Doctoral Thesis

Thesis Title

Group-theoretic Bifurcation Mechanisms

for Economic Agglomerations on a Square Lattice

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Group-theoretic Bifurcation Mechanisms for Economic Agglomerations on a Square Lattice

ABSTRACT : The present thesis aims to elucidate the mechanism of economic agglomerations in two-dimensional economic spaces equipped with square road networks, which prosper worldwide (e.g., Chicago and Kyoto). A series of theoretical approaches provided in the present thesis makes it possible to investigate the spatial patterns of economic agglomerations on such spatial platforms systematically. Theoretical results in the present thesis would contribute greatly to theoretical and numerical investigation of economic agglomerations.

Studies of the spatial patterns of economic activities date back to the classical central place theory in economic geography (Christaller, 1933; Lösch, 1940). This theory suggested that central places, where economic activities are concentrated, would form geometrical patterns under the assumption of completely flat land surface and uniformly distributed consumers. The crude geometric prediction of this theory has come to be supplemented with full-fledged microeconomic foundations. The hexagonal distributions of economic agglomerations on the hexagonal lattice economy have come to be simulated by theoretical and numerical analysis of economic geography models (Ikeda et al., 2012, 2017, 2018). That said, the present thesis focuses on square distributions on the square lattice economy, which has not somewhat been given much attention.

Economic geography models, which encompass a wide range of spatial models in various fields, such as new economic geography, urban economics, and international trade theory, contribute to the understanding of the spatial patterns of economic activities. These models provide a comprehensive knowledge on how the level of transportation costs affects the spatial patterns of agglomerations, such as mono-centric and poly-centric distributions. Possible spatial patterns, however, depend on both economic modeling and spatial platforms. The present thesis mainly focuses on the latter, that is, mathematical mechanisms due to the symmetry of spatial platforms. We introduce appropriate spatial platforms to investigate agglomeration behaviour from the uniform distribution (Chapters 3 and 4), dispersion behaviour from the mono-centric distribution (Chapter 5), and economic interactions between local and global scales (Chapter 6).

Each spatial platform introduced in the present thesis has the symmetry described by a group, such as the dihedral group. Classification of bifurcation behaviour in such symmetric systems is the main subject of group-theoretic bifurcation theory (Golubitsky et al., 1988). We apply group-theoretic predictions to the investigation of bifurcation behavior of economic geography models. Our analysis places a special emphasis on model-independent bifurcation mechanisms behind agglomeration and dispersion behaviour, while the model dependency of such behaviour has come to be elucidated by Akamatsu et al. (2021).

As a whole, the present thesis provides a systematic analysis procedure that is applicable to a wide range of economic geography models. Note, however, that Chapters 4–6 assume the use of the replicator dynamics and, accordingly, are not applicable to the models that prohibit corner solutions, such as Helpman (1998) and Allen and Arkolakis (2014) models.

The present thesis is organized as follows:

Chapter 1 is the introduction that summarizes theoretical background, the contributions of the present thesis, and related studies.

Chapter 2 introduces a general setup of economic geography models. Various kinds of models have been developed in the previous studies. We accordingly discuss the applicability of theoretical results in each chapter to specific models. We employ Forslid and Ottaviano (2003), Helpman (1998), and Pflüger and Südekum (2008) models as representatives of economic geography models for the numerical analyses throughout the present thesis.

Chapter 3 offers a group-theoretic bifurcation theory to explain the mechanism of the self-organization of square patterns in economic agglomerations. As a spatial platform, we introduce an $n \times n$ square lattice that has the symmetry described by the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. We investigate steady-state bifurcation of the spatially uniform equilibrium on the square lattice. We show the existence of bifurcating solutions expressing square and stripe patterns by using two different mathematical methods: (i) the equivariant branching lemma and (ii) the bifurcation equations.

Chapter 4 focuses on the existence of invariant patterns that is a special feature of the replicator dynamics. Invariant patterns are stationary points that retain their spatial distribution even when the value of the bifurcation parameter changes.

We propose a methodology to find invariant patterns exhaustively. In view of invariant patterns, we develop an innovative bifurcation analysis procedure and apply this procedure to economic geography models. We numerically demonstrate the connectivity between the uniform distribution and invariant patterns via bifurcations.

Chapter 5 investigates the bifurcation mechanism of the full agglomeration at the geographical center of a square lattice. We theoretically show the existence of bifurcating solutions that represent a place at the center with large population surrounded by several places with small population. Some of these solutions can be interpreted as the formation of satellite cities around the central city. We numerically demonstrate transition that population emerges from, or is absorbed into, the center as the level of transport costs changes.

Chapter 6 proposes a local-global system, a spatial platform that can represent a hierarchical structure but can retain the insightfulness of bifurcation mechanisms. It consists of the two-level hierarchy comprising local and global systems. Each local system has a particular population size and a geographical configuration such as a square lattice. The global system expresses the geographical distribution of the local systems. We would like to develop the framework of conventional economic geography models to a direction of the qualitative spatial economics.

Chapter 7 summarizes the main results of the present thesis and suggests the directions of future research.

Contents

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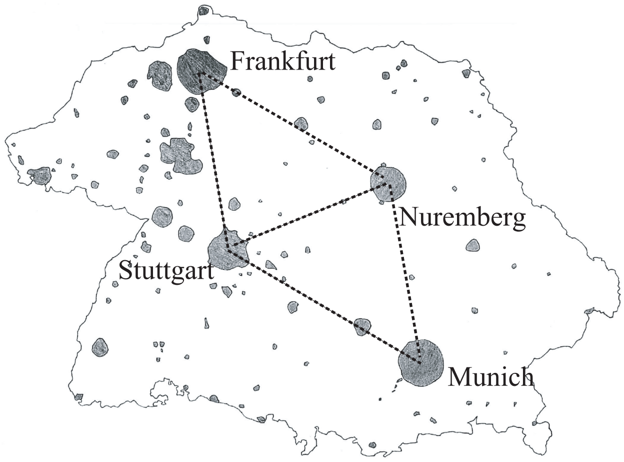


Figure 1.1: Distribution of large cities in southern Germany.

1. Introduction

Hierarchical urbanization of megalopolises, cities, towns, villages, and so on displays characteristic spatial patterns. Figure 1.1 depicts a distribution of large cities in southern Germany with cities of various sizes. In particular, Frankfurt, Stuttgart, Nuremberg, and Munich display a distinctive geometrical pattern. This is hinting at the existence of the underlying geometrical mechanism. Christaller (1933) conducted the first attempt to elucidate such a mechanism for population distribution of Southern Germany to develop central place theory of economic geography. Self-organization of hexagonal market areas of various kinds was proposed. Hexagonal market areas with different sizes were expected to form hierarchical hexagonal distributions of population (cities, towns, villages, etc.). For reviews of central place theory, see Lösch (1940), Lloyd and Dicken (1972), Dicken and Lloyd (1990), Isard (1975), and Beavon (1977), for example.

In economics, central place theory has been exposed to a criticism that it is not based on market equilibrium conditions (Fujita et al., 1999b). To overcome this, Eaton and Lipsey (1975, 1982) made the earliest attempt to provide central place theory with a microeconomic foundation. Clarke and Wilson (1985) and Munz and Weidlich (1990) demonstrated the emergence of spatial patterns in economic agglomerations. Krugman (1996) envisioned that hexagonal distributions envisaged in central place theory are to be self-organized in core-periphery models with a two-dimensional spatial platform.

Core-periphery models, which are based on the Dixit-Stiglitz competition, can express the migration of population among cities with a microeconomic foundation (e.g., Krugman, 1991; Combes et al., 2008). Most studies for these models, however, employed an too much simplified setup of the two-place economy to exploit analytical solvability.

In search of the mechanism of various spatial patterns in economic agglomerations, a proper choice of a spatial platform is vital. To transcend the two-place economy, studies on several spatial platforms have been conducted as reviewed in the *Related Studies* below. These spatial platforms, for example, include a star economy, a line segment economy, a racetrack economy, an equidistant economy, and a lattice economy. The line segment and racetrack economies are one-dimensional, whereas the equidistant and the lattice economies are two-dimensional.

In search of realistic spatial patterns, which are essentially two-dimensional, it is pertinent to employ the square and the hexagonal lattices that are capable of expressing diverse spatial agglomeration patterns:

- The square lattice can engender square, rectangular, and deformed triangular patterns.
- The hexagonal lattice can engender triangular, rectangular, and hexagonal patterns.

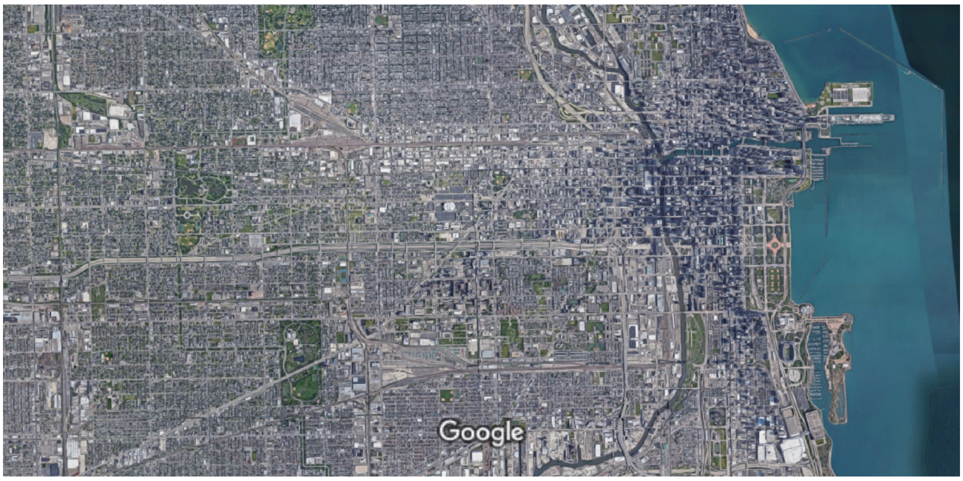
The suitability of these two kinds of lattices varies with cases as explained below:

In the simulation of hexagonal distributions, the use of hexagonal lattices is adequate. In nonlinear mathematics, hexagonal distributions have been shown to exist on planar systems endowed with hexagonal symmetry for several physical problems (Golubitsky and Stewart, 2002). In central place theory, the regular-triangular lattice was suggested for use based on geometrical discussion (Lösch, 1940). In economics, Eaton and Lipsey (1975) displayed the mechanism of the formation of a hexagonal distribution of mobile production factors (e.g., firms and workers) in two dimensions as an economic equilibrium for spatial competition. The bifurcation mechanism of the self-organization of hexagonal distributions on the hexagonal lattice was elucidated (Ikeda et al., 2014; Ikeda and Murota, 2014), as an extension of group-theoretic bifurcation analysis, which is applied mainly to a continuous space (Golubitsky and Schaeffer, 1985; Golubitsky et al., 1988), to a discreteized space.

On the other hand, it is quite noteworthy that square road networks exist worldwide. Chicago (USA) and Kyoto (Japan), for example, are well-known to accommodate such square networks historically (see Fig. 1.2). In fact, several studies of spatial agglomeration have been conducted on square lattices (Clarke and Wilson, 1983, 1985; Weidlich and Haag, 1987; Munz and Weidlich, 1990; Brakman et al., 1999). Yet, bifurcation analysis on a discretized space of a square lattice is very rare, and the study of Ikeda et al. (2018b) is an only exception.

This motivates the study of agglomeration and dispersion mechanisms on a square lattice. The present thesis aims to elucidate such mechanisms on a square lattice by direct and extended group-theoretic bifurcation analyses. We pay attention to the role of boundary conditions: (i) periodic boundary conditions and (ii) ordinary boundary conditions. In search of the square lattice counterpart of hexagonal distributions in central place theory, which considers an infinite and uniform plain, it is pertinent to employ the periodic boundary conditions. We would like to elucidate the mechanism of economic agglomerations on a square lattice with periodic boundary conditions in Chapters 3 and 4. On the other hand, the importance of ordinary (non-periodic) boundary conditions should not be overlooked. The infinite and uniform plain considered in central place theory is an idealization of a finite space with boundary in the real world. We would like to elucidate the mechanism of economic agglomerations on a square lattice with an ordinary boundary condition in Chapters 5 and 6.

Overall, the present thesis provides a systematic analysis procedure that can be applicable to a wide range of economic geography models. While studies of spatial economics are centered mainly on the economic modeling, the present thesis focuses on the spatial structure of economic



(a) Chicago (USA)

(b) Kyoto (Japan)

Figure 1.2: Satellite photographs of cities provided by Google Maps displaying square road networks

agglomerations, which is somewhat overlooked in these studies, despite its vital importance in economic agglomeration. The contributions of the present thesis are summarized as follows:

Chapter 2 introduces a general framework of economic geography models. There is a great number of economic geography models proposed in the previous studies. Accordingly, we discuss the applicability of theoretical analysis to be conducted in each chapter to specific models. We focus on Forslid and Ottaviano (2003), Helpman (1998), and Pflüger and Südekum (2008) models as typical models for numerical analyses in each chapter.

Chapter 3 reveals the mechanism of the self-organization of square agglomeration patterns from a uniform state by relying on group-theoretic bifurcation analysis (Golubitsky et al., 1988). The results of this chapter are applicable to any economic geography models with a single independent variable (expressing mobile population) at each nodal point. As a spatial platform, we introduce an $n \times n$ square lattice with periodic boundary conditions that has the symmetry described by the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, where the group D_4 expresses square symmetry and $\mathbb{Z}_n$ represents translational symmetry. We investigate steady-state bifurcation of the spatially uniform equilibrium and, in turn, to show the existence of bifurcating solutions expressing square and stripe patterns by using two different mathematical methods: (i) the equivariant branching lemma and (ii) the bifurcation equations. The stability of bifurcating solutions is investigated. Square patterns are highlighted as a square lattice counterpart of hexagonal patterns on a hexagonal lattice.

Chapter 4 shows the existence of invariant patterns, which is a special feature of the replicator dynamics. The results of this chapter are applicable to any economic geography models with the replicator dynamics with a single independent variable (expressing mobile population) at each nodal point and the independent variable can possibly become zero at some nodal point (no mobile population at the point). Invariant patterns are stationary points that retain their spatial distribution when the value of the bifurcation parameter changes. In view of invariant patterns, we propose an innovative bifurcation analysis procedure: (i) obtaining invariant patterns and (ii) searching for bifurcating equilibrium curves connecting stable invariant patterns. We apply this procedure to economic geography models. We numerically demonstrate the connectivity between the uniform equilibrium and invariant patterns through the bifurcating solutions.

Chapter 5 elucidates the bifurcation mechanism of the full agglomeration at the geographical center in a square lattice. The results of this chapter are applicable to any economic geography models with invariant patterns, which includes the full agglomeration to a single nodal point. We theoretically show the existence of bifurcating solutions that represent one large central place surrounded several places with small population. Some of these bifurcating solutions can be interpreted as the formation of satellite cities around the center. We numerically demonstrate a transition that population emerges from, or is absorbed into, the center as the transport cost changes.

Chapter 6 proposes a spatial platform that can represent a hierarchical structure but can retain the insightfulness of bifurcation mechanisms. It consists of two-level hierarchy of local and global systems. Each local system has a particular population size and a geographical configuration such as a square lattice. The global system expresses the geographical distribution of the local systems. We would like to develop the framework of conventional economic geography models to a direction of the qualitative spatial economics. As specific examples, we employ two identical square lattices of which the centers are connected directly. The global transport costs between the square lattices and local transport costs within each square lattice are considered.

Related Literature

The present thesis fundamentally relies on group-theoretic bifurcation theory, which has been developed in nonlinear mathematics (e.g., Mitropolsky and Lopatin, 1988; Allgower et al., 1992; Olver, 1995; Marsden and Ratiu, 1999; Hoyle, 2006). Group-theoretic bifurcation theory provides a powerful tool to analyze a system of equations with symmetry described by the group equivariance. Pattern formations in many physical phenomena[1] are often modeled by differential equations with the group equivariance on an infinite plane and is investigated by group-theoretic bifurcation analysis. Systems with the Euclidean group symmetry have been employed for reaction-diffusion systems (Turing, 1952), the Rayleigh-Bénard convection (Busse, 1978), cellular patterns in combustion (Sivashinsky, 1983), and solidification (Coriell and McFadden, 1993). Steady-state bifurcation from the fully symmetric equilibrium (the uniform distribution) on these systems was classified (Melbourne, 1999). The bifurcation behaviour of systems with dihedral group symmetry has been studied in applied mathematics (Sattinger, 1983; Healey, 1988; Dellnitz and Werner, 1989), chemistry (Kim, 1999), and physics (Kettle, 2007). The mechanism of hexagonal patterns[2] are related to the symmetry of the dihedral group D_6, while we focus on the dihedral group D_4 expressing square symmetry in Chapters 5 and 6. As an unified modeling for reaction-diffusion systems, the Navier-Stokes flow, the Bénard problem, and so on, a system with the symmetry of the infinite group $\mathrm{D}_4 \ltimes \mathrm{T}^2$ (T^2 express the two-torus of translation symmetry) has been studied (Dionne et al., 1997; Golubitsky and Stewart, 2002), while we employ the finite group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in Chapters 3 and 4.

The present thesis contributes to an understanding of how group-theoretic bifurcation theory adapts to analysis of economic geography models. Economic geography models, which theoretically describe the spatial patterns of economic activities, include a wide range of spatial models. While a fraction of these models is introduced and classified in Section 2 with reference to the viewpoint of Akamatsu et al. (2021), the readers may refer to standard textbooks, such as Brakman et al. (2001), Fujita and Thisse (2002), Baldwin et al. (2003), and Combes et al. (2008), to name a few.

Various spatial platforms were employed to observe diverse spatial agglomeration patterns. For a long narrow economy on a line segment or an infinite straight line, the literature reports several characteristic agglomeration patterns: the simplest core–satellite pattern for three places (Ago et al., 2006), a chain of spatially repeated core–periphery patterns *a la* Christaller and Lösch (Fujita and Mori, 1997), and a megalopolis which consists of large core cities that are connected by *an industrial belt*, i.e., *a continuum of cities* (Mori, 1997). These patterns were numerically observed by changing agglomeration forces and transport costs (Ikeda et al., 2017a).

[1] For example, in fluid dynamics, the Rayleigh-Bénard convection, which is observed for a horizontal layer of fluid heated from below, displays hexagonal patterns (Bénard, 1900; Koschmieder, 1993). The Couette-Taylor flow, which is a rotating annular fluid in a hollow cylinder, displays various symmetric patterns (Taylor, 1923). In material mechanics, uniform materials, such as cylindrical soils, undergo symmetric deformation patterns (Ikeda and Murota, 2019; Tanaka et al., 2002). Flower patterns of a honeycomb structure, which are observed in uniaxial and biaxial in-plane compression, have drawn keen interest (Saiki et al., 2005).

[2] For example, the mechanism of hexagonal patterns in the Rayleigh-Bénard convection was elucidated by (Kirchgässner, 1979). Competition between hexagonal and triangular patterns on surface waves was studied (Skeldon and Silber, 1998; Silber and Proctor, 1998).

A racetrack economy, which comprises a system of identical cities spread uniformly around the circumference of a circle, was studied extensively as a semi-two-dimensional spatial plarform. Krugman (1993, 1996) carried out local stability analysis of a core-periphery model on the racetrack economy to identify the emergence of several spatial frequencies. For the racetrack economy with 2^k cities, a spatially alternation of a core place with a large population and a peripheral place with a small population was observed for economic geography models (Picard and Tabuchi, 2010; Tabuchi and Thisse, 2011). Such a mechanism was explained in terms of the spatial period doubling bifurcation cascade, which produces fewer larger agglomerations through repeated doubling of the spatial period of agglomerated cities (Ikeda et al., 2012a; Akamatsu et al., 2012; Osawa et al., 2017). Anas (2004) demonstrated the presence of other agglomeration patterns, such as balanced agglomeration, concentrated agglomeration, and de-agglomeration.

The racetrack economy was studied comparatively with an economy on a line segment (a long narrow economy) by Mossay and Picard (2011) in a continuous space to display the difference in agglomeration patterns. Agglomerations in racetrack and star economies were studied comparatively (Barbero and Zofío, 2016). An analogy of the agglomerations in the racetrack economy to a long narrow economy and a square lattice economy was studied in Ikeda et al. (2017a, 2018b), respectively.

2. Economic Geography Models

This chapter provides the general framework of economic geography models, which we employed throughout the present thesis. Section 2.1 explains basic assumptions of the modeling on a symmetric spatial platform. Section 2.2 presents the classification of a series of economic geography models with reference to Akamatsu et al. (2021). Section 2.3 introduces some particular models, which we use for numerical simulations.

2.1. *General Framework*

We explain modeling of the economy to investigate the spatial patterns of economic agglomerations.

2.1.1. *Basic Assumptions*

Let $P = \{1, \ldots, K\}$ be the set of places. A spatial distribution of mobile workers is denoted by $\boldsymbol{\lambda} = (\lambda_i)$ under the normalizing constraint $\sum_{i \in P} \lambda_i = 1$. The payoff for the mobile workers is given by a payoff function $\boldsymbol{v} = (v_i)$.

The economy of economic geography models involves spatial frictions. That is, the payoff function $\boldsymbol{v}$ depends on a proximity matrix $D = [d_{ij}]$ with $d_{ij} = \phi^{m(i,j)}$, where $\phi \in (0, 1)$ is the trade freeness between two consecutive places, and $m(i, j)$ is the shortest distance between places i and j along the transport network of the economy.

Each mobile worker selects a place to locate in response to the payoff v_i. A spatial equilibrium is defined as a spatial distribution $\boldsymbol{\lambda}$ that satisfies the following condition:

$$\begin{cases} v^* - v_i(\boldsymbol{\lambda}, \phi) = 0 & \text{if } \lambda_i > 0, \\ v^* - v_i(\boldsymbol{\lambda}, \phi) \geq 0 & \text{if } \lambda_i = 0, \end{cases} \tag{2.1}$$

where v^* denotes the equilibrium utility level.

To investigate the stability of a spatial equilibrium, we consider an adjustment dynamics:

$$\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \phi) \tag{2.2}$$

with $\boldsymbol{F} = (F_i)$. A stationary point of the adjustment dynamics is given as a spatial distribution $\boldsymbol{\lambda}$ that satisfies the governing equation:

$$\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{0}. \tag{2.3}$$

The stability of a stationary point is classified via eigenanalysis of the Jacobian matrix $J = \partial \boldsymbol{F} / \partial \boldsymbol{\lambda}$ as follows:

$$\begin{cases} \text{linearly stable:} & \text{every eigenvalue has a negative real part,} \\ \text{linearly unstable:} & \text{at least one eigenvalue has a positive real part.} \end{cases}$$

We choose an adjustment dynamics such that a stable stationary point satisfies the equilibrium condition in (2.1).

Example 2.1. A most customary example of the adjustment dynamics in (2.2) is the replicator dynamics:

$$F_i(\boldsymbol{\lambda}, \phi) = (v_i(\boldsymbol{\lambda}, \phi) - \bar{v}(\boldsymbol{\lambda}, \phi))\lambda_i, \tag{2.4}$$

where $\bar{v}$ represents the weighted average utility defined as

$$\bar{v} = \sum_{i \in P} \lambda_i v_i. \tag{2.5}$$

This dynamics is widely used in economics including Krugman's original study (Krugman, 1991).

Another well-known example is the logit dynamics:

$$F_i(\boldsymbol{\lambda}, \phi) = \frac{\exp[\theta v_i(\boldsymbol{\lambda}, \phi)]}{\sum_{j \in P} \exp[\theta v_j(\boldsymbol{\lambda}, \phi)]} - \lambda_i, \tag{2.6}$$

where $\theta \in (0, \infty)$ is a parameter denoting the inverse of variance of idiosyncratic tastes. □

Remark 2.1. Chapter 3 provides general results that are independent of the functional form of adjustment dynamics. Chapters 4–6, however, employ the replicator dynamics, and hence the results are not applicable to other dynamics such as the logit dynamics. □

2.1.2. Group Equivariance

We are interested in the economy defined on a symmetric spatial platform, where places are symmetrically emplaced according to a certain rule. We assume that the payoff function $\boldsymbol{v}$ and thus the adjustment dynamics $\boldsymbol{F}$ introduce no additional asymmetries. Such a condition can be formalized as the group equivariance:

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \phi), \quad g \in G, \tag{2.7}$$

where G is a group describing the symmetry of the underlying spatial platform, and $T(g)$ is the permutation matrix specified by

$$T(g)D = DT(g), \quad g \in G. \tag{2.8}$$

Example 2.2. Models with the replicator dynamics in (2.4) on a symmetric spatial platform that are endowed with the equivariance of a group G satisfy the equivariance to G in the sense of (2.7) as proved in Proposition 2.1. The proof for the logit dynamics can be treated in a similar manner. □

2.2. Classification of the Models

In the classification of economic geography models with the general framework in Section 2.1, we rely on the recent work of Akamatsu et al. (2021). This reference considers a many-region racetrack economy, in which regions with the same characteristics are equidistantly located over a circumference. Proposition 1 in this reference shows that the endogenous spatial patterns that emerge upon the instability of the symmetric equilibrium (the uniform distribution of mobile workers) substantially differ across model classes. That is, this classification depends on the spatial scale of dispersion forces in a model. If a model is of Class I, a multimodal pattern emerges. If a model is of Class II, only a unimodal pattern emerges. If a model is of Class III, both possibilities arise, depending on the bifurcation parameter (trade freeness).

Example 2.3. Each model class includes the following typical models:

- Class I includes models by Krugman (1991), Puga (1999), Forslid and Ottaviano (2003), Pflüger (2004), and Harris and Wilson (1978).
- Class II includes models by Helpman (1998), Murata and Thisse (2005), Redding and Sturm (2008), Allen and Arkolakis (2014), Redding and Rossi-Hansberg (2017), and Beckmann (1976).
- Class III includes modelds by Tabuchi (1998), Pflüger and Südekum (2008), and Takayama and Akamatsu (2011).

□

Remark 2.2. The present thesis basically provides general results that are applicable to any model class. Chapters 4–6, however, are not applicable to some kind of models. We employ the replicator dynamics and focus on a special kind of corner solutions, called invariant patterns, which admit $\boldsymbol{\lambda}$ to have zero components. Hence, it cannot be applied to models that do not take corner solutions due to the existence of the housing market such as Helpman (1998) and Allen and Arkolakis (2014) models. □

2.3. Examples of Economic Geography Models and their Group Equivariance

We briefly introduce some particular models and explain their group equivariance for symmetric spatial platforms.

2.3.1. Examples of Economic Geography Models

We briefly introduce the multi-regional version of Forslid and Ottaviano (2003), Helpman (1998), and Pflüger and Südekum (2008) models as a representative of Class I, II, and III, respectively. We employ these models for numerical bifurcation and stability analysis throughout the present thesis. The fundamental logic of these models are investigated in the work of Akamatsu et al. (2021).

Forslid and Ottaviano (2003) (FO) Model

There are two types of workers: skilled and unskilled workers. The total endowments of skilled and unskilled workers are H and L, respectively. Skilled workers are mobile across K places. The number of skilled workers in place i is denoted by λ_i under the normalizing constraint $\sum_{i=1}^{K} \lambda_i = 1$. Unskilled workers are immobile and are distributed equally across all places with L/K. For simplicity, we assume that $H = 1$ and $L/K = 1$.

There are two industrial sectors: manufacturing (M) and agriculture (A). The A-sector is modeled by perfect competition and requires a unit input of unskilled workers to produce one unit of goods. The M-sector is modeled by Dixit-Stiglitz monopolistic competition and requires both skilled and unskilled workers as the input.

Preferences over the M-sector and A-sector goods are identical across individuals. The utility function U of an individual in place i is defined by

$$U_i = \mu \ln C_i^{\mathrm{M}} + (1 - \mu) \ln C_i^{\mathrm{A}}, \tag{2.9}$$

where $\mu \in (0,1)$ is the constant expenditure share of manufacturing sector goods, C_i^{A} stands for the consumption of the A-sector product in place i, and C_i^{M} represents the manufacturing aggregates in place i, defined as

$$C_i^{\mathrm{M}} \equiv \left(\sum_{j=1}^{K} \int_0^{n_j} q_{ji}(\ell)^{(\sigma-1)/\sigma} d\ell \right)^{\sigma/(\sigma-1)}, \tag{2.10}$$

where $q_{ji}(\ell)$ represents the consumption in place i of a variety $\ell \in [0, n_j]$ produced in place $j \in P$, n_j stands for the number of produced varieties at place j, and $\sigma \in (1, \infty)$ denotes the constant elasticity of substitution between any two varieties.

The transportation cost for the M-sector goods are assumed to take the iceberg form. For each unit of M-sector goods transported from place i to j $(\neq i)$, only a fraction $1/\tau_{ij} < 1$ arrives ($\tau_{ii} = 1$ for all i), and $\tau_{ij} = \tau_{ij}(\tau)$ is a function in the transportation cost parameter $\tau > 0$ defined as

$$\tau_{ij} = \exp[\tau\, m(i,j)], \tag{2.11}$$

where $m(i,j)$ is the shortest distance between places i and j. The trade freeness ϕ is defined as

$$\phi = \exp[-(\sigma-1)\tau], \quad \phi \in (0,1). \tag{2.12}$$

Note that ϕ is inversely proportional to τ. Then, the spatial discounting factor d_{ij} is represented as

$$d_{ij} = \phi^{m(i,j)} = \tau_{ij}{}^{-(\sigma-1)}. \tag{2.13}$$

The indirect utility v_i in place i is given by

$$v_i = \frac{\mu}{\sigma-1} \ln \Delta_i + \ln w_i, \tag{2.14}$$

where $\Delta_j = \sum_{k=1}^{K} d_{kj}\lambda_k$. The market equilibrium wage w_i is given by the equilibrium wage equation:

$$w_i = \frac{\mu}{\sigma} \sum_{j=1}^{K} \frac{d_{ij}}{\Delta_j}(w_j\lambda_j + 1). \tag{2.15}$$

With the notations

$$\boldsymbol{w} = (w_i), \quad D = [d_{ij}], \quad \Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_K), \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_K), \tag{2.16}$$

the equation (2.15) is written as

$$\boldsymbol{w} = \frac{\mu}{\sigma} D\Delta^{-1}(\Lambda\boldsymbol{w} + \mathbf{1}), \tag{2.17}$$

which can be solved for w_i as

$$\boldsymbol{w} = \frac{\mu}{\sigma} \left(I - \frac{\mu}{\sigma} D\Delta^{-1}\Lambda \right)^{-1} D\Delta^{-1}\mathbf{1}. \tag{2.18}$$

Helpman (1998) (Hm) Model

The Hm model assumes that all workers are mobile across K places. The total endowment of mobile workers is H. For simplicity, we set $H = 1$.

There are two industrial sectors: the housing (H) sector and the manufacturing (M) sector. The amount of housing stock in place i is denoted by a_i. The M-sector is based the same assumption as those of the FO model. The utility function U of an individual in place i is defined by

$$u_i = \mu \ln C_i^{\mathrm{M}} + (1 - \mu) \ln C_i^{\mathrm{H}}, \tag{2.19}$$

where C_i^{H} represents the consumption of H-sector goods.

The indirect utility function v_i in place i is given by

$$v_i = \frac{\mu}{\sigma - 1} \ln \tilde{\Delta}_i + \mu \ln \frac{\lambda_i(w_i + r)}{a_i} - \ln \frac{\lambda_i}{a_i}, \tag{2.20}$$

where $\tilde{\Delta}_i = \sum_{j=1}^{K} d_{ji} w_j^{1-\sigma} \lambda_j$. The market equilibrium wage w_i is given by the equilibrium wage equation:

$$w_i \lambda_i = \mu \sum_{j=1}^{K} \frac{d_{ij} w_i^{1-\sigma} \lambda_i}{\sum_{k=1}^{K} d_{kj} w_k^{1-\sigma} \lambda_k} (w_i + r) \lambda_j, \tag{2.21}$$

where r represents dividend of rental revenue. For simplicity, we set $r = 1$ and $a_i = 1$.

Pflüger and Südekum (2008) (PS) Model

The PS model is based on the same assumptions as those of the FO model. This model introduces the H-sector and employs a quasi-linear logarithmic function as the utility function instead of (2.9):

$$U_i = \mu \ln C_i^{\mathrm{M}} + \gamma \ln C_i^{\mathrm{H}} + C_i^{\mathrm{A}}, \tag{2.22}$$

where $\gamma \in (0, 1)$ denotes the constant expenditure share of H-sector goods.

The indirect utility v_i in place i is given by

$$v_i = \frac{\mu}{\sigma - 1} \ln \Delta_i - \gamma \ln \frac{\lambda_i + 1}{a_i} + w_i, \tag{2.23}$$

where the market equilibrium wage w_i is given by

$$w_i = \frac{\mu}{\sigma} \sum_{j=1}^{K} \frac{d_{ij}}{\Delta_j} (\lambda_j + 1). \tag{2.24}$$

For simplicity, we set $a_i = 1$.

2.3.2. Group Equivariance

We have the following proposition for the equivariance of the FO model, the Hm model, and the PS model with the replicator dynamics for symmetric spatial platforms.

Proposition 2.1. *For a spatial platform that has the symmetry described by a group G, the FO model, the Hm model, and the PS model with the replicator dynamics in* (2.4) *are equivariant to G in the sense of* (2.7)*, i.e.,*

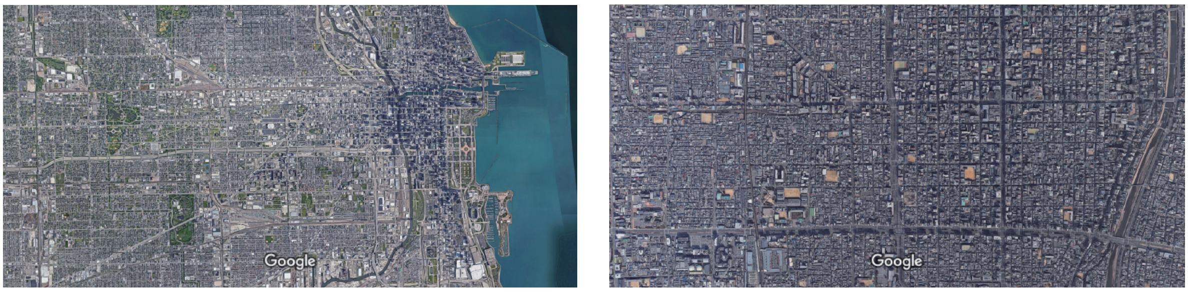
$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \phi), \quad g \in G \tag{2.25}$$

for some permutation matrix $T(g)$ of G.

Proof. We treat the case of an $n \times n$ square lattice with $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ to be introduced in Section 3.2. Note that the concrete form of $T(g)$ is to be given in Section 3.4.1. Each element g of G acts as a permutation of place numbers $(1, \ldots, K)$, and the action of $g \in G$ is expressed as $g : i \mapsto i^*$. For the indirect utility function v_i in (2.14) for the FO model, that in (2.20) for the Hm model, and that in (2.23) for the PS model, we have $v_i(T(g)\boldsymbol{\lambda}, \phi) = v_{i^*}(\boldsymbol{\lambda}, \phi)$ because of the definition of the transport cost parameter in (2.11). We also have $\bar{v}(T(g)\boldsymbol{\lambda}, \phi) = \bar{v}(\boldsymbol{\lambda}, \phi)$ by (2.5). Therefore, we have

$$F_i(T(g)\boldsymbol{\lambda}, \phi) = (v_{i^*}(\boldsymbol{\lambda}, \phi) - \bar{v}(\boldsymbol{\lambda}, \phi))\lambda_{i^*} = F_{i^*}(\boldsymbol{\lambda}, \phi) \tag{2.26}$$

for the function F_i in (2.2). This proves the equivariance in (2.25). □



(a) Chicago (the U.S.) (b) Kyoto (Japan)

Figure 3.1: Satellite photographs of cities provided by Google Maps displaying square road networks.

