S'$ as follows:

$$\begin{aligned} \Delta_{1,1} &= \Delta_{2,2} = \{\{1\},\{2,3\},\{4\}\}, \\ \Delta_{1,2} &= \Delta_{2,1} = \{\{1,2\},\{3\}\}, \\ T_{\sigma_0} &= \{s_0\}, F_{\sigma_0} = \{1,2\}, \\ T_{\sigma_1} &= \{s_1, s_2\}, F_{\sigma_1} = \{1,2\}, \\ T_{\sigma_2} &= \{s_3\}, F_{\sigma_2} = \{1\}, \\ T_{\sigma_3} &= \{s_4\}, F_{\sigma_3} = \{2\}. \end{aligned}$$

Then $\Delta, T_\sigma, F_\sigma$ satisfy the conditions in Theorem 3.2.4 with $c_{i,j}^s = 1$ for all $i,j \in \{1,2\}, s \in S_{i,j}$. In this case, $O_{1,2} = O_{2,1}$ are empty. Thus a fusion configuration with the same fibers as those of $\mathfrak{X}$ is obtained and its eigenmatrix $P' = (P'_{i,j})$ is

$$P'_{1,1} = \begin{array}{c} \\ \sigma_0 \\ \sigma_1 \\ \sigma_2 \end{array} \begin{array}{c} \begin{array}{ccc} \{1\} & \{2,3\} & \{4\} \end{array} \\ \begin{pmatrix} 1 & 50 & 30 \\ 1 & -4 & 3 \\ 1 & 5 & -6 \end{pmatrix} \end{array}, P'_{2,2} = \begin{array}{c} \\ \sigma_0 \\ \sigma_1 \\ \sigma_3 \end{array} \begin{array}{c} \begin{array}{ccc} \{1\} & \{2,3\} & \{4\} \end{array} \\ \begin{pmatrix} 1 & 50 & 30 \\ 1 & -4 & 3 \\ 1 & 5 & -6 \end{pmatrix} \end{array}$$

$$P'_{1,2} = P'_{2,1} = \begin{array}{c} \\ \sigma_0 \\ \sigma_1 \end{array} \begin{array}{c} \begin{array}{cc} \{1,2\} & \{3\} \end{array} \\ \begin{pmatrix} 75 & 6 \\ -3 & 3 \end{pmatrix} \end{array}.$$

Moreover, $\mathfrak{A}$ has the following subalgebra $\mathfrak{A}''$. Since this subalgebra is not closed with respect to the transpose, this subalgebra is not an adjacency algebra of any fusion configuration. For $i = 1, 2$, we construct subalgebras $\mathfrak{A}''_{i,i} \subset \mathfrak{A}_{i,i}$ which is the same as above; i.e.

$$\begin{aligned} \mathfrak{A}''_{i,i} &= \langle A_{i,i,1}, A_{i,i,2} + A_{i,i,3}, A_{i,i,4} \rangle_{\mathbb{C}} \\ &= \langle \varepsilon_{i,i}^{s_0}, \varepsilon_{i,i}^{s_1} + \varepsilon_{i,i}^{s_2}, \varepsilon_{i,i}^{s_\alpha} \rangle_{\mathbb{C}}, \end{aligned}$$

where $\alpha = 3$ if $i = 1$ and $\alpha = 4$ if $i = 2$. Thus the transition matrices with respect to these bases are $P''_{i,i} = P'_{i,i}$ for $i = 1, 2$. On the other hand, in $\mathfrak{A}_{1,2}, \mathfrak{A}_{2,1}$, we construct subspaces $\mathfrak{A}''_{1,2}, \mathfrak{A}''_{2,1}$ as follows;

$$\begin{aligned} \mathfrak{A}''_{1,2} &= \langle A_{1,2,1} + A_{1,2,3}, A_{1,2,2} \rangle_{\mathbb{C}} \\ &= \langle \varepsilon_{1,2}^{s_0}, \frac{1}{\sqrt{2}}(\varepsilon_{1,1}^{s_1} - 2\varepsilon_{1,2}^{s_2}) \rangle_{\mathbb{C}}, \\ \mathfrak{A}''_{2,1} &= \langle A_{2,1,1}, A_{2,1,2} + A_{2,1,3} \rangle_{\mathbb{C}} \\ &= \langle \varepsilon_{2,1}^{s_0}, \frac{1}{\sqrt{2}}(2\varepsilon_{1,1}^{s_1} - \varepsilon_{1,2}^{s_2}) \rangle_{\mathbb{C}}. \end{aligned}$$

