Study on the second derivative of bounded analytic functions

Dissertation

Chen Gangqiang

Computer and Mathematical Sciences
Graduate School of Information Sciences
Tohoku University

Abstract

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Assume that a point $z_0 \in \mathbb{D}$ and f is an analytic self-map of $\mathbb{D}$ fixing 0. Schwarz's lemma asserts that $|f(z_0)| \leq |z_0|$, and Dieudonné's lemma derives an inequality about the derivative $f'(z_0)$, $|f'(z_0)| \leq \min\{1,\ (1+|z_0|^2)/(4|z_0|(1-|z_0|^2))\}$, which is best possible for each value of $|z_0|$. In this paper, we shall obtain a sharp upper bound for the second derivative $f''(z_0)$ depending only on $|z_0|$.

Furthermore, assume that $w_0 \in \mathbb{D}$ with the modulus of z_0 greater than that of w_0 and denote by $\mathcal{H}_0$ the set of all analytic self-maps of $\mathbb{D}$ that fix the origin. For $c \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}(c, r) = \{z \in \mathbb{C} : |z - c| < r\}$ and $\overline{\mathbb{D}}(c, r) = \{z \in \mathbb{C} : |z - c| \leq r\}$. Schwarz's lemma shows that $\{f(z_0) : f \in \mathcal{H}_0\} = \overline{\mathbb{D}}(0, |z_0|)$. Dieudonné's lemma asserts that

$$\{f'(z_0) : f \in \mathcal{H}_0, f(z_0) = w_0\} = \overline{\mathbb{D}}\left(\frac{w_0}{z_0}, \frac{|z_0|^2 - |w_0|^2}{|z_0|(1-|z_0|^2)}\right).$$

We shall determine the variability region $\{f''(z_0) : f(z_0) = w_0\}$ when f ranges over the class of all analytic self-maps of unit disk fixing 0. We also graphically illustrate our main result by using Mathematica.

Keywords: Bounded analytic functions; Schwarz's lemma; Dieudonné's lemma; variability region.

Contents

1 Introduction

The theory of bounded analytic functions is one of the most important subjects in conformal geometry theory. In 1890, Schwarz proved the classcial Schwarz lemma, which states that, for z in the open unit disk $\mathbb{D}$, $|f(z)| \le |z|$ holds for all analytic self-maps f of $\mathbb{D}$ fixing the origin. It also plays a key role in the development of complex analysis such as conformal geometry, hyperbolic geometry, and so on. Schwarz's lemma studies the analytic self-maps of $\mathbb{D}$ with an interior fixed point, which gives sharp estimates of the values of holomorphic self-mappings of $\mathbb{D}$ and the first derivative at the origin.

In 1915, Pick observed that f does not necessarily fix the origin and proved the famous Schwarz-Pick lemma: if f is an analytic self-map of $\mathbb{D}$, then $h(f(z), f(w)) \le h(z, w)$ for $z, w \in \mathbb{D}$, here h is the hyperbolic metric in the unit disk $\mathbb{D}$ (see also [5]).

For many years the classical results, Schwarz's lemma and the Schwarz-Pick lemma, attract a lot of mathematicians and inspire dozens of books and papers about the refined forms on this topic and more and more extensions and generalizations have appeared. In 1934, Rogosinski [34] established an assertion which can be considered as a sharpened version of Schwarz's lemma. It describes the variability region of $f(z)$ for $z \in \mathbb{D}$ for $f : \mathbb{D} \to \mathbb{D}$ holomorphic, fixing 0 and $|f'(0)| < 1$, proved by calculating the envelop of a certain union of disks [19].

It is natural to consider the estimates of the derivatives $f^{(n)}(z)$, $n \in \mathbb{N}$. In fact, the Schwarz-Pick lemma also states that $|f'(z)| \le (1-|f(z)|^2)/(1-|z|^2)$ for a holomorphic $f : \mathbb{D} \to \mathbb{D}$ and $z \in \mathbb{D}$, and equality holds for some $z \in \mathbb{D}$ if and only if f is a conformal automorphism of $\mathbb{D}$. For brevity and our interest, we denote by $\mathcal{H}$ the set of all analytic self-maps of $\mathbb{D}$, and its subspace $\mathcal{H}_0$ consists of those $f \in \mathcal{H}$ such that $f(0) = 0$. There is also a higher order version of the Schwarz-Pick lemma. More precisely, in 1974, Ruscheweyh [35] conjectured a sharp upper bound for $|f^{(n)}|$ depending on $|f(z)|$ and $|z|$ for $f \in \mathcal{H}, z \in \mathbb{D}$ and proved it in 1985 (see [36]). Since then, estimates of $f^{(n)}(z)$, $n \in \mathbb{N}$, have been investigated by many mathematicians ([14], [18], [40]). Particularly, in 2006, Anderson and Rovnyakuse [2] applied a new method to derive Ruscheweyh's results.

The Schwarz-Pick lemma implies the sharp inequality $|f'(z)| \leq 1/(1-|z|^2)$ for $f \in \mathcal{H}$ and $z \in \mathbb{D}$. This inequality has the upper bound depending only on $|z|$ and equality occurs only for conformal automorphisms f of $\mathbb{D}$ such that $f(z) = 0$. In 1920, Szász [37] extended this inequality to odd order derivatives of $f \in \mathcal{H}$ and gave the form of the extremal mapping. In the same paper, he also obtained a sharp upper bound for $|f''|$ (see also [3]). Whereas, finding sharp upper bounds of even order derivatives of $f \in \mathcal{H}$ is still an open problem.

In 1931, Dieudonné [15] firstly gave an improvement for the derivative part of Schwarz's lemma, which gives a sharp estimate for the derivative of $f \in \mathcal{H}_0$ depending only on $|z|$: $|f'(z)| \leq \min\{1, \ (1+|z|^2)/(4|z|(1-|z|^2))\}$, and also describes the variability region of $f'(z)$. In addition, it plays a key role in the so-called multi-point Schwarz-Pick lemma. Another version of Dieudonné's lemma for $f \in \mathcal{H}$ was proved by Kaptanoğlu [22], which is the so-called Dieudonné-Pick lemma and can be reduced to the original Dieudonné's lemma (see also [13], [32] and [33]). In 2012, Cho, Kim and Sugawa [13] obtained a sharp upper bound (depending on $z, f(z), f'(z)$) of the second order derivative f'' for $f \in \mathcal{H}_0$, which can be viewed as the second order Dieudonné's lemma. They also refined Dieudonné's lemma for the first order but involving the term $f'(0)$. Nevertheless, a sharp upper bound in terms of z for the modulus of the higher order derivatives of $f \in \mathcal{H}_0$ has not been investigated.

Furthermore, assume that $w \in \mathbb{D}$ with the modulus of z greater than that of w. Schwarz's lemma describes the variability region $\{f(z) : f \in \mathcal{H}_0\} = \overline{\mathbb{D}}(0, |z|)$. Dieudonné's lemma gives an explicit description of the variability region $\{f'(z) : f \in \mathcal{H}_0, f(z) = w\}$ when f ranges over the class of all analytic self-maps of $\mathbb{D}$ fixing 0. However, the variability regions of the higher order derivatives of $f \in \mathcal{H}_0$ with $f(z) = w$ have not been described yet.

In the historical overview of the refinements of Schwarz' lemma and the Schwarz-Pick lemma, we do not show all the known generalizations (for example, see [4], [9],

[16], [17], [20], [25], [27], [28]). Among others, many important results such as the Landau-Toeplitz theorem [24] proved in 1906, the celebrated Schwarz-Alfors lemma [1] proved in 1938, boundary versions of Schwarz's lemma such as the Julia theorem [21] proved in 1920, the Wolff theorem [38] proved in 1926 and the Julia-Wolff-Carathéodory theorem [8] proved in 1929, as well as the geometric form of Schwarz's lemma such as the Diameter Schwarz lemma and the Area Schwarz lemma [7] proved in 2008 are omitted here. Instead, we deal mainly with the second derivative of $f \in \mathcal{H}_0$. In particular, we pay attention to the estimates of f'' as well as the variability region of $f''(z)$ for $z \in \mathbb{D}$, proving the sharpness by giving the existence of extremal functions.

We start Chapter 1 with the historical overview of the refinements of Schwarz's lemma and the Schwarz-Pick lemma, which includes the contributions of many famous mathematicians such as Rogosinski, Dieudonné and so on.

In Chapter 2, we introduce some properties of subclass of the bounded analytic functions such as conformal automorphisms of the unit disk and Blaschke product. We also give a brief introduction to Peschl's invariant derivative.

Chapter 3 is devoted to the estimates of the second derivative of $f \in \mathcal{H}_0$. We generalize Dieudonné's lemma for the first derivative to the second derivative. At first we recall the estimates of the derivatives of $f \in \mathcal{H}$ and $f \in \mathcal{H}_0$. In particular, we consider the upper bound of f'' depending on $|f(z)|$ and $|z|$ as well as depending only on $|z|$ and give the extremal functions.

In Chapter 4, we are interested in the variability region of the second derivative of $f \in \mathcal{H}_0$. We consider the properties of the boundary of a compact convex domain of the complex plane and give the parameter representation of the boundary of the region of the values of $f''(z)$ for $z \in \mathbb{D}$. The study on the second derivative of bounded analytic functions in this thesis is not exhaustive but could, in our opinion, serve as a basis for further investigations such as the subordination and the extremal problems.

In the last chapter, based on the result of Cho, Kim and Sugawa [13, Theorem 3.5], we also obtain a description of the variability region of $f'''(z)$ in terms of $z \in \mathbb{D}$, $f(z)$, $f'(z)$ and $f''(z)$ and give the form of all the extremal functions.

We should also mention that one part of the original content of this thesis corresponds to articles [11] and [12].

2 Preliminaries

In this chapter, we present some fundamental knowledge needed for a convenient understanding of the proof of all the results. First of all, we introduce some well-studied subclasses of bounded analytic functions. The conformal automorphisms of $\mathbb{D}$ play a decisive role in complex function theory and can also be used to characterize the Blaschke product. We also give a introduction to the definition and properties of Peschl invariant derivatives.

2.1 Blaschke product

In this section we begin with a brief discussion of conformal automorphisms before giving the definition and some properties of the Blaschke product. First we say that a map f is a conformal automorphism of a region $\Omega \in \mathbb{C}$ if and only if f is a holomorphic bijection of Ω onto itself.

Theorem 2.1.1 *A map $f : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism of $\mathbb{D}$ if and only if f is a $\ddot{M}$obius map of the form*

$$f(z) = \frac{az + \overline{c}}{cz + \overline{a}}, \quad a, c \in \mathbb{C}, \quad |a|^2 - |c|^2 = 1,$$

and also if and only if f is of the form

$$f(z) = e^{i\theta} \frac{z - b}{1 - \overline{b}z}, \quad b \in \mathbb{D}, \theta \in \mathbb{R}.$$

The set of conformal automorphisms of $\mathbb{D}$ forms a group $Aut(\mathbb{D})$, under composition. We recall that a group of homeomorphisms acts on a set X transitively on X if for each $x, y \in X$, there is some g in G such that $g(x) = y$. Thus we can say $Aut(\mathbb{D})$ acts transitively on $\mathbb{D}$ in this sense.

Next we would like to use the conformal automorphisms of $\mathbb{D}$ to connect the geometric and the analytic theories and see how it arises as multiplicative factors in Blaschke products. In fact, a function $B : \mathbb{D} \to \mathbb{D}$ is a (finite) Blaschke product if it is holomorphic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}}$ (the closed unit disk), and $|B(z)| = 1$ when $|z| = 1$ (see [?] for details). Thus, we have the equivalent definition of Blaschke product as follows.

Definition 2.1.2 *For $n \in \mathbb{N}$, $\{z_j\}_{j=1}^n \subset \mathbb{D}$ and a point $\theta \in \mathbb{R}$, a Blaschke product of degree n with zeros $\{z_j\}$ takes the form*

$$B(z) = e^{i\theta} \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z_j} z}, \quad z \in \mathbb{D}.$$

2.2 Peschl's invariant derivative

For $f \in \mathcal{H}$, Peschl [30] defined the so-called Peschl's invariant derivatives $D_n f(z)$ with respect to the hyperbolic metric by the Taylor series expansion:

$$z \to g(z) = \frac{f(\frac{z+z_0}{1+\overline{z}_0 z}) - f(z_0)}{1 - \overline{f(z_0)} f(\frac{z+z_0}{1+\overline{z}_0 z})} = \sum_{n=1}^{\infty} \frac{D_n f(z_0)}{n!} z^n, \quad z, z_0 \in \mathbb{D},$$

where

$$D_1 f(z) = g'(0), D_2 f(z) = g''(0), ...$$

For example, precise forms of $D_n f(z)$, $n = 1, 2$, are given by

$$\begin{aligned}
D_1 f(z) &= \frac{(1 - |z|^2) f'(z)}{1 - |f(z)|^2}, \\
D_2 f(z) &= \frac{(1 - |z|^2)^2}{1 - |f(z)|^2} \left[f''(z) - \frac{2\overline{z} f'(z)}{1 - |z|^2} + \frac{2\overline{f(z)} f'(z)^2}{1 - |f(z)|^2} \right], \\
D_3 f(z) &= \frac{(1 - |z|^2)^3}{1 - |f(z)|^2} \left[f'''(z) - \frac{6\overline{z} f''(z)}{1 - |z|^2} + \frac{6\overline{f(z)} f'(z) f''(z)}{1 - |f(z)|^2} + \frac{6\overline{z}^2 f'(z)}{(1 - |z|^2)^2} \right. \\
&\qquad \left. - \frac{12\overline{f(z)} f'(z)^2}{(1 - |z|^2)(1 - |f(z)|^2)} + \frac{6\overline{f(z)}^2 f'(z)^3}{(1 - |f(z)|^2)^2} \right].
\end{aligned}$$

The Schwarz-Pick lemma shows that $|D_1 f(z)| \le 1$. These derivatives $D_n f$ are differential invariants. We can explain this terminology in the sense as follows. In fact, if φ

and ψ are conformal automorphisms of $\mathbb{D}$, we have

$$|D_n(\psi \circ f \circ \varphi)(z)|) = |D_n(f)(\varphi(z))|, \quad z \in \mathbb{D}.$$

In 2007, Kim and Sugawa [23] derived the concrete formula for $D_n f$ in terms of $f^{(n)}$, the ordinary lower-order derivatives of f, the derivatives $D_1 f, ..., D_{n-1} f$ and the Bell polynomials.

3 Estimates of the second derivative of bounded analytic functions

Assume that a point z lies in the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, and f is an analytic self-map of $\mathbb{D}$ fixing 0. Then Schwarz's lemma gives $|f(z)| \leq |z|$, and Dieudonné's lemma asserts that $|f'(z)| \leq \min\{1,\ (1+|z|^2)/(4|z|(1-|z|^2))\}$. In this chapter, we prove a sharp upper bound for $|f''(z)|$ depending only on $|z|$.

3.1 Introduction

Let $\mathbb{D}$ be the open unit disk $\{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. The set of all analytic self-maps of $\mathbb{D}$ is denoted by $\mathcal{H}$, and its subspace $\mathcal{H}_0$ consists of those $f \in \mathcal{H}$ such that $f(0) = 0$. There are a lot of well-known results for the spaces $\mathcal{H}$ and $\mathcal{H}_0$ in the theory of complex analysis, and next we recall some classical growth estimates for the functions in these spaces.