3. Bifurcation Mechanism from the Uniform Distribution on a Square Lattice

3.1. Introduction

Square road networks prosper worldwide. Chicago (the U.S.) and Kyoto (Japan), for example, are well-known to accommodate such square networks historically (see Fig. 3.1). We intend to elucidate the mechanism of economic agglomerations on such square networks as an important contribution of nonlinear mathematics to spatial economics.

In spatial economics, the mechanism of economic agglomerations is highlighted as the most important topic. After a pioneering work by Krugman (1991), bifurcation is welcomed as a catalyst to engender a core place and a peripheral place from two identical places. The study of spatial agglomerations have come to be extended from the two-places economy to a racetrack economy (one-dimension) and, in turn, to explain various poly-centric agglomerations (Tabuchi and Thisse, 2011; Ikeda et al., 2012a; Akamatsu et al., 2012). In economic geography, central place theory (Christaller, 1933; Lösch, 1940) envisaged the emergence of hexagonal agglomerations based on the distribution of cities and towns in Southern Germany. The existence of the hexagonal distribution of mobile production factors (e.g., firms and workers) was shown based on a microeconomic foundation (Eaton and Lipsey, 1975). To explain the mechanism of economic agglomerations in the real world, spatial platforms for economic geography models need to be extended to two-dimensional spaces as conducted in this chapter.

Lattice economies, including hexagonal and square lattices, can accommodate various two-dimensional agglomeration patterns of economic interest. Motivated by hexagonal agglomerations in central place theory, Ikeda and Murota (2014) elucidated the bifurcation mechanism of economic geography models on a hexagonal lattice. The stability of bifurcating solutions from the uniform distribution was investigated to demonstrate that theoretically predicted bifurcating solutions, including hexagonal patterns, are all unstable just after the bifurcation (Ikeda et al., 2018a). Geometrical distributions that are solutions to the governing equation of an economic geography model with the replicator dynamics, irrespective of the value of the bifurcation (transport cost) parameter, are called invariant patterns and were demonstrated to represent economic agglomerations of great economic interest (Ikeda et al., 2019a).

Yet the bifurcation mechanism of economic geography models on a square lattice is not understood to the full extent. Some studies dealt with economic agglomerations on a square lattice (Clarke and Wilson, 1983; Weidlich and Haag, 1987; Munz and Weidlich, 1990; Brakman et al.,

1999) but are not based on economic geography models. As a pioneering study that combined a square lattice with an economic geography model, Ikeda et al. (2018b) investigated a break bifurcation point on the uniform distribution and indicated the occurrence of period-doubling bifurcation. They, however, found just a fraction of bifurcating solutions on a square lattice by relying on an ad hoc procedure.

That said, this chapter aims to develop group-theoretic bifurcation theory for economic geography models on a square lattice that has the symmetry described by the finite group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. We focus on a bifurcation mechanism due to the geometrical symmetry. We present an exhaustive list of bifurcating solutions from the uniform distribution on this lattice. The list of bifurcating solutions advanced in this chapter would be of assistance in the study of economic agglomerations. We futhermore pay a special attention to the symmetry of two half branches at a bifurcation point. We obtain theoretical conditions for the symmetry and the asymmetry of such bifurcating half branches. The present theory is applicable to any economic geography models with a single degree of freedom at each node.

Many pattern-formation phenomena have been modeled by partial differential equations with group equivariance on an infinite plane. As the mathematical model of reaction-diffusion models, Navier-Stokes flow, and the Bénard problem, a system that is equivariant to the infinite group $D_4 \ltimes T^2$ (T^2 expresses the two-torus of translation symmetries) has been studied (Dionne et al., 1997; Golubitsky and Stewart, 2002). As for economic agglomerations described by economic geography models, it is essential to consider a discretized finite plane. For this reason, we employ the finite group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

This chapter is organized as follows. Section 3.2 introduces an $n \times n$ square lattice with symmetry labeled by the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ and classifies square patterns for economic agglomerations on this lattice. Section 3.3 gives derivation of the irreducible representations of the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. Section 3.4 provides the matrix representations of this group. Section 3.5, as well as Appendix A.4, presents group-theoretic bifurcation analysis by using equivariant branching lemma and that by solving bifurcation equations. Section 3.6 summarizes results of the stability of bifurcating solutions. Section 3.7 applies group-theoretic bifurcation analysis to typical economic geography models on this lattice and conducts numerical simulations based on theoretical results elucidated in the previous sections.

3.2. Square Lattice and its Symmetry

In this section, we introduce an $n \times n$ finite square lattice comprising a system of uniformly distributed $n \times n$ places. We allocate discretized degrees-of-freedom to each node of this lattice. We apply periodic boundary conditions to this lattice. It allows us to express infiniteness and to avoid heterogeneity due to boundaries. Periodic repetition of this lattice covers an infinite two-dimensional plane.

Using a group consisting of D_4 and $\mathbb{Z}_n \times \mathbb{Z}_n$, we express the symmetry of this lattice. We consider the compatibility of n with square patterns of interest on this lattice. We present and classify subgroups expressing the symmetry of square patterns. The study conducted in this section is purely geometric and involves no bifurcation mechanism. It forms, however, an important foundation of group-theoretic bifurcation analysis in Section 3.5 and Appendix A.4.

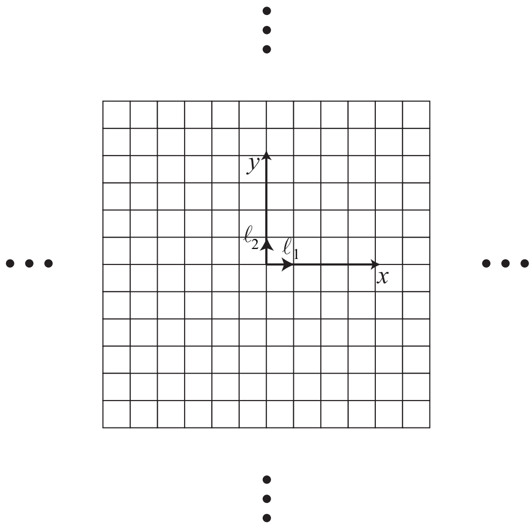


Figure 3.2: An infinite square lattice.

This section is organized as follows. An infinite square lattice is introduced in Section 3.2.1. Square patterns on this lattice are described in Section 3.2.2. The $n \times n$ square lattice is introduced in Section 3.2.3. The group expressing the symmetry of this lattice is given in Section 3.2.4.

3.2.1. Infinite Square Lattice

We introduce an infinite square lattice as a set of integer combinations of oblique basis vectors

$$\boldsymbol{\ell}_1 = d \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{\ell}_2 = d \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.1}$$

where $d > 0$ means the length of these vectors. We denote the infinite square lattice as

$$\mathcal{H} = \{n_1 \boldsymbol{\ell}_1 + n_2 \boldsymbol{\ell}_2 \mid n_1, n_2 \in \mathbb{Z}\}, \tag{3.2}$$

where $\mathbb{Z}$ denotes the set of integers. Figure 3.2 depicts the infinite square lattice.

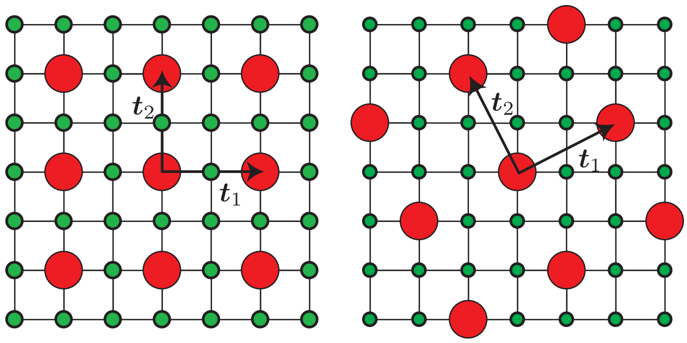
To represent square patterns on the infinite square lattice, we consider a sublattice spanned by basis vectors

$$\boldsymbol{t}_1 = \alpha \boldsymbol{\ell}_1 + \beta \boldsymbol{\ell}_2, \quad \boldsymbol{t}_2 = -\beta \boldsymbol{\ell}_1 + \alpha \boldsymbol{\ell}_2, \tag{3.3}$$

where α and β are integer-valued parameters with $(\alpha, \beta) \neq (0, 0)$. We denote the sublattice by $\mathcal{H}(\alpha, \beta)$, that is,

$$\begin{aligned} \mathcal{H}(\alpha, \beta) &= \{n_1 \boldsymbol{t}_1 + n_2 \boldsymbol{t}_2 \mid n_1, n_2 \in \mathbb{Z}\} \\ &= \{(n_1\alpha - n_2\beta)\boldsymbol{\ell}_1 + (n_1\beta + n_2\alpha)\boldsymbol{\ell}_2 \mid n_1, n_2 \in \mathbb{Z}\} \\ &= \left\{ \begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \middle| \; n_1, n_2 \in \mathbb{Z} \right\}. \end{aligned} \tag{3.4}$$

We see that the angle between $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ is $\pi/2$. In addition, we have $|\boldsymbol{t}_1| = |\boldsymbol{t}_2|$. Thus, the sublattice $\mathcal{H}(\alpha, \beta)$ represents a square pattern (see Fig. 3.3).



(a) $(\alpha, \beta) = (1, 0)$ (b) $(\alpha, \beta) = (1, 1)$

Figure 3.3: Square patterns represented by sublattices.

We define the spatial period L as

$$L = d\sqrt{\alpha^2 + \beta^2}, \tag{3.5}$$

which represents the common length of the basis vectors $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$. We refer to

$$\frac{L}{d} = \sqrt{\alpha^2 + \beta^2} \tag{3.6}$$

as the normalized spatial period, which is an important index for characterizing the size of a square pattern. Although this definition refers to the basis vectors $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$, the spatial period L, as well as the normalized spatial period L/d, is in fact determined by the sublattice $\mathcal{H}(\alpha, \beta)$, as seen from (3.8) with (3.7) below.

The normalized spatial period L/d in (3.6) takes specific values $\sqrt{1}$, $\sqrt{2}$, $\sqrt{4}$, $\sqrt{5}$, ... as a consequence of the fact that α and β are integers. The square pattern with $L/d = 1$ is called the uniform distribution. The normalized spatial period is obtained from (3.6) as

$$\begin{aligned}
\frac{L}{d} &= \sqrt{\alpha^2 + \beta^2} \\
&= \sqrt{1}, \sqrt{2}, \sqrt{4}, \sqrt{5}, \sqrt{8}, \sqrt{9}, \sqrt{10}, \sqrt{13}, \sqrt{16}, \sqrt{17}, \sqrt{18}, \sqrt{20}, \sqrt{25}, \ldots \\
&= \begin{cases} 1, 2, 3, 4, 5, \ldots, \\ \sqrt{2}, \sqrt{5}, \sqrt{8}, \sqrt{10}, \sqrt{13}, \sqrt{17}, \sqrt{18}, \sqrt{20}, \ldots. \end{cases}
\end{aligned}$$

The parameter values are given as follows:

$$(\alpha, \beta) = \begin{cases} (1,0) & (L/d = 1), \\ (1,1) & (L/d = \sqrt{2}), \\ (2,0) & (L/d = 2), \\ (2,1) & (L/d = \sqrt{5}), \\ (2,2) & (L/d = \sqrt{8}), \\ (3,0) & (L/d = 3), \\ (3,1) & (L/d = \sqrt{10}), \\ (3,2) & (L/d = \sqrt{13}), \\ (4,0) & (L/d = 4), \\ (4,1) & (L/d = \sqrt{17}), \\ (3,3) & (L/d = \sqrt{18}), \\ (4,2) & (L/d = \sqrt{20}), \\ (4,3) & (L/d = 5), \\ (5,0) & (L/d = 5), \ldots. \end{cases}$$

3.2.2. Description of Square Patterns

Sublattices introduced in the previous subsection describe square patterns on an infinite square lattice. Using the parameter values of the sublattices, we classify square patterns into several types.

Parameterization of Square Patterns

In the parameterization (α, β) of sublattices, let us note its non-uniqueness that different parameter values of (α, β) can sometimes result in the same sublattice $\mathcal{H}(\alpha, \beta)$. We define

$$D = D(\alpha, \beta) = \alpha^2 + \beta^2, \tag{3.7}$$

which is a positive integer for $(\alpha, \beta) \neq (0, 0)$. It will be shown later in this subsection that D is an invariant in this parameterization, that is, we have the following implication:

$$\mathcal{H}(\alpha, \beta) = \mathcal{H}(\alpha', \beta') \quad \Longrightarrow \quad D(\alpha, \beta) = D(\alpha', \beta'). \tag{3.8}$$

Then, the parameter space for sublattices is given as follows:

Proposition 3.1. *Square sublattices $\mathcal{H}(\alpha, \beta)$ are parameterized, one-to-one, by*

$$\{(\alpha, \beta) \in \mathbb{Z}^2 \mid \alpha > 0,\ \beta \geq 0\}. \tag{3.9}$$

Two sublattices $\mathcal{H}(\alpha, \beta)$ and $\mathcal{H}(\beta, \alpha)$ are not identical in general, but are mirror images with respect to the y-axis. They are regarded as the same essentially. We call two sublattices essentially different if they are neither identical nor mirror images with respect to the y-axis. Essentially different square sublattices are parameterized as follows:

Table 3.1: The values of $D(\alpha, \beta)$ for (α, β) in (3.10).

$\alpha \setminus \beta$	0	1	2	3	4	5	6	7
1	1	2						
2	4	5	8					
3	9	10	13	18				
4	16	17	20	$\boxed{25}$	32			
5	$\boxed{25}$	26	29	34	41	$\boxed{50}$		
6	36	37	40	45	52	61	72	
7	49	$\boxed{50}$	53	58	65	74	85	98

Proposition 3.2. *Essentially different square sublattices $\mathcal{H}(\alpha, \beta)$ are parameterized, one-to-one, by*

$$\{(\alpha, \beta) \in \mathbb{Z}^2 \mid \alpha \geq \beta \geq 0,\ \alpha \neq 0\}. \tag{3.10}$$

Table 3.1 shows the values of $D = D(\alpha, \beta)$ for (α, β) with $7 \geq \alpha \geq \beta \geq 0$, $\alpha \neq 0$. It is worth noting that the values of D in this table are all distinct with the exceptions of $D(5, 0) = D(4, 3) = 25$ and $D(5, 5) = D(7, 1) = 50$. This means, in particular, that smaller square patterns (with $D < 25$) are uniquely determined by their spatial period L, which is related to D as

$$\frac{L}{d} = \sqrt{D} \tag{3.11}$$

by (3.6) and (3.7).

Proofs of (3.8) *and Propositions 3.1 and 3.2*

First, recall that $\mathcal{H}(\alpha, \beta)$ is generated by $(\boldsymbol{t}_1, \boldsymbol{t}_2) = (\boldsymbol{t}_1(\alpha, \beta), \boldsymbol{t}_2(\alpha, \beta))$ in (3.3), which can be expressed as

$$\begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

The determinant of this coefficient matrix coincides with $D(\alpha, \beta)$ in (3.7), i.e.,

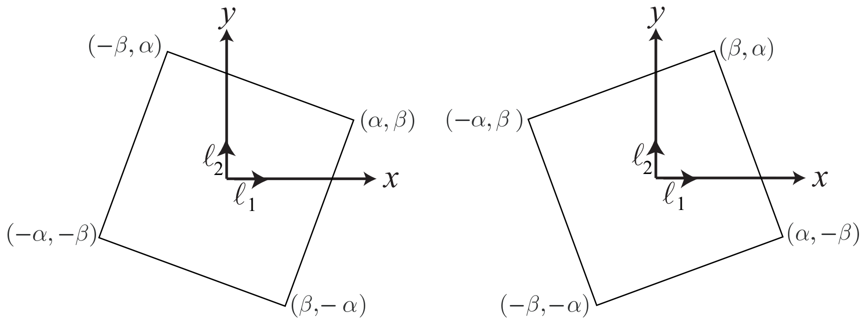
$$D(\alpha, \beta) = \alpha^2 + \beta^2 = \det \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

If $\mathcal{H}(\alpha', \beta') \subseteq \mathcal{H}(\alpha, \beta)$, then

$$\begin{bmatrix} \alpha' & -\beta' \\ \beta' & \alpha' \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

for some integers $x_{11}, x_{12}, x_{21}, x_{22}$. Hence, $D(\alpha', \beta')$ is a multiple of $D(\alpha, \beta)$. Exchanging the roles of (α, β) and (α', β'), we have (3.8).

Next, we derive the parameter spaces (3.9) and (3.10) for $\mathcal{H}(\alpha, \beta)$. We observe geometrically (see Fig. 3.4(a)) that $\mathcal{H}(\alpha', \beta') = \mathcal{H}(\alpha, \beta)$ if and only if $\boldsymbol{t}'_1 = \alpha' \boldsymbol{\ell}_1 + \beta' \boldsymbol{\ell}_2$ is obtained from



(a) A square for (3.12) (b) A square for (3.13)

Figure 3.4: Squares associated with (3.12) and (3.13).

$\boldsymbol{t}_1 = \alpha \boldsymbol{\ell}_1 + \beta \boldsymbol{\ell}_2$ by a rotation at an angle that is a multiple of $\pi/2$, i.e., $\boldsymbol{t}_1' = R_4{}^k\, \boldsymbol{t}_1$ with

$$R_4 = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

for some $k \in \{0, 1, 2, 3\}$. Since

$$R_4 \boldsymbol{t}_1 = R_4(\alpha \boldsymbol{\ell}_1 + \beta \boldsymbol{\ell}_2) = \alpha(\boldsymbol{\ell}_2) + \beta(-\boldsymbol{\ell}_1) = \begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

we have $\mathcal{H}(\alpha', \beta') = \mathcal{H}(\alpha, \beta)$ if and only if

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

for some $k \in \{0, 1, 2, 3\}$. Therefore, we obtain the same lattice for the following four parameter values:

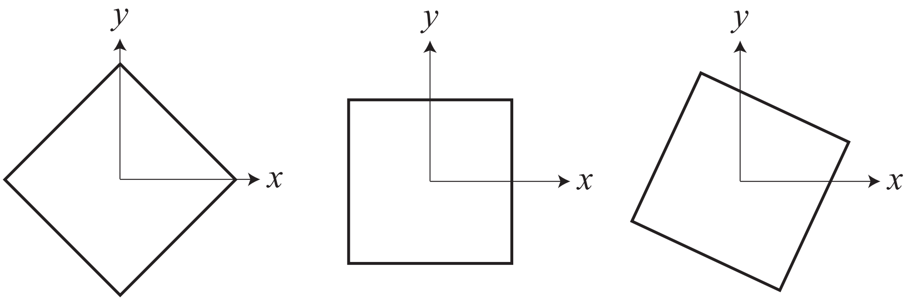
$$(\alpha, \beta),\ (-\beta, \alpha),\ (-\alpha, -\beta),\ (\beta, -\alpha). \tag{3.12}$$

This allows us to adopt (3.9) as the parameter space for $\mathcal{H}(\alpha, \beta)$, by which we mean that, for every $(\alpha', \beta') \neq (0, 0)$ in $\mathbb{Z}^2$, the sublattice $\mathcal{H}(\alpha', \beta')$ is the same as the sublattice $\mathcal{H}(\alpha, \beta)$ for some (uniquely determined) (α, β) in (3.9). It should be mentioned, in particular, that $\mathcal{H}(0, \beta) = \mathcal{H}(\beta, 0)$ by (3.12). Hence, we have $\alpha > 0$ in (3.9).

Geometrically, the sublattices for (α, β) and (β, α) are mirror images with respect to the line $x = y$. In this sense, we regard $\mathcal{H}(\alpha, \beta)$ and $\mathcal{H}(\beta, \alpha)$ as essentially the same. Thus, we regard the following four parameter values as essentially equivalent to (α, β):

$$(\beta, \alpha),\ (-\alpha, \beta),\ (-\beta, -\alpha),\ (\alpha, -\beta). \tag{3.13}$$

See Fig. 3.4(b) for the square of (3.13). If $\beta = 0$ or $\alpha = \beta$, the set of four parameters in (3.13) is identical to the set in (3.12). This is because the lattices for $\beta = 0$ or $\alpha = \beta$ are symmetric with respect to the line $x = y$.



(a) Type V (b) Type M (c) Type T

Figure 3.5: Square patterns of three types that are centered at the origin.

Thus, essentially equivalent parameter values can be summarized as follows:

$$(\alpha,\beta),\ (-\beta,\alpha),\ (-\alpha,-\beta),\ (\beta,-\alpha),\ (\beta,\alpha),\ (-\alpha,\beta),\ (-\beta,-\alpha),\ (\alpha,-\beta). \tag{3.14}$$

which reduces in a special case of $\beta = 0$ to

$$(\alpha,0),\ (0,\alpha),\ (-\alpha,0),\ (0,-\alpha) \tag{3.15}$$

or in another special case of $\alpha = \beta$ to

$$(\alpha,\alpha),\ (-\alpha,\alpha),\ (-\alpha,-\alpha),\ (\alpha,-\alpha). \tag{3.16}$$

On the basis of the observations above, (3.10) can be adopted as the parameter space for essentially different sublattices. This means that every $(\alpha,\beta) \neq (0,0)$ in $\mathbb{Z}^2$ is essentially equivalent to some (uniquely determined) member in (3.10).

Types of Square Patterns

We define the tilt angle φ of $\mathcal{H}(\alpha,\beta)$ as

$$\cos\varphi = \frac{(\boldsymbol{\ell}_1)^\top \boldsymbol{t}_1}{\|\boldsymbol{\ell}_1\| \cdot \|\boldsymbol{t}_1\|}, \tag{3.17}$$

where (α,β) belongs to the parameter space in (3.9) or (3.10). This definition is equivalent to

$$\varphi = \arcsin\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right). \tag{3.18}$$

With reference to the tilt angle φ, we classify square patterns into three types:

$$\begin{cases} \text{type V} & \text{if } \varphi = 0, \\ \text{type M} & \text{if } \varphi = \pi/4, \\ \text{type T} & \text{otherwise.} \end{cases} \tag{3.19}$$

Figure 3.5 depicts square patterns of these types that are centered at the origin, where "V" indicates that the x-axis contains a vertex of the square, "M" denotes that the x-axis contains the midpoint of two neighboring vertices of that square, and "T" means "tilted." Using the parameter (α, β), we also have

$$\begin{cases} \text{type V} & \text{if } (\alpha,\beta) = (\alpha,0)\ (\alpha \geq 1), \\ \text{type M} & \text{if } (\alpha,\beta) = (\beta,\beta)\ (\beta \geq 1), \\ \text{type T} & \text{otherwise,} \end{cases} \tag{3.20}$$

where the parameter space for type T depends on the choice of (3.9) or (3.10) as

$$\text{For (3.9): } \{(\alpha,\beta) \mid \alpha > 0,\ \beta \geq 0,\ \alpha \neq \beta\}, \tag{3.21}$$

$$\text{For (3.10): } \{(\alpha,\beta) \mid \alpha > \beta \geq 0\}. \tag{3.22}$$

Accordingly, the parameter spaces in (3.9) and (3.10) are divided, respectively, into three parts:

$$\{(\alpha,0) \mid \alpha \geq 1\} \ \cup\ \{(\beta,\beta) \mid \beta \geq 1\} \ \cup\ \{(\alpha,\beta) \mid \alpha > 0,\ \beta \geq 0,\ \alpha \neq \beta\}, \tag{3.23}$$

$$\{(\alpha,0) \mid \alpha \geq 1\} \ \cup\ \{(\beta,\beta) \mid \beta \geq 1\} \ \cup\ \{(\alpha,\beta) \mid \alpha > \beta \geq 0\}. \tag{3.24}$$

The types V, M, and T are correlated with the normalized spatial period as

$$L/d = \begin{cases} \sqrt{4}, \sqrt{9}, \sqrt{16}, \sqrt{25}, \ldots & \text{for type V,} \\ \sqrt{2}, \sqrt{8}, \sqrt{18}, \sqrt{32}, \ldots & \text{for type M,} \\ \sqrt{5}, \sqrt{10}, \sqrt{13}, \sqrt{17}, \ldots & \text{for type T.} \end{cases}$$

It should be emphasized, however, that the type does not always determine, nor is determined by, the spatial period. This is demonstrated by the two lattices $\mathcal{H}(5,0)$ and $\mathcal{H}(4,3)$. These lattices share the same normalized spatial period $L/d = \sqrt{25}$ but are different types; the former is of type V and the latter of type T.

3.2.3. Square Lattice with Periodic Boundaries

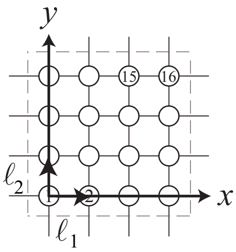
We introduce an $n \times n$ finite square lattice $\mathcal{H}_n$ as a subset of the infinite square lattice $\mathcal{H}$ spreading over the entire plane. We define $\mathcal{H}_n$ as

$$\mathcal{H}_n = \{n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2 \mid n_i \in \mathbb{Z},\ 0 \leq n_i \leq n-1\ (i = 1,2)\}, \tag{3.25}$$

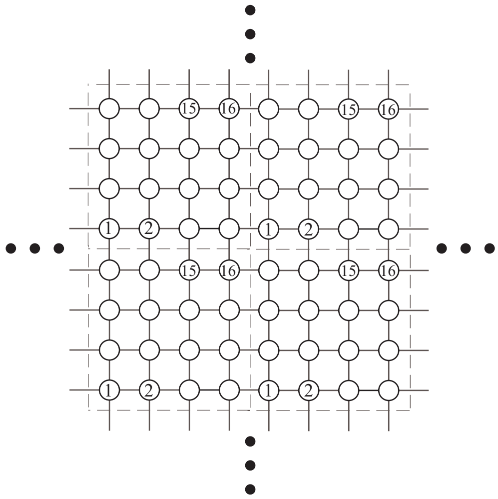
which consists of integer combinations with coefficients between 0 and $n-1$. This is a finite set comprising n^2 elements, where n represents the size of the lattice. Figure 3.6(a) depicts the 4×4 square lattice.

The infinite square lattice $\mathcal{H}$ is regarded as a periodic extension of the $n \times n$ square lattice $\mathcal{H}_n$ with the two-dimensional period of $(n\boldsymbol{\ell}_1, n\boldsymbol{\ell}_2)$. In other words, $\mathcal{H}$ is regarded as being covered by translations of $\mathcal{H}_n$ by vectors of the form $m_1(n\boldsymbol{\ell}_1) + m_2(n\boldsymbol{\ell}_2)$ with integers m_1 and m_2. A point $n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2$ in $\mathcal{H}$ corresponds to $n_1'\boldsymbol{\ell}_1 + n_2'\boldsymbol{\ell}_2$ in $\mathcal{H}_n$ for (n_1', n_2') given by

$$n_1' \equiv n_1 \mod n, \quad n_2' \equiv n_2 \mod n. \tag{3.26}$$



(a) The 4×4 square lattice



(b) Periodic boundaries

Figure 3.6: A system of places on the 4×4 square lattice with periodic boundaries.

Figure 3.6(b) depicts the 4×4 square lattice with periodic boundaries.

For the sublattice $\mathcal{H}(\alpha, \beta)$ of $\mathcal{H}$, we may consider its portion $\mathcal{H}(\alpha, \beta) \cap \mathcal{H}_n$ contained in $\mathcal{H}_n$ and assume that the periodic extension of this portion coincides with $\mathcal{H}(\alpha, \beta)$ itself. If this is the case, we say that (α, β) is compatible with n, or n is compatible with (α, β). Using the Minkowski sum[3] of $\mathcal{H}(\alpha, \beta) \cap \mathcal{H}_n$ and $\mathcal{H}(n, 0)$, we have the condition for compatibility as

$$(\mathcal{H}(\alpha, \beta) \cap \mathcal{H}_n) + \mathcal{H}(n, 0) = \mathcal{H}(\alpha, \beta), \tag{3.27}$$

which is equivalent to

$$\mathcal{H}(n, 0) \subseteq \mathcal{H}(\alpha, \beta). \tag{3.28}$$

We can restate the compatibility condition as follows:

Proposition 3.3. *The size n of $\mathcal{H}_n$ is compatible with (α, β) if and only if n is a multiple of $D(\alpha, \beta)/\gcd(\alpha, \beta)$, that is,*

$$n = m \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}, \quad m = 1, 2, \ldots. \tag{3.29}$$

Proof. By (3.28), the size n is compatible with (α, β) if and only if

$$\begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = n \begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix}$$

for some integers $x_{11}, x_{12}, x_{21}, x_{22}$, where $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are defined in (3.3). Substituting

$$\begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

[3] For two sets $X, Y \subseteq \mathbb{Z}^2$, their Minkowski sum $X + Y$ is defined as $X + Y = \{\boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{x} \in X,\ \boldsymbol{y} \in Y\}$.

into the above equation and multiplying the inverse of $\begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix}$ from the left, we obtain

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

from which

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = n \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}^{-1} = \frac{n}{D(\alpha,\beta)} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \frac{n \ \gcd(\alpha,\beta)}{D(\alpha,\beta)} \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\hat{\beta} & \hat{\alpha} \end{bmatrix},$$

where $\hat{\alpha} = \alpha/\gcd(\alpha,\beta)$ and $\hat{\beta} = \beta/\gcd(\alpha,\beta)$. This shows that $x_{11}, x_{12}, x_{21}, x_{22}$ are integers if and only if n is a multiple of $D(\alpha,\beta)/\gcd(\alpha,\beta)$. □

With the classification of three types in (3.20), the compatibility condition (3.29) in Proposition 3.3 shows the following statements:

- For a pattern $\mathcal{H}(\alpha,\beta)$ of type V, parameterized by $(\alpha,\beta) = (\alpha,0)$ with $\alpha \geq 1$, a compatible n is a multiple of α.
- For a pattern $\mathcal{H}(\alpha,\beta)$ of type M, parameterized by $(\alpha,\beta) = (\beta,\beta)$ with $\beta \geq 1$, a compatible n is a multiple of 2β.
- For a pattern $\mathcal{H}(\alpha,\beta)$ of type T, with (α,β) in (3.21) or (3.22), a compatible n is a multiple of $D(\alpha,\beta)/\gcd(\alpha,\beta)$.

To sum up, we have

$$n = \begin{cases} m\alpha \ (\alpha \geq 1) & \text{for type V,} \\ 2m\beta \ (\beta \geq 1) & \text{for type M,} \\ mD(\alpha,\beta)/\gcd(\alpha,\beta) & \text{for type T,} \end{cases} \tag{3.30}$$

where $m = 1, 2, \ldots$.

3.2.4. Group Expressing Symmetry

We introduce the group expressing the symmetry of the $n \times n$ square lattice. As a first step of bifurcation analysis of the square patterns on the $n \times n$ square lattice, we identify the subgroups expressing the symmetry of these patterns.

Symmetry of the Finite Square Lattice

The symmetry of $\mathcal{H}_n$ in (3.25) is characterized by invariance with respect to

- r: counterclockwise rotation about the origin at an angle of $\pi/2$,
- s: reflection $y \mapsto -y$,
- p_1: periodic translation along the $\boldsymbol{\ell}_1$-axis (i.e., the x-axis), and
- p_2: periodic translation along the $\boldsymbol{\ell}_2$-axis (i.e., the y-axis).

Consequently, the symmetry of the square lattice $\mathcal{H}_n$ is described by the group

$$G = \langle r, s, p_1, p_2 \rangle, \tag{3.31}$$

which is generated by r, s, p_1, and p_2 with the fundamental relations:

$$\begin{aligned} r^4 = s^2 = (rs)^2 = {p_1}^n = {p_2}^n = e, \quad p_2 p_1 = p_1 p_2, \\ r p_1 = p_2 r, \quad r p_2 = {p_1}^{-1} r, \quad s p_1 = p_1 s, \quad s p_2 = {p_2}^{-1} s, \end{aligned} \tag{3.32}$$

where e is the identity element. Each element of G can be represented uniquely in the form of

$$s^l r^m {p_1}^i {p_2}^j, \quad l \in \{0, 1\},\ m \in \{0, 1, 2, 3\},\ i, j \in \{0, 1, \ldots, n-1\}. \tag{3.33}$$

The group G contains the dihedral group

$$\langle r, s \rangle \simeq \mathrm{D}_4$$

and the cyclic groups

$$\langle p_1 \rangle \simeq \mathbb{Z}_n, \quad \langle p_2 \rangle \simeq \mathbb{Z}_n$$

as its subgroups, where $\mathbb{Z}_n$ means the cyclic group of order n, which is denoted as C_n. The group G has the structure of the semidirect product of D_4 by $\mathbb{Z}_n \times \mathbb{Z}_n$, that is, $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

Remark 3.1. A group G is said to be the semidirect product of a subgroup H by another subgroup A, denoted $G = A \rtimes H$, if

- A is a normal subgroup of G, and
- each element $g \in G$ is represented uniquely as $g = ah$ with $a \in A$ and $h \in H$.

Each element $g = ah \in G$ can also be represented uniquely in an alternative form of $g = h'a$ with $h' \in H$ and $a \in A$, since $g = ah = h(h^{-1}ah)$ and $h' = h^{-1}ah \in A$ by the normality of A. Our group $G = \langle r, s, p_1, p_2 \rangle$ is a semidirect product of $H = \mathrm{D}_4$ by $A = \mathbb{Z}_n \times \mathbb{Z}_n$, and we have $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in accordance with $g = s^l r^m {p_1}^i {p_2}^j$ in (3.33) with $s^l r^m \in \mathrm{D}_4$ and ${p_1}^i {p_2}^j \in \mathbb{Z}_n \times \mathbb{Z}_n$. For more details on the definition of semidirect product, see Curtis and Reiner (1962).

□

Subgroups for Square Patterns

The symmetry of $\mathcal{H}(\alpha, \beta) \cap \mathcal{H}_n$ is described by a subgroup of $G = \langle r, s, p_1, p_2 \rangle$, which is denoted by $G(\alpha, \beta)$. With notations[4] With notations

$$\Sigma(\alpha, \beta) = \langle r, s, {p_1}^{\alpha} {p_2}^{\beta}, {p_1}^{-\beta} {p_2}^{\alpha} \rangle, \tag{3.34}$$

$$\Sigma_0(\alpha, \beta) = \langle r, {p_1}^{\alpha} {p_2}^{\beta}, {p_1}^{-\beta} {p_2}^{\alpha} \rangle, \tag{3.35}$$

[4] The subscript "0" to $\Sigma_0(\alpha, \beta)$ indicates the lack of s.

the subgroup $G(\alpha, \beta)$ is given as follows:

$$G(\alpha, \beta) = \begin{cases} \langle r, s, p_1{}^{\alpha}, p_2{}^{\alpha} \rangle = \Sigma(\alpha, 0) & \text{if } \alpha \geq 1,\ \beta = 0 \text{ (type V)}, \\ \langle r, s, p_1{}^{\beta} p_2{}^{\beta}, p_1{}^{-\beta} p_2{}^{\beta} \rangle = \Sigma(\beta, \beta) & \text{if } \alpha = \beta,\ \beta \geq 1 \text{ (type M)}, \\ \langle r, p_1{}^{\alpha} p_2{}^{\beta}, p_1{}^{-\beta} p_2{}^{\alpha} \rangle = \Sigma_0(\alpha, \beta) & \text{otherwise (type T)}, \end{cases} \tag{3.36}$$

where the parameter (α, β) for type T runs over $\{(\alpha, \beta) \mid \alpha > 0,\ \beta \geq 0,\ \alpha \neq \beta\}$ in (3.21) or $\{(\alpha, \beta) \mid \alpha > \beta \geq 0\}$ in (3.22), depending on the adopted parameter space (3.9) or (3.10).