Note that the transition matrices with respect to these bases are

$$P''_{1,2} = \begin{array}{c} \\ \sigma_0 \\ \sigma_1 \end{array} \begin{array}{c} \begin{array}{cc} \{1,3\} & \{2\} \end{array} \\ \begin{pmatrix} 21 & 60 \\ 6\sqrt{2} & -6\sqrt{2} \end{pmatrix} \end{array}, \quad P''_{2,1} = \begin{array}{c} \\ \sigma_0 \\ \sigma_1 \end{array} \begin{array}{c} \begin{array}{cc} \{1\} & \{2,3\} \end{array} \\ \begin{pmatrix} 15 & 66 \\ 6\sqrt{2} & -6\sqrt{2} \end{pmatrix} \end{array},$$

where σ_0, σ_1 correspond to $\{s_0\}, \{s_1, s_2\}$, respectively. Then

$$\mathfrak{A}'' = \bigoplus_{i,j=1}^{2} \mathfrak{A}''_{i,j}$$

is closed with respect to the matrix multiplication and this is a subalgebra of $\mathfrak{A}$ which is not closed with respect to the transpose.

This example shows that adjacency algebras may have a subalgebra which are closed with respect to the Hadamard product but fails to be an adjacency algebra because it is not closed under the transpose.

Chapter 4

Nearly multiplicity-free imprimitive permutation groups

4.1 The multiplicity-free condition

Let G be a permutation group on a finite set X and let Λ_i be all the orbits of G on $X \times X$. Then $\mathfrak{X} = (X, \{\Lambda_i\}_{i=0}^d)$ is a Schurian scheme. In general, $\mathfrak{X}$ may not be commutative.

Fix a point $x \in X$, and let $K = \mathrm{Stab_G(x)}$ be the stabilizer of x in G. Then G/K can be identified with X and the action of G on X corresponds to the coset action of G on G/K.

Lemma 4.1.1 (The multiplicity-free condition, [2, Theorem 1.4 in Chapter II])**.** *Let $\mathbf{1}_K^G$ be the permutation character of G on X. Suppose that $\mathbf{1}_K^G$ is decomposed into $\mathbf{1}_K^G = \sum_{s=0}^r e_s\chi_s$, where χ_s are irreducible characters of G and e_s are positive integers. Then the following are equivalent:*

(i) *for all $i \in \{0, 1, \ldots, d\}, e_s = 1$ hold,*

(ii) *$\mathfrak{X}$ is commutative.*

Moreover, if one of the equivalent conditions holds, then $r = d$ follows.

Let $\mathfrak{A}$ be the adjacency algebra of a Schurian scheme of G on X and $\mathbb{C}^X$ be the permutation module.

Theorem 4.1.2 ([2, Theorem 1.3 in chapter II])**.** *Let $\mathfrak{A}$ be the adjacency algebra of $\mathfrak{X}$. Then $\mathfrak{A} \simeq \mathrm{End}_G(\mathbb{C}^X)$ holds as algebras.*

Suppose that the permutation module $\mathbb{C}^X$ is decomposed into

$$\mathbb{C}^X = \bigoplus_{s=0}^{r} V_s = \bigoplus_{s=0}^{r} \bigoplus_{i=1}^{e_s} V_{s,i},$$

where V_s are isotypic components and $V_{s,i}$ are irreducible modules. By this irreducible decomposition,

$$\mathfrak{A} \simeq \mathrm{End}_G(\mathbb{C}^X) = \bigoplus_{s=0}^{r} \mathrm{End}_G(V_s) = \bigoplus_{s=0}^{r} \bigoplus_{i,j=1}^{e_s} \mathrm{Hom}_G(V_{s,i}, V_{s,j}),$$

where $(\mathrm{Hom}_G(V_{s,i}, V_{s,j}))_{i,j}$ is an $e_s \times e_s$ matrix for each $s \in \{0, 1, \ldots, r\}$. By Schur's lemma, $\mathrm{Hom}_G(V_{s,i}, V_{s,j})$ span 1-dimensional subspaces for all $s \in \{0, 1, \ldots, r\}, i, j \in \{1, 2, \ldots, e_s\}$. Thus we obtain $\mathrm{Hom}_G(V_{s,i}, V_{s,i}) \simeq \mathrm{M}_{e_s}(\mathbb{C})$ and

$$\mathfrak{A} \simeq \mathrm{End}_G(\mathbb{C}^X) = \bigoplus_{s=0}^{r} \mathrm{End}_G(V_s) \simeq \bigoplus_{s=0}^{r} \mathrm{M}_{e_s}(\mathbb{C}). \tag{4.1}$$