Schwarz's lemma asserts that $|f(z)| \leq |z|$ for all $f \in \mathcal{H}_0$ and $z \in \mathbb{D}$. Equality holds if and only if f is an Euclidean rotation about the origin. Rogosinski [34] gave the following generalization for this result:
If $f \in \mathcal{H}_0$ and $f'(0)$ is fixed, then for $z \in \mathbb{D} \setminus \{0\}$, the region of values of $f(z)$ is the closed disk $\{\zeta \in \mathbb{C} : |\zeta - c| \leq r\}$, where

$$c = \frac{zf'(0)(1-z^2)}{1-|z|^2|f'(0)|^2}, \quad r = |z|^2 \frac{1-|f'(0)|^2}{1-|z|^2|f'(0)|^2}.$$

Another version of Rogosinski's lemma for $f \in \mathcal{H}$ was given by Rivard [32] (see also [33]), which is called Rogosinski-Pick lemma. In addition, Schwarz-Pick lemma states that

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}, \quad f \in \mathcal{H}, \quad z \in \mathbb{D},$$

and equality holds for some $z \in \mathbb{D}$ if and only if f is an automorphism of $\mathbb{D}$. Schwarz-Pick lemma has also a higher order version. More precisely, Ruscheweyh [36] proved that, for $f \in \mathcal{H}$ and $n \in \mathbb{N}$, the sharp inequality

$$|f^{(n)}(z)| \leq \frac{n!(1-|f(z)|^2)}{(1-|z|)^n(1+|z|)}, \quad z \in \mathbb{D}, \tag{3.1.1}$$

is valid (see also [3] and [23]).

Schwarz-Pick lemma implies the sharp inequality $|f'(z_0)| \leq 1/(1-|z_0|^2)$ for $f \in \mathcal{H}$ and $z_0 \in \mathbb{D}$. This inequality has the upper bound depending only on $|z_0|$ and equality occurs only for $f(z) = e^{i\theta}(z-z_0)/(1-\overline{z}_0 z)$, $\theta \in \mathbb{R}$. Szász [37] extended this inequality to odd order derivatives of $f \in \mathcal{H}$ and also obtained the sharp upper bound for $|f''|$ (see also [3]):

$$|f''(z_0)| \leq \frac{(8+|z_0|^2)^2}{32(1-|z_0|^2)^2}, \quad f \in \mathcal{H}, \quad z_0 \in \mathbb{D}.$$

Equality occurs only for

$$f(z) = e^{i\theta}\frac{u^2 + \frac{1}{2}z_0 u - \frac{1}{8}z_0^2}{1 + \frac{1}{2}\overline{z}_0 u - \frac{1}{8}\overline{z}_0^2 u^2}, \quad u = \frac{z-z_0}{1-\overline{z}_0 z}, \quad z \in \mathbb{D}, \quad \theta \in \mathbb{R}.$$

Dieudonné [15] proved the following estimate for the derivative of $f \in \mathcal{H}_0$ depending only on $|z|$:

$$|f'(z)| \leq \begin{cases} 1, & \text{if } |z| \leq \sqrt{2}-1; \quad (3.1.2) \\ \dfrac{(1+|z|^2)^2}{4|z|(1-|z|^2)}, & \text{if } |z| > \sqrt{2}-1. \quad (3.1.3) \end{cases}$$

Equality holds in (3.1.2) for some z_0 with $r = |z_0|$ if and only if $f(z) = e^{i\theta}z$ for some real constant θ. Equality holds in (3.1.3) for some z_0 with $r = |z_0|$ if and only if

$$f(z) = e^{i\theta}z\frac{z-a}{1-\overline{a}z},$$

where $a = (3r^2-1)z_0/(r^2(1+r^2))$ and $\theta \in \mathbb{R}$ is arbitrary. This result is known as Dieudonné's lemma, and it can be seen as Schwarz's lemma for f'. Another version of Dieudonné's lemma for $f \in \mathcal{H}$ was proved by Kaptanoğlu [22], which is so-called Dieudonné-Pick lemma (see also [13] and [32]).

Our main result below gives a sharp upper bound for the modulus of the second derivative of $f \in \mathcal{H}_0$.

Theorem 3.1.1 *If $f \in \mathcal{H}_0$, then*

$$|f''(z)| \le \begin{cases} \dfrac{4}{1-9|z|^2+(1+3|z|^2)^{3/2}}, & |z| \le \dfrac{1+\sqrt{3}}{4}; \qquad (3.1.4) \\ \dfrac{(1+8|z|^2)^2}{32|z|^3(1-|z|^2)^2}, & |z| > \dfrac{1+\sqrt{3}}{4}. \qquad (3.1.5) \end{cases}$$

Equality holds in (3.1.4) *for some z_0 with $r = |z_0| \le (1+\sqrt{3})/4$ if and only if*

$$f(z) = e^{i\theta} z \, \frac{z-a}{1-\overline{a}z},$$

where

$$a = \frac{3}{1+\sqrt{1+3r^2}} z_0, \quad \theta \in \mathbb{R}.$$

Equality holds in (3.1.5) *for some z_0 with $r = |z_0| > (1+\sqrt{3})/4$ if and only if*

$$f(z) = e^{i\theta} z \, \frac{z-a_1}{1-\overline{a}_1 z} \cdot \frac{z-a_2}{1-\overline{a}_2 z},$$

where

$$a_1 = \left(\frac{2r^2-1}{r^2} + \frac{2(1-r^2)}{\sqrt{3}r^2} \right) z_0, \quad a_2 = \left(\frac{2r^2-1}{r^2} - \frac{2(1-r^2)}{\sqrt{3}r^2} \right) z_0, \quad \theta \in \mathbb{R}.$$

Remark 3.1.2 *This theorem gives an improvement of Szász's upper bound of $|f''|$ for $f \in \mathcal{H}_0$.*

Remark 3.1.3 *As $r \downarrow (1+\sqrt{3})/4$,*

$$a_1 \to 6(3\sqrt{3}-5)z_0 = a_0 \quad \textit{and} \quad |a_2| \to 1.$$

When $r = (1+\sqrt{3})/4$, we obtain $a = 6(3\sqrt{3}-5)z_0 = a_0$. Thus

$$f(z) = e^{i\theta} z \frac{z-a_1}{1-\overline{a}_1 z} \frac{z-a_2}{1-\overline{a}_2 z} \to e^{i\gamma} z \frac{z-a_0}{1-\overline{a}_0 z}$$

for some $\gamma \in \mathbb{R}$ *as* $r \downarrow (1+\sqrt{3})/4$.

Remark 3.1.4 *The upper bound of* $|f''(z_0)|$ *is continuous but not real analytic, and* $4/(1-9r^2+(1+3r^2)^{3/2})$ *is increasing with respect to* r *on* $[0,(1+\sqrt{3})/4]$, $(1+8r^2)^2/(32r^3(1-r^2)^2)$ *is increasing with respect to* r *on* $((1+\sqrt{3})/4, 1)$.

The remainder of this paper is organized as follows: In Section 2, we present some auxiliary results on the space $\mathcal{H}_0$; and Section 3 consists of the proof of Theorem 1.

3.2 Auxiliary results on the space $\mathcal{H}_0$

In this section, we state and prove some auxiliary results related to the space $\mathcal{H}_0$. These results are needed for the proof of Theorem 3.1.1. Before them we fix some notation. For $c \in \mathbb{C}$ and $\rho > 0$, the discs $\mathbb{D}(c,\rho)$ and $\overline{\mathbb{D}}(c,\rho)$ are defined by

$$\mathbb{D}(c,\rho) := \{\zeta \in \mathbb{C} : |\zeta - c| < \rho\},$$

and

$$\overline{\mathbb{D}}(c,\rho) := \{\zeta \in \mathbb{C} : |\zeta - c| \le \rho\}.$$

In addition, we write

$$T_a(z) = \frac{z+a}{1+\overline{a}z}, \quad z, a \in \mathbb{D},$$

and define

$$\Delta(z_0, w_0) = \overline{\mathbb{D}}\left(\frac{w_0}{z_0}, \frac{|z_0|^2 - |w_0|^2}{|z_0|(1-|z_0|^2)}\right).$$

With these preparations we are ready to state a classical theorem of Dieudonné [15] which gives a description of the region of values of $f'(z_0)$.

Lemma 3.2.1 ([15]) *Suppose that* z_0 *and* w_0 *are points in* $\mathbb{D}$ *with* $|w_0| < |z_0|$. *If* $f \in \mathcal{H}_0$ *satisfies* $f(z_0) = w_0$, *then the region of values of* $f'(z_0)$ *is the closed disk* $\Delta(z_0, w_0)$.

Further, $f'(z_0) \in \partial\Delta(z_0, w_0)$ *if and only if* $f(z) = z\ T_{u_0}(e^{i\theta}T_{-z_0}(z))$, *where* $u_0 = w_0/z_0$ *and* $\theta \in \mathbb{R}$.

Cho, Kim and Sugawa [13] gave a similar result of Lemma 3.2.1 for the second derivative (see also [32]). We refine their original version in an appropriate way. We also characterize f when $|f''(z_0) - c| = \rho$, where z_0, c, and ρ are as in Lemma 3.2.2. This result may look a bit technical but it is needed for the argument of Theorem 3.1.1. Before the statement of Lemma 3.2.2, we define c and ρ by

$$\begin{cases} c = c(z_0, w_0, w_1) = \dfrac{2(r^2 - s^2)\beta(1 - \overline{w_0}\beta)}{z_0^2(1 - r^2)^2}; \\ \rho = \rho(z_0, w_0, w_1) = \dfrac{2(r^2 - s^2)(1 - |\beta|^2)}{r(1 - r^2)^2}. \end{cases}$$

Lemma 3.2.2 ([13]) *Suppose that z_0 and w_0 are points in $\mathbb{D}$ with $|w_0| = s < r = |z_0|$, $w_1 \in \Delta(z_0, w_0)$, and that $f \in \mathcal{H}_0$ satisfies $f(z_0) = w_0$ and $f'(z_0) = w_1$. Let β be given by the relation $w_1 = \dfrac{w_0}{z_0} + \dfrac{r^2 - s^2}{z_0(1 - r^2)}\beta$, $|\beta| \le 1$. Set $u_0 = w_0/z_0$ and $v_0 = \overline{z}_0^2\beta/|z_0|^2$.*

1. *If $|\beta| = 1$, then $f''(z_0) = c$ and $f(z) = zT_{u_0}(e^{i\theta}T_{-z_0}(z))$, where $\theta = \arg(\overline{z_0}^2\beta)$.*

2. *If $|\beta| < 1$, then the region of values of $f''(z_0)$ is the closed disk $\overline{\mathbb{D}}(c, \rho)$. Further, $f''(z_0) \in \partial\mathbb{D}(c, \rho)$ if and only if $f(z) = zT_{u_0}\left(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z))\right)$, where $\theta \in \mathbb{R}$.*

   When $\beta \neq 0$, $f''(z_0) \in \partial\mathbb{D}(c, \rho)$ and $\arg f''(z_0) = \arg c$ if and only if $f(z) = zT_{u_0}\left(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z))\right)$, where $\theta = \arg(\overline{z_0}^3\beta(1 - \overline{w_0}\beta))$.

Proof Although the proof of the assertion that $f''(z_0) \in \overline{\mathbb{D}}(c, \rho)$ can be found in [13, Theorem 3.7] and [32, Corollary 4.2], we reprove it here to present a full discussion for equality conditions and that $\overline{\mathbb{D}}(c, \rho)$ is covered, which are not explicitly given in [13] and [32]. Let $g(z) = f(z)/z$. Then by assumption, $g \in \mathcal{H}$. From [39, Theorem 2], we have

$$|D_2 g(z_0)| \le 2(1 - |D_1 g(z_0)|),$$

which is equivalent to

$$|f''(z_0) - c| \le \rho. \tag{3.2.1}$$

Here equality holds for some point z_0 if and only if $f(z) = zg(z)$, where g is a Blaschke product of degree 1 or 2 satisfying $g(z_0) = u_0$ and $g'(z_0) = (z_0 w_1 - w_0)/z_0{}^2$.

(1) If $|\beta| = 1$, then $f''(z_0) = c$ and $f(z) = zg(z)$, where g is an automorphism of $\mathbb{D}$ satisfying $g(z_0) = u_0$ and $g'(z_0) = (z_0 w_1 - w_0)/{z_0}^2$. Applying this fact, we determine the explicit form of g. Set

$$h(z) = T_{-u_0} \circ g \circ T_{z_0}(z), \quad z \in \mathbb{D}.$$

It is obvious that h is an automorphism of $\mathbb{D}$ depending on g and satisfying

$$h(0) = 0 \quad \text{and} \quad h'(0) = \frac{\overline{z_0}^2}{|z_0|^2}\beta,$$

which means that $h(z) = e^{i\theta}z$ for $z \in \mathbb{D}$ and $\theta = \arg(\overline{z_0}^2\beta)$. Now it is easy to check that

$$g(z) = T_{u_0} \circ h \circ T_{-z_0}(z) = T_{u_0}(e^{i\theta}T_{-z_0}(z)) = e^{i\gamma}z\,\frac{z-a}{1-\overline{a}z},$$

where

$$\gamma = \arg\left(\overline{z}_0^2\beta(1-w_0\overline{\beta})^2\right) \quad \text{and} \quad a = \frac{|z_0|^2 - w_0\overline{\beta}}{\overline{z}_0(1-w_0\overline{\beta})}.$$

This completes the proof of (1).

(2) The inequality (3.2.1) means that $f''(z_0)$ lies in $\overline{\mathbb{D}}(c,\rho)$. To show that $\overline{\mathbb{D}}(c,\rho)$ is covered, let $\alpha \in \overline{\mathbb{D}}$, $u_0 = w_0/z_0$ and $v_0 = \overline{z}_0^2\beta/|z_0|^2$, and set $f(z) = zg(z)$, where

$$g(z) = T_{u_0}\left(T_{-z_0}(z)T_{v_0}(\alpha T_{-z_0}(z))\right).$$

Then $f(0) = 0$ and $f(z_0) = w_0$. Next we show that $f'(z_0) = w_1$. A calculation shows that $f'(z_0) = g(z_0) + z_0 g'(z_0)$. Note that

$$T_{-u_0} \circ g(z) = T_{-z_0}(z)T_{v_0}(\alpha T_{-z_0}(z)).$$

Differentiating both sides, we get

$$\begin{aligned}(T_{-u_0})'(g(z))g'(z) &= T'_{-z_0}(z)T_{v_0}(\alpha T_{-z_0}(z)) \\ &\quad + T_{-z_0}(z)T'_{v_0}(\alpha T_{-z_0}(z))\alpha T'_{-z_0}(z)\end{aligned} \tag{3.2.2}$$

for all $z \in \mathbb{D}$. Substituting $z = z_0$ into this equation, we have

$$(T_{-u_0})'(g(z_0))g'(z_0) = T'_{-z_0}(z_0)T_{v_0}(0),$$

which gives

$$g'(z_0) = \frac{(r^2 - s^2)\overline{z}_0^2\beta}{(1 - r^2)r^4}.$$

Consequently, we prove that f also satisfies

$$f'(z_0) = \frac{w_0}{z_0} + \frac{|z_0|^2 - |w_0|^2}{z_0(1 - |z_0|^2)}\beta = w_1.$$

Next we find the form of $f''(z_0)$. By a straightforward computation, we have

$$f''(z_0) = 2g'(z_0) + z_0 g''(z_0). \tag{3.2.3}$$

Differentiating both sides of (5.3.3), we obtain

$$\begin{aligned}
&(T_{-u_0})''(g(z))(g'(z))^2 + (T_{-u_0})'(g(z))g''(z) \\
&\quad = T''_{-z_0}(z)T_{v_0}(\alpha T_{-z_0}(z)) \\
&\qquad + 2T'_{-z_0}(z)T'_{v_0}(\alpha T_{-z_0}(z))\alpha T'_{-z_0}(z) \\
&\qquad + T_{-z_0}(z)\, T''_{v_0}(\alpha T_{-z_0}(z))(\alpha T'_{-z_0}(z))^2 \\
&\qquad + T_{-z_0}(z)T'_{v_0}(\alpha T_{-z_0}(z))\alpha T''_{-z_0}(z), \quad z \in \mathbb{D}.
\end{aligned}$$

Substituting $z = z_0$ into this equation, we have

$$\begin{aligned}
&(T_{-u_0})''(g(z_0))(g'(z_0))^2 + (T_{-u_0})'(g(z_0))g''(z_0) \\
&= \frac{2\overline{z_0}^3}{(1 - r^2)^2 r^2}\beta + \frac{2(1 - |\beta|^2)\alpha}{(1 - r^2)^2}.
\end{aligned}$$

Consequently, we get

$$g''(z_0) = \frac{2(r^2 - s^2)}{r^2(1 - r^2)^2}\left(\frac{\overline{z_0}^3\beta}{r^2} + \alpha(1 - |\beta|^2) - \frac{\overline{w_0}r^2\beta^2}{z_0^3}\right).$$

Now this together with(5.3.4) gives

$$
\begin{aligned}
f''(z_0) &= \frac{2(r^2-s^2)\beta(1-\overline{w_0}\beta)}{z_0^2(1-r^2)^2} + \frac{2z_0(r^2-s^2)(1-|\beta|^2)}{r^2(1-r^2)^2}\alpha \\
&= c + \rho\frac{z_0\alpha}{r}.
\end{aligned}
$$

Now $\alpha \in \overline{\mathbb{D}}$ is arbitrary, so the closed disk $\overline{\mathbb{D}}(c,\rho)$ is covered.