The parameter (α, β) must be compatible with the lattice size n via (3.30), which restricts (α, β) to stay in a bounded range. Among the square patterns of type V on the $n \times n$ square lattice, we exclude those with $\Sigma(1, 0)$ from our consideration of subgroups since $\Sigma(1, 0) = \langle r, s, p_1, p_2 \rangle$ represents the symmetry of the underlying $n \times n$ square lattice. That is, we consider $\Sigma(\alpha, 0)$ for $2 \leq \alpha \leq n$ since n is divisible by α by (3.30). A square pattern with the symmetry of $\Sigma(n, 0) = \mathrm{D}_4$, which lacks translational symmetry, is included here as a square of type V for theoretical consistency. As for type M, we must have $1 \leq \beta \leq n/2$ in $\Sigma(\beta, \beta)$ since n is divisible by 2β $(\beta \geq 1)$ by (3.30). The parameter for type T, which is dependent on the choice of (3.9) or (3.10), must stay in the range

$$\text{for (3.9): } \{(\alpha, \beta) \mid 1 \leq \alpha \leq n-1,\ 1 \leq \beta \leq n-1,\ \alpha \neq \beta\}, \tag{3.37}$$

$$\text{for (3.10): } \{(\alpha, \beta) \mid 1 \leq \beta < \alpha \leq n-1\}. \tag{3.38}$$

To sum up, the relevant subgroups of our interest are given by

$$\begin{cases} \Sigma(\alpha, 0) = \langle r, s, p_1{}^{\alpha}, p_2{}^{\alpha} \rangle \ (2 \leq \alpha \leq n) & \text{for type V}, \\ \Sigma(\beta, \beta) = \langle r, s, p_1{}^{\beta} p_2{}^{\beta}, p_1{}^{-\beta} p_2{}^{\beta} \rangle \ (1 \leq \beta \leq n/2) & \text{for type M}, \\ \Sigma_0(\alpha, \beta) = \langle r, p_1{}^{\alpha} p_2{}^{\beta}, p_1{}^{-\beta} p_2{}^{\alpha} \rangle \ ((\alpha, \beta) \in \text{(3.37) or (3.38)}) & \text{for type T}. \end{cases} \tag{3.39}$$

Recall that (α, β) must also satisfy the compatibility condition in (3.30).

3.3. Irreducible Representations of the Group for the Square Lattice

In the previous section, we introduced the $n \times n$ square lattice as a two-dimensional discretized space. We identified the symmetry of this lattice by the group in (3.31):

$$G = \langle r, s, p_1, p_2 \rangle, \tag{3.40}$$

which is composed of the dihedral group $\langle r, s \rangle \simeq \mathrm{D}_4$ expressing local square symmetry and the group $\langle p_1, p_2 \rangle \simeq \mathbb{Z}_n \times \mathbb{Z}_n$ (direct product of two cyclic groups of order n) expressing translational symmetry in two directions. In group-theoretic bifurcation analysis in Section 3.5 and Appendix A.4, we will find bifurcating solutions for each irreducible representation of this group, as each irreducible representation is associated with possible bifurcating solutions with certain symmetries. The first step of the analysis is to obtain all the irreducible representations of this group.

It is not difficult to obtain all irreducible representations for groups with simple structures such as the dihedral and cyclic groups. Since the group G in (3.40) has a far more complicated structure, it might be difficult to list all the irreducible representations in an ad hoc way. Fortunately, we can use the method of little groups in group representation theory to obtain all the irreducible representations in a systematic manner. In this section, we describe this method and construct a complete list of the irreducible representations of G. It turns out that the irreducible representations over $\mathbb{R}$ are one-, two-, four-, or eight-dimensional, and all of them are absolutely irreducible. We will use the irreducible representations derived in this manner in group-theoretic bifurcation analysis in Section 3.5 and Appendix A.4 to prove the existence of square patterns.

In this section, the matrix forms of the irreducible representations of the group G in (3.40) are listed. A systematic method using little groups to construct the irreducible representations of G is described in Appendix A.1.2. The method of little groups is applied to G in Appendix A.1.3. The irreducible representations of G are derived in Appendix A.1.4.

List of Irreducible Representations

The irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ over $\mathbb{R}$ are one-, two-, four-, or eight-dimensional. The number N_d of the d-dimensional irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ depends on n, as shown below:

$n \setminus d$	1	2	4	8
	N_1	N_2	N_4	N_8
$2m$	8	6	$3(n-2)$	$(n^2-6n+8)/8$
$2m-1$	4	1	$2(n-1)$	$(n^2-4n+3)/8$

(3.41)

where m denotes a positive integer. For some values of n, the concrete numbers N_d of the d-dimensional irreducible representations are listed in Table 3.2. This table for $n = 1$ shows that $\mathrm{D}_4 \ltimes (\mathbb{Z}_1 \times \mathbb{Z}_1)$, being isomorphic to D_4, has four one-dimensional irreducible representations and one two-dimensional ones. Four-dimensional irreducible representations exist for $n \geq 3$ and eight-dimensional ones appear for $n \geq 5$.

We have the relation

$$\sum_d d^2 N_d = 1^2 N_1 + 2^2 N_2 + 4^2 N_4 + 8^2 N_8 = 8n^2, \tag{3.42}$$

which is a special case of the well-known general identity for the number of irreducible representations over $\mathbb{C}$. This formula applies since all the irreducible representations over $\mathbb{R}$ of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ are absolutely irreducible (see Appendix A.1.4).

In the following subsections, we present the matrix forms of the irreducible representations of respective dimensions together with their characters. Table 3.3 summarizes the irreducible representations. The labels such as $(1; +, +, +)$ and $(8; k, \ell)$ represent the name of the irreducible representations.

One-Dimensional Irreducible Representations

The group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ has eight one-dimensional irreducible representations. These are labeled by

$$\begin{array}{llll} (1; +, +, +), & (1; +, -, +), & (1; -, +, +), & (1; -, -, +), \\ (1; +, +, -), & (1; +, -, -), & (1; -, +, -), & (1; -, -, -) \end{array} \tag{3.43}$$

Table 3.2: The number N_d of the d-dimensional irreducible representations of $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

$n \setminus d$	1 N_1	2 N_2	4 N_4	8 N_8	$\sum N_d$
1	4	1	0	0	5
2	8	6	0	0	14
3	4	1	4	0	9
4	8	6	6	0	20
5	4	1	8	1	14
6	8	6	12	1	27
7	4	1	12	3	20
8	8	6	18	3	35
9	4	1	16	6	27
10	8	6	24	6	44
11	4	1	20	10	35
12	8	6	30	10	54

$n \setminus d$	1 N_1	2 N_2	4 N_4	8 N_8	$\sum N_d$
13	4	1	24	15	44
14	8	6	36	15	65
15	4	1	28	21	54
16	8	6	42	21	77
17	4	1	32	28	65
18	8	6	48	28	90
19	4	1	36	36	77
20	8	6	54	36	104
21	4	1	40	45	90
⋮	⋮	⋮	⋮	⋮	⋮
42	8	6	120	190	324

Table 3.3: The irreducible representations of $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

$n \setminus d$	1	2	4	8
$2m$	$(1;+,+,+), (1;+,-,+)$ $(1;+,-,+), (1;-,-,+)$ $(1;+,+,-), (1;+,-,-)$ $(1;-,+,-), (1;-,-,-)$	$(2;+), (2;-)$ $(2;+,+), (2;+,-)$ $(2;-,+), (2;-,-)$	$(4;k,0,+), (4;k,0,-)$ $(4;k,k,+), (4;k,k,-)$ $(4;n/2,\ell,+), (4;n/2,\ell,-)$	$(8;k,\ell)$
$2m-1$	$(1;+,+,+), (1;+,-,+)$ $(1;+,-,+), (1;-,-,+)$	$(2;+)$	$(4;k,0,+), (4;k,0,-)$ $(4;k,k,+), (4;k,k,-)$	$(8;k,\ell)$

$(4;k,0;+), (4;k,0;-)$ with $1 \le k \le \lfloor (n-1)/2 \rfloor$ in (3.49);
$(4;k,k;+), (4;k,k;-)$ with $1 \le k \le \lfloor (n-1)/2 \rfloor$ in (3.50);
$(4;n/2,\ell;+), (4;n/2,\ell;-)$ with $1 \le \ell \le \lfloor (n-1)/2 \rfloor$ in (3.51);
$(8;k,\ell)$ with $1 \le \ell \le k-1$, $2 \le k \le \lfloor (n-1)/2 \rfloor$ in (3.60)

and are given by

$$
\begin{aligned}
&T^{(1;+,+,+)}(r)=1, && T^{(1;+,+,+)}(s)=1, && T^{(1;+,+,+)}(p_1)=1, && T^{(1;+,+,+)}(p_2)=1,\\
&T^{(1;+,-,+)}(r)=1, && T^{(1;+,-,+)}(s)=-1, && T^{(1;+,-,+)}(p_1)=1, && T^{(1;+,-,+)}(p_2)=1,\\
&T^{(1;-,+,+)}(r)=-1, && T^{(1;-,+,+)}(s)=1, && T^{(1;-,+,+)}(p_1)=1, && T^{(1;-,+,+)}(p_2)=1,\\
&T^{(1;-,-,+)}(r)=-1, && T^{(1;-,-,+)}(s)=-1, && T^{(1;-,-,+)}(p_1)=1, && T^{(1;-,-,+)}(p_2)=1,\\
&T^{(1;+,+,-)}(r)=1, && T^{(1;+,+,-)}(s)=1, && T^{(1;+,+,-)}(p_1)=-1, && T^{(1;+,+,-)}(p_2)=-1,\\
&T^{(1;+,-,-)}(r)=1, && T^{(1;+,-,-)}(s)=-1, && T^{(1;+,-,-)}(p_1)=-1, && T^{(1;+,-,-)}(p_2)=-1,\\
&T^{(1;-,+,-)}(r)=-1, && T^{(1;-,+,-)}(s)=1, && T^{(1;-,+,-)}(p_1)=-1, && T^{(1;-,+,-)}(p_2)=-1,\\
&T^{(1;-,-,-)}(r)=-1, && T^{(1;-,-,-)}(s)=-1, && T^{(1;-,-,-)}(p_1)=-1, && T^{(1;-,-,-)}(p_2)=-1.
\end{aligned}
\tag{3.44}
$$

Two-Dimensional Irreducible Representations

The group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ has six or one two-dimensional irreducible representations depending on whether n is even or odd. Two two-dimensional irreducible representations, which are denoted as $(2;\sigma)$ $(\sigma \in \{+,-\})$, exist for n even and are defined by

$$T^{(2;\sigma)}(r) = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \quad T^{(2;\sigma)}(s) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \tag{3.45}$$

$$T^{(2,\sigma)}(p_1) = T^{(2;\sigma)}(p_2) = \sigma \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \tag{3.46}$$

whereas $(2;-)$ is absent for n odd. The other four two-dimensional irreducible representations, denoted as $(2;\sigma_r,\sigma_s)$ $(\sigma_r,\sigma_s \in \{+,-\})$, exist when n is even and are defined by

$$T^{(2;\sigma_r,\sigma_s)}(r) = \begin{bmatrix} & \sigma_r \\ 1 & \end{bmatrix}, \quad T^{(2;\sigma_r,\sigma_s)}(s) = \sigma_s \begin{bmatrix} 1 & \\ & \sigma_r \end{bmatrix}, \tag{3.47}$$

$$T^{(2;\sigma_r,\sigma_s)}(p_1) = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \quad T^{(2;\sigma_r,\sigma_s)}(p_2) = \begin{bmatrix} -1 & \\ & -1 \end{bmatrix}. \tag{3.48}$$

Four-Dimensional Irreducible Representations

The group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ with $n \geq 3$ has four-dimensional irreducible representations. We can designate them by

$$(4;k,0,\sigma) \ \text{ with } \ 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \ \sigma \in \{+,-\}; \tag{3.49}$$

$$(4;k,k,\sigma) \ \text{ with } \ 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \ \sigma \in \{+,-\}; \tag{3.50}$$

$$(4;n/2,\ell,\sigma) \ \text{ with } \ 1 \leq \ell \leq \frac{n}{2} - 1, \ \sigma \in \{+,-\}. \tag{3.51}$$

Therein, $(4;n/2,\ell,\sigma)$ exists only for n even, and $\lfloor x \rfloor$ denotes the largest integer not larger than x for a real number x. The number of four-dimensional irreducible representations is given by

$$N_4 = \begin{cases} 3n-6 & \text{if } n \text{ is even}, \\ 2n-2 & \text{if } n \text{ is odd}. \end{cases} \tag{3.52}$$

The irreducible representation $(4; k, 0, \sigma)$ is given by

$$T^{(4;k,0,\sigma)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix}, \quad T^{(4;k,0,\sigma)}(s) = \sigma \begin{bmatrix} I & \\ & S \end{bmatrix}, \tag{3.53}$$

$$T^{(4;k,0,\sigma)}(p_1) = \begin{bmatrix} R^k & \\ & I \end{bmatrix}, \quad T^{(4;k,0,\sigma)}(p_2) = \begin{bmatrix} I & \\ & R^k \end{bmatrix}, \tag{3.54}$$

where

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \tag{3.55}$$

The irreducible representation $(4; k, k, \sigma)$ is given by

$$T^{(4;k,k,\sigma)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix}, \quad T^{(4;k,k,\sigma)}(s) = \sigma \begin{bmatrix} & S \\ S & \end{bmatrix}, \tag{3.56}$$

$$T^{(4;k,k,\sigma)}(p_1) = \begin{bmatrix} R^k & \\ & R^{-k} \end{bmatrix}, \quad T^{(4;k,k,\sigma)}(p_2) = \begin{bmatrix} R^k & \\ & R^k \end{bmatrix}. \tag{3.57}$$

The irreducible representation $(4; n/2, \ell, \sigma)$ is given by

$$T^{(4;n/2,\ell,\sigma)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix}, \quad T^{(4;n/2,\ell,\sigma)}(s) = \sigma \begin{bmatrix} S & \\ & I \end{bmatrix}, \tag{3.58}$$

$$T^{(4;n/2,\ell,\sigma)}(p_1) = \begin{bmatrix} -I & \\ & R^{-\ell} \end{bmatrix}, \quad T^{(4;n/2,\ell,\sigma)}(p_2) = \begin{bmatrix} R^{\ell} & \\ & -I \end{bmatrix}. \tag{3.59}$$

Eight-Dimensional Irreducible Representations

The group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ with $n \geq 5$ has eight-dimensional irreducible representations. We can designate them by $(8; k, \ell)$ with

$$1 \leq \ell \leq k - 1, \quad 2 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{3.60}$$

The number of eight-dimensional irreducible representations is given by

$$N_8 = \begin{cases} (n^2 - 6n + 8)/8 & \text{if } n \text{ is even}, \\ (n^2 - 4n + 3)/8 & \text{if } n \text{ is odd}. \end{cases} \tag{3.61}$$

The irreducible representation $(8; k, \ell)$ is defined as

$$T^{(8;k,\ell)}(r) = \left[\begin{array}{cc|cc} & S & & \\ I & & & \\ \hline & & & I \\ & & S & \end{array}\right], \quad T^{(8;k,\ell)}(s) = \left[\begin{array}{cc|cc} & & I & \\ & & & I \\ \hline I & & & \\ & I & & \end{array}\right], \tag{3.62}$$

$$T^{(8;k,\ell)}(p_1) = \left[\begin{array}{cc|cc} R^k & & & \\ & R^{-\ell} & & \\ \hline & & R^k & \\ & & & R^{-\ell} \end{array}\right], \quad T^{(8;k,\ell)}(p_2) = \left[\begin{array}{cc|cc} R^{\ell} & & & \\ & R^k & & \\ \hline & & R^{-\ell} & \\ & & & R^{-k} \end{array}\right] \tag{3.63}$$

with

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \tag{3.64}$$

3.4. *Representation Matrix for the Square Lattice*

In the previous section, we found the irreducible representations of the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ as preparation for group-theoretic analysis in Section 3.5 and Appendix A.4. Note that not all the irreducible representations are involved in mathematical models on the square lattice. The consideration of relevant irreducible representations is essential in group-theoretic analysis that provides accurate information about bifurcating solutions.

In this section, we first identify the irreducible representations μ that are relevant to our analysis on the square lattice. For this purpose, we derive the explicit form of the permutation representation $T(g)$ of the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ and investigate the irreducible decomposition of this permutation representation. We can exclude irreducible representations that are not contained in $T(g)$ from consideration in search of square bifurcating patterns in Section 3.5 and Appendix A.4. It turns out that the only some of the one-, two-, and four-dimensional ones are relevant, and all of the eight-dimensional ones are relevant.

We next present the transformation matrix Q for irreducible decomposition. Since the irreducible representations of the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ have a special feature of multiplicity-free, the orthogonal transformation of the Jacobian matrix $J = \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}$ of the governing equation in (2.3) takes a diagonal form

$$Q^{-1} J Q = \mathrm{diag}(e_1, \dots, e_K).$$

This diagonal form is useful in eigenanalysis of computational bifurcation analysis on the square lattice.

This section is organized as follows. The permutation representation for the square lattice is investigated in Section 3.4.1. The irreducible decomposition of the permutation representation is presented in Section 3.4.2. Transformation matrices for block-diagonalization are derived in Section 3.4.3.

3.4.1. *Representation Matrix*

In our study of a system of $K = n^2$ places on the $n \times n$ square lattice, each element g of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n) = \langle r, s, p_1, p_2 \rangle$ acts as a permutation of place numbers $(1, \dots, K)$. Consequently, the representation matrix $T(g)$ is a permutation matrix for each g. By definition, $T(g)$ has "1" at the (i, j) entry if place j is moved to place i by the action of g.

The representation matrix $T(g)$ for general n can be determined as follows. The coordinate of a place on the $n \times n$ square lattice is given by

$$\boldsymbol{x} = n_1 \boldsymbol{\ell}_1 + n_2 \boldsymbol{\ell}_2, \quad n_1, n_2 = 0, 1, \dots, n-1$$

with $\boldsymbol{\ell}_1 = d(1,0)^\top$, $\boldsymbol{\ell}_2 = d(0,1)^\top$ in (3.1), where d means the length of these vectors. Thus, the n^2 places are indexed by (n_1, n_2), and so are the rows and columns of the representation matrix $T(g)$. The action of r is expressed as

$$r \cdot \boldsymbol{\ell}_1 = \boldsymbol{\ell}_2, \quad r \cdot \boldsymbol{\ell}_2 = -\boldsymbol{\ell}_1.$$

Hence, we have

$$r \cdot \boldsymbol{x} = n_1 (r \cdot \boldsymbol{\ell}_1) + n_2 (r \cdot \boldsymbol{\ell}_2) = n_1 (\boldsymbol{\ell}_2) + n_2 (-\boldsymbol{\ell}_1) = (-n_2) \boldsymbol{\ell}_1 + n_1 \boldsymbol{\ell}_2,$$

which means that the action of r on (n_1, n_2) is given by

$$r \cdot (n_1, n_2) \equiv (-n_2, n_1) \mod n. \tag{3.65}$$

Then, the column of $T(r)$ indexed by (n_1, n_2) has “1” in the row indexed by $(-n_2 \mod n, n_1)$. Similarly, the actions of s, p_1, and p_2 are expressed as

$$s \cdot (n_1, n_2) \equiv (n_1, -n_2) \mod n, \tag{3.66}$$
$$p_1 \cdot (n_1, n_2) \equiv (n_1 + 1, n_2) \mod n, \tag{3.67}$$
$$p_2 \cdot (n_1, n_2) \equiv (n_1, n_2 + 1) \mod n. \tag{3.68}$$

The permutation representation $T(g)$ is specified by (3.65)–(3.68) above.

Example 3.1. The permutation representation for the 4×4 square lattice is given by (3.65)–(3.68) as follows:

$$T(r) = \left[\begin{array}{cccc|cccc|cccc|cccc}
1&&&&&&&&&&&&&&& \\
&&&&&&&&&&&&1&&& \\
&&&&&&&&1&&&&&&& \\
&&&&1&&&&&&&&&&& \\ \hline
&1&&&&&&&&&&&&&& \\
&&&&&&&&&&&&&1&& \\
&&&&&&&&&1&&&&&& \\
&&&&&1&&&&&&&&&& \\ \hline
&&1&&&&&&&&&&&&& \\
&&&&&&&&&&&&&&1& \\
&&&&&&&&&&1&&&&& \\
&&&&&&1&&&&&&&&& \\ \hline
&&&1&&&&&&&&&&&& \\
&&&&&&&&&&&&&&&1 \\
&&&&&&&&&&&1&&&& \\
&&&&&&&1&&&&&&&&
\end{array}\right], \quad
T(s) = \left[\begin{array}{cccc|cccc|cccc|cccc}
1&&&&&&&&&&&&&&& \\
&1&&&&&&&&&&&&&& \\
&&1&&&&&&&&&&&&& \\
&&&1&&&&&&&&&&&& \\ \hline
&&&&&&&&&&&&1&&& \\
&&&&&&&&&&&&&1&& \\
&&&&&&&&&&&&&&1& \\
&&&&&&&&&&&&&&&1 \\ \hline
&&&&&&&&1&&&&&&& \\
&&&&&&&&&1&&&&&& \\
&&&&&&&&&&1&&&&& \\
&&&&&&&&&&&1&&&& \\ \hline
&&&&1&&&&&&&&&&& \\
&&&&&1&&&&&&&&&& \\
&&&&&&1&&&&&&&&& \\
&&&&&&&1&&&&&&&&
\end{array}\right],$$

$$T(p_1) = \left[\begin{array}{cccc|cccc|cccc|cccc}
&&&1&&&&&&&&&&&& \\
1&&&&&&&&&&&&&&& \\
&1&&&&&&&&&&&&&& \\
&&1&&&&&&&&&&&&& \\ \hline
&&&&&&&1&&&&&&&& \\
&&&&1&&&&&&&&&&& \\
&&&&&1&&&&&&&&&& \\
&&&&&&1&&&&&&&&& \\ \hline
&&&&&&&&&&&1&&&& \\
&&&&&&&&1&&&&&&& \\
&&&&&&&&&1&&&&&& \\
&&&&&&&&&&1&&&&& \\ \hline
&&&&&&&&&&&&&&&1 \\
&&&&&&&&&&&&1&&& \\
&&&&&&&&&&&&&1&& \\
&&&&&&&&&&&&&&1&
\end{array}\right], \quad
T(p_2) = \left[\begin{array}{cccc|cccc|cccc|cccc}
&&&&&&&&&&&&1&&& \\
&&&&&&&&&&&&&1&& \\
&&&&&&&&&&&&&&1& \\
&&&&&&&&&&&&&&&1 \\ \hline
1&&&&&&&&&&&&&&& \\
&1&&&&&&&&&&&&&& \\
&&1&&&&&&&&&&&&& \\
&&&1&&&&&&&&&&&& \\ \hline
&&&&1&&&&&&&&&&& \\
&&&&&1&&&&&&&&&& \\
&&&&&&1&&&&&&&&& \\
&&&&&&&1&&&&&&&& \\ \hline
&&&&&&&&1&&&&&&& \\
&&&&&&&&&1&&&&&& \\
&&&&&&&&&&1&&&&& \\
&&&&&&&&&&&1&&&&
\end{array}\right].$$

□

3.4.2. Irreducible Decomposition

The irreducible decomposition of the permutation representation $T(g)$ for the $n\times n$ square lattice is now investigated. The multiplicities of irreducible representations in this decomposition are determined. It is to be emphasized that irreducible representations lacking in the decomposition of $T(g)$ can be excluded from consideration in the search for square bifurcating patterns in Section 3.5 and Appendix A.4.

Simple Examples

Prior to analysis for general n, we present the results for $n = 3$ and $n = 4$. We begin with the case of $n = 3$. The group $\mathrm{D}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)$ has nine irreducible representations (see Section 3.3):

$$\begin{aligned} R(\mathrm{D}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)) = \{&(1;+,+,+),(1;+,-,+),(1;-,+,+),(1;-,-,+),\\ &(2;+),(4;1,0,+),(4;1,0,-),(4;1,1,+),(4;1,1,-)\}. \end{aligned}$$

Among these nine irreducible representations, only three of them, $(1;+,+,+)$, $(4;1,0,+)$, and $(4;1,1,+)$, are contained in $T(g)$ with multiplicity 1, whereas the others are missing in $T(g)$. Indeed we will see in Section 3.4.3 in a general setting that

$$Q^{-1}T(g)Q = T^{(1;+,+,+)}(g) \oplus T^{(4;1,0,+)}(g) \oplus T^{(4;1,1,+)}(g), \quad g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)$$

for some orthogonal matrix Q. Accordingly, the multiplicities a^{μ} for $\mu \in R(\mathrm{D}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3))$ are given as follows:

$$\begin{aligned} &a^{(1;+,+,+)} = 1, \quad a^{(1;+,-,+)} = 0, \quad a^{(1;-,+,+)} = 0, \quad a^{(1;-,-,+)} = 0;\\ &a^{(2;+)} = 0; \quad a^{(4;1,0,+)} = 1, \quad a^{(4;1,0,-)} = 0, \quad a^{(4;1,1,+)} = 1, \quad a^{(4;1,1,-)} = 0. \end{aligned}$$

We next show the case of $n = 4$. Recall the permutation representation $T(g)$ for $n = 4$ from Example 3.1. The group $\mathrm{D}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)$ has 20 irreducible representations (see Section 3.3):

$$\begin{aligned} R(\mathrm{D}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4)) = \{&(1;+,+,+),(1;+,-,+),(1;-,+,+),(1;-,-,+),\\ &(1;+,+,-),(1;+,-,-),(1;-,+,-),(1;-,-,-),\\ &(2;+),(2;-),(2;+,+),(2;+,-),(2;-,+),(2;-,-),\\ &(4;1,0,+),(4;1,0,-),(4;1,1,+),(4;1,1,-),(4;2,1,+),(4;2,1,-)\}. \end{aligned}$$

Among these 20 irreducible representations, only six of them, $(1;+,+,+)$, $(1;+,+,-)$, $(2;+,+)$, $(4;1,0,+)$, $(4;1,1,+)$, and $(4;2,1,+)$, are contained in $T(g)$ with multiplicity 1, whereas the others are missing in $T(g)$, as we will see in Section 3.4.3 in a general setting. This means that

$$\begin{aligned} Q^{-1}T(g)Q = T^{(1;+,+,+)}(g) \oplus T^{(1;+,+,-)}(g) \oplus T^{(2;+,+)}(g) \oplus T^{(4;1,0,+)}(g) \oplus T^{(4;1,1,+)}(g) \oplus T^{(4;2,1,+)}(g),\\ g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4) \end{aligned}$$

Table 3.4: The values of character χ of the permutation representation T.

g		$\chi(g)$	
e		n^2	
${p_1}^i{p_2}^j$	$((i,j) \neq (0,0))$	0	
$r{p_1}^i{p_2}^j$	$(i+j=2k)$	2	1
	$(i+j \neq 2k)$	0	1
		$(n=2m)$	$(n \neq 2m)$
$r^2{p_1}^i{p_2}^j$	$(i,j\text{: even})$	4	1
	(other (i,j))	0	1
		$(n=2m)$	$(n \neq 2m)$
$r^3{p_1}^i{p_2}^j$	$(i+j=2k)$	2	1
	$(i+j \neq 2k)$	0	1
		$(n=2m)$	$(n \neq 2m)$

g		$\chi(g)$	
$s{p_1}^i{p_2}^j$	$(i=0,\ j=2k)$	$2n$	n
	$(i=0,\ j \neq 2k)$	0	n
	$(i \neq 0)$	0	0
		$(n=2m)$	$(n \neq 2m)$
$sr{p_1}^i{p_2}^j$	$(i=j)$	n	
	$(i \neq j)$	0	
$sr^2{p_1}^i{p_2}^j$	$(j=0,\ i=2k)$	$2n$	n
	$(j=0,\ i \neq 2k)$	0	n
	$(j \neq 0)$	0	0
		$(n=2m)$	$(n \neq 2m)$
$sr^3{p_1}^i{p_2}^j$	$(i=n-j)$	n	
	$(i \neq n-j)$	0	

$0 \leq i,j \leq n-1;\ k,m\text{: integers}$

for some orthogonal matrix Q, the concrete form of which is given in Example 3.2 in Section 3.4.3. Accordingly, the multiplicities a^μ for $\mu \in R(\mathrm{D}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_4))$ are given as follows:

$$
\begin{aligned}
&a^{(1;+,+,+)} = 1, \quad a^{(1;+,-,+)} = 0, \quad a^{(1;-,+,+)} = 0, \quad a^{(1;-,-,+)} = 0,\\
&a^{(1;+,+,-)} = 1, \quad a^{(1;+,-,-)} = 0, \quad a^{(1;-,+,-)} = 0, \quad a^{(1;-,-,-)} = 0,\\
&a^{(2;+)} = 0, \quad a^{(2;-)} = 0, \quad a^{(2;+,+)} = 1, \quad a^{(2;+,-)} = 0,\\
&a^{(2;-,+)} = 0, \quad a^{(2;-,-)} = 0,\\
&a^{(4;1,0,+)} = 1, \quad a^{(4;1,0,-)} = 0, \quad a^{(4;1,1,+)} = 1, \quad a^{(4;1,1,-)} = 0,\\
&a^{(4;2,1,+)} = 1, \quad a^{(4;2,1,-)} = 0.
\end{aligned}
$$

Analysis for the Finite Square Lattice

For general n, the permutation representation $T(g)$ is specified by (3.65)–(3.68). We determine the irreducible decomposition of $T(g)$ with the aid of characters. Let $\chi(g)$ be the character of $T(g)$, which is defined by

$$\chi(g) = \mathrm{Tr}\, T(g), \qquad g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n). \tag{3.69}$$

Table 3.4 shows the values of $\chi(g)$ for all $g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, which are dependent on whether n is even or odd. For example, the action of $r{p_1}^i{p_2}^j$ reads

$$r{p_1}^i{p_2}^j \cdot (n_1, n_2) = r \cdot (n_1 + i, n_2 + j) = (-n_2 - j, n_1 + i).$$

Table 3.5: The values of irreducible characters χ^μ appearing in (3.72).

g	$\chi^{(1;+,+,+)}$	$\chi^{(4;k,0,+)}$	$\chi^{(4;k,k,+)}$	$\chi^{(8;k,\ell)}$	$\chi^{(1;+,+,-)}$ $(n=2m)$	$\chi^{(2;+,+)}$ $(n=2m)$	$\chi^{(4;n/2,\ell,+)}$ $(n=2m)$
${p_1}^i{p_2}^j$	1	$2[\cos(ki\theta)$ $+\cos(kj\theta)]$	$2[\cos(k(i+j)\theta)$ $+\cos(k(i-j)\theta)]$	(A.6)	$(-1)^{i+j}$	$(-1)^i+(-1)^j$	$2[(-1)^i\cos(\ell j\theta)$ $+(-1)^j\cos(\ell i\theta)]$
$r{p_1}^i{p_2}^j$	1	0	0	0	$(-1)^{i+j}$	0	0
$r^2{p_1}^i{p_2}^j$	1	0	0	0	$(-1)^{i+j}$	$(-1)^i+(-1)^j$	0
$r^3{p_1}^i{p_2}^j$	1	0	0	0	$(-1)^{i+j}$	0	$(-1)^i+(-1)^j$
$s{p_1}^i{p_2}^j$	1	$2\cos(ki\theta)$	0	0	$(-1)^{i+j}$	$(-1)^i+(-1)^j$	$2(-1)^j\cos(\ell i\theta)$
$sr{p_1}^i{p_2}^j$	1	0	$2\cos(k(i-j)\theta)$	0	$(-1)^{i+j}$	0	0
$sr^2{p_1}^i{p_2}^j$	1	$2\cos(kj\theta)$	0	0	$(-1)^{i+j}$	$(-1)^i+(-1)^j$	$2(-1)^i\cos(\ell j\theta)$
$sr^3{p_1}^i{p_2}^j$	1	0	$2\cos(k(i+j)\theta)$	0	$(-1)^{i+j}$	0	0

$\theta = 2\pi/n$; (A.6) reads:

$$\chi^{(8;k,\ell)}({p_1}^i{p_2}^j) = 2[\cos((ki+\ell j)\theta) + \cos((-\ell i + kj)\theta) + \cos((ki-\ell j)\theta) + \cos((-\ell i - kj)\theta)]$$

Invariant points (n_1, n_2) are those which satisfying $(n_1, n_2) \equiv (-n_2 - j, n_1 + i) \pmod n$. The number of these points, which depend on $i+j$ and n, gives $\chi(r{p_1}^i{p_2}^j)$.

In terms of characters, the irreducible decomposition of $T(g)$ can be expressed as

$$\chi(g) = \sum_\mu a^\mu \chi^\mu(g), \quad g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n), \tag{3.70}$$

where χ^μ is the character of $\mu \in R(\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n))$, and the multiplicity a^μ of μ can be determined by the formula

$$a^\mu = \frac{1}{8n^2} \sum_{g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)} \chi(g)\chi^\mu(g). \tag{3.71}$$

In the case of $n = 2m$, for example, we obtain

$$\chi(g) = \chi^{(1;+,+,+)}(g) + \chi^{(1;+,+,-)}(g) + \chi^{(2;+,+)}(g) + \sum_{k:(3.51)} \chi^{(4;n/2,\ell,+)}(g)$$
$$+ \sum_{k:(3.49)} \chi^{(4;k,0,+)}(g) + \sum_{k:(3.50)} \chi^{(4;k,k,+)}(g) + \sum_{(k,\ell):(3.60)} \chi^{(8;k,\ell)}(g)$$

as the decomposition (3.70). The terms $\chi^{(1;+,+,-)}(g)$, $\chi^{(2;+,+)}(g)$, and $\chi^{(4;n/2,\ell,+)}(g)$ appear only when n is even. Hence, we may represent this succinctly as

$$\chi(g) = \chi^{(1;+,+,+)}(g) \left[+ \chi^{(1;+,+,-)}(g) + \chi^{(2;+,+)}(g) + \sum_{k:(3.51)} \chi^{(4;n/2,\ell,+)}(g) \right]_{\text{if } n=2m} + \sum_{k:(3.49)} \chi^{(4;k,0,+)}(g) + \sum_{k:(3.50)} \chi^{(4;k,k,+)}(g) + \sum_{(k,\ell):(3.60)} \chi^{(8;k,\ell)}(g), \quad g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n), \tag{3.72}$$

where $[\,\cdot\,]_{\text{if } n=2m}$ means that the term is included when n is even. Table 3.5 shows the values of the irreducible characters $\chi^{\mu}(g)$ appearing on the right-hand side of (3.72) (see Section 3.3 for details about $\chi^{\mu}(g)$). The equality in (3.72) can be verified with the aid of Tables 3.4 and 3.5.