Lemma 4.1.3 ([4, Section 3.4])**.** *Let ρ be the permutation representation of G on G/K. Set*

$$E_s = \frac{\deg(\chi_s)}{|G|} \sum_{g \in G} \overline{\chi_s(g)} \rho(g)$$

for every $s = 0, 1, \ldots, r$. Then each E_i is the central idempotent corresponding to V_s. In particular, if the Schurian scheme $\mathfrak{X}$ of G on G/K is commutative, then $E_0, E_1, \ldots, E_r$ are all the primitive idempotents of $\mathfrak{X}$.

4.2 Imprimitive association schemes

Let G be a transitive permutation group on a finite set X and $K = \mathrm{Stab}_G(x)$ be a the stabilizer of G for a fixed point $x \in X$.

Definition 4.2.1. A transitive permutation group G on X is called *imprimitive* if there exists a subset $Y \subset X$ with $1 < |Y| < |X|$ such that $Y^g = Y$ or $Y^g \cap Y = \emptyset$ for all $g \in G$. The subset Y is called a *block*. The set $\{Y^g \mid g \in G\}$ is a partition of X and called *the system of imprimitivity*. If there does not exist no trivial block, then G is called *primitive*.

Proposition 4.2.2 ([2, Proposition 9.1 in Section 2.9])**.** *The stabilizer* $K = \mathrm{Stab}_G(x)$ *is a maximal subgroup of* G *if and only if* G *is primitive.*

In the rest of this section, we assume that G is an imprimitive permutation group on X. Then there exists a subgroup H satisfying $K < H < G$. Since G acts on G/K transitively, G also acts on G/H transitively. By $K < H$, G/H gives a partition on G/K and it is a system of imprimitivity.

Let $\mathfrak{X} = (G/K, \{\Lambda_i\}_{i=0}^{d})$ be the Schurian scheme of G on G/K and $\mathfrak{X}' = (G/H, \{\Lambda_i'\}_{i=0}^{d'})$ be the Schurian scheme of G on G/H. Let $\mathfrak{A} = \langle A_i \mid i \in \{0, 1, \ldots, d\}\rangle_{\mathbb{C}}$ be the adjacency algebra of $\mathfrak{X}$ and $\mathfrak{A}' = \langle A_i' \mid i \in \{0, 1, \ldots, d'\}\rangle_{\mathbb{C}}$ be the adjacency algebra of $\mathfrak{X}'$.

Lemma 4.2.3. *Under the above notation,* $\mathfrak{A}'$ *can be identified with a subalgebra of* $\mathfrak{A}$*. In particular, by a suitable rearrangement of indices,* $\mathfrak{A}' \otimes J_q = \{A \otimes J_q \mid A \in \mathfrak{A}'\}$ *is a subalgebra of* $\mathfrak{A}$*.*

Proof. For $A' \in \mathfrak{A}'$, define $A \in \mathfrak{A}$ as $A_{gK,hK} = A'_{g'H,h'H}$ for $gK \subset g'H, hK \subset h'H$, then the result follows. Or equivalently, by a suitable rearrangement of indices, $A' \otimes J_q \in \mathfrak{A}$ hold, where $q = |H|/|K|$. □

Corollary 4.2.4. *For the adjacency matrices* $A_0, A_1, \ldots, A_d$ *of* $\mathfrak{X}$*, there exists a partition* Δ *of* $\{0, 1, \ldots, d\}$ *such that* $\mathfrak{A}' \otimes J_q = \langle \sum_{i\in\delta} A_i \mid \delta \in \Delta\rangle_{\mathbb{C}}$*, where* $\mathfrak{A}' \otimes J_q = \{A \otimes J_q \mid A \in \mathfrak{A}'\}$*.*