We know that $f''(z_0) \in \partial\mathbb{D}(c,\rho)$ if and only if $f(z) = zg(z)$, where g is a Blaschke product of degree 2 satisfying $g(z_0) = w_0/z_0$ and $g'(z_0) = (z_0w_1 - w_0)/{z_0}^2$. Applying this fact, we determine the precise form of g. Set

$$
h(z) = \frac{T_{-u_0}\circ g\circ T_{z_0}(z)}{z}, \quad z \in \mathbb{D}.
$$

It is clear that h is an automorphism of $\mathbb{D}$ depending on g and satisfying

$$
h(0) = (T_{-u_0}\circ g\circ T_{z_0})'(0) = \frac{(1-|z_0|^2)g'(z_0)}{1-|u_0|^2} = v_0.
$$

Then $T_{-v_0}\circ h$ is an automorphism of $\mathbb{D}$ fixing 0, which means that $T_{-v_0}\circ h(z) = e^{i\theta}z$ for $z \in \mathbb{D}$ and $\theta \in \mathbb{R}$. Now it is easy to check that

$$
g(z) = T_{u_0}\left(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z))\right), \quad z \in \mathbb{D}.
$$

Conversely, if $f(z) = zT_{u_0}\left(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z))\right)$, where $\theta \in \mathbb{R}$, then

$$
f''(z_0) = c + \rho\frac{z_0}{r}e^{i\theta} \in \partial\mathbb{D}(c,\rho).
$$

Next, we prove the last assertion in this case. By basic geometry, we note that $f''(z_0) \in \partial\mathbb{D}(c,\rho)$ and $\arg f''(z_0) = \arg c$ if and only if $f''(z_0) = tc$ for $t = 1 + \rho/|c|$. Hence it suffices to show $f''(z_0) = tc$ for $t = 1+\rho/|c|$ if and only if $f(z) = zT_{u_0}(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z)))$, where $\theta = \arg(\overline{z}_0^3\beta(1-\overline{w}_0\beta))$.

If $f''(z_0) = tc$ for $t = 1 + \rho/|c|$, then

$$f(z) = zg(z) = zT_{u_0}\left(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z))\right), \quad z \in \mathbb{D}.$$

Next we determine the precise value of θ. A calculation shows that

$$f''(z_0) = c + \rho\frac{z_0}{r}e^{i\theta}.$$

Therefore, $f''(z_0) = tc$ implies that

$$e^{i\theta} = \frac{r^3\beta(1-\overline{w_0}\beta)}{z_0^3|\beta||1-\overline{w_0}\beta|}.$$

Conversely, if

$$f(z) = zg(z) = zT_{u_0}(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z))), \quad e^{i\theta} = \frac{r^3\beta(1-\overline{w_0}\beta)}{z_0^3|\beta||1-\overline{w_0}\beta|},$$

then

$$f''(z_0) = c + \rho\frac{z_0}{r}e^{i\theta} = c + \rho\frac{r^2\beta(1-\overline{w_0}\beta)}{z_0^2|\beta||1-\overline{w_0}\beta|} = c + \frac{c}{|c|}\rho = tc.$$

Hence (2) is proved. □

Based on Lemma 3.2.2, we give a sharp upper bound for $|f''(z)|$ depending only on $|z|$ and $|f(z)|$.

Lemma 3.2.3 *Suppose that z_0 and w_0 are points in $\mathbb{D}$ with $|w_0| = s < r = |z_0|$. If $f \in \mathcal{H}_0$ satisfies $f(z_0) = w_0$, then*

$$|f''(z_0)| \leq \begin{cases} \dfrac{2(1+s)(r^2-s^2)}{r^2(1-r^2)^2}, & r - s \leq \dfrac{1}{2}; \quad (3.2.4) \\[2ex] \dfrac{(r+s)(4r^2-4rs+1)}{2r^2(1-r^2)^2}, & r - s > \dfrac{1}{2}. \quad (3.2.5) \end{cases}$$

Equality holds in (3.2.4) *if and only if*

$$f(z) = e^{i\theta}z\frac{z-a}{1-\overline{a}z},$$

where

$$\theta = \arg\left(-\bar{z}_0^2 w_0\right) \ (\textit{If } w_0 = 0, \ \textit{then } \theta \in \mathbb{R} \textit{ is arbitrary}), \quad a = \frac{r^2+s}{r^2(1+s)} z_0.$$

Equality holds in (3.2.5) *if and only if*

$$f(z) = e^{i\theta} z \frac{z-a_1}{1-\overline{a}_1 z} \cdot \frac{z-a_2}{1-\overline{a}_2 z},$$

where

$$\begin{aligned}
&\theta = \arg\left(-\overline{z_0}^3 w_0\right) \ (\textit{If } w_0 = 0, \ \textit{then } \theta \in \mathbb{R} \textit{ is arbitrary}),\\
&a_1 = \frac{-1+3r^2-4rs+(1-r^2)\sqrt{1+16rs}}{2r^2(1-2rs)} z_0,\\
&a_2 = \frac{-1+3r^2-4rs-(1-r^2)\sqrt{1+16rs}}{2r^2(1-2rs)} z_0.
\end{aligned}$$

Proof First we suppose that $w_0 \neq 0$. From Lemma 3.2.1, we know that

$$f'(z_0) = \frac{w_0}{z_0} + \frac{|z_0|^2 - |w_0|^2}{z_0(1-|z_0|^2)} \beta, \quad |\beta| \leq 1.$$

Set $|\beta| = x$. From Lemma 3.2.2, we have

$$\begin{aligned}
|f''(z_0)| \leq |c| + \rho &= \frac{2(r^2-s^2)}{r^2(1-r^2)^2} \left(|\beta||1-\overline{w_0}\beta| + r(1-|\beta|^2)\right)\\
&\leq \frac{2(r^2-s^2)}{r^2(1-r^2)^2} \left(|\beta|(1+s|\beta|) + r(1-|\beta|^2)\right)\\
&= \frac{2(r^2-s^2)\Psi(x)}{r^2(1-r^2)^2},
\end{aligned} \tag{3.2.6}$$

where

$$\Psi(x) = (s-r)x^2 + x + r,$$

and equality holds in the second last inequality if and only if $-\overline{w_0}\beta = s|\beta|$.

Observe that $\Psi(x)$ takes its maximum at $x = 1/(2(r-s))$, which is less than 1 if and only if $r - s > 1/2$. In this case, the sharp upper bound for $|f''(z_0)|$ is

$$\frac{2(r^2-s^2)\Psi(\frac{1}{2(r-s)})}{r^2(1-r^2)^2} = \frac{(r+s)(4r^2-4rs+1)}{2r^2(1-r^2)^2}.$$

Moreover, from Lemma 3.2.2, the sharp upper bound for $|f''(z_0)|$ is obtained if and only if $f(z) = zT_{u_0}\left(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z))\right)$, where $\theta = \arg(\overline{z}_0^3\beta)$, $u_0 = w_0/z_0$ and $\beta = -w_0/(2s(r-s))$. In other words, equality holds in (3.2.5) if and only if the form of f is

$$f(z) = e^{i\theta}z\frac{z-a_1}{1-\overline{a}_1z}\cdot\frac{z-a_2}{1-\overline{a}_2z},$$

where

$$\begin{aligned}\theta &= \arg\left(-\overline{z_0}^3w_0\right),\\ a_1 &= \frac{-1+3r^2-4rs+(1-r^2)\sqrt{1+16rs}}{2r^2(1-2rs)}z_0,\\ a_2 &= \frac{-1+3r^2-4rs-(1-r^2)\sqrt{1+16rs}}{2r^2(1-2rs)}z_0.\end{aligned}$$

If $w_0 = 0$, then we can prove that $\theta \in \mathbb{R}$ is arbitrary.

For $r - s \leq 1/2$, $\Psi(x) \leq \Psi(1) = 1 + s$ in the interval $0 \leq x \leq 1$, so that

$$|f''(z_0)| \leq \frac{2(r^2-s^2)\Psi(1)}{r^2(1-r^2)^2} = \frac{2(1+s)(r^2-s^2)}{r^2(1-r^2)^2}.$$

Equality holds in the above inequality if and only if $f(z) = zT_{u_0}(e^{i\theta}T_{-z_0}(z))$, where $u_0 = w_0/z_0$, $\theta = \arg(-\overline{z_0}^2\beta)$ and $|\beta| = 1$. In another word, equality holds in (3.2.4) if and only if f is a Blaschke product of degree 2 of the following form

$$f(z) = e^{i\theta}z\frac{z-a}{1-\overline{a}z},$$

where

$$\theta = \arg\left(-\overline{z_0}^2w_0\right), \quad a = \frac{r^2+s}{r^2(1+s)}z_0.$$

If $w_0 = 0$, then $\theta \in \mathbb{R}$ is arbitrary. $\square$

We close this section by noting that from Ruscheweyh's inequality (3.1.1), for $f \in \mathcal{H}$:

$$|f''(z_0)| \leq \frac{2(1-|w_0|^2)}{(1+|z_0|)^2(1-|z_0|)},$$

where z_0 and w_0 are as in Lemma 3.2.3. Lemma 3.2.3 offers a smaller bound for $|f''(z_0)|$ when $f \in \mathcal{H}_0$.

3.3 Proof of Theorem 3.1.1

Proof [Proof of Theorem 3.1.1.] Fix $z_0 \in \mathbb{D}$, for $f \in \mathcal{H}_0$, $w_0 = f(z_0)$, $s = |w_0|$, $r = |z_0|$.

Assume that $r = 0$, then equality in (3.1.4) holds if and only if

$$f(z) = e^{i\theta} z^2, \quad \theta \in \mathbb{R}.$$

Assume that $r \neq 0$ and $s < r$ (If $s = r$, then $f(z) = e^{i\theta} z$ and $f''(z) = 0$). From Lemma 3.2.3, we consider two cases for $r - s \leq 1/2$ and $r - s > 1/2$.

Case (i) For $r - s \leq 1/2$, we know that

$$|f''(z_0)| \leq \frac{2(1+s)(r^2-s^2)}{r^2(1-r^2)^2} = \frac{2\varphi(s)}{r^2(1-r^2)^2},$$

where $\varphi(s) = -s^3 - s^2 + r^2 s + r^2$ and $s < r$. Let

$$\varphi'(s) = -3s^2 - 2s + r^2 = 0.$$

Then we have

$$s_1 = \frac{-1-\sqrt{1+3r^2}}{3}, \quad s_2 = \frac{-1+\sqrt{1+3r^2}}{3}.$$

Note that $s_1 < 0$, while $s_2 < r$ is equivalent to $6r^2 + r > 0$. Thus, $\varphi(s)$ is increasing with respect to s on $[0, s_2)$ and is decreasing on $(s_2, r]$. In this case, if $r - s_2 \leq 1/2$, then $r \leq (1+\sqrt{3})/4$, so that

$$|f''(z_0)| \leq \frac{2\varphi(s_2)}{r^2(1-r^2)^2} = \frac{4}{1 - 9r^2 + (1+3r^2)^{3/2}}.$$

In addition, if $r - s_2 > 1/2$, then $r > (1+\sqrt{3})/4$, hence $\varphi(s) \le \varphi(r-1/2)$, therefore

$$|f''(z_0)| \le \frac{2\varphi(r-\frac{1}{2})}{r^2(1-r^2)^2} = \frac{(2r+1)(4r-1)}{4r^2(1-r^2)}.$$

Case (ii) For $r - s > 1/2$, we note that

$$|f''(z_0)| \le \frac{(r+s)(4r^2-4rs+1)}{2r^2(1-r^2)^2} = \frac{\Phi(s)}{2r^2(1-r^2)^2},$$

where

$$\Phi(s) = -4rs^2 + s + r + 4r^3.$$

But $\Phi(s)$ reaches its maximum at $s = 1/(8r)$, which is less than r if and only if $r > \sqrt{2}/4$. In this case, if $r - 1/(8r) > 1/2$, then $r > (1+\sqrt{3})/4$, so that the sharp upper bound for $|f''(z_0)|$ is

$$\frac{\Phi(\frac{1}{8r})}{2r^2(1-r^2)^2} = \frac{(8r^2+1)^2}{32r^3(1-r^2)^2}.$$

Moreover, if $1/(8r) \le r$ but $r - 1/(8r) \le 1/2$, then $1/2 < r \le (1+\sqrt{3})/4$, hence $\Phi(s) < \Phi(r-1/2)$, therefore

$$|f''(z_0)| < \frac{(r+s)\Phi(r-\frac{1}{2})}{2r^2(1-r^2)^2} = \frac{(2r+1)(4r-1)}{4r^2(1-r^2)^2}.$$

From Case (i) and Case (ii), and note that

$$\frac{(2r+1)(4r-1)}{4r^2(1-r^2)} < \frac{(8r^2+1)^2}{32r^3(1-r^2)^2}$$

for $r > (1+\sqrt{3})/4$, and

$$\frac{(2r+1)(4r-1)}{4r^2(1-r^2)} < \frac{4}{1-9r^2+(1+3r^2)^{3/2}}$$

for $1/2 \le r \le (1+\sqrt{3})/4$, we prove that the inequalities (3.1.4) and (3.1.5) hold.

From Lemma 3.2.3, we know that equality holds in (3.1.4) at a point z_0 with $r = |z_0| \le (1+\sqrt{3})/4$ if and only if $f(z) = zT_{u_0}(e^{i\theta}T_{-z_0}(z))$, where $u_0 = w_0/z_0$, $\theta = \arg(-\overline{z_0}^2\beta)$, $\beta = -w_0/s$ and $s = (-1+\sqrt{1+3r^2})/3$. In another word, equality holds in (3.1.4) at a point z_0 with $r = |z_0| \leq (1+\sqrt{3})/4$ if and only if f is of the following form

$$f(z) = e^{i\theta} z \frac{z-a}{1-\overline{a}z},$$

where

$$a = \frac{3}{1+\sqrt{1+3r^2}} z_0, \quad \theta \in \mathbb{R}.$$

Actually, if f is the form of the above, then we compute that

$$|f''(z_0)| = \frac{2(1-|a^2|)}{(1-|a|r)^3} = \frac{4}{1-9r^2+(1+3r^2)^{3/2}}.$$

We also know that equality holds in (3.1.5) at a point z_0 with $r = |z_0| > (1+\sqrt{3})/4$ if and only if $f(z) = zT_{u_0}\left(T_{-z_0}(z)T_{v_0}(e^{i\theta}T_{-z_0}(z))\right)$, where $u_0 = w_0/z_0$, $v_0 = \overline{z}_0^2\beta/|z_0|^2$, $\theta = \arg(\overline{z}_0^3\beta)$, $\beta = -w_0/(2s(r-s))$ and $s = 1/(8r)$. In other words, equality holds in (3.1.5) at a point z_0 with $r = |z_0| > (1+\sqrt{3})/4$ if and only if the form of f is

$$f(z) = e^{i\theta} z \frac{z-a_1}{1-\overline{a}_1 z} \cdot \frac{z-a_2}{1-\overline{a}_2 z},$$

where

$$a_1 = \frac{2-\sqrt{3}+2(\sqrt{3}-1)r^2}{\sqrt{3}r^2} z_0, \quad a_2 = \frac{-(2+\sqrt{3})+2(\sqrt{3}+1)r^2}{\sqrt{3}r^2} z_0, \quad \theta \in \mathbb{R}.$$

In fact, if f is of the above form, then we calculate

$$|f''(z_0)| = \frac{(1+8r^2)^2}{32r^3(1-r^2)^2}.$$

This completes the proof. $\square$

4 Variability region for the second derivative of bounded analytic functions

Let z_0 and w_0 be given points in the open unit disk $\mathbb{D}$ with $|w_0| < |z_0|$. Let $\mathcal{H}_0$ be the class of all analytic self-maps f of $\mathbb{D}$ normalized by $f(0) = 0$, and $\mathcal{H}_0(z_0, w_0) = \{f \in \mathcal{H}_0 : f(z_0) = w_0\}$. In this chapter, we explicitly determine the variability region of $f''(z_0)$ when f ranges over $\mathcal{H}_0(z_0, w_0)$. We also show a geometric view of our main result by Mathematica.