The decomposition (3.72) of the character $\chi(g)$ of $T(g)$ means that some orthogonal matrix Q exists such that

$$Q^{-1}T(g)Q = T^{(1;+,+,+)}(g) \left[\oplus T^{(1;+,+,-)}(g) \oplus T^{(2;+,+)}(g) \oplus \bigoplus_{k:(3.51)} T^{(4;n/2,\ell,+)}(g) \right]_{\text{if } n=2m} \oplus \bigoplus_{k:(3.49)} T^{(4;k,0,+)}(g) \oplus \bigoplus_{k:(3.50)} T^{(4;k,k,+)}(g) \oplus \bigoplus_{(k,\ell):(3.60)} T^{(8;k,\ell)}(g), \quad g \in \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n). \tag{3.73}$$

This gives the irreducible decomposition of $T(g)$. Accordingly, the multiplicities a^{μ} in the irreducible decomposition of $T(g)$ are given as follows:

$$\begin{aligned}
&a^{(1;+,+,+)} = 1, \quad a^{(1;+,-,+)} = 0, \quad a^{(1;-,+,+)} = 0, \quad a^{(1;-,-,+)} = 0,\\
&a^{(1;+,+,-)} = 1, \quad a^{(1;+,-,-)} = 0, \quad a^{(1;-,+,-)} = 0, \quad a^{(1;-,-,-)} = 0,\\
&a^{(2;+)} = 0, \quad a^{(2;-)} = 0,\\
&a^{(2;+,+)} = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad a^{(2;+,-)} = 0, \quad a^{(2;-,+)} = 0, \quad a^{(2;-,-)} = 0,\\
&a^{(4;k,0,+)} = 1, \quad a^{(4;k,0,-)} = 0, \quad 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor,\\
&a^{(4;k,k,+)} = 1, \quad a^{(4;k,k,-)} = 0, \quad 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor,\\
&a^{(4;n/2,\ell,+)} = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad a^{(4;n/2,\ell,-)} = 0, \quad 1 \le \ell \le \frac{n}{2} - 1,\\
&a^{(8;k,\ell)} = 1, \quad 1 \le \ell \le k-1, \ 2 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor.
\end{aligned}$$

It is noteworthy that the multiplicity is either 0 or 1 for each irreducible representation, that is, the permutation representation $T(g)$ in (3.65)–(3.68) is multiplicity-free (see Remark 3.2). Table 3.6 shows a summary.

Table 3.6: Irreducible representations contained in the permutation representation T.

$n \setminus d$	1	2	4	8
$2m$	$(1;+,+,+), (1;+,+,-)$	$(2;+,+)$	$(4;k,0;+), (4;k,k;+), (4;n/2,\ell,+)$	$(8;k,\ell)$
$2m-1$	$(1;+,+,+)$		$(4;k,0;+), (4;k,k;+)$	$(8;k,\ell)$

$(4;k,0;+)$ with $1 \le k \le \lfloor (n-1)/2 \rfloor$;
$(4;k,k;+)$ with $1 \le k \le \lfloor (n-1)/2 \rfloor$;
$(4;n/2,\ell;+)$ with $1 \le \ell \le n/2-1$;
$(8;k,\ell)$ with $1 \le \ell \le k-1,\ 2 \le k \le \lfloor (n-1)/2 \rfloor$

Table 3.7: The number $\tilde{N}_d$ of the d-dimensional irreducible representations of $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ contained in the permutation representation T for the square lattice.

$n \setminus d$	1	2	4	8	
	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_4$	$\tilde{N}_8$	$\sum \tilde{N}_d$
1	1				1
2	2	1			3
3	1		2		3
4	2	1	3		6
5	1		4	1	6
6	2	1	6	1	10
7	1		6	3	10
8	2	1	9	3	15
9	1		8	6	15
10	2	1	12	6	21
11	1		10	10	21
12	2	1	15	10	28

$n \setminus d$	1	2	4	8	
	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_4$	$\tilde{N}_8$	$\sum \tilde{N}_d$
13	1		12	15	28
14	2	1	18	15	36
15	1		14	21	36
16	2	1	21	21	45
17	1		16	28	45
18	2	1	24	28	55
19	1		18	36	55
20	2	1	27	36	66
21	1		20	45	66
⋮	⋮	⋮	⋮	⋮	⋮
42	2	1	30	190	223

By $\tilde{N}_d$, we denote the number of d-dimensional irreducible representations of $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ that exist in the permutation representation $T(g)$. We have the following expressions for $\tilde{N}_d$:

$n \setminus d$	1	2	4	8
	$\tilde{N}_1$	$\tilde{N}_2$	$\tilde{N}_4$	$\tilde{N}_8$
$2m$	2	1	$3(n-2)/2$	$(n^2-6n+8)/8$
$2m-1$	1	0	$n-1$	$(n^2-4n+3)/8$

(3.74)

whereas Table 3.7 shows the values of $\tilde{N}_d$ for several n. Also note the relation

$$\sum_d d\tilde{N}_d = n^2. \tag{3.75}$$

Remark 3.2. It is a basic fact that a permutation representation $T(g)$ representing the action of a group G on a finite set P is multiplicity-free if there exists some $g \in G$ such that $g \cdot p = q$ and $g \cdot q = p$ (e.g., see Proposition 1.4.8 of Ceccherini-Silberstein et al., 2010). The permutation representation $T(g)$ in (3.65)–(3.68) satisfies this condition as follows. By (3.65), (3.67), and (3.68), we have

$$r^2 {p_1}^i {p_2}^j \cdot (n_1, n_2) \equiv (-n_1 - i, n_2 - j) \mod n.$$

Hence, any pair of (n_1, n_2) and (n_1', n_2') can be rewritten as

$$g \cdot (n_1, n_2) \equiv (n_1', n_2') \mod n, \quad g \cdot (n_1', n_2') \equiv (n_1, n_2) \mod n$$

by $g = r^2 {p_1}^i {p_2}^j$ with $i = n_1 - n_1'$ and $j = n_2 - n_2'$.

□

3.4.3. Transformation Matrix for Irreducible Decomposition

Transformation matrix Q for the irreducible decomposition is derived for the square lattice, and examples of this matrix Q are presented.

For the $n \times n$ square lattice with the symmetry of $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, we derive the transformation matrix

$$Q = (Q^\mu \mid \mu \in D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)) \tag{3.76}$$

for the irreducible decomposition. Note that the column set of Q is partitioned into blocks, each associated with an irreducible representation μ contained in $T(g)$ (see Table 3.6). Since such μ has $a^\mu = 1$ (multiplicity-free), we have the relation

$$T(g)Q^\mu = Q^\mu T^\mu(g), \quad g \in D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n), \tag{3.77}$$

where $T(g)$ is the permutation representation given in Section 3.4.1.

The vector $\boldsymbol{\lambda}$ expressing population distribution is defined as

$$\begin{aligned}\boldsymbol{\lambda} &= (\lambda_1, \dots, \lambda_K)^\top \\ &= (\lambda_{00}, \dots, \lambda_{n-1,0};\ \lambda_{01}, \dots, \lambda_{n-1,1};\ \dots\ ;\ \lambda_{0,n-1}, \dots, \lambda_{n-1,n-1})^\top \\ &= (\lambda_{n_1 n_2} \mid n_1, n_2 = 0, \dots, n-1),\end{aligned}$$

where $K = n^2$ and $(\lambda_{n_1 n_2} \mid n_1, n_2 = 0, \ldots, n-1)$ is an K-dimensional column vector. For a vector on this lattice with the (n_1, n_2)-component $g(n_1, n_2)$, we express its normalization as[5]

$$\langle g(n_1, n_2) \rangle = (g(n_1, n_2) / \Big(\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} g(i, j)^2 \Big)^{1/2} \mid n_1, n_2 = 0, \ldots, n-1). \tag{3.78}$$

Recall that the permutation representation $T(g)$ is specified by (3.65)–(3.68) above. The action of r on (n_1, n_2), for example, is expressed by

$$r \cdot (n_1, n_2) \equiv (-n_2, n_1) \mod n$$

in (3.65), which shows that the column of $T(r)$ indexed by (n_1, n_2) has "1" in the row indexed by $(-n_2, n_1) \mod n$. For the present purpose, however, it is convenient to consider $T(g)$ row-wise. It is seen that the row of $T(r)$ indexed by (n_1, n_2) has "1" at the column indexed by $(n_2, -n_1) \mod n$, since

$$(n_1', n_2') \equiv (-n_2, n_1) \mod n$$

can be solved for (n_1, n_2) as

$$(n_1, n_2) \equiv (n_2', -n_1') \mod n.$$

We denote this as

$$r * (n_1, n_2) \equiv (n_2, -n_1) \mod n. \tag{3.79}$$

For s, p_1, and p_2, a similar argument based on (3.66)–(3.68) yields

$$s * (n_1, n_2) \equiv (n_1, -n_2) \mod n, \tag{3.80}$$

$$p_1 * (n_1, n_2) \equiv (n_1 - 1, n_2) \mod n, \tag{3.81}$$

$$p_2 * (n_1, n_2) \equiv (n_1, n_2 - 1) \mod n. \tag{3.82}$$

The submatrices Q^μ for μ are given by the following proposition, where the notation $\langle \cdot \rangle$ for normalization in (3.78) is used.

Proposition 3.4. *The submatrices Q^μ of the transformation matrix Q on the $n \times n$ square lattice*

[5] The notation $\langle \cdot \rangle$ here should not be confused with that for the generators of a group.

are given by

$$Q^{(1;+,+,+)} = \frac{1}{n}(1,\ldots,1)^\top = \langle 1\rangle, \tag{3.83}$$

$$Q^{(1;+,+,-)} = \begin{cases} [\langle\cos(\pi(n_1 - n_2))\rangle] & \text{if } n \text{ is even,} \\ \text{missing} & \text{if } n \text{ is odd,} \end{cases} \tag{3.84}$$

$$Q^{(2;+,+)} = \begin{cases} [\langle\cos(\pi n_1)\rangle,\ \langle\cos(\pi n_2)\rangle] & \text{if } n \text{ is even,} \\ \text{missing} & \text{if } n \text{ is odd,} \end{cases} \tag{3.85}$$

$$Q^{(4;k,0,+)} = [\langle\cos(2\pi k n_1/n)\rangle,\ \langle\sin(2\pi k n_1/n)\rangle,\ \langle\cos(2\pi k n_2/n)\rangle,\ \langle\sin(2\pi k n_2/n)\rangle], \quad 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{3.86}$$

$$\begin{aligned} Q^{(4;k,k,+)} = {} & [\langle\cos(2\pi k(n_1 + n_2)/n)\rangle,\ \langle\sin(2\pi k(n_1 + n_2)/n)\rangle, \\ & \langle\cos(2\pi k(-n_1 + n_2)/n)\rangle,\ \langle\sin(2\pi k(-n_1 + n_2)/n)\rangle], \\ & 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \end{aligned} \tag{3.87}$$

$$Q^{(4;n/2,\ell,+)} = \begin{cases} [\langle\cos(\pi n_1 + 2\pi\ell n_2/n)\rangle,\ \langle\sin(\pi n_1 + 2\pi\ell n_2/n)\rangle, \\ \ \langle\cos(-2\pi\ell n_1/n + \pi n_2)\rangle,\ \langle\sin(-2\pi\ell n_1/n + \pi n_2)\rangle], \\ \qquad 1 \le \ell \le \frac{n}{2} - 1 & \text{if } n \text{ is even,} \\ \text{missing} & \text{if } n \text{ is odd,} \end{cases} \tag{3.88}$$

$$\begin{aligned} Q^{(8;k,\ell)} = {} & [\langle\cos(2\pi(kn_1 + \ell n_2)/n)\rangle,\ \langle\sin(2\pi(kn_1 + \ell n_2)/n)\rangle, \\ & \langle\cos(2\pi(-\ell n_1 + kn_2)/n)\rangle,\ \langle\sin(2\pi(-\ell n_1 + kn_2)/n)\rangle, \\ & \langle\cos(2\pi(kn_1 - \ell n_2)/n)\rangle,\ \langle\sin(2\pi(kn_1 - \ell n_2)/n)\rangle, \\ & \langle\cos(2\pi(-\ell n_1 - kn_2)/n)\rangle,\ \langle\sin(2\pi(-\ell n_1 - kn_2)/n)\rangle], \\ & 1 \le \ell \le k-1,\ 2 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor. \end{aligned} \tag{3.89}$$

Proof. Proof is given in Section 3.4.4. □

An example of the transformation matrix Q for $n = 4$ is presented below by assembling submatrices Q^μ in Proposition 3.4.

Example 3.2. The transformation matrix Q for the 4×4 square lattice reads

$$
\begin{aligned}
Q &= [Q^{(1;+,+,+)},\ Q^{(1;+,+,-)},\ Q^{(2;+,+)},\ Q^{(4;1,0,+)},\ Q^{(4;1,1,+)},\ Q^{(4;2,1,+)}] \\
&= [\langle 1 \rangle \mid \langle \cos(\pi(n_1 - n_2)) \rangle \mid \langle \cos(\pi n_1) \rangle,\ \langle \cos(\pi n_2) \rangle\ \mid \\
&\quad \langle \cos(\pi\, n_1/2) \rangle,\ \langle \sin(\pi\, n_1/2) \rangle,\ \langle \cos(\pi n_2/2) \rangle,\ \langle \sin(\pi n_2/2) \rangle \mid \\
&\quad \langle \cos(\pi(n_1 + n_2)/2) \rangle,\ \langle \sin(\pi(n_1 + n_2)/2) \rangle,\ \langle \cos(\pi(-n_1 + n_2)/2) \rangle,\ \langle \sin(\pi(-n_1 + n_2)/2) \rangle \mid \\
&\quad \langle \cos(\pi n_1 + \pi n_2/2) \rangle,\ \langle \sin(\pi n_1 + \pi n_2/2) \rangle,\ \langle \cos(-\pi n_1/2 + \pi n_2) \rangle,\ \langle \sin(-\pi n_1/2 + \pi n_2) \rangle] \\
&= \frac{1}{4} \left[\begin{array}{c|c|cc|cccc|cccc|cccc}
1 & 1 & 1 & 1 & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & \\
1 & -1 & -1 & 1 & & \sqrt{2} & \sqrt{2} & & & \sqrt{2} & & -\sqrt{2} & -\sqrt{2} & & & -\sqrt{2} \\
1 & 1 & 1 & 1 & -\sqrt{2} & & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & \sqrt{2} & & -\sqrt{2} & \\
1 & -1 & -1 & 1 & & -\sqrt{2} & \sqrt{2} & & & -\sqrt{2} & & \sqrt{2} & -\sqrt{2} & & & \sqrt{2} \\
\hline
1 & -1 & 1 & -1 & \sqrt{2} & & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & & \sqrt{2} & -\sqrt{2} & \\
1 & 1 & -1 & -1 & & \sqrt{2} & & \sqrt{2} & -\sqrt{2} & & \sqrt{2} & & & -\sqrt{2} & & \sqrt{2} \\
1 & -1 & 1 & -1 & -\sqrt{2} & & & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & \sqrt{2} & \sqrt{2} & \\
1 & 1 & -1 & -1 & & -\sqrt{2} & & \sqrt{2} & \sqrt{2} & & -\sqrt{2} & & & -\sqrt{2} & & -\sqrt{2} \\
\hline
1 & 1 & 1 & 1 & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & \sqrt{2} & \\
1 & -1 & -1 & 1 & & \sqrt{2} & -\sqrt{2} & & & -\sqrt{2} & & \sqrt{2} & \sqrt{2} & & & -\sqrt{2} \\
1 & 1 & 1 & 1 & -\sqrt{2} & & -\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & \\
1 & -1 & -1 & 1 & & -\sqrt{2} & -\sqrt{2} & & & \sqrt{2} & & -\sqrt{2} & \sqrt{2} & & & \sqrt{2} \\
\hline
1 & -1 & 1 & -1 & \sqrt{2} & & & -\sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & & -\sqrt{2} & -\sqrt{2} & \\
1 & 1 & -1 & -1 & & \sqrt{2} & & -\sqrt{2} & \sqrt{2} & & -\sqrt{2} & & & \sqrt{2} & & \sqrt{2} \\
1 & -1 & 1 & -1 & -\sqrt{2} & & & -\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & -\sqrt{2} & \sqrt{2} & \\
1 & 1 & -1 & -1 & & -\sqrt{2} & & -\sqrt{2} & -\sqrt{2} & & \sqrt{2} & & & \sqrt{2} & & -\sqrt{2}
\end{array} \right].
\end{aligned}
$$

□

3.4.4. Proof of Proposition 3.4

We will now show that the relation $T(g)Q^{\mu} = Q^{\mu}T^{\mu}(g)$ in (3.77) is satisfied by Q^{μ} in Proposition 3.4 for r, s, p_1, and p_2 that generate the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. Recall the actions of r, s, p_1, and p_2 given in (3.79)–(3.82). We demonstrate the proof for $\mu = (2; +, +)$ and $(8; k, \ell)$, and the other cases can be treated similarly.

Two-Dimensional Irreducible Representation

We shall prove that

$$
Q^{(2;+,+)} = [\langle \cos(\pi n_1) \rangle,\ \langle \cos(\pi n_2) \rangle] \tag{3.90}
$$

satisfies (3.77) for $\mu = (2; +, +)$. Recall that $(2; +, +)$ exists when n is even and $T^{(2;+,+)}(g)$ is defined by (3.47) and (3.48).

The action of r on the wave numbers (n_1, n_2) in (3.90) is given, by a formal calculation using (3.79), as

$$
r * (n_1, n_2) = (r * n_1, r * n_2) \equiv (n_2, -n_1 \mod n).
$$

In the matrix form, this gives

$$\begin{aligned} T(r)Q^{(2;+,+)} &= [\langle\cos(\pi n_2)\rangle,\ \langle\cos(-\pi n_1)\rangle] \\ &= [\langle\cos(\pi n_2)\rangle,\ \langle\cos(\pi n_1)\rangle] \\ &= [\cos(\pi n_1),\ \cos(\pi n_2)] \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \\ &= Q^{(2;+,+)}T^{(2;+,+)}(r). \end{aligned}$$

The action of p_1 on the wave numbers (n_1, n_2) is given by (3.81) as

$$p_1 * (n_1, n_2) \equiv (n_1 - 1 \mod n, n_2),$$

which, in the matrix form, yields

$$\begin{aligned} T(p_1)Q^{(2;+,+)} &= [\langle\cos(\pi(n_1 - 1))\rangle,\ \langle\cos(\pi n_2)\rangle] \\ &= [\langle-\cos(\pi n_1)\rangle,\ \langle\cos(\pi n_2)\rangle] \\ &= [\langle\cos(\pi n_1)\rangle,\ \langle\cos(\pi n_2)\rangle] \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \\ &= Q^{(2;+,+)}T^{(2;+)}(p_1). \end{aligned}$$

The cases of s and p_2 can be treated similarly. Thus, we have

$$T(g)Q^{(2;+,+)} = Q^{(2;+,+)}T^{(2;+,+)}(g), \quad g = r, s, p_1, p_2.$$

This completes the proof for $\mu = (2; +, +)$.

Eight-Dimensional Irreducible Representations

We shall prove that

$$\begin{aligned} Q^{(8;k,\ell)} = [&\langle\cos(2\pi(kn_1 + \ell n_2)/n)\rangle,\ \langle\sin(2\pi(kn_1 + \ell n_2)/n)\rangle, \\ &\langle\cos(2\pi(-\ell n_1 + kn_2)/n)\rangle,\ \langle\sin(2\pi(\ell n_1 + kn_2)/n)\rangle, \\ &\langle\cos(2\pi(kn_1 - \ell n_2)/n)\rangle,\ \langle\sin(2\pi(kn_1 - \ell n_2)/n)\rangle, \\ &\langle\cos(2\pi(-\ell n_1 - kn_2)/n)\rangle,\ \langle\sin(2\pi(-\ell n_1 - kn_2)/n)\rangle], \\ &1 \le \ell \le k - 1,\ 2 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor \end{aligned} \tag{3.91}$$

satisfies (3.77) for $(8; k, \ell)$ where $n \geq 5$. Recall the definition of $T^{(8;k,\ell)}(g)$ for $g = r, s, p_1, p_2$ in (3.62) and (3.63), as well as the notations

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \tag{3.92}$$

The action of r on the four wave numbers in (3.91) is given by (3.79) as

$$r * \begin{bmatrix} kn_1 + \ell n_2 \\ -\ell n_1 + kn_2 \\ kn_1 - \ell n_2 \\ -\ell n_1 - kn_2 \end{bmatrix} \equiv \begin{bmatrix} -\ell n_1 + kn_2 \\ -(kn_1 + \ell n_2) \\ -(-\ell n_1 - kn_2) \\ kn_1 - \ell n_2 \end{bmatrix} \quad \text{mod } n,$$

which permutes and changes the sign of the column vectors of $Q^{(8;k,\ell)}$ in (3.91) as

$$T(r)Q^{(8;k,\ell)} = Q^{(8;k,\ell)} \left[\begin{array}{cc|cc} & S & & \\ I & & & \\ \hline & & & I \\ & & S & \end{array}\right] = Q^{(8;k,\ell)}T^{(8;k,\ell)}(r).$$

The action of s on the four wave numbers in (3.91) is given by (3.80) as

$$s * \begin{bmatrix} kn_1 + \ell n_2 \\ -\ell n_1 + kn_2 \\ kn_1 - \ell n_2 \\ -\ell n_1 - kn_2 \end{bmatrix} \equiv \begin{bmatrix} kn_1 - \ell n_2 \\ -\ell n_1 - kn_2 \\ kn_1 + \ell n_2 \\ -\ell n_1 + kn_2 \end{bmatrix} \quad \text{mod } n,$$

which gives

$$T(s)Q^{(8;k,\ell)} = Q^{(8;k,\ell)} \left[\begin{array}{cc|cc} & & I & \\ & & & I \\ \hline I & & & \\ & I & & \end{array}\right] = Q^{(8;k,\ell)}T^{(8;k,\ell)}(s).$$

The action of p_1 on the four wave numbers in (3.91) is given by (3.81) as

$$p_1 * \begin{bmatrix} kn_1 + \ell n_2 \\ -\ell n_1 + kn_2 \\ kn_1 - \ell n_2 \\ -\ell n_1 - kn_2 \end{bmatrix} \equiv \begin{bmatrix} kn_1 + \ell n_2 - k \\ -\ell n_1 + kn_2 + \ell \\ kn_1 - \ell n_2 - k \\ -\ell n_1 - kn_2 + \ell \end{bmatrix} \quad \text{mod } n,$$

which gives

$$T(p_1)Q^{(8;k,\ell)} = Q^{(8;k,\ell)} \left[\begin{array}{cc|cc} R^k & & & \\ & R^{-\ell} & & \\ \hline & & R^k & \\ & & & R^{-\ell} \end{array}\right] = Q^{(8;k,\ell)}T^{(8;k,\ell)}(p_1).$$

The action of p_2 on the four wave numbers in (3.91) is given by (3.82) as

$$p_2 * \begin{bmatrix} kn_1 + \ell n_2 \\ -\ell n_1 + kn_2 \\ kn_1 - \ell n_2 \\ -\ell n_1 - kn_2 \end{bmatrix} \equiv \begin{bmatrix} kn_1 + \ell n_2 - \ell \\ -\ell n_1 + kn_2 - k \\ kn_1 - \ell n_2 + \ell \\ -\ell n_1 - kn_2 + k \end{bmatrix} \quad \text{mod } n,$$

which gives

$$T(p_2)Q^{(8;k,\ell)} = Q^{(8;k,\ell)} \left[\begin{array}{cc|cc} R^{\ell} & & & \\ & R^{k} & & \\ \hline & & R^{-\ell} & \\ & & & R^{-k} \end{array}\right] = Q^{(8;k,\ell)}T^{(8k,\ell)}(p_2).$$

Thus, we have the following relation to complete the proof for $\mu = (8; k, \ell)$:

$$T(g)Q^{(8;k,\ell)} = Q^{(8;k,\ell)}T^{(8;k,\ell)}(g), \quad g = r, s, p_1, p_2.$$

3.5. Existence of Bifurcating Solutions with Square Symmetry

We presented fundamental facts about the square lattice in Sections 3.2–3.4. We introduced the $n \times n$ square lattice with periodic boundary conditions as a spatial platform for agglomeration (Section 3.2). We labeled the symmetry of this lattice by the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, and obtained the irreducible representations of this group (Section 3.3). We decomposed the representation matrix for the square lattice into irreducible components to determine the multiplicity a^{μ} of each irreducible representation μ (Section 3.4).

We would like to investigate the existence of square patterns as bifurcating solutions on the square lattice. For each irreducible representation μ with $a^{\mu} \geq 1$, we study bifurcation from a critical point associated with μ by using group-theoretic bifurcation analysis procedures under group symmetry. The following two different methods are available:

(i) The equivariant branching lemma is applied to the bifurcation equation associated with μ to show the existence of bifurcating solutions with a specified symmetry. This analysis is algebraic or group-theoretic, which focuses on the symmetry of solutions. The concrete form of the bifurcation equation need not be derived, and isotropy subgroups play a key role in this analysis.

(ii) The bifurcation equation is obtained in the form of power series expansions and is solved asymptotically. This method is more complicated, treating nonlinear terms directly, but is more informative, giving asymptotic forms of the bifurcating solutions and their directions in addition to their existence.

In this section, we apply the first method (i), using the equivariant branching lemma, to the economy on the $n \times n$ square lattice with the symmetry of $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. We obtain possible bifurcating square patterns and associated lattice sizes for all the irreducible representations, which are related to group-theoretic critical points with multiplicity $M = 1, 2, 4, 8$.

The second method (ii), solving the bifurcation equation, is not based on the equivariant branching lemma and capable of capturing all bifurcating solutions by dealing with the bifurcation equation explicitly. The first method conducted in this section demands less analytical effort than this method and fits to pinpoint the targeted square patterns among many other bifurcating solutions.

This section is organized as follows. Theoretically predicted bifurcating square patterns are previewed in Section 3.5.1. Fundamentals of bifurcation analysis are recapitulated in Section 3.5.2. Bifurcation points of multiplicity $M = 1, 2, 4, 8$ are respectively studied in Sections 3.5.3–3.5.6.

3.5.1. Summary of Theoretical Results

As a preview of group-theoretic bifurcation analysis to be conducted in Sections 3.5.4–3.5.6, we present possible bifurcations that produce bifurcating square patterns. Note that all critical points are assumed to be group-theoretic as explained in Section 3.5.2.

Symmetry of Bifurcating Square Patterns

Recall that the symmetry of the $n \times n$ square lattice is labeled by the group

$$G = \langle r, s, p_1, p_2 \rangle = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n). \tag{3.93}$$

in (3.31). The fundamental relations are given as

$$\begin{aligned} &r^4 = s^2 = (rs)^2 = {p_1}^n = {p_2}^n = e, \quad p_2 p_1 = p_1 p_2, \\ &r p_1 = p_2 r, \quad r p_2 = {p_1}^{-1} r, \quad s p_1 = p_1 s, \quad s p_2 = {p_2}^{-1} s \end{aligned} \tag{3.94}$$

in (3.32), where e is the identity element.

Let us consider the governing equation

$$\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \mathbf{0}, \tag{3.95}$$

in (2.3), where $\boldsymbol{\lambda} = (\lambda, \dots, \lambda_K)^\top$ with $K(= n^2)$ is an K-dimensional independent variable vector, and ϕ is the bifurcation parameter. Among many possible solutions $\boldsymbol{\lambda}$ to the governing equation in (3.95), we are particularly interested in those bifurcating solutions that represent square patterns.

To describe square patterns, we introduced a sublattice

$$\begin{aligned} \mathcal{H}(\alpha, \beta) &= \{ n_1(\alpha \boldsymbol{\ell}_1 + \beta \boldsymbol{\ell}_2) + n_2(-\beta \boldsymbol{\ell}_1 + \alpha \boldsymbol{\ell}_2) \mid n_1, n_2 \in \mathbb{Z} \} \\ &= \{ \begin{bmatrix} \boldsymbol{\ell}_1 & \boldsymbol{\ell}_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \mid n_1, n_2 \in \mathbb{Z} \}, \end{aligned} \tag{3.96}$$

in (3.4), where

$$\boldsymbol{\ell}_1 = d \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{\ell}_2 = d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.97}$$

are basis vectors of length d of the underlying infinite square lattice

$$\mathcal{H} = \{ n_1 \boldsymbol{\ell}_1 + n_2 \boldsymbol{\ell}_2 \mid n_1, n_2 \in \mathbb{Z} \} \tag{3.98}$$

in (3.2). In this chapter, we adopt the parameter space

$$\{ (\alpha, \beta) \in \mathbb{Z}^2 \mid \alpha > 0, \ \beta \geq 0 \} \tag{3.99}$$

in (3.9) of Proposition 3.1, instead of $\{(\alpha, \beta) \in \mathbb{Z}^2 \mid \alpha \geq \beta \geq 0, \ \alpha \neq 0\}$ in (3.10), unless otherwise stated. We characterized the size of a square pattern $\mathcal{H}(\alpha, \beta)$ by

$$D = D(\alpha, \beta) = \alpha^2 + \beta^2 \tag{3.100}$$

in (3.7).

The $n \times n$ square lattice is described by

$$\mathcal{H}_n = \{n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2 \mid n_i \in \mathbb{Z},\ 0 \leq n_i \leq n-1\ (i=1,2)\} \tag{3.101}$$

in (3.25). The symmetry of a square pattern $\mathcal{H}(\alpha, \beta) \cap \mathcal{H}_n$ on this lattice is represented by the subgroup $G(\alpha, \beta)$ of G. This subgroup is classified into three types:

$$G(\alpha,\beta) = \begin{cases} \langle r, s, p_1{}^{\alpha}, p_2{}^{\alpha}\rangle = \Sigma(\alpha, 0) & \text{if } \alpha \geq 1,\ \beta = 0 \text{ (type V)}, \\ \langle r, s, p_1{}^{\beta}p_2{}^{\beta}, p_1{}^{-\beta}p_2{}^{\beta}\rangle = \Sigma(\beta,\beta) & \text{if } \alpha = \beta,\ \beta \geq 1 \text{ (type M)}, \\ \langle r, p_1{}^{\alpha}p_2{}^{\beta}, p_1{}^{-\beta}p_2{}^{\alpha}\rangle = \Sigma_0(\alpha,\beta) & \text{otherwise (type T)}, \end{cases} \tag{3.102}$$

in (3.36). It is convenient to introduce a convention

$$\Sigma_0(0,0) = \langle r\rangle, \quad \Sigma(0,0) = \langle r, s\rangle, \quad \Sigma(1,0) = \langle r, s, p_1, p_2\rangle. \tag{3.103}$$

We have the compatibility condition in (3.30) between (α, β) and n given as

$$n = \begin{cases} m\alpha\ (\alpha \geq 1) & \text{for type V}, \\ 2m\beta\ (\beta \geq 1) & \text{for type M}, \\ mD(\alpha,\beta)/\gcd(\alpha,\beta) & \text{for type T}, \end{cases} \tag{3.104}$$

where $m = 1, 2, \ldots$.

The objective of this section is to look for a solution $\boldsymbol{\lambda}$ to (3.95) such that the isotropy subgroup $\Sigma(\boldsymbol{\lambda})$ for the symmetry of $\boldsymbol{\lambda}$ coincides with one of the subgroups in (3.102).

Square Patterns Engendered by Direct Bifurcations

The main message of this section is that bifurcating solutions for square patterns do arise from the mathematical model on the square lattice with pertinent lattice sizes, and therefore these patterns can be understood within the framework of group-theoretic bifurcation theory. The major results to be derived in Sections 3.5.4–3.5.6 are summarized as follows:

Proposition 3.5. *A bifurcating solution with the square symmetry expressed by the subgroup in* (3.102) *exists for pertinent lattice sizes n. To be specific, we have the following, where m denotes a positive integer.*

- *For* $(\alpha, \beta; n) = (\alpha, 0; \alpha m)$ $(2 \leq \alpha \leq n)$*, a square pattern of type V with symmetry* $\Sigma(\alpha, 0)$ *branches at a bifurcation point with multiplicity* $M = 2$ $(\alpha = 2)$*,* $M = 4$ $(\alpha \geq 3)$*, or* $M = 8$ $(\alpha \geq 5)$*.*
- *For* $(\alpha, \beta; n) = (\beta, \beta; 2\beta m)$ $(1 \leq \beta \leq n/2)$*, a square pattern of type M with symmetry* $\Sigma(\beta, \beta)$ *branches at a bifurcation point with multiplicity* $M = 1$ $(\beta = 1)$*,* $M = 4$ $(\beta \geq 2)$*, or* $M = 8$ $(\beta \geq 4)$*.*
- *For* $(\alpha, \beta; n) = (\alpha, \beta; mD(\alpha, \beta)/\gcd(\alpha, \beta))$*, where* $1 \leq \alpha \leq n-1$*,* $1 \leq \beta \leq n-1$*, and* $\alpha \neq \beta$*, a square pattern of type T with symmetry* $\Sigma_0(\alpha, \beta)$ *branches at a bifurcation point with multiplicity* $M = 8$*.*

Proof. This is proved in Sections 3.5.4–3.5.6. □

Possible square patterns for each value of $(\alpha, \beta; n)$ in Proposition 3.5 are summarized as follows:

$(\alpha, \beta; n)$		M	Type
	$\alpha = 2$	2	
$(\alpha, 0; \alpha m)$	$\alpha \geq 3$	4	V
	$\alpha \geq 5$	8	
	$\beta = 1$	1	
$(\beta, \beta; 2\beta m)$	$\beta \geq 2$	4	M
	$\beta \geq 4$	8	
$(\alpha, \beta; \dfrac{mD(\alpha,\beta)}{\gcd(\alpha,\beta)})$	$1 \leq \alpha \leq n-1,\ 1 \leq \beta \leq n-1,\ \alpha \neq \beta$	8	T

where $m = 1, 2, \ldots$.

The following proposition plays a pivotal role in the search for square patterns.

Proposition 3.6. *The existence of square patterns depends on the divisors of the lattice size n as follows:*

(i) *If n has a divisor* α $(2 \leq \alpha \leq n)$*, a square pattern of type V with symmetry* $\Sigma(\alpha, 0)$ *exists.*

(ii) *If n has a divisor* 2β $(1 \leq \beta \leq n/2)$*, a square pattern of type M with symmetry* $\Sigma(\beta, \beta)$ *exists.*

(iii) *If n has a divisor* $D(\alpha, \beta)/\gcd(\alpha, \beta)$*, where* $1 \leq \alpha \leq n-1$*,* $1 \leq \beta \leq n-1$*, and* $\alpha \neq \beta$*, a square pattern of type T with symmetry* $\Sigma_0(\alpha, \beta)$ *exists.*

Proof. This follows from Proposition 3.5. □

Possible square patterns emerging via direct bifurcations for several values of n, obtained from Proposition 3.6, are listed in Tables 3.8 and 3.9.

3.5.2. Analysis Procedure Using Equivariant Branching Lemma

We summarize a bifurcation analysis procedure resorting to the equivariant branching lemma.