Proof. By Lemma 4.2.3, for any adjacency matrix A_i' of $\mathfrak{X}'$ is in $\mathfrak{A}$. Since $A_i' \otimes J_q$ is a $\{0, 1\}$-matrix, there exists a subset $\delta_i \subset \{0, 1, \ldots, d\}$ such that $A_i' \otimes J_q = \sum_{j\in\delta_i} A_j$. By $\sum_{i=0}^{d'} A_i' \otimes J_q = \sum_{j=0}^{d} A_j = J$, $\{\delta_i \mid i \in \{0, 1, \ldots, d'\}\}$ is a partition of $\{0, 1, \ldots, d\}$. □

Definition 4.2.5. For an association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$, $\mathfrak{X}'$ is called *a quotient scheme of* $\mathfrak{X}$ if there exists a partition of $\{0, 1, \ldots, d\}$ and a positive integer q such that, for the adjacency algebra $\mathfrak{A}'$ of $\mathfrak{X}$, $\mathfrak{A}' \otimes J_q = \langle \sum_{i\in\delta} A_i \mid \delta \in \Delta\rangle_{\mathbb{C}}$ holds. Moreover, $\mathfrak{X}$ is called an *imprimitive association scheme.*

Note that quotient and imprimitive association schemes are defined regardless of whether association schemes are Schurian or not.

4.3 The nearly multiplicity-free condition

Let G be a imprimitive permutation group on a finite set X with subsets $K < H < G$, where $K = \mathrm{Stab}_G(x)$ for a fixed point $x \in X$.

Definition 4.3.1. An imprimitive permutation group G with subgroups $K < H < G$ is called *nearly multiplicity-free* if Both $\mathbf{1}_H^G$ and $\mathbf{1}_K^G - \mathbf{1}_H^G$ are multiplicity-free.

In this section, suppose that G with $K < H < G$ is nearly multiplicity-free. Let $\mathfrak{X} = (G/K, \{\Lambda_i\}_{i=0}^d)$ be the Schurian scheme of G on G/K and $\mathfrak{X}' = (G/H, \{\Lambda_i'\}_{i=0}^{d'})$ be the Schurian scheme of G on G/H. Since $\mathbf{1}_H^G$ is multiplicity-free, $\mathfrak{X}'$ is commutative. Thus the adjacency algebra $\mathfrak{A}'$ of $\mathfrak{X}'$ has primitive idempotents. On the other hand, $\mathfrak{X}$ in a non-commutative association scheme.

Let

$$\mathbf{1}_K^G = \sum_{s=0}^{r} e_s \chi_s$$

be the permutation character of G. Since G is nearly multiplicity-free, $e_s \in \{1, 2\}$ for all $s \in \{0, 1, \ldots, r\}$, and if $e_s = 2$, then $(\mathbf{1}_H^G, \chi_s) = (\mathbf{1}_K^G - \mathbf{1}_H^G, \chi_s) = 1$.

By (4.1), if $e_s = 2$, then

$$\mathrm{End}_G(V_s) = \bigoplus_{i,j=1}^{2} \mathrm{Hom}_G(V_{k,i}, V_{s,j}).$$

Since each $\mathrm{Hom}_G(V_{s,i}, V_{s,j})$ is a 1-dimensional subspace, we can take $\varepsilon_{i,i}^s \in \mathfrak{A}$ corresponding to the basis of $\mathrm{End}_G(V_{s,i})$ for $i = 1, 2$ as follows: $\mathrm{End}_G(V_{s,i}) \ni \mathrm{Id}_{V_{s,i}} \mapsto \varepsilon_{i,i}^s \in \mathfrak{A}$.