4.1 Introduction

First we fix some notation. For $c \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}(c, r) = \{z \in \mathbb{C} : |z - c| < r\}$ and $\overline{\mathbb{D}}(c, r) = \{z \in \mathbb{C} : |z - c| \leq r\}$. In particular we denote the open and closed unit disks $\mathbb{D}(0, 1)$ and $\overline{\mathbb{D}}(0, 1)$ by $\mathbb{D}$ and $\overline{\mathbb{D}}$, respectively. Let z_0 and w_0 be given points in the open unit disk $\mathbb{D}$ with $|w_0| < |z_0|$. We denote by $\mathcal{H}_0$ the set of all analytic self-maps f of $\mathbb{D}$ normalized by $f(0) = 0$ and set $\mathcal{H}_0(z_0, w_0) = \{f \in \mathcal{H}_0 : f(z_0) = w_0\}$. Schwarz's Lemma states that $\{f(z_0) : f \in \mathcal{H}_0\} = \overline{\mathbb{D}}(0, |z_0|)$ for any $z_0 \in \mathbb{D}$, and $f(z_0) \in \partial\overline{\mathbb{D}}(0, |z_0|)$ if and only if $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

In 1934, Rogosinski [34] explicitly described the region of values of $f(z_0)$ when f ranges over $\mathcal{H}_0$ satisfying $f'(0) = t$ for some prescribed value $t \in \overline{\mathbb{D}}$ (see also [4], [16], [27]). This refinement of Schwarz's lemma asserts that for $z_0 \in \mathbb{D}\backslash\{0\}$,

$$\{f(z_0) : f \in \mathcal{H}_0 \text{ with } f'(0) = t\} = \overline{\mathbb{D}}(c, r),$$

where

$$c = \frac{z_0 t(1 - |z_0|^2)}{1 - |z_0|^2|t|^2}, \qquad r = \frac{(1 - |t|^2)|z_0|^2}{1 - |z_0|^2|t|^2}.$$

Notice that the variability region $\overline{\mathbb{D}}(c, r)$ is strictly contained in $\overline{\mathbb{D}}(0, |z_0|)$.

In 1931, Dieudonné [15] determined the variability region of $f'(z_0)$ at a fixed point

$z_0 \in \mathbb{D} \setminus \{0\}$ when f ranges over $\mathcal{H}_0(z_0, w_0)$. If we write

$$T_a(z) = \frac{z+a}{1+\overline{a}z}, \quad z, a \in \mathbb{D}, \tag{4.1.1}$$

and define

$$\Delta(z_0, w_0) = \overline{\mathbb{D}}\left(\frac{w_0}{z_0}, \frac{|z_0|^2 - |w_0|^2}{|z_0|(1-|z_0|^2)}\right),$$

then Dieudonné's lemma asserts that

$$\{f'(z_0) : f \in \mathcal{H}_0(z_0, w_0)\} = \Delta(z_0, w_0)$$

For $f \in \mathcal{H}_0(z_0, w_0)$ consider the function $\tilde{f}$ defined implicitly by

$$\frac{z - z_0}{1 - \overline{z_0}z}\tilde{f}(z) = \frac{\frac{f(z)}{z} - \frac{w_0}{z_0}}{1 - \overline{\left(\frac{w_0}{z_0}\right)}\frac{f(z)}{z}}. \tag{4.1.2}$$

Notice that $|\tilde{f}(z)| \le 1$, $z \in \mathbb{D}$. Differentiating both sides shows

$$\frac{1-|z_0|^2}{(1-\overline{z_0}z)^2}\tilde{f}(z) + \frac{z-z_0}{1-\overline{z_0}z}\tilde{f}'(z) = \frac{1 - \left|\frac{w_0}{z_0}\right|^2}{\left(1 - \overline{\left(\frac{w_0}{z_0}\right)}\frac{f(z)}{z}\right)^2}\frac{zf'(z) - f(z)}{z^2}. \tag{4.1.3}$$

By substituting $z = z_0$, we have

$$\frac{\tilde{f}(z_0)}{1-|z_0|^2} = \frac{z_0 f'(z_0) - w_0}{z_0^2\left(1 - |\frac{w_0}{z_0}|^2\right)}, \tag{4.1.4}$$

and hence

$$f'(z_0) = \frac{w_0}{z_0} + \frac{|z_0|^2 - |w|^2}{\overline{z}_0(1-|z_0|^2)}\tilde{f}(z_0). \tag{4.1.5}$$

Combining this and the estimate $|\tilde{f}(z_0)| \le 1$ we easily obtain $\{f'(z_0) : f \in \mathcal{H}_0(z_0, w_0)\} \subset \Delta(z_0, w_0)$. The reverse inclusion relation follows from considering the function $f_\lambda \in \mathcal{H}_0(z_0, w_0)$ defined by

$$f_\lambda(z) = zT_{\frac{w_0}{z_0}}\left(\lambda T_{-z_0}(z)\right).$$

Notice that f_λ can be obtained by putting $f_\lambda = \lambda$ in (4.1.2). Furthermore, $f'(z_0) \in \partial\Delta(z_0, w_0)$ if and only if $f(z) = z\, T_{u_0}(e^{i\theta}T_{-z_0}(z))$, where $u_0 = w_0/z_0$ and $\theta \in \mathbb{R}$ and also if and only if f is implicitly defined by

$$\frac{\frac{f(z)}{z} - \frac{w_0}{z_0}}{1 - \overline{\left(\frac{w_0}{z_0}\right)}\frac{f(z)}{z}} = e^{i\theta}\frac{z - z_0}{1 - \overline{z_0}z}$$

for some $\theta \in \mathbb{R}$. The result is nowadays called Dieudonné's lemma.

In 2013, Rivard [32] proved a Dieudonné's lemma of the second order (see also [13]). We modify the original version appropriately as follows. Let $\lambda \in \overline{\mathbb{D}}$. Then

$$\left\{ f''(z_0) : f \in \mathcal{H}_0(z_0, w_0) \text{ with } f'(z_0) = \frac{w_0}{z_0} + \frac{|z_0|^2 - |w|^2}{\overline{z}_0(1 - |z_0|^2)}\lambda \right\} \tag{4.1.6}$$
$$= A(z_0, w_0)\overline{\mathbb{D}}(c(\lambda), \rho(\lambda)),$$

where

$$A(z_0, w_0) = \frac{2\left(|z_0|^2 - |w_0|^2\right)}{|z_0|^2(1 - |z_0|^2)^2}, \quad c(\lambda) = \lambda\left(1 - \frac{z_0\overline{w}_0}{\overline{z}_0}\lambda\right), \quad \rho(\lambda) = |z_0|(1 - |\lambda|^2).$$

For completeness, in the second section, we shall give an elementary proof of the second order Dieudonné's lemma and determine all the extremal functions.

Based on this result, the first author [11] gave the sharp estimate for $|f''(z_0)|$. In this paper, we are interested in providing an explicit description of the variability region

$$V(z_0, w_0) = \{f''(z_0) : f \in \mathcal{H}_0(z_0, w_0)\}. \tag{4.1.7}$$

From the second order Dieudonné's lemma it easily follows that

$$V(z_0, w_0) = A(z_0, w_0) \bigcup_{\lambda \in \overline{\mathbb{D}}} \overline{\mathbb{D}}(c(\lambda), \rho(\lambda)). \tag{4.1.8}$$

We first note some basic properties of the set $V(z_0, w_0)$. Since the class $\mathcal{H}_0(z_0, w_0)$ is a compact subset of the linear space of all analytic functions f in $\mathbb{D}$ endowed with the topology of locally uniformly convergence on $\mathbb{D}$ and the functional $\ell : \mathcal{H}_0(z_0, w_0) \ni f \mapsto f''(z_0) \in \mathbb{C}$ is continuous. Therefore the image $V(z_0, w_0) = \ell(\mathcal{H}_0(z_0, w_0))$ is also a compact subset of $\mathbb{C}$.

Next, we take $f_1, f_2 \in \mathcal{H}_0(z_0, w_0)$ and assume that $f(z) = (1-t)f_1(z) + tf_2(z)$, $0 \leq t \leq 1$. It is easy to see $f(0) = 0$, $|f(z)| \leq (1-t)|f_1(z)| + t|f_2(z)| \leq 1$. Then we get $f \in \mathcal{H}_0(z_0, w_0)$. Since $f''(z_0) = (1-t)f_1''(z_0) + tf''(z_0) \in V(z_0, w_0)$, then the convexity of $V(z_0, w_0)$ is proved.

Therefore we prove the set $V(z_0, w_0)$ given in(4.1.7) is a compact convex subset of $\mathbb{C}$. Furthermore the origin is an interior point of $V(z_0, w_0)$, because

$$A(z_0, w_0)\overline{\mathbb{D}}(0, |z_0|) = A(z_0, w_0)\overline{\mathbb{D}}(c(0), \rho(0)) \subset V(z_0, w_0).$$

Recall that a compact convex subset in $\mathbb{C}$ with nonempty interior is a Jordan closed domain (for a proof see [6, §11.2]). Therefore $\partial V(z_0, w_0)$ is a Jordan curve and $V(z_0, w_0)$ is the convex closed domain enclosed by $\partial V(z_0, w_0)$.
Recall that a compact and convex subset in $\mathbb{R}^2$ with nonempty interior is a Jordan closed domain (for a proof see [6, §11.2]). Therefore $\partial V(z_0, w_0)$ is a Jordan curve and $V(z_0, w_0)$ is the convex closed domain enclosed by $\partial V(z_0, w_0)$.

Moreover, for $z_0 = re^{i\varphi}, w_0 = se^{i\xi} \in \mathbb{D}$ with $s < r$, define $\tilde{f}(z) = e^{-i\xi} f(e^{i\varphi} z)$, then we have $\tilde{f}'(r) = e^{i(\varphi-\xi)} f'(z_0) \in \Delta(r, s)$ and $\tilde{f}''(r) = e^{i(2\varphi-\xi)} f''(z_0)$. Thus we obtain the relation

$$V(r, s) = e^{i(2\varphi-\xi)} V(z_0, w_0).$$

It suffices to determine $\partial V(r, s)$ for $0 \leq s < r < 1$. Define

$$c_s(\zeta) = \zeta(1 - s\zeta), \quad \rho_r(\zeta) = r(1 - |\zeta|^2).$$

We state our main result as follows.

Theorem 4.1.1 *Let* $0 \leq s < r < 1$.

(i) *If* $r - s \geq \frac{1}{2}$, *then* $\partial V(r,s)$ *coincides with the circle given by*

$$\partial\mathbb{D} \ni \zeta \mapsto \frac{1}{2r^2(1-r^2)^2}\left[\left\{1+4(r^2-s^2)\right\}r\zeta - s\right]. \tag{4.1.9}$$

(ii) *If* $r + s \leq \frac{1}{2}$, *then* $\partial V(r,s)$ *coincides with the Jordan curve given by*

$$\partial\mathbb{D} \ni \zeta \mapsto \frac{2(r^2-s^2)}{r^2(1-r^2)^2}c_s(\zeta). \tag{4.1.10}$$

(iii) *If* $r + s > \frac{1}{2}$ *and* $r - s < \frac{1}{2}$, *then* $\partial V(r,s)$ *consists of the circular arc given by*

$$(-\theta_0, \theta_0) \ni \theta \mapsto \frac{1}{2r^2(1-r^2)^2}\left[\left\{1+4(r^2-s^2)\right\}re^{i\theta} - s\right] \tag{4.1.11}$$

and the simple arc given by

$$J \ni \zeta \mapsto \frac{2(r^2-s^2)}{r^2(1-r^2)^2}c_s(\zeta), \tag{4.1.12}$$

where

$$\theta_0 = \cos^{-1}\frac{r^2+s^2-4(r^2-s^2)^2}{2sr} \in (0,\pi) \tag{4.1.13}$$

and J *is the closed subarc of* $\partial\mathbb{D}$ *which has end points* $\zeta_{\theta_0} = \dfrac{re^{i\theta_0}-s}{2(r^2-s^2)}$ *and* $\zeta_{-\theta_0} = \dfrac{re^{-i\theta_0}-s}{2(r^2-s^2)}$ *and contains* -1.

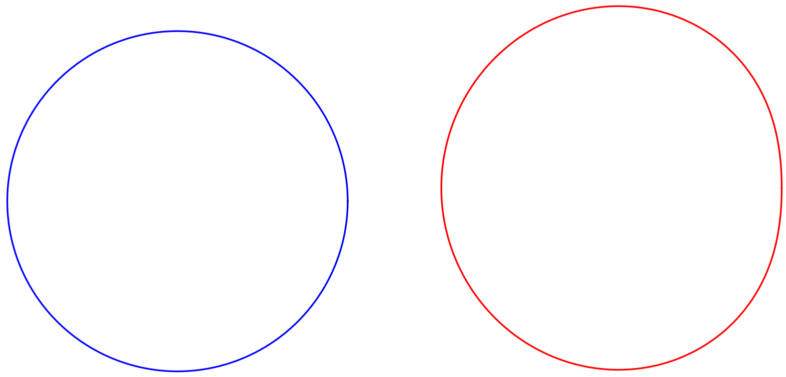
We show these three cases of $\partial V(r,s)$ in Figure 4.1(a), 4.1(b) and 4-2. In fact, Theorem 4.1.1 is a direct consequence of the following result which gives the parametric representation of $\partial V(r,s)$.

Theorem 4.1.2 *Let* $0 \leq s < r < 1$. *For* $\theta \in \mathbb{R}$ *let* r_θ *be the unique solution to the equation*

$$|xe^{i\theta} - s| = 2(x^2 - s^2), \quad x > s, \tag{4.1.14}$$

if $|re^{i\theta} - s| \geq 2(r^2 - s^2)$, *and let* $r_\theta = r$, *if* $|re^{i\theta} - s| < 2(r^2 - s^2)$. *Set*

$$\zeta_\theta = \frac{r_\theta e^{i\theta} - s}{2(r_\theta^2 - s^2)} \in \overline{\mathbb{D}}. \tag{4.1.15}$$



(a) r=3/4,s=1/4 (b) r=1/4, s=1/5

Figure 4-1 If $r = 3/4, s = 1/4$, $\partial V(r, s)$ is a circle; if $r = 1/4, s = 1/5$, $\partial V(r, s)$ is a convex Jordan curve.

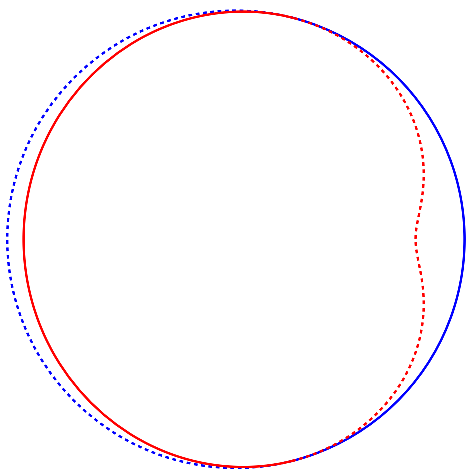


Figure 4-2 If $r = 2/3, s = 1/3$, $\partial V(r, s)$ consists of a circular arc (blue solid) and a simple arc (red solid).