Bifurcation and Symmetry of Solutions

Let us consider the governing equation

$$\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{0} \tag{3.105}$$

in (2.3) endowed with the equivariance to $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ as

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \phi), \quad g \in G \tag{3.106}$$

in (2.7). Recall that ϕ, being the trade freeness, serves as the bifurcation parameter, $\boldsymbol{\lambda} \in \mathbb{R}^K$ is an independent variable vector of dimension $K = n^2$ expressing a pattern of mobile population, $\boldsymbol{F} : \mathbb{R}^K \times \mathbb{R} \to \mathbb{R}^K$ is the nonlinear function, and T is the K-dimensional permutation representation in Section 3.4.1 of the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

Table 3.8: Possible square patterns for several lattice sizes n (n = 2–17).

n	(α, β)	D	Type	$G(\alpha, \beta)$	M
2	(2, 0)	4	V	$\Sigma(2,0)$	2
	(1, 1)	2	M	$\Sigma(1,1)$	1
3	(3, 0)	9	V	$\Sigma(3,0)$	4
4	(2, 0)	4	V	$\Sigma(2,0)$	2
	(4, 0)	16	V	$\Sigma(4,0)$	4
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(2, 2)	8	M	$\Sigma(2,2)$	4
5	(5, 0)	25	V	$\Sigma(5,0)$	4 or 8
	(2, 1)	5	T	$\Sigma_0(5,0)$	4
6	(2, 0)	4	V	$\Sigma(2,0)$	2
	(3, 0)	9	V	$\Sigma(3,0)$	4
	(6, 0)	36	V	$\Sigma(6,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(3, 3)	18	M	$\Sigma(3,3)$	4
7	(7, 0)	49	V	$\Sigma(7,0)$	4 or 8
8	(2, 0)	4	V	$\Sigma(2,0)$	2
	(4, 0)	16	V	$\Sigma(4,0)$	4
	(8, 0)	64	V	$\Sigma(8,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(2, 2)	8	M	$\Sigma(2,2)$	4
	(4, 4)	32	M	$\Sigma(4,4)$	4 or 8
9	(3, 0)	9	V	$\Sigma(3,0)$	4
	(9, 0)	81	V	$\Sigma(9,0)$	4 or 8
10	(2, 0)	4	V	$\Sigma(2,0)$	2
	(5, 0)	25	V	$\Sigma(5,0)$	4 or 8
	(10, 0)	100	V	$\Sigma(10,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(5, 5)	50	M	$\Sigma(5,5)$	4 or 8
	(2, 1)	5	T	$\Sigma_0(2,1)$	8
	(3, 1)	10	T	$\Sigma_0(3,1)$	8
	(4, 2)	20	T	$\Sigma_0(4,2)$	8
11	(11, 0)	121	V	$\Sigma(11,0)$	4 or 8
12	(2, 0)	4	V	$\Sigma(2,0)$	2
	(3, 0)	9	V	$\Sigma(3,0)$	4
	(4, 0)	16	V	$\Sigma(4,0)$	4
	(6, 0)	36	V	$\Sigma(6,0)$	4 or 8
	(12, 0)	144	V	$\Sigma(12,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(2, 2)	8	M	$\Sigma(2,2)$	4
	(3, 3)	18	M	$\Sigma(3,3)$	4
	(6, 6)	72	M	$\Sigma(6,6)$	4 or 8
13	(13, 0)	169	V	$\Sigma(13,0)$	4 or 8
	(3, 2)	13	T	$\Sigma_0(3,2)$	8
14	(2, 0)	4	V	$\Sigma(2,0)$	2
	(7, 0)	49	V	$\Sigma(7,0)$	4 or 8
	(14, 0)	196	V	$\Sigma(14,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(7, 7)	98	M	$\Sigma(7,7)$	8 or 8
15	(3, 0)	9	V	$\Sigma(3,0)$	4
	(5, 0)	25	V	$\Sigma(5,0)$	4 or 8
	(15, 0)	225	V	$\Sigma(15,0)$	4 or 8
	(2, 1)	5	T	$\Sigma_0(2,1)$	8
	(6, 3)	45	T	$\Sigma_0(6,3)$	8
16	(2, 0)	4	V	$\Sigma(2,0)$	2
	(4, 0)	16	V	$\Sigma(4,0)$	4
	(8, 0)	64	V	$\Sigma(8,0)$	4 or 8
	(16, 0)	256	V	$\Sigma(16,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(2, 2)	8	M	$\Sigma(2,2)$	4
	(4, 4)	32	M	$\Sigma(4,4)$	4 or 8
	(8, 8)	128	M	$\Sigma(8,8)$	4 or 8
17	(17, 0)	289	V	$\Sigma(17,0)$	4 or 8
	(4, 1)	17	T	$\Sigma_0(4,1)$	8

Table 3.9: Possible square patterns for several lattice sizes n (n = 18–30).

n	(α,β)	D	Type	$G(\alpha,\beta)$	M
18	(2, 0)	4	V	$\Sigma(2,0)$	2
	(3, 0)	9	V	$\Sigma(3,0)$	4
	(6, 0)	36	V	$\Sigma(6,0)$	4 or 8
	(9, 0)	81	V	$\Sigma(9,0)$	4 or 8
	(18, 0)	324	V	$\Sigma(18,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(3, 3)	18	M	$\Sigma(3,3)$	4
	(9, 9)	162	M	$\Sigma(9,9)$	4 or 8
19	(19, 0)	361	V	$\Sigma(19,0)$	4 or 8
20	(2, 0)	4	V	$\Sigma(2,0)$	2
	(4, 0)	16	V	$\Sigma(4,0)$	4
	(5, 0)	25	V	$\Sigma(5,0)$	4 or 8
	(10, 0)	100	V	$\Sigma(10,0)$	4 or 8
	(20, 0)	400	V	$\Sigma(20,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(2, 2)	8	M	$\Sigma(2,2)$	4
	(5, 5)	50	M	$\Sigma(5,5)$	4 or 8
	(10, 10)	200	M	$\Sigma(10,10)$	4 or 8
	(2, 1)	5	T	$\Sigma_0(2,1)$	8
	(3, 1)	10	T	$\Sigma_0(3,1)$	8
	(4, 2)	20	T	$\Sigma_0(4,2)$	8
	(6, 2)	40	T	$\Sigma_0(6,2)$	8
	(8, 4)	80	T	$\Sigma_0(8,4)$	8
21	(3, 0)	9	V	$\Sigma(3,0)$	4
	(7, 0)	49	V	$\Sigma(7,0)$	4 or 8
	(21, 0)	441	V	$\Sigma(21,0)$	4 or 8
22	(2, 0)	4	V	$\Sigma(2,0)$	2
	(11, 0)	121	V	$\Sigma(11,0)$	4 or 8
	(22, 0)	484	V	$\Sigma(22,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(11, 11)	242	M	$\Sigma(11,11)$	4 or 8
23	(23, 0)	529	V	$\Sigma(23,0)$	4 or 8
24	(2, 0)	4	V	$\Sigma(2,0)$	2
	(3, 0)	9	V	$\Sigma(3,0)$	4
	(4, 0)	16	V	$\Sigma(4,0)$	4
	(6, 0)	36	V	$\Sigma(6,0)$	4 or 8
	(12, 0)	144	V	$\Sigma(12,0)$	4 or 8
	(24, 0)	576	V	$\Sigma(24,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(2, 2)	8	M	$\Sigma(2,2)$	4
	(3, 3)	18	M	$\Sigma(3,3)$	4
	(4, 4)	32	M	$\Sigma(4,4)$	4 or 8
	(6, 6)	72	M	$\Sigma(6,6)$	4 or 8
	(12, 12)	288	M	$\Sigma(12,12)$	4 or 8

n	(α,β)	D	Type	$G(\alpha,\beta)$	M
25	(5, 0)	25	V	$\Sigma(5,0)$	4 or 8
	(25, 0)	625	V	$\Sigma(25,0)$	4 or 8
	(2, 1)	5	T	$\Sigma(2,1)$	8
	(4, 3)	25	T	$\Sigma(4,3)$	8
	(10, 5)	125	T	$\Sigma(10,5)$	8
26	(2, 0)	4	V	$\Sigma(2,0)$	2
	(13, 0)	169	V	$\Sigma(13,0)$	4 or 8
	(26, 0)	676	V	$\Sigma(26,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(13, 13)	338	M	$\Sigma(13,13)$	4 or 8
	(3, 2)	13	T	$\Sigma_0(3,2)$	8
	(5, 1)	26	T	$\Sigma_0(5,1)$	8
	(6, 4)	52	T	$\Sigma_0(6,4)$	8
27	(3, 0)	9	V	$\Sigma(3,0)$	4
	(9, 0)	81	V	$\Sigma(9,0)$	4 or 8
	(27, 0)	729	V	$\Sigma(27,0)$	4 or 8
28	(2, 0)	4	V	$\Sigma(2,0)$	2
	(4, 0)	16	V	$\Sigma(4,0)$	4
	(7, 0)	49	V	$\Sigma(7,0)$	4 or 8
	(14, 0)	196	V	$\Sigma(14,0)$	4 or 8
	(28, 0)	784	V	$\Sigma(28,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(2, 2)	8	M	$\Sigma(2,2)$	4
	(7, 7)	98	M	$\Sigma(7,7)$	4 or 8
	(14, 14)	392	M	$\Sigma(14,14)$	4 or 8
29	(29, 0)	841	V	$\Sigma(29,0)$	4 or 8
	(5, 2)	29	T	$\Sigma_0(5,2)$	8
30	(2, 0)	4	V	$\Sigma(2,0)$	2
	(3, 0)	9	V	$\Sigma(3,0)$	4
	(5, 0)	25	V	$\Sigma(5,0)$	4 or 8
	(6, 0)	36	V	$\Sigma(6,0)$	4 or 8
	(10, 0)	100	V	$\Sigma(10,0)$	4 or 8
	(15, 0)	225	V	$\Sigma(15,0)$	4 or 8
	(30, 0)	900	V	$\Sigma(30,0)$	4 or 8
	(1, 1)	2	M	$\Sigma(1,1)$	1
	(3, 3)	18	M	$\Sigma(3,3)$	4
	(5, 5)	50	M	$\Sigma(5,5)$	4 or 8
	(15, 15)	450	M	$\Sigma(15,15)$	4 or 8
	(2, 1)	5	T	$\Sigma_0(2,1)$	8
	(3, 1)	10	T	$\Sigma_0(3,1)$	8
	(4, 2)	20	T	$\Sigma_0(4,2)$	8
	(6, 3)	45	T	$\Sigma_0(6,3)$	8
	(9, 3)	90	T	$\Sigma_0(9,3)$	8
	(12, 6)	180	T	$\Sigma_0(12,6)$	8

Let (λ_c, ϕ_c) be a critical point of multiplicity M (≥ 1), at which the Jacobian matrix of $\boldsymbol{F}$ has a rank deficiency M. The critical point (λ_c, ϕ_c) is assumed to be G-symmetric in the sense of

$$T(g)\lambda_c = \lambda_c, \quad g \in G. \tag{3.107}$$

Moreover, it is assumed to be group-theoretic, which means, by definition, that the M-dimensional kernel space of the Jacobian matrix at (λ_c, ϕ_c) is irreducible with respect to the representation T. Then the critical point (λ_c, ϕ_c) is associated with an irreducible representation μ of G, and the multiplicity M corresponds to the dimension of the irreducible representation μ. We denote the representation matrix for μ by $T^{\mu}(g)$.

By the Liapunov–Schmidt reduction with symmetry,[6] the full system in (3.105) is reduced, in a neighborhood of the critical point (λ_c, ϕ_c), to a system of M equations

$$\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi}) = \boldsymbol{0} \tag{3.108}$$

in $\boldsymbol{w} \in \mathbb{R}^M$, where $\widetilde{\boldsymbol{F}}\colon \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$ is a function and $\widetilde{\phi} = \phi - \phi_c$ denotes the increment of ϕ. This reduced system is called the bifurcation equation.

In the reduction process, the equivariance in (3.106) of the full system is inherited by the reduced system in (3.108). With the use of the representation matrix $T^{\mu}(g)$ for the associated irreducible representation μ, the equivariance of $\widetilde{\boldsymbol{F}}$ can be expressed as

$$T^{\mu}(g)\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi}) = \widetilde{\boldsymbol{F}}(T^{\mu}(g)\boldsymbol{w}, \widetilde{\phi}), \quad g \in G. \tag{3.109}$$

This inherited symmetry plays a key role in determining the symmetry of bifurcating solutions.

The reduced system in (3.108) can possibly admit multiple solutions $\boldsymbol{w} = \boldsymbol{w}(\widetilde{\phi})$ with $\boldsymbol{w}(0) = \boldsymbol{0}$ since $(\boldsymbol{w}, \widetilde{\phi}) = (\boldsymbol{0}, 0)$ is a singular point of (3.108). This gives rise to bifurcation. Each $\boldsymbol{w}$ uniquely determines a solution $\boldsymbol{\lambda}$ to the full system in (3.105), and moreover the symmetry of $\boldsymbol{w}$ is identical with that of $\boldsymbol{\lambda}$. Indeed, we have the following relation:

$$G^{\mu} \subseteq \Sigma^{\mu}(\boldsymbol{w}) = \Sigma(\boldsymbol{\lambda}), \tag{3.110}$$

where G^{μ} is a subgroup of G as

$$G^{\mu} = \{g \in G \mid T^{\mu}(g) = I\}, \tag{3.111}$$

and $\Sigma(\boldsymbol{\lambda})$ and $\Sigma^{\mu}(\boldsymbol{w})$ are isotropy subgroups defined respectively as

$$\Sigma(\boldsymbol{\lambda}) = \Sigma(\boldsymbol{\lambda}; G, T) = \{g \in G \mid T(g)\boldsymbol{\lambda} = \boldsymbol{\lambda}\}, \tag{3.112}$$
$$\Sigma^{\mu}(\boldsymbol{w}) = \Sigma(\boldsymbol{w}; G, T^{\mu}) = \{g \in G \mid T^{\mu}(g)\boldsymbol{w} = \boldsymbol{w}\}. \tag{3.113}$$

The significance of the relation in (3.110) is twofold. First, unless a subgroup Σ is large enough to contain G^{μ}, no bifurcating solution $\boldsymbol{\lambda}$ exists such that $\Sigma = \Sigma(\boldsymbol{\lambda})$. Second, the symmetry of a bifurcating solution $\boldsymbol{\lambda}$ is known as $\Sigma(\boldsymbol{\lambda}) = \Sigma^{\mu}(\boldsymbol{w})$ through analysis of the bifurcation equation in $\boldsymbol{w}$.

[6] For details on the Liapunov–Schmidt reduction, see Sattinger (1979), Chow and Hale (1982), and Golubitsky et al. (1988).

Table 3.10: The irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ to be considered in bifurcation analysis.

$n \setminus d$	1	2	4	8
$2m$	$(1;+,+,+), (1;+,+,-)$	$(2;+,+)$	$(4;k,0;+), (4;k,k;+), (4;n/2,\ell,+)$	$(8;k,\ell)$
$2m-1$	$(1;+,+,+)$		$(4;k,0;+), (4;k,k;+)$	$(8;k,\ell)$

$(4;k,0;+)$ with $1 \le k \le \lfloor (n-1)/2 \rfloor$;
$(4;k,k;+)$ with $1 \le k \le \lfloor (n-1)/2 \rfloor$;
$(4;n/2,\ell;+)$ with $1 \le \ell \le \lfloor (n-1)/2 \rfloor$;
$(8;k,\ell)$ with $1 \le \ell \le k-1, 2 \le k \le \lfloor (n-1)/2 \rfloor$

Remark 3.3. We define the variables $\boldsymbol{w} = (w_1, \ldots, w_M)^\top$ in the bifurcation equation in (3.108) with the matrix Q derived in Section 3.4.3. That is, the components of $\boldsymbol{w} = (w_1, \ldots, w_M)^\top$ are assumed to correspond to the column vectors of $Q^\mu = [\boldsymbol{q}_1^\mu, \ldots, \boldsymbol{q}_M^\mu]$. Then, the equivariance condition in (3.109) holds for the matrix representations T^μ of the irreducible representations μ derived in Appendix A.1.4.

□

Bifurcation Equation and the Associated Irreducible Representation

To investigate the existence of a bifurcating solution $\boldsymbol{\lambda}$ with a specified symmetry Σ to the governing equation $\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{0}$ in (3.105), it suffices to apply the equivariant branching lemma to the bifurcation equation $\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi})$ in (3.108). This is justified by the fact that the isotropy subgroup $\Sigma(\boldsymbol{\lambda})$ expressing the symmetry of a bifurcating solution $\boldsymbol{\lambda}$ is identical to the isotropy subgroup $\Sigma^\mu(\boldsymbol{w})$ of the corresponding solution $\boldsymbol{w}$ for the bifurcation equation, i.e., $\Sigma(\boldsymbol{\lambda}) = \Sigma^\mu(\boldsymbol{w})$ as shown in (3.110).

The bifurcation equation is associated with an irreducible representation μ of $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ as in (3.109). The associated irreducible representation μ is restricted to

$$\mu = (1;+,+,+), (1;+,+,-), (2;+,+), (4;k,0;+), (4;k,k;+), (4;n/2,\ell;+), (8;k,\ell)$$

with k for $(4;k,0;+)$ in (3.49), k for $(4;k,k;+)$ in (3.50), ℓ for $(4;n/2,\ell;+)$ in (3.51), and (k,ℓ) for $(8;k,\ell)$ in (3.60), as a consequence of the irreducible decomposition (3.73) of the permutation representation T for the economy on the $n \times n$ square lattice. The unit representation $(1;+,+,+)$ has been excluded since it does not correspond to a symmetry-breaking bifurcation point. Thus we have to deal with critical points of multiplicity $M = 1, 2, 4, 8$. As a modified form of Table 3.6, therefore, we obtain Table 3.10, where the multiplicity M of a critical point is equal to the dimension d of the associated irreducible representation.

Isotropy Subgroup and Fixed-Point Subspace

In analysis by the equivariant branching lemma, the isotropy subgroup of $\boldsymbol{w}$ with respect to T^μ:

$$\Sigma^\mu(\boldsymbol{w}) = \{g \in G \mid T^\mu(g)\boldsymbol{w} = \boldsymbol{w}\} \tag{3.114}$$

in (3.113) and the fixed-point subspace of Σ for T^μ:

$$\mathrm{Fix}^\mu(\Sigma) = \{\boldsymbol{w} \in \mathbb{R}^M \mid T^\mu(g)\boldsymbol{w} = \boldsymbol{w} \text{ for all } g \in \Sigma\} \tag{3.115}$$

play the major roles. The following facts, though immediate from the definitions, are important and useful.

- By definition, Σ is an isotropy subgroup if and only if $\Sigma = \Sigma^{\mu}(\boldsymbol{w})$ for some $\boldsymbol{w} \neq \boldsymbol{0}$.
- If $\Sigma = \Sigma^{\mu}(\boldsymbol{w})$, then $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$ and $\dim\ \mathrm{Fix}^{\mu}(\Sigma) \geq 1$.
- Not every Σ with the property of $\dim\ \mathrm{Fix}^{\mu}(\Sigma) \geq 1$ is an isotropy subgroup.
- $\Sigma \subseteq \Sigma^{\mu}(\boldsymbol{w})$ for every $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$.
- Σ is an isotropy subgroup if and only if $\Sigma = \Sigma^{\mu}(\boldsymbol{w})$ for some $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$ with $\boldsymbol{w} \neq \boldsymbol{0}$.
- Unless Σ is an isotropy subgroup, there exists no bifurcating solution $\boldsymbol{w}$ with symmetry Σ.

Analysis Procedure Using Equivariant Branching Lemma

Equivariant branching lemma is a useful mathematical means to prove the existence of a bifurcating solution with a specified symmetry without actually solving the bifurcation equation in (3.108). By the equivariant branching lemma, we shall demonstrate the emergence of square patterns.

Analysis for the $n \times n$ square lattice based on the equivariant branching lemma follows the steps below.

1. Specify an irreducible representation μ of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in Table 3.10.
2. Specify a subgroup Σ as a candidate of an isotropy subgroup of a possible bifurcating solution.
3. Obtain the fixed-point subspace $\mathrm{Fix}^{\mu}(\Sigma)$ in (3.115) for the subgroup Σ with respect to the irreducible representation μ.
4. Search for some $\boldsymbol{w} \in \mathrm{Fix}^{\mu}(\Sigma)$ such that $\Sigma^{\mu}(\boldsymbol{w}) = \Sigma$. If no such $\boldsymbol{w}$ exists, then Σ is not an isotropy subgroup, and hence there exists no solution with the specified symmetry Σ for the bifurcation equation associated with μ. If such $\boldsymbol{w}$ exists, then we can ensure that Σ is an isotropy subgroup, and can proceed to the next step.
5. Calculate the dimension $\dim\ \mathrm{Fix}^{\mu}(\Sigma)$ of the fixed-point subspace.
6. If $\dim\ \mathrm{Fix}^{\mu}(\Sigma) = 1$, a bifurcating solution with symmetry Σ is guaranteed to exist generically by the equivariant branching lemma. If $\dim\ \mathrm{Fix}^{\mu}(\Sigma) \geq 2$, no definite conclusion can be reached by means of the equivariant branching lemma.

Remark 3.4. The equivariant branching lemma assumes two technical conditions: (i) absolute irreducibility and (ii) genericity (see Section 2.4.5 of Ikeda and Murota, 2014). The former condition is satisfied by the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ since all the irreducible representations over $\mathbb{R}$ of this group are absolutely irreducible (see Appendix A.1.4). The latter condition is a matter of modeling, and we assume this condition throughout this chapter. For details on the equivariant branching lemma, see Cicogna (1981), Vanderbauwhede (1982), and Golubitsky et al. (1988). □

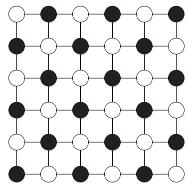


Figure 3.7: A pattern on the 6×6 square lattice expressed by the column vector of $Q^{(1;+,+,-)}$. A black circle denotes a positive component, and a white circle denotes a negative component.

3.5.3. *Bifurcation Point of Multiplicity 1*

As shown by Table 3.10 in Section 3.5.2, a critical point of multiplicity 1 is associated with the two-dimensional irreducible representation $(1; +, +, -)$, which exists only when n is even. Recall from (3.44) that this irreducible representation is given by

$$T^{(1;+,+,-)}(r) = 1, \quad T^{(1;+,+,-)}(s) = 1, \quad T^{(1;+,+,-)}(p_1) = -1, \quad T^{(1;+,+,-)}(p_2) = -1. \tag{3.116}$$

In view of Remark 3.3 in Section 3.5.2, let us assume that the variable $\boldsymbol{w} = w$ for the bifurcation equation (3.108) corresponds to the column vectors of

$$Q^{(1;+,+,-)} = \boldsymbol{q} = [\langle \cos(\pi(n_1 - n_2)) \rangle] \tag{3.117}$$

in (3.84). The spatial pattern for this vector is depicted in Fig. 3.7 for $n = 6$. This is the smallest square pattern.

Proposition 3.7. *When n is even, a bifurcating solution in the direction of $\boldsymbol{q}$ with the symmetry of $\langle r, s, p_1p_2, p_1{}^{-1}p_2 \rangle$ arises from a critical point of multiplicity 1 associated with the irreducible representation* $(1; +, +, -)$.

Proof. The general procedure in Section 3.5.2 is applied to $\mu = (1; +, +, -)$ and $\Sigma = \langle r, s \rangle \ltimes \langle p_1p_2, p_1{}^{-1}p_2 \rangle$. We have

$$\mathrm{Fix}^{(1;+,+,-)}(\Sigma) = \{w \in \mathbb{R}\}$$

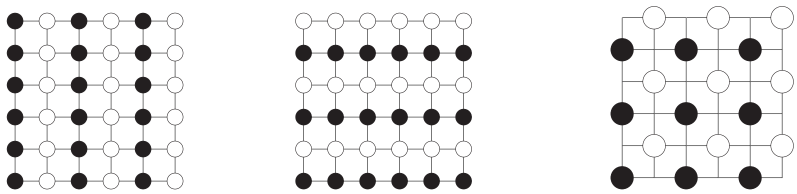
since

$$T^{(1;+,+,-)}(r)w = w, \quad T^{(1;+,+,-)}(s)w = w, \quad T^{(1;+,+,-)}(p_1p_2)w = w, \quad T^{(1;+,+,-)}(p_1{}^{-1}p_2)w = w$$

by (3.116). Thus the targeted symmetry Σ is an isotropy subgroup with

$$\dim\ \mathrm{Fix}^{(1;+,+,-)}(\Sigma) = 1.$$

The equivariant branching lemma then guarantees the existence of a bifurcating path with symmetry Σ. □



(a) $\boldsymbol{q}_1$ $(\langle r^2, s, {p_1}^2, p_2\rangle)$ (b) $\boldsymbol{q}_2$ $(\langle r^2, s, p_1, {p_2}^2\rangle)$ (c) $\boldsymbol{q}_1 + \boldsymbol{q}_2$ $(\langle r, s, {p_1}^2, {p_2}^2\rangle)$

Figure 3.8: Patterns on the 6×6 square lattice expressed by the column vectors of $Q^{(2;+,+)}$. A black circle denotes a positive component, and a white circle denotes a negative component.

3.5.4. Bifurcation Point of Multiplicity 2

As shown by Table 3.10 in Section 3.5.2, a critical point of multiplicity 2 is associated with the two-dimensional irreducible representation $(2;+,+)$, which exists only when n is even. Recall from (3.47) and (3.48) that this irreducible representation is given by

$$T^{(2;+,+)}(r) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad T^{(2;+,+)}(s) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \tag{3.118}$$

$$T^{(2;+,+)}(p_1) = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \quad T^{(2;+,+)}(p_2) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \tag{3.119}$$

In view of Remark 3.3 in Section 3.5.2, let us assume that the variable $\boldsymbol{w} = (w_1, w_2)^\top$ for the bifurcation equation (3.108) corresponds to the column vectors of

$$Q^{(2;+,+)} = [\boldsymbol{q}_1,\ \boldsymbol{q}_2] = [\langle\cos(\pi n_1)\rangle,\ \langle\cos(\pi n_2)\rangle] \tag{3.120}$$

in (3.85). The spatial patterns for these vectors are depicted in Fig. 3.8 for $n = 6$. The vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ represent stripe patterns but $\boldsymbol{q}_1 + \boldsymbol{q}_2$ expresses a square pattern.

Proposition 3.8. *When n is even, bifurcating solutions from a critical point of multiplicity 2 associated with the irreducible representation* $(2;+,+)$ *exist in the following directions:*

(i) $\boldsymbol{q}_1 + \boldsymbol{q}_2$ *with the symmetry of* $\langle r, s, {p_1}^2, {p_2}^2\rangle$,

(ii) $\boldsymbol{q}_1$ *with the symmetry of* $\langle r^2, s, {p_1}^2, p_2\rangle$, *and*

(iii) $\boldsymbol{q}_2$ *with the symmetry of* $\langle r^2, s, p_1, {p_2}^2\rangle$.

Proof. (i) The general procedure in Section 3.5.2 is applied to $\mu = (2;+,+)$ and $\Sigma = \langle r, s\rangle \ltimes \langle {p_1}^2, {p_2}^2\rangle$. Note

$$\mathrm{Fix}^{(2;+,+)}(\Sigma) = \mathrm{Fix}^{(2;+,+)}(\langle r\rangle) \cap \mathrm{Fix}^{(2;+,+)}(\langle s, {p_1}^2, {p_2}^2\rangle).$$

Here we have

$$\mathrm{Fix}^{(2;+,+)}(\langle r\rangle) = \{c(1,1)^\top \mid c \in \mathbb{R}\}$$

since $T^{(2;+,+)}(r)(w_1, w_2)^\top = (w_2, w_1)^\top$ by (3.118), whereas

$$\mathrm{Fix}^{(2;+,+)}(\langle s, {p_1}^2, {p_2}^2\rangle) = \mathbb{R}^2$$

since $T^{(2;+,+)}(s) = T^{(2;+,+)}({p_1}^2) = T^{(2;+,+)}({p_2}^2) = I$ by (3.118) and (3.119). Therefore,

$$\mathrm{Fix}^{(2;+,+)}(\Sigma) = \{c(1,1)^\top \mid c \in \mathbb{R}\},$$

that is, $\Sigma = \Sigma^{(2;+,+)}(\boldsymbol{w}_0)$ for $\boldsymbol{w}_0 = (1,1)^\top$. Thus the targeted symmetry Σ is an isotropy subgroup with

$$\dim\ \mathrm{Fix}^{(2;+,+)}(\Sigma) = 1.$$

The equivariant branching lemma then guarantees the existence of a bifurcating path with symmetry Σ.

(ii) Next the general procedure is applied to $\mu = (2;+,+)$ and $\Sigma = \langle r^2, s, {p_1}^2, p_2\rangle$. Note

$$\mathrm{Fix}^{(2;+,+)}(\Sigma) = \mathrm{Fix}^{(2;+,+)}(\langle p_2\rangle)\ \cap\ \mathrm{Fix}^{(2;+,+)}(\langle r^2, s, {p_1}^2\rangle).$$

Here we have

$$\mathrm{Fix}^{(2;+,+)}(\langle p_2\rangle) = \{c(1,0)^\top \mid c \in \mathbb{R}\}$$

since $T^{(2;+,+)}(p_2)(w_1,w_2)^\top = (w_1,-w_2)^\top$ by (3.118), whereas

$$\mathrm{Fix}^{(2;+,+)}(\langle r^2, s, {p_1}^2\rangle) = \mathbb{R}^2$$

since $T^{(2;+,+)}(r^2) = T^{(2;+,+)}(s) = T^{(2;+,+)}({p_1}^2) = I$ by (3.118) and (3.119). Therefore,

$$\mathrm{Fix}^{(2;+,+)}(\Sigma) = \{c(1,0)^\top \mid c \in \mathbb{R}\},$$

that is, $\Sigma = \Sigma^{(2;+,+)}(\boldsymbol{w}_0)$ for $\boldsymbol{w}_0 = (1,0)^\top$. Thus the targeted symmetry Σ is an isotropy subgroup with

$$\dim\ \mathrm{Fix}^{(2;+,+)}(\Sigma) = 1.$$

The equivariant branching lemma then guarantees the existence of a bifurcating path with symmetry Σ. The case of (iii) can be treated similarly. □

3.5.5. Bifurcation Point of Multiplicity 4

We investigate square patterns branching from bifurcation points of multiplicity 4.

Representation in Complex Variables

As shown by Table 3.10 in Section 3.5.2, a critical point of multiplicity 4 is associated with one of the four-dimensional irreducible representations

$$(4;k,0,+)\ \text{ with }\ 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{3.121}$$

$$(4;k,k,+)\ \text{ with }\ 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{3.122}$$

$$(4;n/2,\ell,+)\ \text{ with }\ 1 \le \ell \le \frac{n}{2} - 1, \tag{3.123}$$

where $n \ge 3$ and $(4;n/2,\ell,+)$ exists only when n is even.

The irreducible representation $(4;k,0,+)$ is given by

$$T^{(4;k,0,+)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix},\quad T^{(4;k,0,+)}(s) = \begin{bmatrix} I & \\ & S \end{bmatrix}, \tag{3.124}$$

$$T^{(4;k,0,+)}(p_1) = \begin{bmatrix} R^k & \\ & I \end{bmatrix},\quad T^{(4;k,0,+)}(p_2) = \begin{bmatrix} I & \\ & R^k \end{bmatrix}, \tag{3.125}$$

where

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \tag{3.126}$$

The irreducible representation $(4; k, k, +)$ is given by

$$T^{(4;k,k,+)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix}, \quad T^{(4;k,k,+)}(s) = \begin{bmatrix} & S \\ S & \end{bmatrix}, \tag{3.127}$$

$$T^{(4;k,k,+)}(p_1) = \begin{bmatrix} R^k & \\ & R^{-k} \end{bmatrix}, \quad T^{(4;k,k,+)}(p_2) = \begin{bmatrix} R^k & \\ & R^k \end{bmatrix}. \tag{3.128}$$

The irreducible representation $(4; n/2, \ell, +)$ is given by

$$T^{(4;n/2,\ell,+)}(r) = \begin{bmatrix} & S \\ I & \end{bmatrix}, \quad T^{(4;n/2,\ell,+)}(s) = \begin{bmatrix} S & \\ & I \end{bmatrix}, \tag{3.129}$$

$$T^{(4;n/2,\ell,+)}(p_1) = \begin{bmatrix} -I & \\ & R^{-\ell} \end{bmatrix}, \quad T^{(4;n/2,\ell,+)}(p_2) = \begin{bmatrix} R^\ell & \\ & -I \end{bmatrix}. \tag{3.130}$$

Let us assume that, for $(4; k, 0, +)$, the variable $\boldsymbol{w} = (w_1, w_2, w_3, w_4)^\top$ for the bifurcation equation (3.108) corresponds to the column vectors of

$$Q^{(4;k,0,+)} = [\langle\cos(2\pi k\, n_1/n)\rangle,\ \langle\sin(2\pi k\, n_1/n)\rangle,\ \langle\cos(2\pi k n_2/n)\rangle,\ \langle\sin(2\pi k n_2/n)\rangle] \tag{3.131}$$

in (3.86). The variables $\boldsymbol{w}$ for $(4; k, k, +)$ and $(4; n/2, \ell, +)$ can be defined similarly. The spatial patterns for these vectors are depicted in Fig. 3.9 for $n = 6$.

Using complex variables

$$(z_1, z_2) = (w_1 + \mathrm{i}w_2, w_3 + \mathrm{i}w_4),$$

we can express the actions in $(4; k, 0, +)$, given in (3.124) and (3.125) for the 4-dimensional vectors $(w_1, \ldots, w_4)$, as

$$\begin{aligned} r &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_2 \\ z_1 \end{bmatrix}, & s &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \overline{z}_2 \end{bmatrix}, \\ p_1 &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ z_2 \end{bmatrix}, & p_2 &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \omega^k z_2 \end{bmatrix}, \end{aligned} \tag{3.132}$$

where $\omega = \exp(\mathrm{i}2\pi/n)$. The actions in $(4; k, k, +)$, given in (3.127) and (3.128), are

$$\begin{aligned} r &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_2 \\ z_1 \end{bmatrix}, & s &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_2 \\ \overline{z}_1 \end{bmatrix}, \\ p_1 &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^{-k} z_2 \end{bmatrix}, & p_2 &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^k z_2 \end{bmatrix}. \end{aligned} \tag{3.133}$$

The actions in $(4; n/2, \ell, +)$, given in (3.129) and (3.130), are

$$\begin{aligned} r &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_2 \\ z_1 \end{bmatrix}, & s &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_1 \\ z_2 \end{bmatrix}, \\ p_1 &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} -z_1 \\ \omega^{-\ell} z_2 \end{bmatrix}, & p_2 &: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^\ell z_1 \\ -z_2 \end{bmatrix}. \end{aligned} \tag{3.134}$$

The actions of p_1 and p_2 in $(4; k, \ell, +)$ are expressed in a unified form as

$$p_1 : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^{-\ell} z_2 \end{bmatrix}, \quad p_2 : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^\ell z_1 \\ \omega^k z_2 \end{bmatrix}. \tag{3.135}$$

Isotropy Subgroups

To apply the method of analysis in Section 3.5.2, we identify isotropy subgroups for $(4; k, 0, +)$, $(4; k, k, +)$, and $(4; n/2, \ell, +)$ that are relevant to square patterns. We denote the isotropy subgroup of $z = (z_1, z_2)$ and the fixed-point subspace of Σ with respect to $T^{(4;k,\ell,+)}$ with $\ell \in \{0, k\}$ as

$$\Sigma^{(4;k,\ell,+)}(z) = \{g \in G \mid T^{(4;k,\ell,+)}(g) \cdot z = z\}, \tag{3.136}$$

$$\mathrm{Fix}^{(4;k,\ell,+)}(\Sigma) = \{z \mid T^{(4;k,\ell,+)}(g) \cdot z = z \text{ for all } g \in \Sigma\}, \tag{3.137}$$

where $T^{(4;k,\ell,+)}(g) \cdot z$ means the action of $g \in G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ on z given in (3.132) and (3.133). We also define

$$\check{n} = \frac{n}{\gcd(n,k)}, \quad \check{k} = \frac{k}{\gcd(n,k)}, \quad \tilde{n} = \frac{n}{\gcd(n,\ell)}, \quad \tilde{\ell} = \frac{\ell}{\gcd(n,\ell)}, \tag{3.138}$$

where $\gcd(\cdot,\cdot)$ means the greatest common divisor of the integers therein.