Since $(\mathbf{1}_H^G, \chi_s) = 1$, we can assume that $V_{s,1} \subset \mathbb{C}^{G/H} \subset \mathbb{C}^{G/K} = \mathbb{C}^X$. Moreover, since $\mathfrak{X}'$ is commutative, there is a primitive idempotent E_s' of $\mathfrak{X}'$ corresponding to $V_{s,1}$. By Lemma 4.2.3, $E_s' \otimes J_q \in \mathfrak{A}$ holds where $q = |H|/|K|$ and we can write

$$\varepsilon_{1,1}^s = \frac{1}{q}(E_s' \otimes J_q).$$

On the other hand, by Lemma 4.1.3, $E_s \in \mathfrak{A}$ is the central idempotent corresponding to V_s. Thus we obtain

$$\varepsilon_{2,2}^s = E_s - \varepsilon_{1,1}^s.$$

Moreover, for $i \neq j$, since $\mathrm{Hom}_G(V_{s,i}, V_{s,j})$ is a 1-dimensional subspace, $\varepsilon_{i,i}^s \mathfrak{A} \varepsilon_{j,j}^s$ is also a 1-dimensional subspace. Thus $\varepsilon_{i,j}^s \in \mathfrak{A}$ satisfying $\varepsilon_{i,j}^s {\varepsilon_{i,j}^s}^* = \varepsilon_{i,i}^s$ is uniquely determined up to complex scalar multiple with absolute 1. Note that ${\varepsilon_{i,j}^s}^* = \varepsilon_{j,i}^s$ holds.

If $e_s = 1$, then by Lemma 4.1.3, we write $\varepsilon_{1,1}^s = E_s$.

Definition 4.3.2. The set $\{\varepsilon^s_{i,j} \mid 0 \le s \le r, 1 \le i, j \le e_s\}$ is called *bases of matrix units* for $\mathfrak{A}$ and this is a basis of $\mathfrak{A}$.

Note that, by the construction of the bases of matrix units, $\varepsilon^s_{i,i}$ are determined uniquely and $\varepsilon^s_{i,j}$ $(i \neq j)$ are determined uniquely up to complex scalars with its absolute 1 for $0 \le s \le r, 1 \le i, j \le e_s$.

Thus we may define the first and second eigenmatrices for the non-commutative association scheme $\mathfrak{X}$.

Definition 4.3.3. Let

$$A_k = \sum_{s=0}^{r} \sum_{i,j=1}^{e_s} p_k(s,i,j)\varepsilon^s_{i,j},$$

$$\varepsilon^s_{i,j} = \frac{1}{|G/K|} \sum_{k=0}^{d} q_{i,j,s}(k) A_k.$$

The *first and second eigenmatrices* P, Q *of* $\mathfrak{X}$ are defined by

$$P = (p_k(s,i,j)), Q = (q_{i,j,s}(k)).$$

It is clear that $PQ = QP = |G/K| I_{d+1}$.

By Lemma 1.2.4, the Krein conditions for these imprimitive association schemes are as follows.

Theorem 4.3.4. *Let $G > H > K$ be an imprimitive permutation group satisfying nearly multiplicity-free and $\mathfrak{A}$ be the adjacency algebra of the Schurian scheme $(G/K, \{\Lambda_i\}_{i=0}^d)$. Let $\{\varepsilon^s_{i,j} \mid 0 \le s \le r, 1 \le i, j \le e_s\}$ be the bases of matrix units for $\mathfrak{A}$. Then, for any $1 \le s, t \le r$,*

$$\left(\sum_{i,j=1}^{e_s} b_{i,j}\varepsilon^s_{i,j} \right) \circ \left(\sum_{i,j=1}^{e_t} c_{i,j}\varepsilon^t_{i,j} \right)$$

is positive semidefinite, where $B = (b_{i,j}) \in \mathrm{M}_{e_s}(\mathbb{C}) C = (c_{i,j}) \in \mathrm{M}_{e_t}(\mathbb{C})$ are positive semidefinite matrices.

4.4 Examples

In this section, we construct two examples satisfying the nearly multiplicity-free condition: the symmetric groups acting on ordered pairs and the dihedral groups acting on themselves.

4.4.1 Symmetric groups on ordered pairs

Let S_n be the symmetric group on n letters and consider $S_{n-2} < S_2 \times S_{n-2} < S_n$. Then S_n acts on $S_n/S_2 \times S_{n-2}$ and S_n/S_{n-2} transitively and, by the action on S_n/S_{n-2}, we have a Schurian scheme $(X, \{\Lambda_i\}_{i=0}^6)$, where $X = \{(i,j) \mid i,j \in \{1,2,\ldots,n\}, i \neq j\}$ and