Then a parametric representation $(-\pi, \pi] \ni \theta \mapsto \gamma(\theta)$ *of the Jordan curve* $\partial V(r,s)$ *is given by*

$$\gamma(\theta) = A(r,s)\left(c_s(\zeta_\theta) + \rho_r(\zeta_\theta)e^{i\theta}\right) \in \partial V(r,s).$$

Furthermore, the equality

$$f''(r) = A(r,s)\left(c_s(\zeta_\theta) + \rho_r(\zeta_\theta)e^{i\theta}\right) \in \partial V(r,s),$$

holds for some $\theta \in \mathbb{R}$ *with* $\zeta_\theta \in \mathbb{D}$ *if and only if*

$$f(z) = zT_{\frac{s}{r}}\left(T_{-r}(z)T_{\zeta_\theta}(e^{i\theta}T_{-\zeta_\theta}(z))\right), \quad z \in \mathbb{D}. \tag{4.1.16}$$

Here T_a *is defined by* (4.1.1)*. Similarly the equality*

$$f''(r) = A(r,s)c_s(\zeta_\theta) \in \partial V(r,s),$$

holds for some $\theta \in \mathbb{R}$ *with* $\zeta_\theta \in \partial\mathbb{D}$ *if and only if*

$$f(z) = zT_{\frac{s}{r}}\left(\zeta_\theta T_{-r}(z)\right), \quad z \in \mathbb{D}. \tag{4.1.17}$$

4.2 Envelop of a family of circles

In this section, we first state and prove some auxiliary results related to the compact convex domain. Let $E \subset \mathbb{C}$ be a compact convex domain containing a neighborhood of the origin. For $\theta \in \mathbb{R}$ let $t_\theta = \sup\{t > 0 : te^{i\theta} \in E\}$. Then the mapping $(-\pi, \pi] \ni \theta \mapsto t_\theta e^{i\theta} \in \partial E$ is a continuous bijection ($= 1 : 1$ and onto mapping). Particulary ∂E is a Jordan curve and the map gives a parametric representation of ∂E, and E is the union of ∂E and the domain enclosed by ∂E. Refer to [10], [26] and [29] for details.

We give the proof of the second order Dieudonné's lemma as follows, which is needed to determine the extremal functions in Theorem 4.1.2.

Proof [Proof of second order Dieudonné's lemma]

Let $f \in \mathcal{H}_0(z_0, w_0)$ and define $\tilde{f}$ by (4.1.2). By differentiating both sides of (4.1.3) and substituting $z = z_0$ we have

$$\begin{aligned}
&\frac{2\overline{z_0}\tilde{f}(z_0)}{(1-|z_0|^2)^2} + \frac{2\tilde{f}'(z_0)}{1-|z_0|^2} \\
&= \frac{2\overline{\left(\frac{w_0}{z_0}\right)}}{\left(1-\left|\frac{w_0}{z_0}\right|^2\right)^2}\left(\frac{z_0 f'(z_0) - w_0}{z_0^2}\right)^2 \\
&\quad + \frac{1}{1-\left|\frac{w_0}{z_0}\right|^2}\frac{z_0^2 f''(z_0) - 2(z_0 f'(z_0) - f(z_0))}{z_0^3}.
\end{aligned} \tag{4.2.1}$$

Combining this and (4.1.4), we have

$$f''(z_0) = \frac{2\left(1-\left|\frac{w_0}{z_0}\right|^2\right)}{(1-|z_0|^2)^2}\tilde{f}(z_0)\left(1 - z_0\overline{\left(\frac{w_0}{z_0}\right)\tilde{f}(z_0)}\right) + \frac{2\left(1-\left|\frac{w_0}{z_0}\right|^2\right)}{1-|z_0|^2} z_0\tilde{f}'(z_0).$$

By (4.1.5), $f'(z_0) = \dfrac{w_0}{z_0} + \dfrac{|z_0|^2 - |w_0|^2}{\overline{z}_0(1-|z_0|^2)}\lambda$ holds if and only if $\tilde{f}(z_0) = \lambda$. The Schwarz-Pick inequality $|\tilde{f}'(z_0)| \leq \dfrac{1-|\tilde{f}(z_0)|^2}{1-|z_0|^2} = \dfrac{1-|\lambda|^2}{1-|z_0|^2}$ implies $f''(z_0) \in A(z_0, w_0)\overline{\mathbb{D}}(c(\lambda), \rho(\lambda))$.

Conversely for $\lambda \in \mathbb{D}$ and $\alpha \in \overline{\mathbb{D}}$ define analytic functions $\tilde{f}_{\lambda,\alpha}$ and $f_{\lambda,\alpha}$ in $\mathbb{D}$ by

$$\tilde{f}_{\lambda,\alpha}(z) = T_\lambda\left(\frac{|z_0|}{z_0}\alpha T_{-z_0}(z)\right), \quad f_{\lambda,\alpha}(z) = zT_{\frac{w_0}{z_0}}\left(T_{-z_0}(z)\tilde{f}_{\lambda,\alpha}(z)\right).$$

Then $f_{\lambda,\alpha} \in \mathcal{H}_0(z_0, w_0)$, $\tilde{f}_{\lambda,\alpha}(z_0) = \lambda$ and $f''_{\lambda,\alpha}(z_0) = A(z_0, w_0)\{c(\lambda) + \rho(\lambda)\alpha\}$. It follows that $A(z_0, w_0)\overline{\mathbb{D}}(c(\lambda), \rho(\lambda))$ is contained in the variability region. Furthermore by the uniqueness part of the Schwarz lemma $f''(z_0) = A(z_0, w_0)\{c(\lambda) + \rho(\lambda)e^{i\theta}\}$ for some $f \in \mathcal{H}_0(z_0, w_0)$ if and only if $f = f_{\lambda, e^{i\theta}}$.

Similarly for $\lambda \in \partial\mathbb{D}$ define f_λ by

$$f_\lambda(z) = zT_{\frac{w_0}{z_0}}\left(\lambda T_{-z_0}(z)\right).$$

Then $f_\lambda \in \mathcal{H}_0(z_0, w_0)$ and $f''_\lambda(z_0) = A(z_0, w_0)c(\lambda)$. Again by the uniqueness part of the Schwarz lemma $f''(z_0) = A(z_0, w_0)c(\lambda)$ for some $f \in \mathcal{H}_0(z_0, w_0)$ if and only if

$f = f_\lambda$. Thus the proof is completed. $\square$

First we have the following property of

$$V = \bigcup_{\zeta \in \overline{\mathbb{D}}} \overline{\mathbb{D}}(c_s(\zeta), \rho_r(\zeta)). \tag{4.2.2}$$

Lemma 4.2.1 *The set V is a compact convex subset of $\mathbb{C}$ containing $\overline{\mathbb{D}}(0, r)$.*

Proof Recall that $V(r, s) = \dfrac{2\,(r^2 - s^2)}{r^2(1 - r^2)^2} V$. Thus the set V is a compact and convex subset of $\mathbb{C}$ with $\mathbb{D}(0, r) \subset V$. Therefore V is a convex closed Jordan domain enclosed by the Jordan curve $\partial \tilde{V}(r, s)$. $\square$

We can find that the determination of $\partial V(r, s)$ is reduced to that of V. Next we consider the monotonicity of

$$h(x) = \frac{|xe^{i\theta} - s|}{2(x^2 - s^2)}$$

for $x > s$.

Lemma 4.2.2 *For any $\theta \in \mathbb{R}$ and $s \geq 0$, define a positive and continuous function h_θ by*

$$h(x) = \frac{|xe^{i\theta} - s|}{2(x^2 - s^2)}$$

Then his strictly decreasing in $x > s$ for each fixed θ and $\lim_{x \to \infty} h(x) = 0.$.

Proof Take the logarithmic derivative of $h(x)$, say $g(x) = \log h(x)$, we have

$$g'(x) = \frac{-x^3 - 3xs^2 + 3sx^2 \cos\theta + s^3 \cos\theta}{(x^2 + s^2 - 2sx\cos\theta)(x^2 - s^2)}.$$

Since $-x^3 - 3xs^2 + 3sx^2\cos\theta + s^3\cos\theta \leq -x^3 - 3xs^2 + 3sx^2 + s^3 = -(x - s)^3 < 0$, therefore $g'(x) < 0$, $g(x)$ is strictly decreasing in $x > s$, which implies that $h(x)$ is strictly decreasing in $x > s$. $\square$

Before giving the parameter representation of ∂V, we give the general result for a convex set.

Lemma 4.2.3 *For a compact set $V \subset \mathbb{C}$, the function*

$$g(\theta) = \max_{v \in V} \mathrm{Re}(ve^{-i\theta}),$$

is continuous in $\theta \in \mathbb{R}$.

Proof Since V is compact, then there exists a $v_\theta \in V$ such that

$$g(\theta) = \max_{v \in V} \mathrm{Re}(ve^{-i\theta}) = \mathrm{Re}(v_\theta e^{-i\theta}).$$

For $\theta_0 \in \mathbb{R}$, take a sequence θ_n which satisfies $\theta_n \to \theta_0$, then there are a $v^* \in V$ and a sequence v_{θ_n}, such that $v_{\theta_n} \to v^*$, and we also have

$$\overline{\lim_{\theta \to \theta_0}} g(\theta) = \lim_{n \to \infty} g(\theta_n) = \lim_{n \to \infty} \mathrm{Re}(v_{\theta_n} e^{-i\theta_n}) = \mathrm{Re}(v^* e^{-i\theta_0}) \leq \max_{v \in V} \mathrm{Re}(ve^{-i\theta_0}) = g(\theta_0).$$

Since

$$g(\theta) = \max_{v \in V} \mathrm{Re}(ve^{-i\theta}) \geq \mathrm{Re}(ve^{-i\theta})$$

for any $v \in V$, we obtain

$$\underline{\lim_{\theta \to \theta_0}} g(\theta) \geq \lim_{\theta \to \theta_0} \mathrm{Re}(ve^{-i\theta}) = \mathrm{Re}(ve^{-i\theta_0}).$$

Noting that v is arbitrary, we have

$$\underline{\lim_{\theta \to \theta_0}} g(\theta) \geq \max_{v \in V} \mathrm{Re}(ve^{-i\theta_0}) = g(\theta_0),$$

it follows that

$$\overline{\lim_{\theta \to \theta_0}} g(\theta) = \underline{\lim_{\theta \to \theta_0}} g(\theta) = g(\theta_0),$$

thus we prove the continuity of $g(\theta)$.

$\square$

We recall a basic notion, the corner point, used in conformal geometry, referring to [31, Section 3.4] by Ch. Pommerenke for details. Notice that half-plane H is a supporting half-plane of V if it intersects V on its border and such that $V \subset H$, and ∂H is called the supporting line (see Figure 4-3). For a convex domain $W \subset \mathbb{C}$, the boundary point is a corner point if and only if there are at least two supporting lines of W at w.

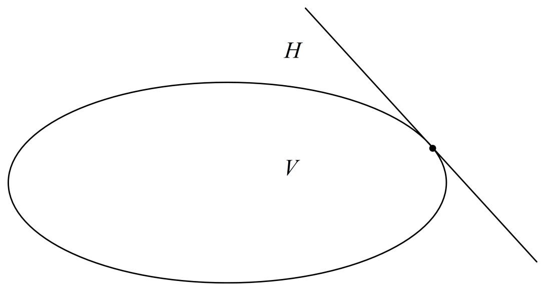


Figure 4-3 The supporting line.

Lemma 4.2.4 *Let V be a compact convex set without corner point in $\mathbb{C}$, and suppose that for each $\theta \in \mathbb{R}$, there is a unique point $v_\theta \in \partial V$ such that*

$$\mathrm{Re}(v_\theta e^{-i\theta}) = \max_{v \in V} \mathrm{Re}(v e^{-i\theta}). \tag{4.2.3}$$

Then the mapping

$$(-\pi, \pi] \ni \theta \mapsto v_\theta \tag{4.2.4}$$

gives a continuous bijection of $(-\pi, \pi]$ onto ∂V.

Proof First we show the mapping $(-\pi, \pi] \ni \theta \mapsto v_\theta$ is continuous. For $\theta_0 \in \mathbb{R}$, we take a sequence θ_n which satisfies $\theta_n \to \theta_0$. Since V is compact, there exists a $v^* \in V$ and a subsequence $v_{\theta_{n_k}}$, such that $v_{\theta_{n_k}} \to v^*$. As $g(\theta_n) = \mathrm{Re}(v_{\theta_n} e^{-i\theta_n})$, we have

$$\mathrm{Re}(v_{\theta_0} e^{-i\theta_0}) = g(\theta_0) = \lim_{k\to\infty} g(\theta_{n_k}) = \lim_{k\to\infty} \mathrm{Re}(v_{\theta_{n_k}} e^{-i\theta_{n_k}}) = \mathrm{Re}(v^* e^{-i\theta_0}).$$

From the uniqueness of v_θ, we have $v_{\theta_0} = v^*$.

Since V is a compact convex subset of $\mathbb{C}$ and has non-empty interior, the boundary ∂V is a simple closed curve. Note that v_θ is injective continuous from $\partial \mathbb{D}$ to ∂V, and recall that a simple closed curve cannot contain any simple closed curve other than itself. Thus, ∂V is given by

$$(-\pi, \pi] \ni \theta \mapsto v_\theta \in \partial V.$$

□

Now we turn to the form of the boundary point of V.

Lemma 4.2.5 *For $\theta \in \mathbb{R}$, take $v_\theta \in \partial V$ such that $\mathrm{Re}(v_\theta e^{-i\theta}) = \max_{v \in V} \mathrm{Re}(ve^{-i\theta})$, then there is only one $\zeta_\theta \in \overline{\mathbb{D}}$ such that $v_\theta = c_a(\zeta_\theta) + \rho_b(\zeta_\theta)e^{i\theta}$.*

Proof For $\theta \in \mathbb{R}$, take $v_\theta \in V$, $\mathrm{Re}(v_\theta e^{-i\theta}) = \max_{v \in V} \mathrm{Re}(ve^{-i\theta})$. Then $\exists \zeta_\theta, \varepsilon_\theta \in \overline{\mathbb{D}}$, such that $v_\theta = c_s(\zeta_\theta) + \rho_r(\zeta_\theta)\varepsilon_\theta$. From the hypotheses of the lemma, we have

$$\mathrm{Re}\{(c_s(\zeta_\theta) + \rho_r(\zeta_\theta)\varepsilon_\theta)e^{-i\theta}\} \geq \mathrm{Re}\{(c_s(\zeta) + \rho_r(\zeta)\varepsilon)e^{-i\theta}\}, \quad \forall \zeta, \varepsilon \in \overline{\mathbb{D}}.$$

Substitute $\zeta = \zeta_\theta$ into this equation, we have $\mathrm{Re}\{\rho_r(\zeta_\theta)\varepsilon_\theta e^{-i\theta}\} \geq \mathrm{Re}\{\rho_r(\zeta_\theta)\varepsilon e^{-i\theta}\}$. Let $\varepsilon = e^{i\theta}$, we obtain $\mathrm{Re}\{\rho_r(\zeta_\theta)\varepsilon_\theta e^{-i\theta}\} \geq \rho_r(\zeta_\theta)$. Thus $\varphi_\theta = e^{i\theta}$ and $v_\theta = c_a(\zeta_\theta) + \rho_b(\zeta_\theta)e^{i\theta}$.

Therefore, we have

$$\mathrm{Re}\{c_s(\zeta)e^{-i\theta}\} + \rho_r(\zeta) \leq \mathrm{Re}\{c_s(\zeta_\theta)e^{-i\theta}\} + \rho_r(\zeta_\theta),$$

and

$$g_\theta(\zeta) = \mathrm{Re}\left(c_s(\zeta) \cdot e^{-i\theta}\right) + \rho_r(\zeta)$$

takes maximum at ζ_θ.