The symmetries of $\langle r \rangle$ and $\langle r, s \rangle$ and the translational symmetry of ${p_1}^a {p_2}^b$ are dealt with in Propositions 3.9, 3.10, and 3.11 below. In this connection, the isotropy subgroups of $z = (z_1, z_2) = (1, 1)$ (i.e., $\boldsymbol{w} = (1, 0, 1, 0)^\top$) play a crucial role. Remark 3.5 given later should be consulted with regard to the geometrical interpretation of the propositions below.

Proposition 3.9. *For* $(4; k, 0, +)$ *in* (3.49)*, we have the following statements:*

(i) $\mathrm{Fix}^{(4;k,0,+)}(\langle r \rangle) = \mathrm{Fix}^{(4;k,0,+)}(\langle r, s \rangle) = \{c(1, 1) \mid c \in \mathbb{R}\}$ *for each* k.

(ii) ${p_1}^a {p_2}^b \in \Sigma^{(4;k,0,+)}((1, 1))$ *if and only if*

$$\check{k}a \equiv 0, \quad \check{k}b \equiv 0 \mod \check{n}. \tag{3.139}$$

(iii) $\Sigma^{(4;k,0,+)}((1, 1)) = \Sigma(\check{n}, 0)$ *and* $\mathrm{Fix}^{(4;k,0,+)}(\Sigma(\check{n}, 0)) = \{c(1, 1) \mid c \in \mathbb{R}\}$. *That is,* $\Sigma(\check{n}, 0)$ *is the isotropy subgroup of* $z = (1, 1)$ *with* $\dim \mathrm{Fix}^{(4;k,0,+)}(\Sigma(\check{n}, 0)) = 1$.

(iv) *If* $\Sigma(\alpha, \beta)$ *is an isotropy subgroup (for some* z*), then* $(\alpha, \beta) = (\check{n}, 0)$ *and it is the isotropy subgroup of* $z = (1, 1)$.

(v) $\Sigma_0(\alpha, \beta)$ *is not an isotropy subgroup (for any* z*) for any value of* (α, β).

Proof. (i) By (3.132), $z = (z_1, z_2)$ is invariant to r if and only if $(\overline{z}_2, z_1) = (z_1, z_2)$, which is equivalent to $z_1 = z_2 \in \mathbb{R}$. Such z is also invariant to s.

(ii) By (3.135) for $(4; k, \ell, +)$, the invariance of $z = (1, 1)$ to ${p_1}^a {p_2}^b$ is expressed as

$$ka + \ell b \equiv 0, \quad -\ell a + kb \equiv 0 \mod n, \tag{3.140}$$

For $\ell = 0$, this condition reduces to

$$ka \equiv 0, \quad kb \equiv 0 \mod n,$$

which is equivalent to (3.139).

(iii) (a, b) satisfies (3.139) if and only if both a and b are multiples of $\check{n}$. The subgroup of G generated by ${p_1}^a {p_2}^b$ for such (a, b), together with r and s, coincides with $\Sigma(\check{n}, 0)$.

(iv) This follows from (i) and (iii).

(v) This follows from (v). □

Proposition 3.10. *For* $(4; k, k, +)$ *in* (3.50)*, we have the following statements:*

(i) $\mathrm{Fix}^{(4;k,k,+)}(\langle r\rangle) = \mathrm{Fix}^{(4;k,k,+)}(\langle r, s\rangle) = \{c(1, 1) \mid c \in \mathbb{R}\}$ *for each* k.

(ii) ${p_1}^a {p_2}^b \in \Sigma^{(4;k,k,+)}((1, 1))$ *if and only if*

$$\check{k}(a + b) \equiv 0, \quad \check{k}(-a + b) \equiv 0 \mod \check{n}. \tag{3.141}$$

(iii) *If* $\check{n}$ *is even, then we have*

$$\Sigma^{(4;k,k,+)}((1, 1)) = \Sigma(\check{n}/2, \check{n}/2),$$
$$\mathrm{Fix}^{(4;k,k,+)}(\Sigma(\check{n}/2, \check{n}/2)) = \{c(1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(\check{n}/2, \check{n}/2)$ *is the isotropy subgroup of* $z = (1, 1)$ *with* $\dim\ \mathrm{Fix}^{(4;k,k,+)}(\Sigma(\check{n}/2, \check{n}/2)) = 1$. *If* $\check{n}$ *is odd, then we have*

$$\Sigma^{(4;k,k,+)}((1, 1)) = \Sigma(\check{n}, 0),$$
$$\mathrm{Fix}^{(4;k,k,+)}(\Sigma(\check{n}, 0)) = \{c(1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(\check{n}, 0)$ *is the isotropy subgroup of* $z = (1, 1)$ *with* $\dim\ \mathrm{Fix}^{(4;k,k,+)}(\Sigma(\check{n}, 0)) = 1$.

(iv) *If* $\Sigma(\alpha, \beta)$ *is an isotropy subgroup (for some* z*), then*

$$(\alpha, \beta) = \begin{cases} (\check{n}/2, \check{n}/2) & \text{if } \check{n} \text{ is even,} \\ (\check{n}, 0) & \text{if } \check{n} \text{ is odd.} \end{cases}$$

(v) $\Sigma_0(\alpha, \beta)$ *is not an isotropy subgroup (for any* z*) for any value of* (α, β).

Proof. (i) By (3.133), $z = (z_1, z_2)$ is invariant to r if and only if $(\bar{z}_2, z_1) = (z_1, z_2)$, which is equivalent to $z_1 = z_2 \in \mathbb{R}$. Such z is also invariant to s.

(ii) The condition (3.140) for $\ell = k$ reduces to

$$k(a + b) \equiv 0, \quad k(-a + b) \equiv 0 \mod n,$$

which is equivalent to (3.141).

(iii) The condition (3.141) is equivalent to the existence of integers p and q such that

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \check{n} \begin{bmatrix} p \\ q \end{bmatrix}.$$

Hence a and b satisfy (3.141) if and only if they are integers expressed as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \check{n} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix} = \frac{\check{n}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

for some integers p and q. When $\check{n}$ is odd, this is equivalent to $(a, b) = \check{n}(p', q')$ for integers p' and q'. Therefore, the subgroup of G generated by ${p_1}^a {p_2}^b$ with such (a, b), together with r and s, is given by $\Sigma(\check{n}, 0)$ with $(p', q') = (1, 0)$ or $\Sigma(\check{n}/2, \check{n}/2)$ with $(p, q) = (1, 0)$ according to whether $\check{n}$ is odd or even .

(iv) This follows from (i) and (iii).

(v) This follows from (i). □

Proposition 3.11. *For* $(4; n/2, \ell, +)$ *in* (3.51)*, we have the following statements.*

(i) $\mathrm{Fix}^{(4;n/2,\ell,+)}(\langle r \rangle) = \mathrm{Fix}^{(4;n/2,\ell,+)}(\langle r, s \rangle) = \{c(1, 1) \mid c \in \mathbb{R}\}$ *for each* ℓ.

(ii) ${p_1}^a {p_2}^b \in \Sigma^{(4;n/2,\ell,+)}((1, 1))$ *if and only if*

$$\frac{1}{2}\tilde{n}a + \tilde{\ell}b \equiv 0, \quad -\tilde{\ell}a + \frac{1}{2}\tilde{n}b \equiv 0 \mod \tilde{n}. \tag{3.142}$$

(iii) *If $\tilde{n}$ is odd, then we have*

$$\Sigma^{(4;n/2,\ell,+)}((1, 1)) = \Sigma(2\tilde{n}, 0),$$
$$\mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(2\tilde{n}, 0)) = \{c(1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(2\tilde{n}, 0)$ is the isotropy subgroup of $z = (1, 1)$ with $\dim\ \mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(2\tilde{n}, 0)) = 1$. *If $\tilde{n}$ is even and $\tilde{n}/2$ is odd, then we have*

$$\Sigma^{(4;n/2,\ell,+)}((1, 1)) = \Sigma(\tilde{n}/2, \tilde{n}/2),$$
$$\mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(\tilde{n}/2, \tilde{n}/2)) = \{c(1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(\tilde{n}/2, \tilde{n}/2)$ is the isotropy subgroup of $z = (1, 1)$ with $\dim\ \mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(\tilde{n}/2, \tilde{n}/2)) = 1$. *If $\tilde{n}$ is even and $\tilde{n}/2$ is even, then we have*

$$\Sigma^{(4;n/2,\ell,+)}((1, 1)) = \Sigma(\tilde{n}, 0),$$
$$\mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(\tilde{n}, 0)) = \{c(1, 1) \mid c \in \mathbb{R}\};$$

that is, $\Sigma(\tilde{n}, 0)$ is the isotropy subgroup of $z = (1, 1)$ with $\dim\ \mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(\tilde{n}, 0)) = 1$.

(iv) *If $\Sigma(\alpha, \beta)$ is an isotropy subgroup (for some z), then*

$$(\alpha, \beta) = \begin{cases} (2\tilde{n}, 0) & \text{if } \tilde{n} \text{ is odd,} \\ (\tilde{n}, 0) & \text{if } \tilde{n} \text{ is even, and } \tilde{n}/2 \text{ is even,} \\ (\tilde{n}/2, \tilde{n}/2) & \text{if } \tilde{n} \text{ is even, and } \tilde{n}/2 \text{ is odd.} \end{cases}$$

(v) *$\Sigma_0(\alpha, \beta)$ is not an isotropy subgroup (for any z) for any value of (α, β).*

Proof. (i) By (3.134), $z = (z_1, z_2)$ is invariant to r if and only if $(\overline{z}_2, z_1) = (z_1, z_2)$, which is equivalent to $z_1 = z_2 \in \mathbb{R}$. Such z is also invariant to s.

(ii) The condition (3.140) for $k = n/2$ reduces to

$$\frac{n}{2}a + \ell b \equiv 0, \quad -\ell a + \frac{n}{2}b \equiv 0 \mod n,$$

which is equivalent to (3.142).

(iii) When $\tilde{n}$ is odd, (3.143) gives p and q are even, that is, $(p, q) = (2p', 2q')$ for integers p' and q'. Then, we have $(a, b) = \tilde{n}(2p', 2q') = 2\tilde{n}(p', q')$. When $\tilde{n}$ is even, from (3.142), we have $(a, b) = (\tilde{n}/2)(p, q)$ for integers p and q and this equation is rewritten as

$$\frac{\tilde{n}}{2}p + \tilde{\ell}q \equiv 0, \quad -\tilde{\ell}p + \frac{\tilde{n}}{2}q \equiv 0 \mod 2. \tag{3.143}$$

When $\tilde{n}$ is even and $\tilde{n}/2$ is even, we have $\tilde{\ell}$ odd and $(p, q) = (2p'', 2q'')$. Hence, we have $(a, b) = (\tilde{n}/2)(2p'', 2q'') = \tilde{n}(p'', q'')$ for integers p'' and q''. When $\tilde{n}$ is even and $\tilde{n}/2$ is odd ($\tilde{\ell}$ odd), we have $(a, b) = (\tilde{n}/2)(p, q)$ for $p + q$ even. Therefore, the subgroup of G generated by ${p_1}^a {p_2}^b$ with such (a, b), together with r and s, is given by $\Sigma(2\tilde{n}, 0)$ with $(p', q') = (1, 0)$, $\Sigma(\tilde{n}, 0)$ with $(p'', q'') = (1, 0)$, or $\Sigma(\tilde{n}/2, \tilde{n}/2)$ with $(p, q) = (1, 1)$, according to whether $\tilde{n}$ is odd, $\tilde{n}/2$ is even, or $\tilde{n}/2$ is odd.

(iv) This follows from (i) and (iii).

(v) This follows from (i). □

The above propositions show that, in either case of $(4; k, 0, +)$, $(4; k, k, +)$, and $(4; n/2, \ell, +)$, any isotropy subgroup Σ containing $\langle r \rangle$, which is of our interest, can be represented as $\Sigma = \Sigma^{(4;k,\ell,+)}(z)$ for $z = (1, 1)$ and that $\dim \mathrm{Fix}^{(4;k,\ell,+)}(\Sigma) = 1$. On the basis of this fact, we will investigate possible occurrences of square patterns for each of the three types V, M, and T in the remaining of this section.

Remark 3.5. The four-dimensional space of $\boldsymbol{w} = (w_1, w_2, w_3, w_4)^\top$ for the bifurcation equation (3.108) is spanned by the column vectors of

$$Q^{(4;k,\ell,+)} = [\boldsymbol{q}_1,\ \boldsymbol{q}_2,\ \boldsymbol{q}_3,\ \boldsymbol{q}_4], \tag{3.144}$$

the concrete form of which is given in (3.86)–(3.88). For example, the spatial patterns for these vectors with $n = 6$ are depicted in Fig. 3.9. The two vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_3$ represent stripe patterns in different directions. The sum $\boldsymbol{q}_{\mathrm{sum}} = \boldsymbol{q}_1 + \boldsymbol{q}_3$ of these two vectors, which is associated with $z = (1, 1)$, represents square patterns.

□

Square Patterns of Type V

Square patterns of type V are here shown to branch from critical points of multiplicity 4. Recall that a square pattern of type V is characterized by the symmetry of $\Sigma(\alpha, 0)$ with $2 \le \alpha \le n$ compatible with n (see 3.102 and 3.104) and that $D(\alpha, 0) = \alpha^2$.

The following proposition is concerned with the square patterns of type V.

Proposition 3.12. *Square patterns of type V with the symmetry of $\Sigma(\alpha, 0)$ ($\alpha \ge 3$) arise as bifurcating solutions from critical points of multiplicity 4 for specific values of n and associated*

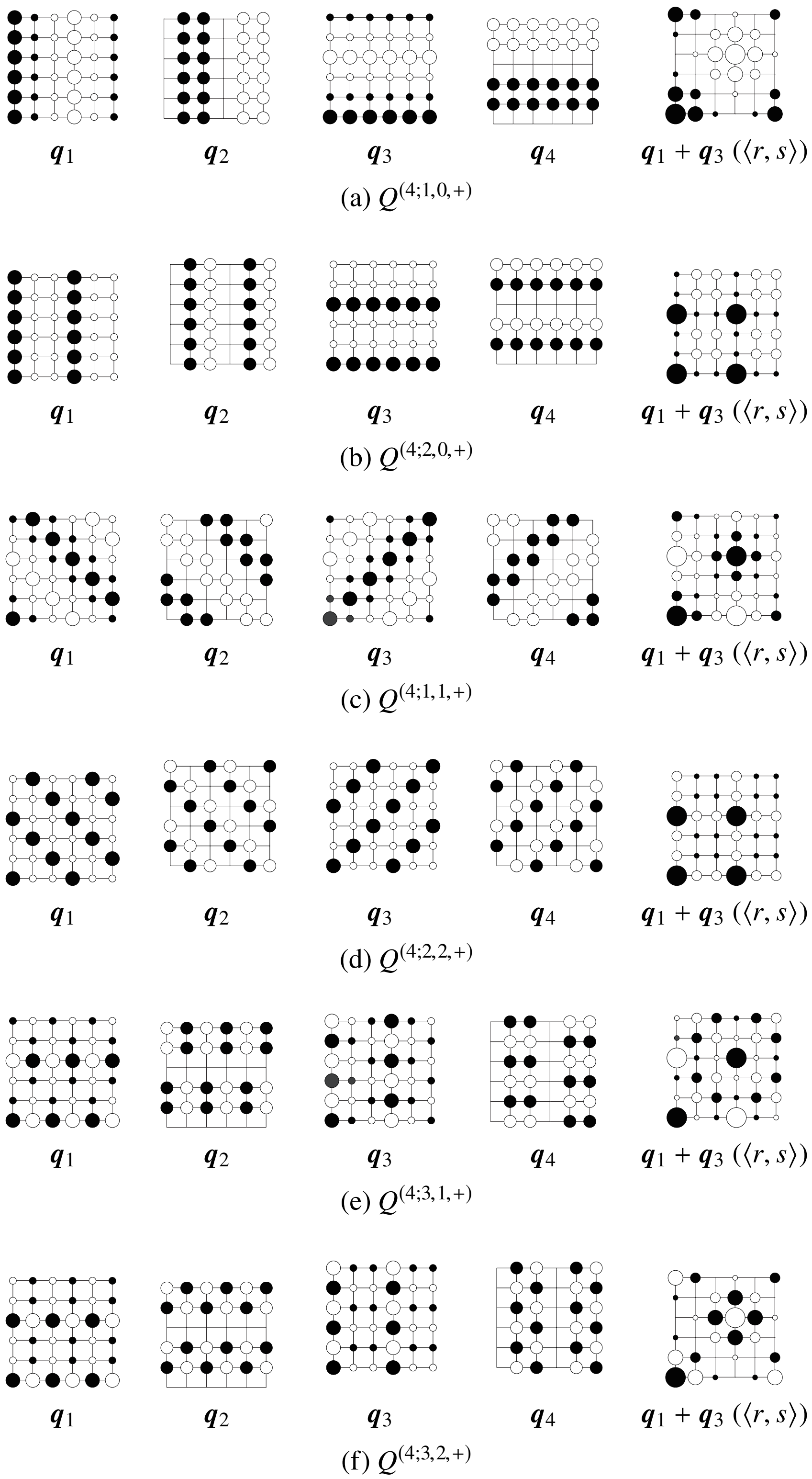


Figure 3.9: Patterns on the 6×6 square lattice expressed by the column vectors of Q^{μ} for four-dimensional irreducible representations. A black circle denotes a positive component, and a white circle denotes a negative component.

irreducible representations given by

$$
\begin{array}{llll l}
\hline
(\alpha,\beta) & D & n & (k,\ell) \text{ in } (4;k,\ell,+) & \\
\hline
(\alpha,0) & \alpha^2 & \alpha m & (pm,0) & \\
(\alpha,0) & \alpha^2 & \alpha m & (pm,pm) & (\alpha \text{ is odd}) \\
(\alpha,0) & \alpha^2 & \alpha m & (\alpha m/2,pm) & (\alpha \text{ is even, and } \alpha/2 \text{ is even}) \\
(\alpha,0) & \alpha^2 & \alpha m & (\alpha m/2,2p'm) & (\alpha \text{ is even, and } \alpha/2 \text{ is odd}) \\
\hline
\end{array}
\tag{3.145}
$$

where $m \geq 1$ and

$$\gcd(p,\alpha) = 1, \quad 1 \leq p < \alpha/2, \tag{3.146}$$

$$\gcd(p',\alpha/2) = 1, \quad 1 \leq p' < \alpha/4. \tag{3.147}$$

Proof. By Propositions 3.9, 3.10, and 3.11, we have three possibilities: $(4;k,0,+)$, $(4;k,k,+)$, and $(4;n/2,\ell,+)$. For $(4;k,0,+)$, we fix α and look for (k,n) that satisfies (3.121) and $\check{n} = \alpha$. For such (k,n), $\Sigma(\alpha,0) = \Sigma(\check{n},0)$ is an isotropy subgroup with $\dim \mathrm{Fix}^{(4;k,0,+)}(\Sigma(\alpha,0)) = 1$ by Proposition 3.9. Then, the equivariant branching lemma (Section 3.5.2) guarantees the existence of a bifurcating solution with symmetry $\Sigma(\alpha,0)$.

For $(4;k,k,+)$, we fix α that is odd and look for (k,n) that satisfies (3.122) and $\tilde{n} = \alpha$, and proceed in a similar manner using Proposition 3.10.

For $(4;n/2,\ell,+)$, we fix α that is even and look for (ℓ,n) that satisfies (3.123) and $\tilde{n} = \alpha/2$ for $\alpha/2$ odd and $\tilde{n} = \alpha$ for $\alpha/2$ even, and proceed in a similar manner using Proposition 3.11.

Suppose that (k,n) for $(k,\ell) = (pm,0)$ and (pm,pm) is given by (3.145) with (3.146). Then, $m = \gcd(k,n)$ by $\gcd(p,\alpha) = 1$ and $\check{n} = n/\gcd(k,n) = n/m = \alpha$. We have $k = pm \geq 1$ and $k/n = p/\alpha < 1/2$, thereby showing $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ in (3.121) for $(4;pm,0,+)$ and (3.122) for $(4;pm,pm,+)$.

Suppose that (ℓ,n) for $(k,\ell) = (\alpha m/2,pm)$ is given by (3.145) with (3.146). Then $m = \gcd(n,\ell)$ by $\gcd(p,\alpha) = 1$ and $\tilde{n} = n/\gcd(\ell,n) = n/m = \alpha$. We have $\ell = pm \geq 1$ and $\ell/n = p/\alpha < 1/2$, thereby showing (3.123).

Suppose that (ℓ,n) for $(k,\ell) = (\alpha m/2,2p'm)$ is given by (3.145) with (3.147). Then $2m = \gcd(n,\ell)$ by $\gcd(p',\alpha/2) = 1$ and $\tilde{n} = n/\gcd(\ell,n) = n/(2m) = \alpha/2$. We have $\ell = 2p'm \geq 1$ and $\ell/n = 2p'/\alpha < 1/2$, thereby showing (3.123).

Conversely, suppose that (k,n) satisfies $\check{n} = \alpha$, and (3.121) or (3.122). Then we have $\alpha = \check{n} = n/\gcd(k,n)$, which shows $\gcd(k,n) = n/\alpha$ is an integer, say m. We also have $k = \check{k}\gcd(k,n) = mp$ for $p = \check{k}$. Then $\gcd(p,\alpha) = \gcd(\check{k},\check{n}) = 1$, $p = \check{k} \geq 1$, and $p/\alpha = k/n < 1/2$ by (3.121) or (3.122), thereby showing (3.146).

Suppose that $\alpha/2$ is even and (ℓ,n) satisfies $\tilde{n} = \alpha$, and (3.123). Then we have $\alpha = \tilde{n} = n/\gcd(\ell,n)$, which shows $\gcd(\ell,n) = n/\alpha$ is an integer, say m. We also have $\ell = \tilde{\ell}\gcd(\ell,n) = mp$ for $p = \tilde{\ell}$. Then $\gcd(p,\alpha) = \gcd(\tilde{\ell},\tilde{n}) = 1$, $p = \tilde{\ell} \geq 1$, and $p/\alpha = \ell/n < 1/2$ by (3.123), thereby showing (3.146).

Suppose that $\alpha/2$ is odd and (ℓ,n) satisfies $2\tilde{n} = \alpha$ and (3.123). Then we have $\alpha = 2\tilde{n} = 2n/\gcd(\ell,n)$, which shows $\gcd(\ell,n) = 2n/\alpha$ is an even integer, say $2m$. We also have $\ell = \tilde{\ell}\gcd(\ell,n) = 2mp'$ for $p' = \tilde{\ell}$. Then $\gcd(p',\alpha/2) = \gcd(\tilde{\ell},\tilde{n}) = 1$, $p' = \tilde{\ell} \geq 1$, and $p'/\alpha = \ell/(2n) < 1/4$ by (3.123), thereby showing (3.147).

The above argument is in fact valid for $\alpha \geq 2$. For $\alpha = 2$, however, the condition $1 \leq p < \alpha/2$ or $1 \leq p' < \alpha/4$ is already a contradiction, which proves the nonexistence of the square pattern with $D = 4$ ($\alpha = 2$). □

Example 3.3. The parameter values of (3.145) in Proposition 3.12 give

(α, β)	D	n	(k, ℓ) in $(4; k, \ell, +)$		
$(3, 0)$	9	$3m$	$(m, 0)$;	(m, m)	
$(4, 0)$	16	$4m$	$(m, 0)$;		$(2m, m)$
$(5, 0)$	25	$5m$	$(m, 0), (2m, 0)$;	$(m, m), (2m, 2m)$	
$(6, 0)$	36	$6m$	$(m, 0)$;		$(3m, 2m)$
$(7, 0)$	49	$7m$	$(m, 0), (2m, 0), (3m, 0)$;	$(m, m), (2m, 2m), (3m, 3m)$	
$(8, 0)$	64	$8m$	$(m, 0), (3m, 0)$;		$(4m, m), (4m, 3m)$

where $m \geq 1$. For each $\alpha \geq 3$, there exists at least one eligible (k, n) for $(4; k, 0, +)$ in (3.145); for instance, $(k, n) = (m, \alpha m)$, which corresponds to $p = 1$.

□

Square Patterns of Type M

Square patterns of type M are shown here to branch from critical points of multiplicity 4. Recall that a square pattern of type M is characterized by the symmetry of $\Sigma(\beta, \beta)$ with $1 \leq \beta \leq n/2$ compatible with n (see (3.102) and (3.104)) and $D(\beta, \beta) = 2\beta^2$.

The following proposition is concerned with the square patterns of type M.

Proposition 3.13. *Square patterns of type M with the symmetry of $\Sigma(\beta, \beta)$ ($\beta \geq 2$) arise as bifurcating solutions from critical points of multiplicity 4 for specific values of n and associated irreducible representations given by*

(α, β)	D	n	(k, ℓ) in $(4; k, \ell, +)$	
(β, β)	$2\beta^2$	$2\beta m$	(pm, pm)	
(β, β)	$2\beta^2$	$2\beta m$	$(\beta m, pm)$	(β is odd)

(3.148)

where $m \geq 1$ and

$$\gcd(p, 2\beta) = 1, \quad 1 \leq p < \beta. \tag{3.149}$$

Proof. By Propositions 3.9, 3.10, and 3.11, we have two possibilities: $(4; k, k, +)$ and $(4; n/2, \ell, +)$ and look for (k, n) that satisfies (3.122) or (3.123) and the condition that

$$\text{for } (4; k, k, +)\text{: } \check{n} \text{ is even, and } \beta = \check{n}/2, \tag{3.150}$$

$$\text{for } (4; n/2, \ell, +)\text{: } \check{n} \text{ is even, } \tilde{n}/2 \text{ is odd, and } \beta = \tilde{n}/2. \tag{3.151}$$

For such parameter value (k, n) in (3.150), $\Sigma(\beta, \beta) = \Sigma(\check{n}/2, \check{n}/2)$ is an isotropy subgroup with

$$\dim\ \text{Fix}^{(4;k,k,+)}(\Sigma(\beta, \beta)) = 1 \tag{3.152}$$

by Proposition 3.10. For such parameter value (ℓ, n) in (3.151), $\Sigma(\beta,\beta) = \Sigma(\tilde{n}/2, \tilde{n}/2)$ is an isotropy subgroup with

$$\dim\ \mathrm{Fix}^{(4;n/2,\ell,+)}(\Sigma(\beta,\beta)) = 1 \tag{3.153}$$

by Proposition 3.11. Then, the equivariant branching lemma (Section 3.5.2) guarantees the existence of a bifurcating solution with symmetry $\Sigma(\beta,\beta)$ for both $(4; k, k, +)$ and $(4; n/2, \ell, +)$.

For $(4; k, k, +)$, suppose that (k, n) is given by (3.148). Then $m = \gcd(k, n)$ by $\gcd(p, 2\beta) = 1$, and $\check{n} = n/\gcd(k, n) = 2\beta$, which shows (3.150). As for the condition (3.122), we first observe that $k/n = p/(2\beta) < 1/2$, which shows $k < n/2$. The case of $(4; n/2, \ell, +)$ can be treated similarly.

Conversely, for $(4; k, k, +)$, suppose that (k, n) satisfies (3.122) and (3.150). Put $m' = \gcd(k, n)$ to obtain $n = m'\check{n} = 2m'\beta$ and $k = m'\check{k} = m'p$ for $p = \check{k}$. Hence we have $(k, n) = (pm', 2\beta m')$, where $\gcd(p, 2\beta) = \gcd(\check{k}, \check{n}) = 1$ and $p/(2\beta) = k/n < 1/2$, thereby showing (3.149). The case of $(4; n/2, \ell, +)$ can be treated similarly.

The above argument is valid also for $\beta = 1$. For $\beta = 1$, however, no p satisfies $1 \le p < \beta$. This proves the nonexistence of the square pattern with $D = 2$. □

Example 3.4. The parameter values of (3.148) in Proposition 3.13 give

(α,β)	D	n	(k,ℓ) in $(4;k,\ell,+)$	
(2, 2)	8	$4m$	(m,m)	
(3, 3)	18	$6m$	(m,m);	$(3m,m)$
(4, 4)	32	$8m$	$(m,m), (3m,3m)$,	
(5, 5)	50	$10m$	$(m,m), (3m,3m)$;	$(5m,m), (5m,3m)$
(6, 6)	72	$12m$	$(m,m), (5m,5m)$	

where $m \ge 1$. For each $\beta \ge 2$, there exists at least one eligible (k, n) in (3.148); for instance, $(k, n) = (m, 2\beta m)$, which corresponds to $p = 1$.

□

Square Patterns of Type T

It is shown that square patterns of type T do not appear from critical points of multiplicity 4. Recall that a square pattern of type T is characterized by the symmetry of $\Sigma_0(\alpha,\beta)$ with $1 \le \alpha \le n-1$, $1 \le \beta \le n-1$, $\alpha \ne \beta$ (see (3.39)). The following proposition denies the existence of square patterns of type T.

Proposition 3.14. *Square patterns of type T with the symmetry of* $\Sigma_0(\alpha,\beta)$ $1 \le \alpha \le n-1$, $1 \le \beta \le n-1$, $\alpha \ne \beta$ *do not arise as bifurcating solutions from critical points of multiplicity 4 for any* n.

Proof. By Propositions 3.9, 3.10, and 3.11, $\Sigma_0(\alpha,\beta)$ is not an isotropy subgroup with respect to neither $(4; k, 0, +)$, nor $(4; k, k, +)$, nor $(4; , n/2, \ell)$. □

Possible Square Patterns for Several Lattice Sizes

We have investigated possible occurrences of square patterns for each of the three types V, M, and T, and enumerated all possible parameter values of n for the lattice size and k for the associated irreducible representations $(4; k, 0, +)$, $(4; k, k, +)$, and/or $(4; n/2, \ell, +)$. By compiling the obtained facts, we can capture, for each n, all square patterns that can potentially arise from critical points of multiplicity 4. The result is given in Tables 3.11–3.14 for several lattice sizes n.