$$\begin{aligned}
\Lambda_0 &= \{((i,j),(i,j)) \mid i,j \in \{1,2,\ldots,n\}, i \neq j\},\\
\Lambda_1 &= \{((i,j),(j,i)) \mid i,j \in \{1,2,\ldots,n\}, i \neq j\},\\
\Lambda_2 &= \{((i,j),(i,k)) \mid i,j,k \in \{1,2,\ldots,n\}, i,j,k : \text{distinct}\},\\
\Lambda_3 &= \{((i,j),(k,i)) \mid i,j,k \in \{1,2,\ldots,n\}, i,j,k : \text{distinct}\},\\
\Lambda_4 &= \{((i,j),(k,j)) \mid i,j,k \in \{1,2,\ldots,n\}, i,j,k : \text{distinct}\},\\
\Lambda_5 &= \{((i,j),(j,k)) \mid i,j,k \in \{1,2,\ldots,n\}, i,j,k : \text{distinct}\},\\
\Lambda_6 &= \{((i,j),(k,l)) \mid i,j,k,l \in \{1,2,\ldots,n\}, i,j,k,l : \text{distinct}\}.
\end{aligned}$$

The permutation characters are decomposed into

$$\begin{aligned}
\mathbf{1}_{S_2\times S_{n-2}}^{S_n} &= \chi_n + \chi_{n-1,1} + \chi_{n-2,2}\\
\mathbf{1}_{S_{n-2}}^{S_n} &= \chi_n + 2\chi_{n-1,1} + \chi_{n-2,2} + \chi_{n-1,1,1}\\
&= (\chi_n + \chi_{n-1,1} + \chi_{n-2,2}) + (\chi_{n-1,1} + \chi_{n-1,1,1}),
\end{aligned}$$

where χ_λ is the irreducible character of S_n corresponding to the Young tableau λ. Note that the Schurian scheme of S_n on $S_n/S_2 \times S_{n-2}$ is the Johnson scheme $J(n,2)$. By the irreducible decompositions, S_n with $S_{n-2} < S_2 \times S_{n-2} < S_n$ is nearly multiplicity-free.

Thus the first eigenmatrix of the Schurian scheme of S_n on S_n/S_{n-2} is

$$P = \begin{array}{c} \\ \varepsilon^0_{i,j} \\ \varepsilon^1_{(1,1)} \\ \varepsilon^1_{1,2} \\ \varepsilon^1_{2,1} \\ \varepsilon^1_{2,2} \\ \varepsilon^2_{1,1} \\ \varepsilon^3_{1,1} \end{array}
\begin{array}{c} \begin{array}{ccccccc} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array} \\
\begin{pmatrix}
1 & 1 & n-2 & n-2 & n-2 & n-2 & (n-2)(n-3)\\
1 & 1 & \frac{n-4}{2} & \frac{n-4}{2} & \frac{n-4}{2} & \frac{n-4}{2} & -2(n-3)\\
0 & 0 & m & -m & -m & m & 0\\
0 & 0 & m & m & -m & -m & 0\\
1 & -1 & \frac{n-2}{2} & -\frac{n-2}{2} & \frac{n-2}{2} & -\frac{n-2}{2} & 0\\
1 & 1 & -1 & -1 & -1 & -1 & 2\\
1 & -1 & -1 & 1 & -1 & 1 & 0
\end{pmatrix}\end{array},$$

where $m = \sqrt{n(n-2)}/2$.

4.4.2 Thin schemes of dihedral groups

Let D_n be the dihedral group on the regular n-gon:

$$D_n = \langle \sigma, \tau \mid \sigma\tau = \tau\sigma^{n-1}, \tau^2 = \sigma^n = 1\rangle,$$

where σ, τ denote the rotation and reflection, respectively. Let $H = \langle \tau \rangle$. Then D_n/H is isomorphic to the cyclic group of order n. In this subsection, we prove that D_n with subgroups $\{1\} < H < D_n$ is nearly multiplicity-free and construct the bases of matrix units.

Let $\mathfrak{A} = \mathbb{C}D_n$ be the group ring of D_n over $\mathbb{C}$. In other words, $\mathfrak{A}$ is the adjacency algebra of the thin scheme D_n. Since D_n is a non-commutative group, $\mathfrak{A}$ is not commutative. Then the set of all elements of D_n can be identified as a basis of $\mathfrak{A}$.

Since irreducible characters of D_n are different whether n is odd or even, we consider each case individually.