(1) For θ satisfies $|re^{i\theta} - s| < 2(r^2 - s^2)$, if $g_\theta(\zeta)$ attains maximum at $\zeta_\theta \in \partial\mathbb{D}$, for $\zeta = \rho e^{i\varphi}$, we have

$$\frac{\partial g_\theta}{\partial \varphi}(\zeta)|_{\zeta=\zeta_\theta} = -\mathrm{Im}(\zeta_\theta(1 - 2s\zeta_\theta)e^{-i\theta}) = 0, \tag{4.2.5}$$

$$\frac{\partial g_\theta}{\partial \rho}(\zeta)|_{\zeta=\zeta_\theta} = \mathrm{Re}(\zeta_\theta(1 - 2s\zeta_\theta)e^{-i\theta}) - 2r \geq 0. \tag{4.2.6}$$

Multiplying both sides with $\overline{\zeta_\theta}$, we have

$$1 - 2s\zeta_\theta = 2r'e^{i\theta}\overline{\zeta_\theta}.$$

Thus we obtain that

$$\zeta_\theta = \frac{r'e^{i\theta} - s}{2(r'^2 - s^2)}.$$

Since the function $h(x)$ is strictly decreasing, we have

$$|\zeta_\theta| < \frac{|re^{i\theta} - s|}{2(r^2 - s^2)} < 1,$$

which is s contradiction to $\zeta_\theta \in \partial\mathbb{D}$. It follows that $g_\theta(\zeta)$ attains its maximum at $\zeta_\theta \in \mathbb{D}$ and satisfies

$$\frac{\partial g(\zeta)}{\partial \zeta}|_{\zeta=\zeta_\theta} = 0, \tag{4.2.7}$$

we have

$$\zeta_\theta = \frac{re^{i\theta} - s}{2(r^2 - s^2)} \in \mathbb{D},$$

which shows that ζ_θ is unique and depends only on θ.

(2)For θ satisfies $|re^{i\theta} - s| \geq 2(r^2 - s^2)$, if $g_\theta(\zeta)$ takes maximum at $\zeta_\theta \in \mathbb{D}$, then (4.2.7) holds, hence $\zeta_\theta = \dfrac{re^{i\theta} - s}{2(r^2 - s^2)} \notin \mathbb{D}$, which is a contradiction. Thus $\zeta_\theta \in \partial\mathbb{D}$, $v_\theta = c_s(\zeta_\theta)$ and $g_\theta(\zeta)$ satisfies (4.2.5) and (4.2.6). Therefore, there is a $\tilde{r} \geq r$, such that $\zeta_\theta(1 - 2s\zeta_\theta)e^{-i\theta} = 2\tilde{r}$, we have

$$\zeta_\theta = \frac{\tilde{r}e^{i\theta} - s}{2(\tilde{r}^2 - s^2)}.$$

Since $h(x)$ is strictly decreasing for $x > s$, $\tilde{r}$ is the unique solution of

$$\frac{|xe^{i\theta} - s|}{2(x^2 - s^2)} = 1,$$

which implies the uniqueness of ζ_θ. The proof is completed. □

Lemma 4.2.6 *For $\theta \in \mathbb{R}$, there is only one $v_\theta \in \partial V$ such that*

$$\mathrm{Re}(v_\theta e^{-i\theta}) = \max_{v \in V} \mathrm{Re}(ve^{-i\theta}) \tag{4.2.8}$$

and the mapping $(-\pi, \pi] \ni \theta \mapsto v_\theta \in \partial V$ is a continuous bijection giving a parametric representation of ∂V.

Proof We just need to show that ∂V has no corner points. Suppose, on the contrary, $v^* \in \partial V$ is a corner point, then there are two supporting half plane H_1, H_2 such that $v^* \in \partial V \subset H_1 \cap H_2$, $v^* \in \partial H_1 \cap \partial H_2$ and the opening angle α of $H_1 \cap H_2$ is less than π (see Figure 4-4).

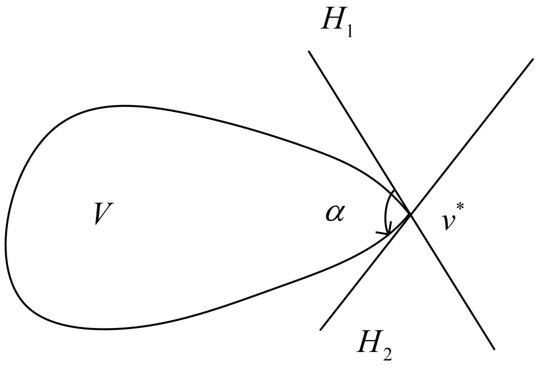


Figure 4-4 The supporting line.

Take $\zeta^* \in \overline{\mathbb{D}}$ and $\theta^* \in \mathbb{R}$ such that $v^* = c_s(\zeta^*) + \rho_r(\zeta^*)e^{i\theta^*}$.

If $\zeta^* \in \mathbb{D}$, then $\partial\mathbb{D}(c_s(\zeta^*), \rho_r(\zeta^*)) \subset V$ and $v^* \in \partial V$, which contradict $\alpha < \pi$.

Assume $\zeta^* \in \partial\mathbb{D}$. Then ∂V passes by $c_s(\zeta^*)$. Note that V contains the curve $\{c_s(\zeta) : \zeta \in \partial\mathbb{D}\}$. If $c_s'(\zeta^*) \neq 0$, then we have a contradiction as before.

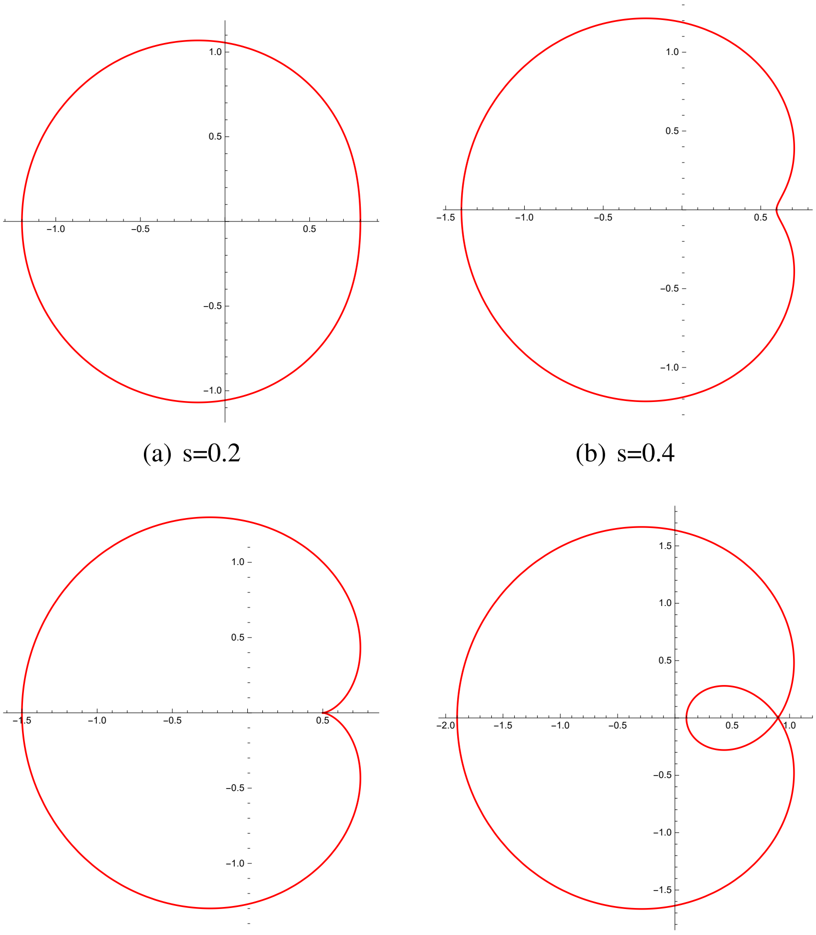
Notice that $c_s'(\zeta^*) = 0$ if and only if $s = \frac{1}{2}$ and $\zeta^* = 1$. In this case, since $r > s = \frac{1}{2}$, $v^* = c_{\frac{1}{2}}(1) = \frac{1}{2} \in \mathbb{D}(0, r) \subset \operatorname{Int} V$, which is a contradiction.

Note that if $0 \leq s \leq 1/4$, then Γ_s is convex; if $1/4 < s < 1/2$, then Γ_s is smooth and non-convex; if $s = 1/2$, then Γ_s has a cusp; if $1/2 < s < 1$, then Γ_s has a self-intersection point. We illustrate the four cases with pictures of the curve $\Gamma_s = \{c_s(\zeta) : \zeta \in \partial\mathbb{D}\}$, see Figure 4-5.

We can also prove the lemma in the following way. Firstly, we show the mapping $(-\pi, \pi] \ni \theta \mapsto v_\theta$ is continuous. By (4.1.15) and $r_{-\theta} = r_\theta$ it suffices to show the mapping $(-\pi, \pi] \ni \theta \mapsto r_\theta$ is continuous at any $\theta_0 \in [0, \pi]$.

(I). Assume $|re^{i\theta_0} - s| < 2(r^2 - s^2)$. Then $|re^{i\theta} - s| < 2(r^2 - s^2)$ holds on some neighborhood I of θ_0 and hence $r_\theta \equiv r$ on I. Thus r_θ is continuous at θ_0.

(II). Assume $|re^{i\theta_0} - s| > 2(r^2 - s^2)$. Then $|re^{i\theta} - s| > 2(r^2 - s^2)$ holds on some neighborhood I of θ_0. Hence r_θ is the unique solution to the equation (4.1.14), which



(a) s=0.2 (b) s=0.4

(c) s=0.5 (d) s=0.9

Figure 4-5 If $s = 0.2$, then Γ_s is convex; if $s = 0.4$, then Γ_s is smooth and non-convex; if $s = 0.5$, then Γ_s has a cusp; if $s = 0.9$, then Γ_s has a self-intersection point.

is equivalent to $h(x) - 1 = 0$. In this case the continuity of r_θ at θ_0 is a consequence of the inequality $\dfrac{dh}{dx}(x) < 0$ (see Lemma 4.2.2) and the implicit function theorem.

(III). Assume $|re^{i\theta_0} - s| = 2(r^2 - s^2)$. As in the case (II) there exists a neighborhood I of θ_0 the equation $h_\theta(x) - 1 = 0$ has the unique solution $x(\theta)$ which is continuous in θ and $x(\theta_0) = r$. Since $|re^{i\theta} - s| < 2(r^2 - s^2)$ for $\theta \in I_1 := I \cap [0, \theta_0)$, we have $r_\theta \equiv r$ on I_1. Similarly since $|re^{i\theta} - s| > 2(r^2 - s^2)$ for $\theta \in I_2 := I \cap (\theta_0, \pi]$, we have $r_\theta \equiv x(\theta)$ on I_2. Therefore r_θ is continuous at $\theta = \theta_0$, as required.

Next we prove $(-\pi, \pi] \ni \theta \mapsto v(\theta)$ is injective. Suppose, on the contrary, $v_{\theta_1} = v_{\theta_2} = v^*$ for some $-\pi < \theta_1 < \theta_2 \leq \pi$. For $j = 1, 2$ define a half plane H_j by

$$H_j = \{w \in \mathbb{C} : \mathrm{Re}(we^{-i\theta_j}) \leq \mathrm{Re}(v^* e^{-i\theta_j})\}.$$

Then $v^* \in \tilde{V}(r, s) \subset H_1 \cap H_2$ and $v^* \in \partial H_1 \cap \partial H_2$. By a geometric consideration $v_\theta \equiv v^*$ for $\theta_1 \leq \theta \leq \theta_2$.

By taking a subinterval, if necessary, we may assume that $|re^{i\theta} - s| > 2(r^2 - s^2)$ on $[\theta_1, \theta_2]$ or $|re^{i\theta} - s| < 2(r^2 - s^2)$ on $[\theta_1, \theta_2]$. First we consider the former case. In this case $\zeta_\theta \in \partial\mathbb{D}$ and $c_s(\zeta_\theta) = v_\theta \equiv v^*$ on $[\theta_1, \theta_2]$. Since c_s is a nonconstant analytic function and v_θ is continuous in θ, this implies that there exists $\zeta^* \in \partial\mathbb{D}$ with $\zeta_\theta \equiv \zeta^*$ on $[\theta_1, \theta_2]$. Let

$$\Phi(z) = \frac{z - s}{2(|z|^2 - s^2)}, \quad |z| > s.$$

Then $\zeta^* \equiv \zeta_\theta = \Phi(r_\theta e^{i\theta})$ on $[\theta_1, \theta_2]$. Since the Jacobian J_Φ of Φ

$$\begin{aligned} J_\Phi(\zeta) :=& \left|\frac{\partial \Phi}{\partial \zeta}(\zeta)\right|^2 - \left|\frac{\partial \Phi}{\partial \bar{\zeta}}(\zeta)\right|^2 \\ =& \frac{|s|^2 |\zeta - s|^2}{4(|\zeta|^2 - s^2)^2} - \frac{|\zeta|^2 |\zeta - s|^2}{4(|\zeta|^2 - s^2)^2} < 0 \quad \text{for} \quad |\zeta| > s, \end{aligned}$$

Φ is locally injective and hence there exists z^* $(\in \Phi^{-1}(\zeta^*))$ with $r_\theta e^{i\theta} \equiv z^*$ on $[\theta_1, \theta_2]$, which is apparently a contradiction.

Next we assume $|re^{i\theta} - s| < 2(r^2 - s^2)$ on $[\theta_1, \theta_2]$. In this case we have on $[\theta_1, \theta_2]$

$$\begin{aligned} \zeta_\theta &= \frac{re^{i\theta} - s}{2(r^2 - s^2)}, \\ v_\theta &= c_s(\zeta_\theta) + \rho_r(\zeta_\theta)e^{i\theta} \equiv v^*. \end{aligned}$$

Then by an elementary calculation we have

$$\begin{aligned} \frac{d}{d\theta}\{v_\theta\} &= c_s'(\zeta_\theta)\frac{d\zeta_\theta}{d\theta} - r\left(\frac{d\zeta_\theta}{d\theta}\overline{\zeta_\theta} + \overline{\frac{d\zeta_\theta}{d\theta}}\zeta_\theta\right)e^{i\theta} + ir(1 - |\zeta_\theta|^2)e^{i\theta} \\ &= \frac{ire^{i\theta}}{4(r^2 - s^2)}\{1 + 4(r^2 - s^2)\} \neq 0, \end{aligned}$$

which is a contradiction.

We remain to consider the case $\theta_1 < \theta_2$, $|re^{i\theta_1} - s| < 2(r^2 - s^2)$ and $|re^{i\theta_2} - s| > 2(r^2 - s^2)$, such that $v_{\theta_1} = v_{\theta_2}$, then for $\theta_1 \leq \theta \leq \theta_2$, we have $v_\theta \equiv v_{\theta_1}$. Thus there exists $\theta_1 < \theta_1' < \theta_2$ such that $|re^{i\theta_1'} - s| < 2(r^2 - s^2)$, $v_{\theta_1} = v_{\theta_1'}$, which is a contradiction. Above all, we prove that $\theta \mapsto v(\theta)$ is injective.

Since ∂V is a Jordan curve, by making use of the intermediate value theorem, one can easily conclude the mapping is also surjective. Therefore the mapping gives a parametric representation of ∂V. □

4.3 Proof of Theorems 4.1.1 and 4.1.2

Proof [Proof of Theorem 4.1.2] Recall that $V(r, s) = A(r, s)\tilde{V}(r, s)$. Then by Lemma 4.2.5 we conclude that the mapping

$$\gamma(\theta) = A(r, s)v_\theta = A(r, s)(c_s(\zeta_\theta) + \rho_r(\zeta_\theta)e^{i\theta})$$

gives the parametric representation of $\partial V(r, s)$.