Table 3.11: Square patterns arising from critical points of multiplicity 4 for several lattice sizes n ($\check{n}$ is given for $(4; k, 0, +)$ and $(4; k, k, +)$, and $\tilde{n}$ is given for $(4; n/2, \ell, +)$).

n	(k, ℓ) in $(4; k, \ell, +)$	$\check{n}$	$\tilde{n}$	(α, β)	D	Type
3	(1, 0)	3		(3, 0)	9	V
	(1, 1)					
4	(1, 0)	4		(4, 0)	16	V
	(2, 1)		4			
	(1, 1)	4		(2, 2)	8	M
5	(1, 0), (2, 0)	5		(5, 0)	25	V
	(1, 1), (2, 2)					
6	(2, 0)	3		(3, 0)	9	V
	(2, 2)					
	(1, 0)	6		(6, 0)	36	V
	(3, 2)		3			
	(1, 1)	6		(3, 3)	18	M
	(3, 1)		6			
7	(1, 0), (2, 0), (3, 0)	7		(7, 0)	49	V
	(1, 1), (2, 2), (3, 3)					
8	(2, 0)	4		(4, 0)	16	V
	(4, 2)		4			
	(1, 0), (3, 0)	8		(8, 0)	64	V
	(4, 1), (4, 3)		8			
	(2, 2)	4		(2, 2)	8	M
	(1, 1), (3, 3)	8		(4, 4)	32	M
9	(3, 0)	3		(3, 0)	9	V
	(3, 3)					
	(1, 0), (2, 0), (4, 0)	9		(9, 0)	81	V
	(1, 1), (2, 2), (4, 4)					
10	(2, 0), (4, 0)	5		(5, 0)	25	V
	(2, 2), (4, 4)					
	(1, 0), (3, 0)	10		(10, 0)	100	V
	(5, 2), (5, 4)		5			
	(1, 1), (3, 3)	10		(5, 5)	50	M
	(5, 1), (5, 3)		10			
11	(1, 0), (2, 0), (3, 0), (4, 0), (5, 0)	11		(11, 0)	121	V
	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)					
12	(4, 0)	3		(3, 0)	9	V
	(4, 4)					
	(3, 0)	4		(4, 0)	16	V
	(6, 3)		4			
	(2, 0)	6		(6, 0)	36	V
	(6, 4)		3			
	(1, 0), (5, 0)	12		(12, 0)	144	V
	(6, 1), (6, 5)		12			
	(3, 3)	4		(2, 2)	8	M
	(2, 2)	6		(3, 3)	18	M
	(6, 2)		6			
	(1, 1), (5, 5)	12		(6, 6)	72	M
13	(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0)	13		(13, 0)	169	V
	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)					

Table 3.12: Square patterns arising from critical points of multiplicity 4 for several lattice sizes n ($\check{n}$ is given for $(4; k, 0, +)$ and $(4; k, k, +)$, and $\tilde{n}$ is given for $(4; n/2, \ell, +)$).

n	(k, ℓ) in $(4; k, \ell, +)$	$\check{n}$	$\tilde{n}$	(α, β)	D	Type
14	(2, 0), (4, 0), (6, 0)	7		(7, 0)	49	V
	(2, 2), (4, 4), (6, 6)					
	(1, 0), (3, 0), (5, 0)	14		(14, 0)	196	V
	(7, 2), (7, 4), (7, 6)		7			
	(1, 1), (3, 3), (5, 5)	14		(7, 7)	98	M
	(7, 1), (7, 3), (7, 5)		14			
15	(5, 0)	3		(3, 0)	9	V
	(5, 5)					
	(3, 0), (6, 0)	5		(5, 0)	25	V
	(3, 3), (6, 6)					
	(1, 0), (2, 0), (4, 0), (7, 0)	15		(15, 0)	225	V
	(1, 1), (2, 2), (4, 4), (7, 7)					
16	(4, 0)	4		(4, 0)	16	V
	(8, 4)		4			
	(2, 0), (6, 0)	8		(8, 0)	64	V
	(8, 2), (8, 6)		8			
	(1, 0), (3, 0), (5, 0), (7, 0)	16		(16, 0)	256	V
	(8, 1), (8, 3), (8, 5), (8, 7)		16			
	(4, 4)	4		(2, 2)	8	M
	(2, 2), (6, 6)	8		(4, 4)	32	M
	(1, 1), (3, 3), (5, 5), (7, 7)	16		(8, 8)	72	M
17	(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0), (8, 0)	17		(17, 0)	289	V
	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8)					
18	(6, 0)	3		(3, 0)	9	V
	(6, 6)					
	(3, 0)	6		(6, 0)	36	V
	(9, 6)		3			
	(2, 0), (4, 0), (8, 0)	9		(9, 0)	81	V
	(2, 2), (4, 4), (8, 8)					
	(1, 0), (5, 0), (7, 0)	18		(18, 0)	324	V
	(9, 2), (9, 4), (9, 8)		9			
	(3, 3)	6		(3, 3)	18	M
	(9, 3)		6			
	(1, 1), (5, 5), (7, 7)	18		(9, 9)	162	M
	(9, 1), (9, 5), (9, 7)		18			
19	(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0), (8, 0), (9, 0)	19		(19, 0)	361	V
	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)					
20	(5, 0)	4		(4, 0)	16	V
	(10, 5)		4			
	(4, 0), (8, 0)	5		(5, 0)	25	V
	(4, 4), (8, 8)					
	(2, 0), (6, 0)	10		(10, 0)	100	V
	(10, 4), (10, 8)		5			
	(1, 0), (3, 0), (7, 0), (9, 0)	20		(20, 0)	400	V
	(10, 1), (10, 3), (10, 7), (10, 9)		20			
	(5, 5)	4		(2, 2)	8	M
	(2, 2), (6, 6)	10		(5, 5)	50	M
	(10, 2), (10, 6)		10			
	(1, 1), (3, 3), (7, 7), (9, 9)	20		(10, 10)	200	M

Table 3.13: Square patterns arising from critical points of multiplicity 4 for several lattice sizes n ($\check{n}$ is given for $(4; k, 0, +)$ and $(4; k, k, +)$, and $\tilde{n}$ is given for $(4; n/2, \ell, +)$).

n	(k, ℓ) in $(4; k, \ell, +)$	$\check{n}$	$\tilde{n}$	(α, β)	D	Type
21	(7, 0)	3		(3, 0)	9	V
	(7, 7)					
	(3, 0), (6, 0), (9, 0)	7		(7, 0)	49	V
	(3, 3), (6, 6), (9, 9)					
	(1, 0), (2, 0), (4, 0), (5, 0), (8, 0), (10, 0)	21		(21, 0)	441	V
	(1, 1), (2, 2), (4, 4), (5, 5), (8, 8), (10, 10)					
22	(2, 0), (4, 0), (6, 0), (8, 0), (10, 0)	11		(11, 0)	121	V
	(2, 2), (4, 4), (6, 6), (8, 8), (10, 10)					
	(1, 0), (3, 0), (5, 0), (7, 0), (9, 0)	22		(22, 0)	484	V
	(11, 2), (11, 4), (11, 6), (11, 8), (11, 10)		11			
	(1, 1), (3, 3), (5, 5), (7, 7), (9, 9)	22		(11, 11)	242	M
	(11, 1), (11, 3), (11, 5), (11, 7), (11, 9)		22			
23	(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0), (8, 0), (9, 0), (10, 0)	23		(23, 0)	529	V
	(11, 0)					
	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9), (10, 10)					
	(11, 11)					
24	(8, 0)	3		(3, 0)	9	V
	(8, 8)					
	(6, 0)	4		(4, 0)	16	V
	(12, 6)		4			
	(4, 0)	6		(6, 0)	36	V
	(12, 8)		3			
	(3, 0), (9, 0)	8		(8, 0)	64	V
	(12, 3), (12, 9)		8			
	(2, 0), (10, 0)	12		(12, 0)	144	V
	(12, 2), (12, 10)		12			
	(1, 0), (5, 0), (7, 0), (11, 0)	24		(24, 0)	576	V
	(12, 1), (12, 5), (12, 7)(12, 11)		24			
	(6, 6)	4		(2, 2)	8	M
	(4, 4)	6		(3, 3)	18	M
	(12, 4)		6			
	(3, 3), (9, 9)	8		(4, 4)	32	M
	(2, 2), (10, 10)	12		(6, 6)	72	M
	(1, 1), (5, 5), (7, 7), (11, 11)	24		(12, 12)	288	M
25	(5, 0), (10, 0)	5		(5, 0)	25	V
	(5, 5), (10, 10)					
	(1, 0), (2, 0), (3, 0), (4, 0), (6, 0), (7, 0), (8, 0), (9, 0), (11, 0), (12, 0)	25		(25, 0)	625	V
	(1, 1), (2, 2), (3, 3), (4, 4), (6, 6), (7, 7), (8, 8), (9, 9), (11, 11), (12, 12)					
26	(2, 0), (4, 0), (6, 0), (8, 0), (10, 0), (12, 0)	13		(13, 0)	169	V
	(2, 2), (4, 4), (6, 6), (8, 8), (10, 10), (12, 12)					
	(1, 0), (3, 0), (5, 0), (7, 0), (9, 0), (11, 0)	26		(26, 0)	676	V
	(13, 2), (13, 4), (13, 6), (13, 8), (13, 10), (13, 12)		13			
	(1, 1), (3, 3), (5, 5), (7, 7), (9, 9), (11, 11)	26		(13, 13)	338	M
	(13, 1), (13, 3), (13, 5), (13, 7), (13, 9), (13, 11)		26			

Table 3.14: Square patterns arising from critical points of multiplicity 4 for several lattice sizes n ($\check{n}$ is given for $(4; k, 0, +)$ and $(4; k, k, +)$, and $\tilde{n}$ is given for $(4; n/2, \ell, +)$).

n	(k, ℓ) in $(4; k, \ell, +)$	$\check{n}$	$\tilde{n}$	(α, β)	D	Type
27	(9, 0)	3		(3, 0)	9	V
	(9, 9)					
	(3, 0), (6, 0), (12, 0)	9		(9, 0)	81	V
	(3, 3), (6, 6), (12, 12)					
	(1, 0), (2, 0), (4, 0), (5, 0), (7, 0), (8, 0), (10, 0), (11, 0), (13, 0)	27		(27, 0)	729	V
	(1, 1), (2, 2), (4, 4), (5, 5), (7, 7), (8, 8), (10, 10), (11, 11), (13, 13)		27			
28	(7, 0)	4		(4, 0)	16	V
	(14, 7)		4			
	(4, 0), (8, 0), (12, 0)	7		(7, 0)	49	V
	(4, 4), (8, 8), (12, 12)					
	(2, 0), (6, 0), (10, 0)	14		(14, 0)	392	V
	(14, 4), (14, 8), (14, 12)		7			
	(1, 0), (3, 0), (5, 0), (9, 0), (11, 0), (13, 0)	28		(28, 0)	784	V
	(14, 1), (14, 3), (14, 5), (14, 9), (14, 11), (14, 13)		28			
	(7, 7)	24		(2, 2)	8	M
	(2, 2), (6, 6), (10, 10)	14		(7, 7)	98	M
	(14, 2), (14, 6), (14, 10)		14			
	(1, 1), (3, 3), (5, 5), (9, 9), (11, 11), (13, 13)	28		(14, 14)	392	M
29	(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0), (8, 0), (9, 0), (10, 0)	29		(29, 0)	841	V
	(11, 0), (12, 0), (13, 0), (14, 0)					
	(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9), (10, 10)					
	(11, 11), (12, 12), (13, 13), (14, 14)					
30	(10, 0)	3		(3, 0)	9	V
	(10, 10)					
	(6, 0), (12, 0)	5		(5, 0)	25	V
	(6, 6), (12, 12)					
	(5, 0)	6		(6, 0)	36	V
	(15, 10)		3			
	(3, 0), (9, 0)	10		(10, 0)	100	V
	(15, 6), (15, 12)		5			
	(2, 0), (4, 0), (8, 0), (14, 0)	15		(15, 0)	225	V
	(2, 2), (4, 4), (8, 8), (14, 14)					
	(1, 0), (7, 0), (11, 0), (13, 0)	30		(30, 0)	900	V
	(15, 2), (15, 4), (15, 8), (15, 14)		15			
	(5, 5)	6		(3, 3)	18	M
	(15, 5)		6			
	(3, 3), (9, 9)	10		(5, 5)	50	M
	(15, 3), (15, 9)		10			
	(1, 1), (7, 7), (11, 11), (13, 13)	30		(15, 15)	450	M
	(15, 1), (15, 7), (15, 11), (15, 13)		30			

3.5.6. Bifurcation Point of Multiplicity 8

We investigate square patterns branching from critical points of multiplicity 8. The emergence of tilted square patterns of type T is the most phenomenal finding of this section. In addition, larger square patterns of type V and type M also branch.

Representation in Complex Variables

As shown by Table 3.10 in Section 3.5.2, a critical point of multiplicity 8 is associated with the eight-dimensional irreducible representation $(8; k, \ell)$ with

$$1 \le \ell \le k-1, \quad 2 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{3.154}$$

where $n \ge 5$.

Recall from (3.62)–(3.63) that the irreducible representation $(8; k, \ell)$ is given by

$$T^{(8;k,\ell)}(r) = \left[\begin{array}{cc|cc} & S & & \\ I & & & \\ \hline & & & I \\ & & S & \end{array}\right], \quad T^{(8;k,\ell)}(s) = \left[\begin{array}{cc|cc} & & I & \\ & & & I \\ \hline I & & & \\ & I & & \end{array}\right], \tag{3.155}$$

$$T^{(8;k,\ell)}(p_1) = \left[\begin{array}{cc|cc} R^k & & & \\ & R^{-\ell} & & \\ \hline & & R^k & \\ & & & R^{-\ell} \end{array}\right], \quad T^{(8;k,\ell)}(p_2) = \left[\begin{array}{cc|cc} R^\ell & & & \\ & R^k & & \\ \hline & & R^{-\ell} & \\ & & & R^{-k} \end{array}\right] \tag{3.156}$$

with

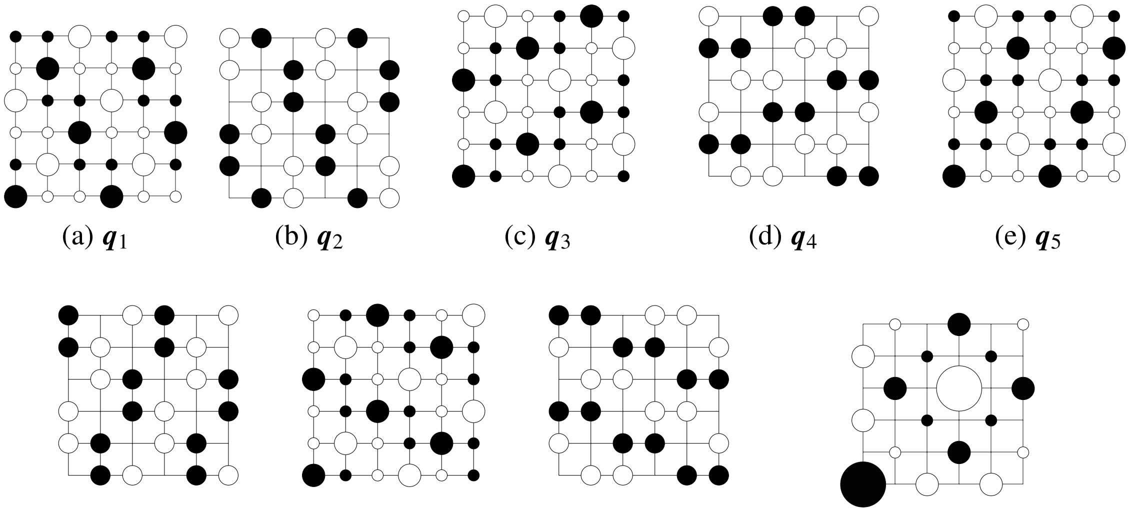
$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \tag{3.157}$$

Let us assume that the variable $\boldsymbol{w} = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8)^\top$ for the bifurcation equation in (3.108) corresponds to the column vectors of

$$\begin{aligned} Q^{(8;k,\ell)} = [&\langle \cos(2\pi(kn_1 + \ell n_2)/n)\rangle, \ \langle \sin(2\pi(kn_1 + \ell n_2)/n)\rangle, \\ &\langle \cos(2\pi(-\ell n_1 + kn_2)/n)\rangle, \ \langle \sin(2\pi(-\ell n_1 + kn_2)/n)\rangle, \\ &\langle \cos(2\pi(kn_1 - \ell n_2)/n)\rangle, \ \langle \sin(2\pi(kn_1 - \ell n_2)/n)\rangle, \\ &\langle \cos(2\pi(-\ell n_1 - kn_2)/n)\rangle, \ \langle \sin(2\pi(-\ell n_1 - kn_2)/n)\rangle], \\ &\qquad 1 \le \ell \le k-1, \ 2 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor. \end{aligned} \tag{3.158}$$

The spatial patterns for these vectors are depicted in Fig. 3.10 for $n = 6$.

The action given in (3.155) and (3.156) on 8-dimensional vectors $(w_1, \ldots, w_8)$ can be expressed



(a) $\boldsymbol{q}_1$ (b) $\boldsymbol{q}_2$ (c) $\boldsymbol{q}_3$ (d) $\boldsymbol{q}_4$ (e) $\boldsymbol{q}_5$

(f) $\boldsymbol{q}_6$ (g) $\boldsymbol{q}_7$ (h) $\boldsymbol{q}_8$ (i) $\boldsymbol{q}_1 + \boldsymbol{q}_3 + \boldsymbol{q}_5 + \boldsymbol{q}_7$ ($\langle r, s\rangle$)

Figure 3.10: Patterns on the 6×6 square lattice expressed by the column vectors of $Q^{(8;2,1)}$. A black circle denotes a positive component, and a white circle denotes a negative component.

for complex variables $z_j = w_{2j-1} + \mathrm{i}w_{2j}$ ($j = 1, \ldots, 4$) as

$$r : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_2 \\ z_1 \\ z_4 \\ \overline{z}_3 \end{bmatrix}, \quad s : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} z_3 \\ z_4 \\ z_1 \\ z_2 \end{bmatrix}, \tag{3.159}$$

$$p_1 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k \ z_1 \\ \omega^{-\ell} \ z_2 \\ \omega^k \ z_3 \\ \omega^{-\ell} \ z_4 \end{bmatrix}, \quad p_2 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \omega^\ell \ z_1 \\ \omega^k \ z_2 \\ \omega^{-\ell} \ z_3 \\ \omega^{-k} \ z_4 \end{bmatrix}, \tag{3.160}$$

where $\omega = \exp(\mathrm{i}2\pi/n)$.

Summary of the Theoretical Results

We preview the major ingredients of our analysis for critical points of multiplicity 8 associated with $(8; k, \ell)$.

We denote the isotropy subgroup of $z = (z_1, \ldots, z_4)$ with respect to $(8; k, \ell)$ as

$$\Sigma^{(8;k,\ell)}(z) = \{g \in G \mid T^{(8;k,\ell)}(g) \cdot z = z\}, \tag{3.161}$$

where $T^{(8;k,\ell)}(g) \cdot z$ means the action of $g \in G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ on z given in (3.159) and (3.160). It turns out that the isotropy subgroup of $z = (1, 1, 0, 0)$ plays a crucial role in our analysis and that

$$\Sigma^{(8;k,\ell)}((1, 1, 0, 0)) = \Sigma_0(\alpha, \beta) \tag{3.162}$$

irred. rep. (k, ℓ)	$\rightarrow(\Phi)\rightarrow$	symmetry (α, β)	$\rightarrow$(Eq. Br. Lemma)$\rightarrow$	bifurcating solution

Figure 3.11: Two stages of bifurcation analysis at a critical point of multiplicity 8.

for a uniquely determined (α, β) with $0 \leq \beta \leq \alpha \leq n$ (see Proposition 3.21). We denote this correspondence $(k, \ell) \mapsto (\alpha, \beta) = (\alpha(k, \ell, n), \beta(k, \ell, n))$ by

$$\Phi(k, \ell, n) = (\alpha, \beta). \tag{3.163}$$

In a sense, (k, ℓ) and (α, β) are dual to each other; (k, ℓ) prescribes the action of the translations p_1 and p_2, and (α, β) describes the symmetry preserved under this action.[7]

Whereas the concrete form of the correspondence Φ is discussed in detail in Appendix A.2, the following proposition shows the most fundamental formulas connecting (k, ℓ) and (α, β). We use the notations:

$$\hat{k} = \frac{k}{\gcd(k, \ell, n)}, \quad \hat{\ell} = \frac{\ell}{\gcd(k, \ell, n)}, \quad \hat{n} = \frac{n}{\gcd(k, \ell, n)}, \tag{3.164}$$

where $\gcd(k, \ell, n)$ means the greatest common divisor of k, ℓ, and n.

Proposition 3.15. *Let* $(\alpha, \beta) = \Phi(k, \ell, n)$.

(i)

$$\hat{n} = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}. \tag{3.165}$$

(ii)

$$\frac{\hat{n}}{\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n})} = \gcd(\alpha, \beta). \tag{3.166}$$

Proof. The proof is given in Appendix A.3; see Propositions A.7(ii) and A.8. It is mentioned here that the proof relies on the Smith normal form for integer matrices. □

Our analysis of bifurcation consists of two stages (see Fig. 3.11):

1. Connect the irreducible representation (k, ℓ) to the associated symmetry represented by (α, β) by obtaining the function $\Phi : (k, \ell) \mapsto (\alpha, \beta)$.
2. Connect the symmetry represented by (α, β) to the existence of bifurcating solutions on the basis of the equivariant branching lemma.

Proposition 3.16 below is a preview of a major result (Proposition 3.24) in a simplified form. For classification of bifurcation into several cases, we consider the condition

$$\textbf{GCD-div}: 2\gcd(\hat{k}, \hat{\ell}) \text{ is not divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}), \tag{3.167}$$

and the negation of this condition is referred to as $\overline{\textbf{GCD-div}}$. The set of even integers is denoted by $2\mathbb{Z}$ below.

[7] In an analogy with physics we may compare (k, ℓ) to frequency and (α, β) to wave length.

Proposition 3.16. *For a critical point of multiplicity 8, let* $(8; k, \ell)$ *be the associated irreducible representation and* $(\alpha, \beta) = \Phi(k, \ell, n)$*. The bifurcation at this point is classified as follows.*

Case 1: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$: *A bifurcating solution with symmetry* $\Sigma(\hat{n}, 0)$ *exists. This solution is of type V.*

Case 2: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$: *A bifurcating solution with symmetry* $\Sigma(\hat{n}/2, \hat{n}/2)$ *exists. This solution is of type M.*

Case 3: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$: *Bifurcating solutions with symmetries* $\Sigma(\hat{n}, 0)$*,* $\Sigma_0(\alpha, \beta)$*, and* $\Sigma_0(\beta, \alpha)$ *exist.*[8] *The first solution is of type V, and the other two solutions are of type T.*

Case 4: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$: *Bifurcating solutions with symmetries* $\Sigma(\hat{n}/2, \hat{n}/2)$*,* $\Sigma_0(\alpha, \beta)$*, and* $\Sigma_0(\beta, \alpha)$ *exist. The first solution is of type M, and the other two solutions are of type T.*

The classification criteria for the above four cases become more transparent when expressed in terms of (α, β) $(= \Phi(k, \ell, n))$ rather than (k, ℓ). The expressions in terms of (α, β) can be obtained from Proposition 3.17 below, where

$$\hat{\alpha} = \frac{\alpha}{\gcd(\alpha, \beta)}, \quad \hat{\beta} = \frac{\beta}{\gcd(\alpha, \beta)}, \tag{3.168}$$

$$\hat{D} = \hat{\alpha}^2 + \hat{\beta}^2 = \frac{D(\alpha, \beta)}{(\gcd(\alpha, \beta))^2}. \tag{3.169}$$

It is noted in passing that an alternative expression

$$\hat{D} = \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}) \tag{3.170}$$

results from (3.165), (3.166), and (3.169).

Proposition 3.17. *Let* $(\alpha, \beta) = \Phi(k, \ell, n)$*.*

(i) $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z} \iff \hat{D} \in 2\mathbb{Z}$*.*

(ii) **GCD-div** *in* (3.167) $\iff \beta = 0$ *or* $\alpha = \beta$*.*

Proof. The proof is given in Appendix A.3; see Proposition A.7(i) and Proposition A.12. It is mentioned here that the proof of the equivalence in (ii) relies on the Smith normal form for integer matrices and the integer analogue of the Farkas lemma. □

Propositions 3.16 and 3.17 together yield Table 3.15 that summarizes the classification of bifurcation phenomena into the four cases in terms of both (k, ℓ) and (α, β).

An important observation here is that the classification into the four cases in Proposition 3.16, as well as in Table 3.15, can also be described in terms of the subgroup $\Sigma_0(\alpha, \beta)$. The following proposition shows how the conditions "$\beta = 0$ or $\alpha = \beta$" and "$\hat{D} \in 2\mathbb{Z}$" can be replaced by conditions for $\Sigma_0(\alpha, \beta)$.

[8] To be precise, $\Sigma_0(\beta, \alpha)$ should be denoted as $\Sigma_0(\alpha', \beta')$ with (α', β') in (3.178), which lies in the parameter space of (3.99).

Table 3.15: The classification of bifurcation at a critical point associated with $(8; k, \ell)$ with $(\alpha, \beta) = \Phi(k, \ell, n)$.

	$\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$	$\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$
	$\hat{D} \notin 2\mathbb{Z}$	$\hat{D} \in 2\mathbb{Z}$
GCD-div	Case 1:	Case 2:
$\beta = 0$ or $\alpha = \beta$	type V	type M
$\overline{\textbf{GCD-div}}$	Case 3:	Case 4:
$\beta \neq 0$ and $\alpha \neq \beta$	type V and type T	type M and type T

Table 3.16: Bifurcation at a critical point associated with $(8; k, \ell)$ classified in terms of the subgroup $\Sigma_0(\alpha, \beta)$ for $(\alpha, \beta) = \Phi(k, \ell, n)$.

	$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\beta, \alpha)$ $= \Sigma_0(\alpha'', 0)$	$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\beta, \alpha)$ $= \Sigma_0(\beta'', \beta'')$
$\Sigma_0(\alpha, \beta) = \Sigma_0(\beta, \alpha)$	Case 1: type V	Case 2: type M
$\Sigma_0(\alpha, \beta) \neq \Sigma_0(\beta, \alpha)$	Case 3: type V and type T	Case 4: type M and type T

Proposition 3.18.

(i) $\Sigma_0(\alpha, \beta) = \Sigma_0(\beta, \alpha) \iff \beta = 0$ *or* $\alpha = \beta$.

(ii)

$$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\beta, \alpha) = \begin{cases} \Sigma_0(\alpha'', 0) & \text{if } \hat{D} \notin 2\mathbb{Z}, \\ \Sigma_0(\beta'', \beta'') & \text{if } \hat{D} \in 2\mathbb{Z} \end{cases} \tag{3.171}$$

with

$$\alpha'' = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}, \quad \beta'' = \frac{D(\alpha, \beta)}{2 \gcd(\alpha, \beta)}. \tag{3.172}$$

Proof. (i) This is obvious from the definition of $\Sigma_0(\alpha, \beta)$ in (3.102).

(ii) The proof is given in Proposition A.4 in Appendix A.3. □

By Proposition 3.18 above, we can rewrite Table 3.15 as Table 3.16. In particular, solutions of type T exist if and only if $\Sigma_0(\alpha, \beta)$ is asymmetric in the sense of $\Sigma_0(\alpha, \beta) \neq \Sigma_0(\beta, \alpha)$. Not only is this statement intuitively appealing, but it plays a crucial role in our technical arguments in Appendix A.3.

Remark 3.6. Some comments are in order about (3.171) in each case corresponding to type V, type M, or type T.

- If $\beta = 0$, we have $\hat{D} = 1$ and $\alpha'' = D(\alpha, 0)/\gcd(\alpha, 0) = \alpha^2/\alpha = \alpha$.
- If $\alpha = \beta$, we have $\hat{D} = 2$ and $\beta'' = D(\beta, \beta)/(2\gcd(\beta, \beta)) = (2\beta^2)/(2\beta) = \beta$.

- For (α, β) with $1 \leq \beta < \alpha$, we have $\hat{D} = 5, 10, 13, 17, 20$, and so on, some of which satisfy $\hat{D} \in 2\mathbb{Z}$, while others do not.

It should be also mentioned that the identity (3.171) is purely geometric in that it is valid for all (α, β) that may or may not be related to irreducible representation $(8; k, \ell)$. If (α, β) is associated with $(8; k, \ell)$, we have $\alpha'' = \hat{n}$ and $\beta'' = \hat{n}/2$ by (3.165) and (3.172), respectively.

□

Isotropy Subgroups

To apply the method of analysis described in Section 3.5.2, we identify isotropy subgroups for $(8; k, \ell)$ related to square patterns.

We denote the fixed-point subspace of Σ in terms of $z = (z_1, \ldots, z_4)$ as

$$\mathrm{Fix}^{(8;k,\ell)}(\Sigma) = \{z \mid T^{(8;k,\ell)}(g) \cdot z = z \text{ for all } g \in \Sigma\}, \tag{3.173}$$

where $T^{(8;k,\ell)}(g) \cdot z$ means the action of $g \in G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ on z given in (3.159) and (3.160). Also recall from (3.161) the notation $\Sigma^{(8;k,\ell)}(z)$ for the isotropy subgroup of z.

The symmetries of $\langle r \rangle$ and $\langle r, s \rangle$ are dealt with in Proposition 3.19 below, and the translational symmetry ${p_1}^a {p_2}^b$ is considered thereafter. Remark 3.10 below should be consulted with regard to the geometrical interpretation of the following discussion.

Proposition 3.19.
(i) $\mathrm{Fix}^{(8;k,\ell)}(\langle r \rangle) = \{c(1,1,0,0) + c'(0,0,1,1) \mid c, c' \in \mathbb{R}\}$.
(ii) $\mathrm{Fix}^{(8;k,\ell)}(\langle r, s \rangle) = \{c(1,1,1,1) \mid c \in \mathbb{R}\}$.

Proof. (i) By (3.159), z is invariant to r if and only if $(\overline{z_2}, z_1, z_4, \overline{z_3}) = (z_1, z_2, z_3, z_4)$, which is equivalent to $z_1 = z_2 \in \mathbb{R}$ and $z_3 = z_4 \in \mathbb{R}$.

(ii) By (3.159), z is invariant to s if and only if $(z_3, z_4, z_1, z_2) = (z_1, z_2, z_3, z_4)$, which is equivalent to $z_1 = z_3$ and $z_2 = z_4$. Hence z is invariant to both r and s if and only if $z_1 = z_2 = z_3 = z_4 \in \mathbb{R}$. □

The above proposition implies that any isotropy subgroup Σ containing $\langle r \rangle$, which is of our interest, can be represented as $\Sigma = \Sigma^{(8;k,\ell)}(z)$ for some vector z of the form

$$z = c(1,1,0,0) + c'(0,0,1,1), \quad c, c' \in \mathbb{R}, \tag{3.174}$$

and that $\dim\ \mathrm{Fix}^{(8;k,\ell)}(\Sigma) \leq 2$.

We now turn to the invariance to the translational symmetry ${p_1}^a {p_2}^b$.

Proposition 3.20.
(i) ${p_1}^a {p_2}^b \in \Sigma^{(8;k,\ell)}((1,1,0,0))$ *if and only if*

$$\hat{k}a + \hat{\ell}b \equiv 0, \quad \hat{\ell}a - \hat{k}b \equiv 0 \mod \hat{n}. \tag{3.175}$$

(ii) ${p_1}^a {p_2}^b \in \Sigma^{(8;k,\ell)}((0,0,1,1))$ *if and only if*

$$\hat{k}a - \hat{\ell}b \equiv 0, \quad \hat{\ell}a + \hat{k}b \equiv 0 \mod \hat{n}. \tag{3.176}$$

Proof. (i) By (3.160), the invariance of $z = (1,1,0,0)$ to ${p_1}^a{p_2}^b$ is expressed as

$$k a + \ell b \equiv 0, \quad \ell a - k b \equiv 0 \mod n,$$

which is equivalent to (3.175) with the notations in (3.164).

(ii) By (3.160) the invariance of $z = (0,0,1,1)$ to ${p_1}^a{p_2}^b$ is expressed as

$$k a - \ell b \equiv 0, \quad \ell a + k b \equiv 0 \mod n,$$

which is equivalent to (3.176). □

The isotropy subgroup of $z = c(1,1,0,0) + c'(0,0,1,1)$ of the form of (3.174) is identified in the following two propositions: the case with $cc' = 0$ in Proposition 3.21 and the case with $cc' \neq 0$ in Proposition 3.22.

Proposition 3.21.

(i) *For each* (k,ℓ)*, we have*

$$\Sigma^{(8;k,\ell)}((1,1,0,0)) = \Sigma_0(\alpha,\beta) \tag{3.177}$$

for a uniquely determined (α,β) *with* $0 \leq \beta < n$*,* $0 < \alpha \leq n$*.*

(ii) *For the* (α,β) *associated with* (k,ℓ) *as in* (i) *above, define*

$$(\alpha',\beta') = \begin{cases} (\beta,\alpha) & \text{if } \beta > 0, \\ (\alpha,0) & \text{if } \beta = 0. \end{cases} \tag{3.178}$$

Then we have

$$\Sigma^{(8;k,\ell)}((0,0,1,1)) = \Sigma_0(\alpha',\beta'). \tag{3.179}$$

Proof. (i) By (3.159), $\Sigma^{(8;k,\ell)}((1,1,0,0))$ contains r and not s. To investigate the translation symmetry, denote by $\mathcal{A}(k,\ell,n)$ the set of all (a,b) satisfying (3.175). That is,

$$\mathcal{A}(k,\ell,n) = \{(a,b) \in \mathbb{Z}^2 \mid \hat{k}a + \hat{\ell}b \equiv 0,\ \hat{\ell}a - \hat{k}b \equiv 0 \mod \hat{n}\}. \tag{3.180}$$

Then $\mathcal{A}(k,\ell,n)$ is closed under integer combination, i.e., if $(a_1,b_1), (a_2,b_2) \in \mathcal{A}(k,\ell,n)$, then $n_1(a_1,b_1) + n_2(a_2,b_2) \in \mathcal{A}(k,\ell,n)$ for any $n_1,n_2 \in \mathbb{Z}$. Next, if $(a,b) \in \mathcal{A}(k,\ell,n)$, then $(a',b') = (-b,a)$ also belongs to $\mathcal{A}(k,\ell,n)$ since

$$\begin{aligned} \hat{k}a' + \hat{\ell}b' &= \hat{k}(-b) + \hat{\ell}a = \hat{\ell}a - \hat{k}b \equiv 0 \mod \hat{n}, \\ \hat{\ell}a' - \hat{k}b' &= \hat{\ell}(-b) - \hat{k}a = -(\hat{k}a + \hat{\ell}b) \equiv 0 \mod \hat{n}. \end{aligned}$$

The above argument shows that $\mathcal{A}(k,\ell,n)$ coincides with a set of the form

$$\mathcal{L}(\alpha,\beta) = \{(a,b) \in \mathbb{Z}^2 \mid (a,b) = n_1(\alpha,\beta) + n_2(-\beta,\alpha),\ n_1,n_2 \in \mathbb{Z}\} \tag{3.181}$$

for some appropriately chosen integers α and β. For such (α,β) we have

$$\Sigma^{(8;k,\ell)}((1,1,0,0)) = \langle r \rangle \ltimes \langle {p_1}^{\alpha}{p_2}^{\beta}, {p_1}^{-\beta}{p_2}^{\alpha} \rangle = \Sigma_0(\alpha,\beta).$$

To see the uniqueness of (α, β) we note the obvious correspondence between $\mathcal{L}(\alpha, \beta)$ and the square sublattice $\mathcal{H}(\alpha, \beta)$ in (3.4). By Proposition 3.1, $\mathcal{H}(\alpha, \beta)$ is uniquely parameterized by (α, β) with $0 \leq \beta < \alpha$. Furthermore, we have $\alpha \leq n$ as a consequence of the fact that $\mathcal{L}(\alpha, \beta)$ contains no point (a, b) of the form of $(a, b) = x(\alpha, \beta) + y(-\beta, \alpha)$ with $0 < x < 1$ and $0 < y < 1$, which lies in the interior of the parallelogram formed by its basis vectors (α, β) and $(-\beta, \alpha)$. To prove this by contradiction, suppose that $\alpha > n$ and consider the point $(a, b) = (\alpha - n, \beta)$. This point belongs to $\mathcal{L}(\alpha, \beta)$, satisfying the defining conditions in of $\mathcal{A}(k, \ell, n)$ in (3.180), whereas the corresponding (x, y) satisfies $0 < x < 1$ and $0 < y < 1$, which is a contradiction.

(ii) Since

$$(0, 0, 1, 1) = T^{(8;k,\ell)}(s) \cdot (1, 1, 0, 0),$$

it follows using the relation for the orbit $\Sigma(T(g)\boldsymbol{u}) = g \cdot \Sigma(\boldsymbol{u}) \cdot g^{-1}$ $(g \in G)$, (3.177), (3.32), and (3.178) in this order that

$$\Sigma^{(8;k,\ell)}((0, 0, 1, 1)) = s \cdot \Sigma^{(8;k,\ell)}((1, 1, 0, 0)) \cdot s^{-1} = s \cdot \Sigma_0(\alpha, \beta) \cdot s^{-1} = \Sigma_0(\beta, \alpha) = \Sigma_0(\alpha', \beta').$$

□

We denote the correspondence $(k, \ell) \mapsto (\alpha, \beta) = (\alpha(k, \ell, n), \beta(k, \ell, n))$ defined by (3.177) in Proposition 3.21 as

$$\Phi(k, \ell, n) = (\alpha, \beta). \tag{3.182}$$

Remark 3.7. A preliminary explanation is presented here about how the value of $(\alpha, \beta) = \Phi(k, \ell, n)$ can be determined, whereas a systematic method is given in Appendix A.2.

The condition for $(a, b) \in \mathcal{A}(k, \ell, n)$ in (3.180) is equivalent to the existence of integers p and q satisfying

$$\begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \hat{n} \begin{bmatrix} p \\ q \end{bmatrix}. \tag{3.183}$$

Hence a pair of integers (a, b) belongs to $\mathcal{A}(k, \ell, n)$ if and only if

$$\begin{bmatrix} a \\ b \end{bmatrix} = \hat{n} \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix} = \frac{\hat{n}}{\hat{k}^2 + \hat{\ell}^2} \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{3.184}$$

for some integers p and q. There are two cases to consider.

- If $\hat{n}/(\hat{k}^2 + \hat{\ell}^2)$ is an integer, a simpler method works. In this case, the right-hand side of (3.184) gives a pair of integers for any integers p and q. Therefore, we set $(p, q) = (1, 0)$ to obtain an integer vector

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\hat{n}}{\hat{k}^2 + \hat{\ell}^2} \begin{bmatrix} \hat{k} \\ \hat{\ell} \end{bmatrix} \tag{3.185}$$

  and note that the vectors $(a, b)^\top$ of integers satisfying (3.183) form a lattice spanned by $(\alpha, \beta)^\top$ and $(\beta, -\alpha)^\top$. For $(k, \ell, n) = (3, 1, 20)$, for example, we have $(\hat{k}, \hat{\ell}, \hat{n}) = (3, 1, 20)$ and $\hat{n}/(\hat{k}^2 + \hat{\ell}^2) = 20/10 = 2$, and hence (3.184) reads

$$\begin{bmatrix} a \\ b \end{bmatrix} = 2 \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

  This shows $\Phi(3, 1, 20) = (\alpha, \beta) = (6, 2)$, corresponding to $(p, q) = (1, 0)$.