The case n is odd:

The set of representative of conjugacy classes of D_n is $\{1, \sigma, \sigma^2, \dots, \sigma^{(n-1)/2}, \tau\}$ and the character table for D_n is given by as follows:

	1	σ^i	τ
size	1	2	n
$\mathbf{1}$	1	1	1
ρ	1	1	-1
θ_k	2	$\omega^{ik} + \omega^{-ik}$	0

for $k = 1, 2, \dots, (n-1)/2$ and $i = 1, 2, \dots, (n-1)/2$, where $\mathbf{1}$ is the trivial character, ρ is the signed character and θ_k are all the other irreducible characters and, $\omega = e^{2\pi\sqrt{-1}/n}$.

For subgroups $\{1\} < H < D_n$, D_n acts on D_n and D_n/H. Since the action on itself is regular, the permutation character is

$$\mathbf{1}^{D_n} = \mathbf{1} + \rho + 2\sum_{k=1}^{(n-1)/2} \theta_k.$$

Moreover

$$\mathbf{1}_H^{D_n} = \mathbf{1} + \sum_{k=1}^{(n-1)/2} \theta_k.$$

Thus D_n with $\{1\} < H =\rangle < D_n$ is nearly multiplicity-free.

By the decomposition of irreducible characters, the adjacency algebra $\mathfrak{A}$ is decomposed into

$$\mathfrak{A} = \bigoplus_{k=0}^{(n+1)/2} \mathfrak{C}_k,$$

where $\mathfrak{C}_0 \simeq \mathbb{C}$ corresponding to $\mathbf{1}$, $\mathfrak{C}_i \simeq \mathrm{M}_2(\mathbb{C})$ corresponding to θ_k for $k = 1, \ldots, (n-1)/2$ and $\mathfrak{C}_{(n+1)/2} \simeq \mathbb{C}$ corresponding to ρ. By Lemma 4.1.3, the central idempotents are

$$E_0 = \frac{1}{2n}\sum_{i=0}^{n-1}(\sigma^i + \sigma^i\tau),$$

$$E_k = \frac{1}{n}\sum_{i=0}^{n-1}(\omega^{ik} + \omega^{-ik})\sigma^i \quad (k = 1, \ldots, (n-1)/2),$$

$$E_{(n+1)/2} = \frac{1}{2n}\sum_{i=0}^{n-1}(\sigma^i - \sigma^i\tau),$$

where E_k corresponds to $\mathfrak{C}_k$ for $k = 0, 1, \ldots, (n+1)/2$.

By the multiplicities of irreducible characters, for $k = 0, (n+1)/2$, we can write $\varepsilon^k_{1,1} = E_k$. By the action of D_n on D_n/H, we have

$$\varepsilon^k_{1,1} = \frac{1}{2n}\sum_{i=0}^{n-1}(\omega^{ik} + \omega^{-ik})(\sigma^i + \sigma^i\tau)$$

$$\varepsilon^k_{2,2} = \frac{1}{2n}\sum_{i=0}^{n-1}(\omega^{ik} + \omega^{-ik})(\sigma^i - \sigma^i\tau),$$

for $k = 1, \ldots, (n-1)/2$.

Moreover, by these elements, we also have

$$\varepsilon^k_{1,2} = \frac{1}{2n}\frac{1}{\sqrt{\omega^{2k} + \omega^{-2k} + 2}}\sum_{i=0}^{n-1}(\omega^{(i-1)k} + \omega^{-(i-1)k} + \omega^{(i+1)k} + \omega^{-(i+1)k})(\sigma^i - \omega^i\tau)$$

$$\varepsilon^k_{2,1} = \frac{1}{2n}\frac{1}{\sqrt{\omega^{2k} + \omega^{-2k} + 2}}\sum_{i=0}^{n-1}(\omega^{(i-1)k} + \omega^{-(i-1)k} + \omega^{(i+1)k} + \omega^{-(i+1)k})(\sigma^i + \omega^i\tau),$$

for $k = 1, \ldots, (n-1)/2$.