From the argument of the proof of the second order Dieudonné's lemma (see [13, Lemma 2.2]), we can easily get the all extremal functions as required. □

Proof [Proof of Theorem 4.1.1] Note that

$$\frac{1}{2(r+s)} = \frac{r-s}{2(r^2-s^2)} \le \frac{|re^{i\theta}-s|}{2(r^2-s^2)} \le \frac{r+s}{2(r^2-s^2)} = \frac{1}{2(r-s)},$$

we consider the following three cases.

(i) If $r - s \ge \frac{1}{2}$, then $\frac{|re^{i\theta}-s|}{2(r^2-s^2)} \le 1$ always hold for $\theta \in (-\pi, \pi]$. Thus $\zeta_\theta = \frac{re^{i\theta}-s}{2(r^2-s^2)}$, $v_\theta = c_s(\zeta_\theta) + \rho_r(\zeta_\theta)e^{i\theta}$. And $\partial V(r,s)$ is given by

$$A(r,s)v_\theta, \quad \theta \in (-\pi, \pi].$$

Since

$$\begin{aligned} v_\theta &= \frac{re^{i\theta}-s}{2(r^2-s^2)}\left(1 - \frac{re^{i\theta}-s}{2(r^2-s^2)}s\right) + r\left(1 - \frac{re^{i\theta}-s}{2(r^2-s^2)}\frac{re^{-i\theta}-s}{2(r^2-s^2)}\right)e^{i\theta} \\ &= re^{i\theta} + \frac{re^{i\theta}-s}{2(r^2-s^2)} - \frac{(re^{i\theta}-s)(s(re^{i\theta}-s)+r(r-se^{i\theta}))}{4(r^2-s^2)^2} \\ &= re^{i\theta} + \frac{re^{i\theta}-s}{2(r^2-s^2)} - \frac{(re^{i\theta}-s)(r^2-s^2)}{4(r^2-s^2)^2} \\ &= \frac{\{1+4(r^2-s^2)\}\, re^{i\theta}-s}{4(r^2-s^2)}, \end{aligned}$$

we conclude that $\partial V(r,s)$ coincides with the circle given by (4.1.9).

(ii) If $s + r \le \frac{1}{2}$, then $\frac{|re^{i\theta}-s|}{2(r^2-s^2)} \ge 1$ always hold for $\theta \in (-\pi, \pi]$. Thus $\zeta_\theta \in \partial\mathbb{D}$, $v_\theta = c_s(\zeta_\theta)$ and $\partial V(r,s)$ is given by

$$A(r,s)c_s(\zeta_\theta), \quad \theta \in \mathbb{R}.$$

As a function of θ, ζ_θ is continuous on $(-\pi, \pi]$ and injective since v_θ is injective. Thus, the mapping $\partial\mathbb{D} \ni e^{i\theta} \mapsto \zeta_\theta \in \partial\mathbb{D}$ is surjective and hence homeomorphic. Therefore, the map $\partial\mathbb{D} \ni \zeta \mapsto \frac{2\,(r^2-s^2)}{r^2(1-r^2)^2}c_s(\zeta)$ is an another parametric representation of $\partial V(r,s)$.

(iii) If $r + s > \frac{1}{2}$ and $r - s < \frac{1}{2}$, then $|re^{i\theta} - s| = 2(r^2 - s^2)$ has the unique solution $\theta_0 = \cos^{-1}\frac{r^2+s^2-4(r^2-s^2)^2}{2sr} \in (0, \pi)$. For $|\theta| < \theta_0$, we have $\frac{|re^{i\theta}-s|}{2(r^2-s^2)} < 1$.

Thus $\zeta_\theta = \dfrac{re^{i\theta} - s}{2(r^2 - s^2)} \in \mathbb{D}$ and $v_\theta = c_s(\zeta_\theta) + \rho_r(\zeta_\theta)e^{i\theta}$. For $\theta_0 \leq |\theta| \leq \pi$, we have $\dfrac{|re^{i\theta} - s|}{2(r^2 - s^2)} \geq 1$. Thus $\zeta_\theta \in \partial\mathbb{D}$ and $v_\theta = c_s(\zeta_\theta)$. Therefore, $\partial V(r, s)$ consists of the following two curves:

(a) If $|\theta| < |\theta_0|$, then

$$\begin{aligned}\gamma(\theta) &= \frac{2\left(r^2 - s^2\right)}{r^2(1 - r^2)^2}\left(c_s(\zeta_\theta) + \rho_r(\zeta_\theta)e^{i\theta}\right) \\ &= \frac{1}{2r^2(1 - r^2)^2}\left[\left\{1 + 4(r^2 - s^2)\right\} re^{i\theta} - s\right],\end{aligned}$$

coincides with (4.1.11).

(b) If $|\theta| \geq |\theta_0|$, then $\gamma(\theta) = \dfrac{2\left(r^2 - s^2\right)}{r^2(1 - r^2)^2} c_s(\zeta_\theta)$, where ζ_θ is defined as in Theorem 4.1.1. Notice that the map ζ_θ is continuous and injective with respect to $\theta \in (-\pi, \pi]$ and $\zeta_\pi = -1$. Therefore, the set $\{\zeta_\theta : |\theta_0| \leq |\theta| \leq \pi\}$ coincides with the close subarc J of $\partial\mathbb{D}$ which has end points $\zeta_{\theta_0} = \dfrac{re^{i\theta_0} - s}{2(r^2 - s^2)}$ and $\zeta_{-\theta_0} = \dfrac{re^{-i\theta_0} - s}{2(r^2 - s^2)}$ and contains -1. $\square$

5 Properties of the third derivative of bounded analytic functions

Let z_0 and w_0 be given points in the open unit disk $\mathbb{D}$ with $|w_0| < |z_0|$. Let $\mathcal{H}_0$ be the class of all analytic self-maps f of $\mathbb{D}$ normalized by $f(0) = 0$, and $\mathcal{H}_0(z_0, w_0) = \{f \in \mathcal{H}_0 : f(z_0) = w_0\}$. Define

$$\Delta(z_0, w_0) = \overline{\mathbb{D}}\left(\frac{w_0}{z_0}, \frac{|z_0|^2 - |w_0|^2}{|z_0|(1 - |z_0|^2)}\right).$$

Suppose that $w_1 \in \Delta(z_0, w_0)$ and w_2 is in a certain disk. In this chapter, we explicitly determine the variability region of $\{f'''(z_0) : f \in H(z_0, w_0), f'(z_0) = w_1, f''(z_0) = w_2\}$ and give the form of the extremal functions.

5.1 Introduction

Schwarz's Lemma shows that $\{f(z_0) : f \in \mathcal{H}_0\} = \overline{\mathbb{D}}(0,\ |z_0|)$ for any $z_0 \in \mathbb{D}$. In 1931, Dieudonné[15] described the variability region of $f'(z)$, $f \in \mathcal{H}_0$, at a fixed point $z_0 \in \mathbb{D}$. His statement is as follows. For $z_0, w_0 \in \mathbb{D}$ with $|w_0| = R < r = |z_0|$, define

$$\Delta(z_0, w_0) = \overline{\mathbb{D}}\left(\frac{w_0}{z_0}, \frac{r^2 - R^2}{r(1 - r^2)}\right).$$

Dieudonné's Lemma asserts that $\{f'(z_0) : f \in \mathcal{H}_0, f(z_0) = w_0\} = \Delta(z_0, w_0)$. In 2012, K. H. Cho, S. Kim and T. Sugawa [13] proved Dieudonné's Lemma of the second order which can be refined in the following appropriate way. Let β be given by the relation

$$w_1 = \frac{w_0}{z_0} + \frac{r^2 - R^2}{z_0(1 - r^2)}\beta, \quad \beta \in \overline{\mathbb{D}},$$

then

$$\{f''(z_0) : f \in \mathcal{H}_0, f(z_0) = w_0, f'(z_0) = w_1\} = \frac{2(r^2 - s^2)}{r^2(1 - r^2)^2}\overline{\mathbb{D}}(c(\beta), \rho(\beta)),$$

where

$$c(\beta) = \frac{\overline{z}_0}{z_0}\beta(1 - \overline{w_0}\beta), \quad \rho(\beta) = r(1 - |\beta|^2).$$

In the same paper, Cho, Kim and Sugawa [13] also proved the following inequality in terms of Peschl's invariant derivatives.

Lemma 5.1.1 ([13]) *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then*

$$\left| \frac{D_3 f(z)}{6}(1 - |D_1 f(z)^2| + \overline{D_1 f(z)} \left(\frac{D_2 f(z)}{2} \right)^2 \right| + \left| \frac{D_2 f(z)}{2} \right|^2 \leq (1 - |D_1 f(z)|^2)^2,$$

equality holds for a point $z \in \mathbb{D}$ if and only if f is a blaschke product of degree at most 3.

5.2 The third order Dieudonné's lemma

We would first like to give a similar result of [11, Lemma 2] for the third derivative. We are interested in the variability region

$$V(z_0, w_0, w_1, w_2) = \{f'''(z_0) : f \in \mathcal{H}(z_0, w_0), f'(z_0) = w_1, f''(z_0) = w_2\}.$$

For brevity, we assume that $z_0 = re^{i\theta}$, $w_0 = se^{i\phi}$. Set $\tilde{f}(z) = e^{-i\phi}f(e^{i\theta}z)$, then we obtain

$$\begin{aligned}
\tilde{f}(r) &= e^{-i\phi}f(e^{i\theta}r) = e^{-i\phi}f(z_0) = s,\\
\tilde{f}'(r) &= e^{-i\phi}f'(e^{i\theta}r)e^{i\theta} = e^{-i\phi}e^{i\theta}w_1 \in \Delta(r, s),\\
\tilde{f}''(r) &= e^{-i\phi}e^{2i\theta}f''(e^{i\theta}r) = e^{-i\phi}e^{2i\theta}w_2,\\
\tilde{f}'''(r) &= e^{-i\phi}e^{3i\theta}f'''(e^{i\theta}r) = e^{-i\phi}e^{3i\theta}w_3.
\end{aligned}$$

Without generality, we can relabel $\tilde{f}$ as f, and assume that

$$z_0 = r, w_0 = s, w_1 = \frac{s}{r} + \frac{r^2 - s^2}{r(1 - r^2)}\sigma, \quad \sigma \in \overline{\mathbb{D}},$$

$$w_2 = \frac{2(r^2-s^2)\sigma(1-s\sigma)}{r^2(1-r^2)^2} + \frac{2(r^2-s^2)(1-|\sigma|^2)}{r(1-r^2)^2}\delta = \frac{2(r^2-s^2)}{r^2(1-r^2)^2}(\sigma(1-s\sigma)+r(1-|\sigma|^2)\delta).$$

It is sufficient to investigate $V(r,s,\sigma,\delta)$ for $0 \le s < r < 1$ and $\sigma, \delta \in \overline{\mathbb{D}}$. Before the statement of our main result, we define c_0 and ρ_0 by

$$\begin{cases} c_0 = c_0(z_0, w_0, w_1, w_2) = A\left(B + r\delta(1-|\sigma|^2)(1+r^2-2s\sigma-r\overline{\sigma}\delta)\right); \\ \rho_0 = \rho_0(z_0, w_0, w_1, w_2) = Ar^2(1-|\sigma|^2)(1-|\delta|^2), \end{cases}$$

where

$$A = \frac{6(r^2-s^2)}{r^3(1-r^2)^3}, \quad B = s^2\sigma^3 - s(1+r^2)\sigma^2 + r^2\sigma.$$

We also characterize f when $|f'''(r)-c_0| = \rho_0$ in the following Lemma. The mentioned above together with Lemma 5.1.1 leads to the following “third order Dieudonnés' lemma”.

Theorem 5.2.1 *Let* $0 \le s < r < 1$, $\sigma, \delta \in \overline{\mathbb{D}}$ *with*

$$w_1 = \frac{s}{r} + \frac{r^2-s^2}{r(1-r^2)}\sigma,$$

and

$$w_2 = \frac{2(r^2-s^2)}{r^2(1-r^2)^2}(\sigma(1-s\sigma) + r(1-|\sigma|^2)\delta).$$

Suppose that $f \in \mathcal{H}_0(r,s)$, $f'(r) = w_1$ *and* $f''(r) = w_2$. *Set* $u_0 = s/r$ *and* $v_0 = \sigma$.

1. *If* $|\sigma| = 1$, *then* $f'''(r) = c_0$ *and* $f(z) = zT_{u_0}(\sigma T_{-r}(z))$.
2. *If* $|\sigma| < 1$, $|\delta| = 1$, *then* $f'''(r) = c_0$ *and* $f(z) = zT_{u_0}\left(T_{-r}(z)T_\sigma(\delta T_{-r}(z))\right)$.
3. *If* $|\sigma| < 1$, $|\delta| < 1$, *then the region of values of* $f'''(z_0)$ *is the closed disk* $\overline{\mathbb{D}}(c_0, \rho_0)$. *Further,* $f'''(z_0) \in \partial\mathbb{D}(c_0, \rho_0)$ *if and only if* $f(z) = zT_{u_0}\left(T_{-r}(z)T_\sigma(T_{-r}(z)T_\delta(e^{i\theta}T_{-r}(z)))\right)$, *where* $\theta \in \mathbb{R}$.

5.3 Proof of Lemma 5.2.1

In this section we go directly to the proof of Lemma 5.2.1.

Proof [Proof of Lemma 5.2.1] Assume that $g(z) = f(z)/z$, then we can easily compute that

$$\begin{aligned}
g'(z) &= \frac{f'(z)}{z} - \frac{f(z)}{z^2} = \frac{zf'(z) - f(z)}{z^2}, \\
g''(z) &= \frac{2f(z)}{z^3} - \frac{2f'(z)}{z^2} + \frac{f''(z)}{z}, \\
g'''(z) &= -\frac{6f(z)}{z^4} + \frac{6f'(z)}{z^3} - \frac{3f''(z)}{z^2} + \frac{f'''(z)}{z} = \frac{z^3 f'''(z) - 3(z^2 f''(z) - 2zf'(z) + 2f(z))}{z^4}.
\end{aligned}$$

Thus we obtain that

$$\begin{aligned}
D^1 g(r) &= \frac{1-r^2}{r^2 - s^2}(rw_1 - s) = \sigma, \\
D^2 g(r) &= \frac{r^2(1-r^2)^2}{r^2-s^2}\left(\frac{r^2 w_2 - 2rw_1 + s}{r^3} - \frac{2(rw_1 - s)}{r(1-r^2)} + \frac{2s(rw_1-s)^2}{r^3(r^2-s^2)}\right) = 2\delta(1-|\sigma|^2),
\end{aligned}$$

From [13, Corollary 3.5], we have

$$\left|\frac{D_3 g(r)}{6} + \overline{\sigma}\delta^2(1-|\sigma|^2)\right| \le (1-|\sigma|^2)(1-|\delta|^2),$$

Set

$$\begin{aligned}
a &= -R\sigma^2 + r^2\sigma + r\delta(1-|\sigma|^2), \\
b &= (-1 - r^2 + 2R\sigma)a + \sigma(r^2 - R\sigma)^2 \\
&= -R^2\sigma^3 + R(1+r^2)\sigma^2 - r^2\sigma + (-1 - r^2 + 2R\sigma)r\delta(1-|\sigma|^2).
\end{aligned}$$

Noting that

$$D_3 g(r) = \frac{r(1-r^2)^3}{r^3(r^2-R^2)} f'''(r) + \frac{6b}{r^2},$$

we obtain

$$\left|f'''(r) + \frac{6(r^2-R^2)}{r^3(1-r^2)^3}(b + r^2\overline{\sigma}\delta^2(1-|\sigma|^2))\right| \le \frac{6(r^2-R^2)}{r(1-r^2)^3}(1-|\sigma|^2)(1-|\delta|^2),$$

which is

$$|f'''(r) - c_0| \le \rho_0. \tag{5.3.1}$$

Equality holds in the above equation if and only if $f(z) = zg(z)$, where g is a Blaschke product of degree $1, 2$ or 3 and satisfies

$$\begin{cases} g(r) = \dfrac{s}{r}; \\ g'(r) = \dfrac{rw_1 - s}{r^2} = \dfrac{r^2 - s^2}{r^2(1-r^2)}\sigma; \\ g''(r) = \dfrac{r^2w_2 - 2rw_1 + 2s}{r^3} = \dfrac{2(r^2-s^2)}{r^3(1-r^2)^2}a. \end{cases} \tag{5.3.2}$$

(1) If $|\sigma| = 1$,then $f'''(r) = c_0$ and $f(z) = zg(z)$, where g is an automorphism of $\mathbb{D}$ and satisfies (5.3.2). In this case, $c_0 = -\dfrac{6(r^2-s^2)}{r^3(1-r^2)^3}b$, where $a = -s\sigma^2 + r^2\sigma$, $b = (1+r^2)s\sigma^2 - s^2\sigma^3 - r^2\sigma$.