The case n is even:

The set of representative of conjugacy classes of D_n is $\{1, \sigma, \sigma^2, \ldots, \sigma^{n/2-1}, \tau, \sigma\tau\}$ and the character table for D_n is given by as follows:

	1	σ^i	τ	$\sigma\tau$
size	1	α	$n/2$	$n/2$
$\mathbf{1}$	1	1	1	1
ρ_1	1	1	-1	-1
ρ_2	1	-1	1	-1
ρ_3	1	-1	-1	1
θ_k	2	$\omega^{ik} + \omega^{-ik}$	0	0

for $k = 1, 2, \ldots, n/2 - 1$ and $i = 1, 2, \ldots, n/2$, where $\alpha = 1$ if $i = n/2$ and $\alpha = 2$ otherwise.

As is the case with n odd, the permutation characters are

$$\mathbf{1}^{D_n} = \mathbf{1} + \rho_1 + \rho_2 + \rho_3 + 2 \sum_{k=1}^{n/2-1} \theta_k,$$

$$\mathbf{1}_H^{D_n} = \mathbf{1} + \rho_2 + \sum_{k=1}^{n/2-1} \theta_k.$$

Then the adjacency algebra $\mathfrak{A}$ is decomposed into

$$\mathfrak{A} = \bigoplus_{k=0}^{n/2+2} \mathfrak{C}_k,$$

where $\mathfrak{C}_0 \simeq \mathbb{C}$ corresponding to $\mathbf{1}$, $\mathfrak{C}_i \simeq \mathrm{M}_2(\mathbb{C})$ corresponding to θ_k for $k = 1, \ldots, n/2 - 1$, $\mathfrak{C}_{n/2} \simeq \mathbb{C}$ corresponding to ρ_1, $\mathfrak{C}_{n/2+1} \simeq \mathbb{C}$ corresponding to ρ_2 and $\mathfrak{C}_{n/2+2} \simeq \mathbb{C}$ corresponding to ρ_3.

By Lemma 4.1.3, the central idempotents are

$$
\begin{aligned}
E_0 &= \frac{1}{2n}\sum_{i=0}^{n-1}(\sigma^i + \sigma^i\tau),\\
E_k &= \frac{1}{n}\sum_{i=0}^{n-1}(\omega^{ik} + \omega^{-ik})\sigma^i \quad (k = 1, \dots, n/2-1),\\
E_{n/2} &= \frac{1}{2n}\sum_{i=0}^{n-1}(\sigma^i - \sigma^i\tau),\\
E_{n/2+1} &= \frac{1}{2n}\left(1 + \sum_{i=1}^{n-1}(-\sigma^i + (-1)^i\sigma^i\tau)\right),\\
E_{n/2+2} &= \frac{1}{2n}\left(1 + \sum_{i=1}^{n-1}(-\sigma^i + (-1)^{i+1}\sigma^i\tau)\right),
\end{aligned}
$$

where E_k corresponds to $\mathfrak{C}_k$ for $k = 0, 1, \dots, n/2+2$.

The following are essentially the same as the case of odd: By the multiplicities of irreducible characters, for $k = 0, (n+1)/2$, we have $\varepsilon^k_{1,1} = E_k$. By the action of D_n on D_n/H, we have

$$
\begin{aligned}
\varepsilon^k_{1,1} &= \frac{1}{2n}\sum_{i=0}^{n-1}(\omega^{ik} + \omega^{-ik})(\sigma^i + \sigma^i\tau)\\
\varepsilon^k_{2,2} &= \frac{1}{2n}\sum_{i=0}^{n-1}(\omega^{ik} + \omega^{-ik})(\sigma^i - \sigma^i\tau),
\end{aligned}
$$

for $k = 1, \dots, n/2-1$.

Moreover, by these elements, we also have

$$
\begin{aligned}
\varepsilon^k_{1,2} &= \frac{1}{2n}\frac{1}{\sqrt{\omega^{2k} + \omega^{-2k} + 2}}\sum_{i=0}^{n-1}(\omega^{(i-1)k} + \omega^{-(i-1)k} + \omega^{(i+1)k} + \omega^{-(i+1)k})(\sigma^i - \omega^i\tau)\\
\varepsilon^k_{2,1} &= \frac{1}{2n}\frac{1}{\sqrt{\omega^{2k} + \omega^{-2k} + 2}}\sum_{i=0}^{n-1}(\omega^{(i-1)k} + \omega^{-(i-1)k} + \omega^{(i+1)k} + \omega^{-(i+1)k})(\sigma^i + \omega^i\tau),
\end{aligned}
$$

for $k = 1, \dots, n/2-1$.

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