Applying this fact, we determine the explicit form of g. Set

$$h(z) = T_{-u_0} \circ g \circ T_r(z), \quad z \in \mathbb{D}.$$

It is obvious that h is an automorphism of $\mathbb{D}$ depending on g and satisfying $h(0) = T_{-u_0} \circ g(r) = 0$ and

$$\begin{aligned} v_0 = h'(0) &= T'_{-u_0}(gT_r(0)) \cdot g'(T_r(0)) \cdot T'_r(0) \\ &= T'_{-u_0}(u_0) \cdot g'(r) \cdot T'_r(0) \\ &= \sigma, \end{aligned}$$

which means that $h(z) = \sigma z$ for $z \in \mathbb{D}$. Now it is easy to check that

$$g(z) = T_{u_0} \circ h \circ T_{-r}(z) = T_{u_0}(\sigma T_{-r}(z)) = e^{i\gamma}\frac{z-a}{1-\overline{a}z},$$

where

$$\gamma = \arg\left(\sigma(1-s\overline{\sigma})^2\right) \quad \text{and} \quad a = \frac{r^2 - s\overline{\sigma}}{r(1-s\overline{\sigma})}.$$

We check that $g''(r) = \dfrac{2(r^2-s^2)}{r^3(1-r^2)^2}a$. This completes the proof of (1).

(2)For $|\sigma| < 1, |\delta| = 1$, we know that $f''(z_0) \in \partial\mathbb{D}(c_0, \rho_0)$ if and only if $f(z) = zg(z)$, where g is a Blaschke product of degree 2 satisfying (5.3.2).

Applying this fact, then we can determine the precise form of g. Set

$$h(z) = \frac{T_{-u_0} \circ g \circ T_r(z)}{z}, \quad z \in \mathbb{D}.$$

It is clear that h is an automorphism of $\mathbb{D}$ depending on g and satisfying

$$h(0) = (T_{-u_0} \circ g \circ T_{z_0})'(0) = \sigma.$$

Then $T_{-\sigma} \circ h$ is an automorphism of $\mathbb{D}$ fixing 0, which means that $T_{-\sigma} \circ h(z) = e^{i\theta} z$ for $z \in \mathbb{D}$ and $\theta \in \mathbb{R}$. Now it is easy to check that

$$g(z) = T_{u_0}\left(T_{-r}(z) T_\sigma(e^{i\theta} T_{-r}(z))\right), \quad z \in \mathbb{D}.$$

We compute that

$$f''(r) = \frac{2(r^2 - s^2)\sigma(1 - s\sigma)}{r^2(1 - r^2)^2} + \frac{2(r^2 - s^2)(1 - |\sigma|^2)}{r(1 - r^2)^2} e^{i\theta}.$$

Noting that

$$w_2 = \frac{2(r^2 - s^2)\sigma(1 - s\sigma)}{r^2(1 - r^2)^2} + \frac{2(r^2 - s^2)(1 - |\sigma|^2)}{r(1 - r^2)^2}\delta,$$

which indicates that $e^{i\theta} = \delta$. Therefore,

$$f(z) = z T_{u_0}\left(T_{-r}(z) T_\sigma(\delta T_{-r}(z))\right).$$

The proof of (2) is completed.

(3)The inequality (5.3.1) means that $f'''(r)$ lies in $\overline{\mathbb{D}}(c_0, \rho_0)$. To show that $\overline{\mathbb{D}}(c_0, \rho_0)$ is covered, let $\alpha \in \overline{\mathbb{D}}$, $u_0 = s/r$ and set $f(z) = zg(z)$, where

$$g(z) = T_{u_0}\left(T_{-r}(z) T_\delta(T_{-r}(z) T_\delta(\alpha T_{-r}(z)))\right).$$

Then $f(0) = 0$ and $f(r) = R$. Next we show that $f'(r) = w_1$. A simple calculation shows that $f'(z) = g(z) + zg'(z)$. Note that

$$T_{-u_0} \circ g(z) = T_{-r}(z) T_\delta(T_{-r}(z) T_\delta(\alpha T_{-r}(z))).$$

Differentiating both sides, we get

$$\begin{aligned}(T_{-u_0})'(g(z))g'(z) = {} & T'_{-r}(z) T_\delta(T_{-r}(z) T_\delta(\alpha T_{-r}(z))) \\ & + T_{-r}(z) T'_\delta(T_{-r}(z) T_\delta(\alpha T_{-r}(z))) \\ & \cdot (T'_{-r}(z) T_\delta(\alpha T_{-r}(z)) + T_{-r}(z) T'_\delta(\alpha T_{-r}(z)) \alpha T'_{-r}(z))),\end{aligned} \tag{5.3.3}$$

for all $z \in \mathbb{D}$. Substituting $z = r$ into this equation, we have

$$(T_{-u_0})'(g(r))g'(r) = T'_{-r}(z_0) T_\delta(0),$$

which gives

$$g'(r) = \frac{(r^2 - s^2)\delta}{r^2(1 - r^2)}.$$

Consequently, we show that f also satisfies

$$f'(r) = g(r) + rg'(r) = w_1.$$

Next we have to prove $f''(r) = w_2$. By a straightforward computation, we have

$$f''(z) = 2g'(z) + zg''(z). \tag{5.3.4}$$

Differentiating both sides of (5.3.3), we obtain

$$
\begin{aligned}
&(T_{-u_0})''(g(z))(g'(z))^2 + (T_{-u_0})'(g(z))g''(z) \\
&\quad = T''_{-r}(z)T_\sigma[T_{-r}(z)T_\delta(\alpha T_{-r}(z))] \\
&\qquad + 2T'_{-r}(z)T'_\sigma(T_{-r}(z)T_\delta(\alpha T_{-r}(z)))(T'_{-r}(z)T_\delta(\alpha T_{-r}(z)) + T_{-r}(z)T'_\delta(\alpha T_{-r}(z))\alpha T'_{-r}(z))) \\
&\qquad + T_{-r}(z)T''_\sigma(T_{-r}(z)T_\delta(\alpha T_{-r}(z)))(T'_{-r}(z)T_\delta(\alpha T_{-r}(z)) + T_{-r}(z)T'_\delta(\alpha T_{-r}(z))\alpha T'_{-r}(z))^2 \\
&\qquad + T_{-r}(z)T'_\sigma(T_{-r}(z)T_\delta(\alpha T_{-r}(z))) \cdot \big[T''_{-r}(z)T_\delta(\alpha T_{-r}(z)) + 2T'_{-r}(z)T'_\delta(\alpha T_{-r}(z))\alpha T'_{-r}(z)) \\
&\qquad\quad + T_{-r}(z)T''_\delta(\alpha T_{-r}(z))(\alpha T'_{-r}(z))^2 + T_{-r}(z)T'_\delta(\alpha T_{-r}(z))\alpha T''_{-r}(z)\big], \quad z \in \mathbb{D}.
\end{aligned}
\tag{5.3.5}
$$

Substituting $z = r$ into this equation,

$$
\begin{aligned}
&(T_{-u_0})''(g(r))(g'(r))^2 + (T_{-u_0})'(g(r))g''(r) \\
&\quad = T''_{-r}(r)T_\delta(0) + 2T'_{-r}(r)T'_\sigma(0)(T'_{-r}(z)T_\sigma(\alpha T_{-r}(z)).
\end{aligned}
$$

Thus,

$$g''(r) = \frac{2(r^2 - R^2)}{r^3(1-r^2)^2}a.$$

Therefore, $f''(r) = w_2$.

Next we determine the form of $f'''(r)$. A straightforward computation shows that

$$f'''(z) = 3g''(z) + zg'''(z). \tag{5.3.6}$$

Differentiating both sides of (5.3.5),

$$\begin{aligned}
&(T_{-u_0})'''(g(z))(g'(z))^3 + 3(T_{-u_0})''(g(z))g''(z) + T_{-u_0}'''(g(z))g'''(z) \\
&\quad = T_{-r}'''(z)T_\sigma(T_{-r}(z)T_\delta(\alpha T_{-r}(z))) \\
&\qquad + 3T_{-r}''(z)T_\sigma'(T_{-r}(z)T_\delta(\alpha T_{-r}(z)))(T_{-r}'(z)T_\delta(\alpha T_{-r}(z)) + T_{-r}(z)T_\delta'(\alpha T_{-r}(z))\alpha T_{-r}'(z))) \\
&\qquad + 3T_{-r}'(z)T_\sigma''(T_{-r}(z)T_\delta(\alpha T_{-r}(z)))(T_{-r}'(z)T_\delta(\alpha T_{-r}(z)) + T_{-r}(z)T_\delta'(\alpha T_{-r}(z))\alpha T_{-r}'(z))^2 \\
&\qquad + 3T_{-r}'(z)T_\sigma'(T_{-r}(z)T_\delta(\alpha T_{-r}(z)))\cdot \\
&\qquad\qquad \big(T_{-r}''(z)T_\delta(\alpha T_{-r}(z))) + 2T_{-r}'(z)T_\delta'(\alpha T_{-r}(z))\alpha T_{-r}'(z)) \\
&\qquad\qquad + T_{-r}(z)T_\delta''(\alpha T_{-r}(z))(\alpha T_{-r}'(z))^2 + T_{-r}(z)T_\delta'(\alpha T_{-r}(z))\alpha T_{-r}''(z)\big) + \\
&\qquad T_{-r}(z)T_\sigma'''(T_{-r}(z)T_\delta(\alpha T_{-r}(z)))(T_{-r}'(z)T_\delta(\alpha T_{-r}(z)) + T_{-r}(z)T_\delta'(\alpha T_{-r}(z))\alpha T_{-r}'(z)))^3 \\
&\qquad 3T_{-r}(z)T_\sigma''(T_{-r}(z)T_\delta(\alpha T_{-r}(z)))(T_{-r}'(z)T_\delta(\alpha T_{-r}(z)) + T_{-r}(z)T_\delta'(\alpha T_{-r}(z))\alpha T_{-r}'(z)) \\
&\qquad\qquad \big(T_{-r}''(z)T_\delta(\alpha T_{-r}(z))) + 2T_{-r}'(z)T_\delta'(\alpha T_{-r}(z))\alpha T_{-r}'(z)) \\
&\qquad\qquad + T_{-r}(z)T_\delta''(\alpha T_{-r}(z))(\alpha T_{-r}'(z))^2 + T_{-r}(z)T_\delta'(\alpha T_{-r}(z))\alpha T_{-r}''(z)\big) \\
&\qquad T_{-r}(z)T_\sigma'(T_{-r}(z)T_\delta(\alpha T_{-r}(z)))\cdot \\
&\qquad \big[T_{-r}'''(z)T_\delta(\alpha T_{-r}(z)) + 3T_{-r}''(z)T_\delta'(\alpha T_{-r}(z))\alpha T_{-r}'(z) + 3T_{-r}'(z)T_\delta''(\alpha T_{-r}(z))(\alpha T_{-r}'(z))^2 \\
&\qquad\qquad + 3T_{-r}'(z)T_\delta'(\alpha T_{-r}(z))\alpha T_{-r}''(z) + T_{-r}(z)T_\delta'''(\alpha T_{-r}(z))(\alpha T_{-r}'(z))^3 \\
&\qquad\qquad\qquad + 3T_{-r}(z)T_\delta''(\alpha T_{-r}(z))\alpha^2 T_{-r}'(z)T_{-r}''(z) + T_{-r}(z)T_\delta'(\alpha T_{-r}(z))\alpha T_{-r}''''(z)\big], \quad z \in \mathbb{D}.
\end{aligned} \tag{5.3.7}$$

Substituting $z = r$ into this equation,

$$\begin{aligned}
&(T_{-u_0})'''(g(r))(g'(r))^3 + 3(T_{-u_0})''(g(r))g''(r) + T_{-u_0}'''(g(r))g'''(r) \\
&\quad = T_{-r}'''(r)T_\sigma(0) + 3T_{-r}''(r)T_\sigma'(0)(T_{-r}'(r)T_\delta(0)) \\
&\qquad + 3T_{-r}'(r)T_\sigma''(0)(T_{-r}'(r)T_\delta(0))^2 + 3T_{-r}'(r)T_\sigma'(0)\big(T_{-r}''(r)T_\delta(0) + 2(T_{-r}'(r))^2T_\delta'(0)\alpha\big).
\end{aligned}$$

Together with(5.3.6), this gives

$$f'''(r) = c_0 + \rho_0\alpha.$$

Now $\alpha \in \overline{\mathbb{D}}$ is arbitrary, it follows that the closed disk $\overline{\mathbb{D}}(c_0, \rho_0)$ is covered.

We know that $f'''(r) \in \partial\mathbb{D}(c_0, \rho_0)$ if and only if $f(z) = zg(z)$, where g is a Blaschke product of degree 3 satisfying (5.3.2). Applying this fact, we determine the precise form of g. Set

$$h(z) = \frac{T_{-u_0} \circ g \circ T_r(z)}{z}, \quad z \in \mathbb{D}.$$

Clearly, h is a Blaschke Product of degree 2 depending on g and satisfying

$$h(0) = (T_{-u_0} \circ g \circ T_{z_0})'(0) = v_0 = \sigma.$$

Then $H(z) = T_{-v_0} \circ h(z)$ is a Blaschke Product of degree 2 fixing 0. Set

$$G(z) = \frac{H(z)}{z}.$$

It is obvious that G is an automorphism of $\mathbb{D}$ depending on g and satisfying

$$G(0) = H'(0) = T'_{-v_0}(v_0)h'(0) = \delta.$$

Then $T_{-\delta} \circ G$ is an automorphism of $\mathbb{D}$ fixing 0, which means that $T_{-\delta} \circ G(z) = e^{i\theta} z$ for $z \in \mathbb{D}$ and $\theta \in \mathbb{R}$. Now it is easy to check that

$$g(z) = T_{u_0}\left(T_{-r}(z)T_\sigma(T_{-r}(z)T_\delta(e^{i\theta}T_{-r}(z)))\right), \quad z \in \mathbb{D}.$$

Conversely, if $f(z) = zT_{u_0}\left(T_{-r}(z)T_\sigma(T_{-r}(z)T_\delta(e^{i\theta}T_{-r}(z)))\right)$, where $\theta \in \mathbb{R}$, then

$$f'''(r) = c_0 + \rho_0 e^{i\theta} \in \partial\mathbb{D}(c_0, \rho_0).$$

We complete the proof of this theorem. $\square$

Remark 5.3.1 *For* $|\sigma| = 1$,

$$|f'''(r)| = \frac{6(r^2 - s^2)}{r^3(1 - r^2)^3}|(1 + r^2)s\sigma^2 - s^2\sigma^3 - r^2\sigma|$$

$$\leq \frac{6(r^2-s^2)}{r^3(1-r^2)^3}[(1+r^2)s+s^2+r^2],$$

and equality holds if and only if $\sigma=-1$*, or if and only if*

$$f(z)=-\frac{z-a}{1-az},$$

where $a=\dfrac{r^2+s}{r(1+s)}$.

We end this chapter by asking the meaningful question: is it possible to obtain a sharp upper bound for $|f'''(r)|$ depending only on r?

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1. G. Q. Chen, *Estimates of the second derivative of bounded analytic functions*, Bull. Aust. Math. Soc. **100** (2019), no. 3, 458–469.

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