- If $\hat{n}/(\hat{k}^2 + \hat{\ell}^2)$ is not an integer, number-theoretic considerations are needed to determine $(\alpha, \beta) = \Phi(k, \ell, n)$. For $(k, \ell, n) = (18, 2, 42)$, for instance, we have $(\hat{k}, \hat{\ell}, \hat{n}) = (9, 1, 21)$ and $\hat{k}^2 + \hat{\ell}^2 = 82$, and (3.184) reads

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{21}{82} \begin{bmatrix} 9 & 1 \\ 1 & -9 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

With some inspection we could arrive at $\Phi(18, 2, 42) = (\alpha, \beta) = (21, 0)$, which corresponds to $(p, q) = (9, 1)$. A systematic procedure based on the Smith normal form is given in Appendix A.2.

□

Remark 3.8. In the following arguments we shall make use of Propositions 3.15, 3.17, and 3.18 presented in Section 3.5.6. The readers may take these propositions for granted in the first reading, but those who are interested in mathematical issues are advised to have a look at their proofs given in Appendix A.3.

□

Proposition 3.22. *Let $(\alpha, \beta) = \Phi(k, \ell, n)$, and define*

$$\alpha'' = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}, \quad \beta'' = \frac{D(\alpha, \beta)}{2 \gcd(\alpha, \beta)}. \tag{3.186}$$

For distinct nonzero real numbers c and c' ($c \neq c'$), we have the following statements:

(i)

$$\Sigma^{(8;k,\ell)}((c, c, c, c)) = \begin{cases} \Sigma(\alpha'', 0) & \text{if } \hat{D} \notin 2\mathbb{Z}, \\ \Sigma(\beta'', \beta'') & \text{if } \hat{D} \in 2\mathbb{Z}, \end{cases}$$

where $\hat{D}$ is defined in (3.169) and $\hat{D} \in 2\mathbb{Z}$ means that $\hat{D}$ is even.

(ii)

$$\Sigma^{(8;k,\ell)}((c, c, c', c')) = \begin{cases} \Sigma_0(\alpha'', 0) & \text{if } \hat{D} \notin 2\mathbb{Z}, \\ \Sigma_0(\beta'', \beta'') & \text{if } \hat{D} \in 2\mathbb{Z}. \end{cases}$$

Proof. We first prove (ii). By (3.159), $\Sigma^{(8;k,\ell)}((c, c, c', c'))$ contains r and not s. We have

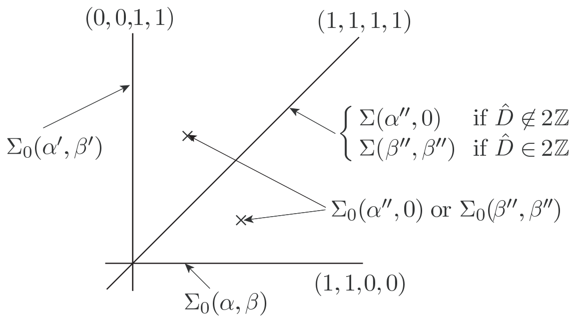
$$\begin{aligned} \Sigma^{(8;k,\ell)}((c, c, c', c')) &= \Sigma^{(8;k,\ell)}((1, 1, 0, 0)) \cap \Sigma^{(8;k,\ell)}((0, 0, 1, 1)) \\ &= \Sigma_0(\alpha, \beta) \cap \Sigma_0(\alpha', \beta'), \end{aligned}$$

where the second equality is due to Proposition 3.21. Then the claim follows from Proposition 3.18(ii).

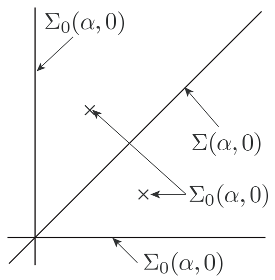
Next we prove (i). By (3.159), $\Sigma^{(8;k,\ell)}((c, c, c, c))$ contains both r and s. We can proceed in a similar manner as above while including the element s. Therefore

$$\Sigma^{(8;k,\ell)}((c, c, c, c)) = \Sigma(\alpha, \beta) \cap \Sigma(\alpha', \beta'),$$

which implies the claim. □



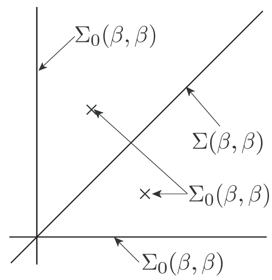
A general result



Case 1. $(\alpha, \beta) = (\alpha, 0)$

dim $\text{Fix}\Sigma(\alpha, 0) = 1$: type V, $z = (1, 1, 1, 1)$

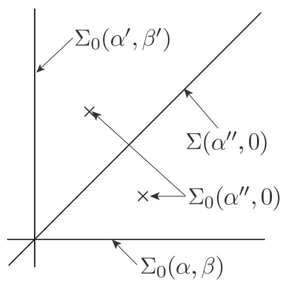
dim $\text{Fix}\Sigma_0(\alpha, 0) = 2$: non-targeted



Case 2. $(\alpha, \beta) = (\beta, \beta)$

dim $\text{Fix}\Sigma(\beta, \beta) = 1$: type M, $z = (1, 1, 1, 1)$

dim $\text{Fix}\Sigma_0(\beta, \beta) = 2$: non-targeted



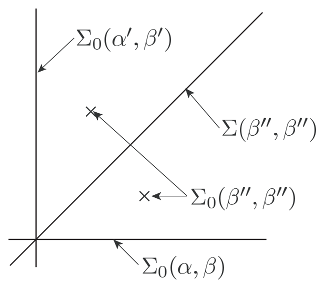
Case 3. (α, β) : $\alpha \neq \beta$,
$1 \leq \alpha \leq n-1$, $1 \leq \beta \leq n-1$, $\hat{D} \notin 2\mathbb{Z}$

dim $\text{Fix}\Sigma(\alpha'', 0) = 1$: type V, $z = (1, 1, 1, 1)$

dim $\text{Fix}\Sigma_0(\alpha, \beta) = 1$: type T, $z = (1, 1, 0, 0)$

dim $\text{Fix}\Sigma_0(\alpha', \beta') = 1$: type T, $z = (0, 0, 1, 1)$

dim $\text{Fix}\Sigma_0(\alpha'', 0) = 2$: non-targeted



Case 4. (α, β) : $\alpha \neq \beta$,
$1 \leq \alpha \leq n-1$, $1 \leq \beta \leq n-1$, $\hat{D} \in 2\mathbb{Z}$

dim $\text{Fix}\Sigma(\beta'', \beta'') = 1$: type M, $z = (1, 1, 1, 1)$

dim $\text{Fix}\Sigma_0(\alpha, \beta) = 1$: type T, $z = (1, 1, 0, 0)$

dim $\text{Fix}\Sigma_0(\alpha', \beta') = 1$: type T, $z = (0, 0, 1, 1)$

dim $\text{Fix}\Sigma_0(\beta'', \beta'') = 2$: non-targeted

Figure 3.12: Isotropy subgroups for $(8; k, \ell)$ with $(\alpha, \beta) = \Phi(k, \ell, n)$, (α', β') in (3.178), and (α'', β'') in (3.186).

In Proposition 3.23, we can present the isotropy subgroups containing $\langle r \rangle$, with a classification of the irreducible representations $(8; k, \ell)$ in terms of $(\alpha, \beta) = \Phi(k, \ell, n)$. See Fig. 3.12 for the classification.

Proposition 3.23. *For an irreducible representation $(8; k, \ell)$, let $(\alpha, \beta) = \Phi(k, \ell, n)$, and define (α', β'), α'' and β'' by (3.178) and (3.186), respectively. Then the isotropy subgroups containing $\langle r \rangle$ are given by Σ listed below.*

Case 1: $(\alpha, \beta) = (\alpha, 0)$ *with* $1 \leq \alpha \leq n$.

$$\begin{cases} \text{(a) } \Sigma = \Sigma(\alpha, 0) = \Sigma^{(8;k,\ell)}((1,1,1,1)), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c,c) \mid c \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(\alpha, 0) = \Sigma^{(8;k,\ell)}((c,c,c',c')) \ (c \neq c', c \neq 0, c' \neq 0), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c',c') \mid c, c' \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 2. \end{cases}$$

Case 2: $(\alpha, \beta) = (\beta, \beta)$ *with* $1 \leq \beta \leq n/2$.

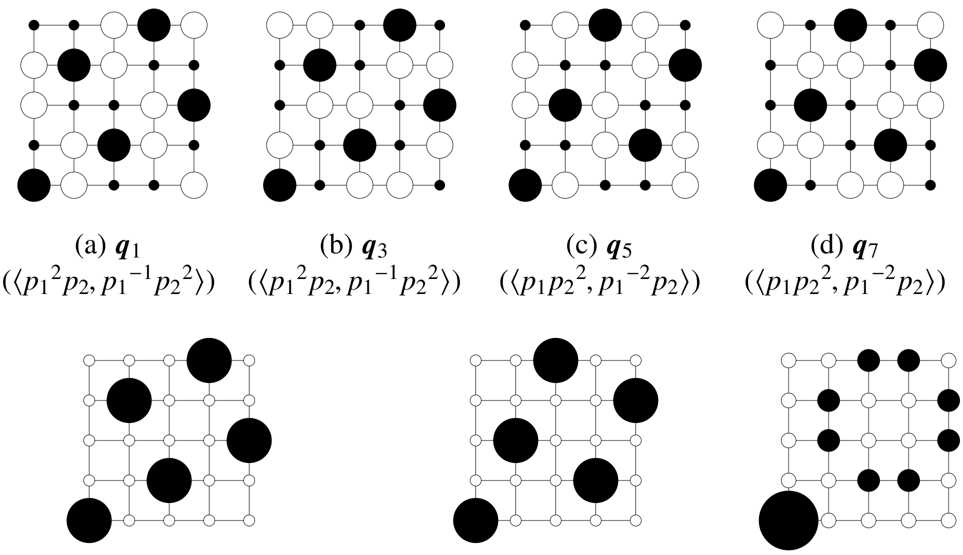
$$\begin{cases} \text{(a) } \Sigma = \Sigma(\beta, \beta) = \Sigma^{(8;k,\ell)}((1,1,1,1)), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c,c) \mid c \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(\beta, \beta) = \Sigma^{(8;k,\ell)}((c,c,c',c')) \ (c \neq c', c \neq 0, c' \neq 0), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c',c') \mid c, c' \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 2. \end{cases}$$

Case 3: (α, β) *with* $1 \leq \alpha \leq n-1, \ 1 \leq \beta \leq n-1, \ \alpha \neq \beta$, *and* $\hat{D} \notin 2\mathbb{Z}$.

$$\begin{cases} \text{(a) } \Sigma = \Sigma(\alpha'', 0) = \Sigma^{(8;k,\ell)}((1,1,1,1)), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c,c) \mid c \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(\alpha, \beta) = \Sigma^{(8;k,\ell)}((1,1,0,0)), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,0,0) \mid c \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(c) } \Sigma = \Sigma_0(\alpha', \beta') = \Sigma^{(8;k,\ell)}((0,0,1,1)), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(0,0,c',c') \mid c' \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(d) } \Sigma = \Sigma_0(\alpha'', 0) = \Sigma^{(8;k,\ell)}((c,c,c',c')) \ (c \neq c', c \neq 0, c' \neq 0), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c',c') \mid c, c' \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 2. \end{cases}$$

Case 4: (α, β) *with* $1 \leq \alpha \leq n-1, \ 1 \leq \beta \leq n-1, \ \alpha \neq \beta$, *and* $\hat{D} \in 2\mathbb{Z}$.

$$\begin{cases} \text{(a) } \Sigma = \Sigma(\beta'', \beta'') = \Sigma^{(8;k,\ell)}((1,1,1,1)), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c,c) \mid c \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(b) } \Sigma = \Sigma_0(\alpha, \beta) = \Sigma^{(8;k,\ell)}((1,1,0,0)), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,0,0) \mid c \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(c) } \Sigma = \Sigma_0(\alpha', \beta') = \Sigma^{(8;k,\ell)}((0,0,1,1)), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(0,0,c',c') \mid c' \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 1. \\ \text{(d) } \Sigma = \Sigma_0(\beta'', \beta'') = \Sigma^{(8;k,\ell)}((c,c,c',c')) \ (c \neq c', c \neq 0, c' \neq 0), \\ \quad \text{Fix}^{(8;k,\ell)}(\Sigma) = \{(c,c,c',c') \mid c, c' \in \mathbb{R}\}, \ \dim \text{Fix}^{(8;k,\ell)}(\Sigma) = 2. \end{cases}$$



(a) $\boldsymbol{q}_1$ $(\langle {p_1}^2 p_2, {p_1}^{-1} {p_2}^2 \rangle)$ (b) $\boldsymbol{q}_3$ $(\langle {p_1}^2 p_2, {p_1}^{-1} {p_2}^2 \rangle)$ (c) $\boldsymbol{q}_5$ $(\langle p_1 {p_2}^2, {p_1}^{-2} p_2 \rangle)$ (d) $\boldsymbol{q}_7$ $(\langle p_1 {p_2}^2, {p_1}^{-2} p_2 \rangle)$

(e) $\boldsymbol{q}_1 + \boldsymbol{q}_3$ $(\Sigma_0(2,1) = \langle r, {p_1}^2 p_2, {p_1}^{-1} {p_2}^2 \rangle)$ (f) $\boldsymbol{q}_5 + \boldsymbol{q}_7$ $(\Sigma_0(1,2) = \langle r, p_1 {p_2}^2, {p_1}^{-2} p_2 \rangle)$ (g) $\sum_{i=1,3,5,7} \boldsymbol{q}_i$ $(\mathrm{D}_4 = \langle r, s \rangle)$

Figure 3.13: Patterns on the 5×5 square lattice expressed by the column vectors of $Q^{(8;2,1)}$. A white circle denotes a positive component, and a black circle denotes a negative component.

Proof. With an observation that $\Sigma_0(\alpha, \beta) \neq \Sigma_0(\alpha', \beta')$ in Cases 3 and 4, the above classification follows immediately from Propositions 3.21 and 3.22. □

Remark 3.9. In Case 1 of Proposition 3.23, we may have $\alpha = n$, in which case $\Sigma(\alpha, 0) = \Sigma(0, 0) = \langle r, s \rangle$ and $\Sigma_0(\alpha, 0) = \Sigma_0(0, 0) = \langle r \rangle$, and the translational symmetry is absent.

□

Remark 3.10. The isotropy subgroups in Proposition 3.23 can be understood quite naturally with reference to the column vectors of the matrix

$$Q^{(8;k,\ell)} = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_8]$$

given in (3.89). The spatial patterns for these vectors are depicted in Fig. 3.13, for example, for $(8; 2, 1)$ with $n = 5$. Although the four vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_3$, $\boldsymbol{q}_5$, and $\boldsymbol{q}_7$ do not represent square patterns (Figs. 3.13(a)–(f)), the sum of these four vectors, which is associated with $z = (1, 1, 1, 1)$ ($\boldsymbol{w} = (1, 0, 1, 0, 1, 0, 1, 0)^\top$), represents a square pattern of type V with $D = 25$ (Fig. 3.13(g)). Moreover, the sum $\boldsymbol{q}_1 + \boldsymbol{q}_3$, which is associated with $z = (1, 1, 0, 0)$, represents square pattern of type T with $D = 5$ (Fig. 3.13(e)). On the other hand, the pattern in Fig. 3.13(f), which is associated with $z = (0, 0, 1, 1)$, represents another square pattern of type T with $D = 5$.

□

Existence of Square Patterns

A combination of Proposition 3.23 with the equivariant branching lemma (Section 3.5.2) shows the existence of solutions with the targeted symmetry bifurcating from a critical point associated with $(8; k, \ell)$.

Bifurcating solutions can be classified in accordance with number-theoretic properties of (k, ℓ). To be specific, it depends on the following two properties:

$$2\gcd(\hat{k}, \hat{\ell}) \text{ is divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}), \tag{3.187}$$

$$\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}. \tag{3.188}$$

We refer to the condition (3.187) as **GCD-div** and its negation as $\overline{\textbf{GCD-div}}$. It should be mentioned that a simplified version of the following proposition has already been presented as Proposition 3.16 in Section 3.5.6. See also Table 3.15.

Proposition 3.24. *From a critical point associated with the irreducible representation* $(8; k, \ell)$*, solutions with the following symmetries emerge as bifurcating solutions, where* $(\alpha, \beta) = \Phi(k, \ell, n)$ *and* (α', β') *is defined in* (3.178)*. We have four cases.*

Case 1: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha, \beta) = (\hat{n}, 0)$*. A bifurcating solution with symmetry* $\Sigma(\hat{n}, 0)$*, which corresponds to* $z^{(1)} = c(1, 1, 1, 1)$*, exists. This solution is of type V.*

Case 2: **GCD-div** *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha, \beta) = (\hat{n}/2, \hat{n}/2)$*. A bifurcating solution with symmetry* $\Sigma(\hat{n}/2, \hat{n}/2)$*, corresponding to* $z^{(1)} = c(1, 1, 1, 1)$*, exists. This solution is of type M.*

Case 3: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha, \beta)$ *with* $1 \le \alpha \le n-1$*,* $1 \le \beta \le n-1$*,* $\alpha \ne \beta$*, and* $\hat{D} \notin 2\mathbb{Z}$*. Bifurcating solutions with symmetries* $\Sigma(\hat{n}, 0)$*,* $\Sigma_0(\alpha, \beta)$*, and* $\Sigma_0(\alpha', \beta')$*, corresponding to* $z^{(1)} = c(1, 1, 1, 1)$*,* $z^{(2)} = c(1, 1, 0, 0)$*, and* $z^{(3)} = c(0, 0, 1, 1)$*, respectively, exist. The first solution is of type V, and the other two solutions are of type T.*

Case 4: $\overline{\textbf{GCD-div}}$ *and* $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$: *We have* $\Phi(k, \ell, n) = (\alpha, \beta)$ *with* $1 \le \alpha \le n-1$*,* $1 \le \beta \le n-1$*,* $\alpha \ne \beta$*, and* $\hat{D} \in 2\mathbb{Z}$*. Bifurcating solutions with symmetries* $\Sigma(\hat{n}/2, \hat{n}/2)$*,* $\Sigma_0(\alpha, \beta)$*, and* $\Sigma_0(\alpha', \beta')$*, corresponding to* $z^{(1)} = c(1, 1, 1, 1)$*,* $z^{(2)} = c(1, 1, 0, 0)$*, and* $z^{(3)} = c(0, 0, 1, 1)$*, respectively, exist. The first solution is of type M, and the other two solutions are of type T.*

Proof. By Proposition 3.17, as well as Remark 3.6 in Section 3.5.6, the above four cases correspond to those in Proposition 3.23. In all cases, the relevant subgroup Σ is an isotropy subgroup with $\dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 1$ by Proposition 3.23. Then the equivariant branching lemma (Section 3.5.2) guarantees the existence of a bifurcating solution with symmetry Σ. □

Remark 3.11. The subgroup $\Sigma = \Sigma_0(\alpha, 0)$, $\Sigma_0(\beta, \beta)$, $\Sigma_0(\hat{n}, 0)$ or $\Sigma_0(\hat{n}/2, \hat{n}/2)$ appearing in Proposition 3.23 is an isotropy subgroup with $\dim \mathrm{Fix}^{(8;k,\ell)}(\Sigma) = 2$, for which the equivariant branching lemma is not effective. It is emphasized that Proposition 3.24 does not assert the nonexistence of solutions of these symmetries. Nonetheless, we do not have to deal with these subgroups since none of these symmetries corresponds to square patterns (see (3.102)).

□

Square Patterns of Type V

Square patterns of type V (with $D \ge 25$) are predicted to branch from critical points of multiplicity 8, whereas smaller square patterns of type V with $D = 4, 9, 16$ do not exist. Recall that a square pattern of type V is characterized by the symmetry of $\Sigma(\alpha, 0)$ with $2 \leq \alpha \leq n$ (see (3.102)) and that $D(\alpha, 0) = \alpha^2$.

The following propositions show such nonexistence and existence of square patterns of type V.

Proposition 3.25. *Square patterns of type V with* $D = 4, 9, 16$ *do not arise as bifurcating solutions from critical points of multiplicity 8 for any n.*

Proof. The proof is given at the end of the proof of Proposition 3.26. □

Proposition 3.26. *Square patterns of type V with the symmetry of* $\Sigma(\alpha, 0)$ $(5 \leq \alpha \leq n)$ *arise as bifurcating solutions from critical points of multiplicity 8 for specific values of n and irreducible representations given by*

$$\begin{array}{cccc} (\alpha,\beta) & D & n & (k,\ell) \text{ in } (8;k,\ell) \\ \hline (\alpha,0) & \alpha^2 & \alpha m & ((p+q)m, qm) \\ \hline \end{array} \tag{3.189}$$

with $m \geq 1$ *and*

$$\begin{aligned} &p \geq 1, \quad q \geq 1, \quad \gcd(p,q,\alpha) = 1, \quad \gcd(p,\alpha) \notin 2\mathbb{Z}, \\ &\begin{cases} 2(p+q+1) \leq \alpha & n \text{ is even, and } m = 1, \\ 2(p+q)+1 \leq \alpha & \text{otherwise.} \end{cases} \end{aligned} \tag{3.190}$$

Proof. Type V occurs in Case 1 and Case 3 in Proposition 3.24, characterized by the condition of $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$. Put $\hat{k} = p + q$ and $\hat{\ell} = q$ for some $p, q \in \mathbb{Z}$ and note $\hat{n} = \alpha$. Since $\gcd(\hat{k} - \hat{\ell}, \hat{n}) = \gcd(p, \alpha)$, the condition $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$ holds if and only if $\gcd(p, \alpha) \notin 2\mathbb{Z}$. We have $(k, \ell, n) = ((p + q)m, qm, \alpha m)$ for $m = \gcd(k, \ell, n)$. Here we must have

$$1 = \gcd(\hat{k}, \hat{\ell}, \hat{n}) = \gcd(p + q, q, \alpha) = \gcd(p, q, \alpha).$$

The inequality constraint in (3.154) is translated as

$$1 \leq \ell \leq k-1, \ 2 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \quad \Longleftrightarrow$$
$$p \geq 1, \ q \geq 1, \ \begin{cases} 2(p+q+1) \leq \alpha & \text{if } n \text{ is even and } m = 1, \\ 2(p+q)+1 \leq \alpha & \text{otherwise.} \end{cases}$$

Proposition 3.26 is thus obtained.

To prove Proposition 3.25, we note that, for $\alpha = 2, 3, 4$, no (p, q) satisfies (3.190), which proves the nonexistence of the smaller square patterns claimed in Proposition 3.25. □

Example 3.5. The parameter values of (3.189) in Proposition 3.26 give Table 3.17. The asterisk $(\cdot)^*$ indicates coexistence of type T (see (3.193)), i.e., Case 3 of Proposition 3.24, whereas unmarked cases correspond to Case 1 of Proposition 3.24, where no solution of type T coexists.

□

Remark 3.12. In all cases in (3.189), the compatibility condition (3.104) is satisfied for $\Sigma(\alpha, 0)$ as $n = m\alpha$ with $m = \gcd(k, \ell, n)$, since we have

$$\gcd(k, \ell, n) = ((p + q)m, qm, \alpha m) = m \gcd(p + q, q, \alpha) = m \gcd(p, q, \alpha) = m$$

by (3.189) and (3.190).

□

Table 3.17: Correspondence of irreducible representation $(8; k, \ell)$ to (α, β) for square patterns of type V.

(α, β)	D	n	(k, ℓ) in $(8; k, \ell)$
$(5, 0)$	25	$5m$	$(2m, m)^*$
$(6, 0)$	36	$6m$	$(2m, m)$
$(7, 0)$	49	$7m$	$(2m, m), (3m, m), (3m, 2m)$
$(8, 0)$	64	$8m$	$(2m, m), (3m, 2m)$
$(9, 0)$	81	$9m$	$(2m, m), (3m, m), (3m, 2m), (4m, m), (4m, 2m), (4m, 3m)$
$(10, 0)$	100	$10m$	$(2m, m)^*, (3m, 2m), (4m, m), (4m, 3m)^*$
$(11, 0)$	121	$11m$	$(2m, m), (3m, m), (3m, 2m), (4m, m), (4m, 2m), (4m, 3m),$ $(5m, m), (5m, 2m), (5m, 3m), (5m, 4m)$
$(12, 0)$	144	$12m$	$(2m, m), (3m, 2m), (4m, m), (4m, 3m), (5m, 2m), (5m, 4m)$

$m = 1, 2, \ldots$; $(\cdot)^*$ indicates coexistence of type T (Case 3)

Square Patterns of Type M

Larger square patterns of type M (with $D \geq 32$) are predicted to branch from critical points of multiplicity 8, whereas smaller square patterns of type M with $D = 2, 8, 18$ do not exist. Recall that a square pattern of type M is characterized by the symmetry of $\Sigma(\beta, \beta)$ with $1 \leq \beta \leq n/2$ (see (3.102)) and that $D(\beta, \beta) = 2\beta^2$.

The following propositions show such nonexistence and existence of square patterns of type M.

Proposition 3.27. *Square patterns of type M with* $D = 2, 8, 18$ *do not arise as bifurcating solutions from critical points of multiplicity 8 for any n.*

Proof. The proof is given at the end of the proof of Proposition 3.28. □

Proposition 3.28. *Square patterns of type M with the symmetry of* $\Sigma(\beta, \beta)$ $(4 \leq \beta \leq n/2)$ *arise as bifurcating solutions from critical points of multiplicity 8 for specific values of n and irreducible representations given by*

(α, β)	D	n	(k, ℓ) in $(8; k, \ell)$
(β, β)	$2\beta^2$	$2\beta m$	$((2p + q)m, qm)$

(3.191)

where $m \geq 1$ *and*

$$p \geq 1, \quad q \geq 1, \quad 2p + q \leq \beta - 1, \quad q \notin 2\mathbb{Z}, \quad \gcd(p, q, \beta) = 1. \tag{3.192}$$

Proof. Type M occurs in Case 2 and Case 4 in Proposition 3.24, characterized by the condition of $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$. For $\hat{k} - \hat{\ell} \in 2\mathbb{Z}$ to be true, we can put $\hat{k} = 2p + q$ and $\hat{\ell} = q$ for some $p, q \in \mathbb{Z}$. Then $(k, \ell, n) = ((2p + q)m, qm, 2\beta m)$ for $m = \gcd(k, \ell, n)$. Since

$$1 = \gcd(\hat{k}, \hat{\ell}, \hat{n}) = \gcd(2p + q, q, 2\beta) = \gcd(2p, q, 2\beta),$$

Table 3.18: Correspondence of irreducible representation $(8; k, \ell)$ to (α, β) for square patterns of type M.

(α, β)	D	n	(k, ℓ) in $(8; k, \ell)$
$(4, 4)$	32	$8m$	$(3m, m)$
$(5, 5)$	50	$10m$	$(3m, m)^*$
$(6, 6)$	72	$12m$	$(3m, m), (5m, m), (5m, 3m)$
$(7, 7)$	98	$14m$	$(3m, m), (5m, m), (5m, 3m)$
$(8, 8)$	128	$16m$	$(3m, m), (5m, m), (5m, 3m), (7m, m), (7m, 3m), (7m, 5m)$
$(9, 9)$	162	$18m$	$(3m, m), (5m, m), (5m, 3m), (7m, m), (7m, 3m), (7m, 5m)$
$(10, 10)$	162	$20m$	$(3m, m)^*, (5m, m), (5m, 3m), (7m, m)^*, (7m, 3m), (7m, 5m),$ $(9m, m), (9m, 3m)^*, (9m, 5m), (9m, 7m)^*$
$(11, 11)$	242	$22m$	$(3m, m), (5m, m), (5m, 3m), (7m, m), (7m, 3m), (7m, 5m),$ $(9m, m), (9m, 3m), (9m, 5m), (9m, 7m)$
$(12, 12)$	288	$24m$	$(3m, m), (5m, m), (5m, 3m), (7m, m), (7m, 3m), (7m, 5m),$ $(9m, m), (9m, 5m), (9m, 7m), (11m, m), (11m, 3m), (11m, 5m),$ $(11m, 7m), (11m, 9m)$

$m = 1, 2, \ldots$; $(\cdot)^*$ indicates coexistence of type T (Case 3)

we must have $q \notin 2\mathbb{Z}$ and $\gcd(p, q, \beta) = 1$. The inequality constraint in (3.154) is translated as

$$1 \le \ell \le k - 1,\ 2 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor \qquad \Longleftrightarrow \qquad p \ge 1,\ q \ge 1,\ 2p + q \le \beta - 1.$$

Proposition 3.28 is thus proved.

Finally, for $\beta = 1, 2, 3$, no (p, q) satisfies (3.192), which proves the nonexistence of the smaller square patterns claimed in Proposition 3.27. □

Example 3.6. The parameter values of (3.191) in Proposition 3.28 give Table 3.17. The asterisk $(\cdot)^*$ indicates the coexistence of type T (see (3.193)), i.e., Case 4 of Proposition 3.24. The other (unmarked) cases correspond to Case 2 of Proposition 3.24, where no solution of type T coexists. The coexistence of type T is a relatively rare event; it does not occur for $n = 8m, 12m, 14m$, but it recurs for $n = 10m$.

□

Remark 3.13. In all cases in (3.191), the compatibility condition (3.104) for $\Sigma(\beta, \beta)$ is satisfied as $n = 2m\beta$ with $m = \gcd(k, \ell, n)$, since

$$\gcd(k, \ell, n) = \gcd((2p + q)m, qm, 2\beta m) = m \gcd(2p + q, q, 2\beta) = m \gcd(2p, q, 2\beta) = m$$

by (3.191) and (3.192).

□

Square Patterns of Type T

Square patterns of type T are shown here to branch from critical points of multiplicity 8. Recall that a square pattern of type T is characterized by the symmetry of $\Sigma_0(\alpha, \beta)$ with $1 \leq \alpha \leq n-1$, $1 \leq \beta \leq n-1$, and $\alpha \neq \beta$ (see (3.102)).

The following proposition is concerned with the five square patterns of type T with $D = 5$, 10, 13, 17 and 20 among ten smallest square patterns.

Proposition 3.29. *Square patterns of type T with* $D = 5,\ 10,\ 13,\ 17,$ *and* 20 *arise as bifurcating solutions from critical points of multiplicity 8 for specific values of n and irreducible representations given by*

(α, β)	D	n	(k, ℓ) in $(8; k, \ell)$	
			$z^{(2)} = c(1, 1, 0, 0)$	$z^{(3)} = c(0, 0, 1, 1)$
$(2, 1)$	5	$5m$	$(2m, m)$	none
$(1, 2)$			none	$(2m, m)$
$(3, 1)$	10	$10m$	$(3m, m)$	none
$(1, 3)$			none	$(3m, m)$
$(3, 2)$	13	$13m$	$(3m, m), (6m, 4m)$	$(5m, m)$
$(2, 3)$			$(5m, m)$	$(3m, m), (6m, 4m)$
$(4, 1)$	17	$17m$	$(4m, m), (7m, 6m), (8m, 2m)$	$(5m, 3m)$
$(1, 4)$			$(5m, 3m)$	$(4m, m), (7m, 6m)(8m, 2m)$
$(4, 2)$	20	$20m$	$(4m, 2m)$	$(8m, 6m)$
$(2, 4)$			$(8m, 6m)$	$(4m, 2m)$

(3.193)

where $m \geq 1$ *is an integer.*

Proof. By Proposition 3.24 (Case 3 and 4), a bifurcating solution with symmetry $\Sigma_0(\alpha, \beta)$ exists for (k, ℓ) such that $\Phi(k, \ell, n) = (\alpha, \beta)$, where the bifurcating solution corresponds to $z = c(1, 1, 0, 0)$. For such (k, ℓ), another bifurcating solution exists, which corresponds to $z = c(0, 0, 1, 1)$ and is endowed with the symmetry $\Sigma_0(\alpha', \beta')$ for (α', β') given by (3.178). The list of parameters in (3.193) is obtained by searching for such (k, ℓ) in the range of (3.154) using the method given in Appendix A.2, which was previewed in Remark 3.7. Alternatively, we can search for such (k, ℓ) in the range of (3.154) satisfying (3.175) for a given (a, b). □

For square patterns of type T, in general, the above statement extends as follows.

Proposition 3.30. *Assume* $1 \leq \alpha \leq n-1$, $1 \leq \beta \leq n-1$, *and* $\alpha \neq \beta$ *for* (α, β).

(i) *Square patterns of type T with the symmetry of* $\Sigma_0(\alpha, \beta)$ *arise as bifurcating solutions from critical points of multiplicity 8 associated with the irreducible representation* $(8; k, \ell)$ *such that* $\Phi(k, \ell, n) = (\alpha, \beta)$ *or* (α', β'), *where* (α', β') *is defined by* (3.178).

(ii) *Some* (k, ℓ, n) *exist such that* $\Phi(k, \ell, n) = (\alpha, \beta)$ *or* (α', β').

Proof. (i) The proof is the same as the proof of Proposition 3.29.

(ii) We can assume $\alpha > \beta$ by replacing (α, β) by (α', β') if necessary. Take

$$(k, \ell, n) = m(\hat{\alpha}, \hat{\beta}, D(\alpha, \beta)/\gcd(\alpha, \beta)),$$

for instance. Then $m = \gcd(k,\ell,n)$ and $(\hat{k},\hat{\ell},\hat{n}) = (\hat{\alpha},\hat{\beta},D(\alpha,\beta)/\gcd(\alpha,\beta))$, and therefore

$$\hat{k}^2 + \hat{\ell}^2 = \hat{\alpha}^2 + \hat{\beta}^2 = \hat{n}/\gcd(\alpha,\beta).$$

This shows that the simpler method of computing $\Phi(k,\ell,n)$, described in Remark 3.7, is applicable. The right-hand side of (3.185) is calculated as

$$\frac{\hat{n}}{\hat{k}^2 + \hat{\ell}^2}\begin{bmatrix}\hat{k}\\ \hat{\ell}\end{bmatrix} = \gcd(\alpha,\beta)\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\end{bmatrix} = \begin{bmatrix}\alpha\\ \beta\end{bmatrix},$$

which shows $\Phi(k,\ell,n) = (\alpha,\beta)$.

We also note that the chosen parameter (k,ℓ) lies in the range of (3.154). The inequality $1 \le \ell \le k-1$ is immediate from $\beta \ge 1$ and $\alpha > \beta$, whereas $2 \le k \le \lfloor \frac{n-1}{2} \rfloor$ is shown as follows. The inequality $k \ge 2$ holds since $\hat{\alpha} \ge 2$. When n is odd,

$$\begin{aligned}\frac{2}{m}\left(\left\lfloor\frac{n-1}{2}\right\rfloor - k\right) &= \frac{1}{m}(n-1-2k) = \gcd(\alpha,\beta)(\hat{\alpha}^2+\hat{\beta}^2) - \frac{1}{m} - 2\hat{\alpha}\\ &\ge (\hat{\alpha}^2+\hat{\beta}^2) - 1 - 2\hat{\alpha} = \hat{\alpha}(\hat{\alpha}-2) + \hat{\beta}^2 - 1 \ge 0.\end{aligned}$$

where $\hat{\alpha} \ge 2$ and $\hat{\beta} \ge 1$ is used in the last inequality. When n is even,

$$\frac{2}{m}\left(\left\lfloor\frac{n-1}{2}\right\rfloor - k\right) = \frac{1}{m}(n-2-2k) = \gcd(\alpha,\beta)(\hat{\alpha}^2+\hat{\beta}^2) - \frac{2}{m} - 2\hat{\alpha}.$$

If $\hat{n}$ is odd, we have m even since n is even and

$$\frac{2}{m}\left(\left\lfloor\frac{n-1}{2}\right\rfloor - k\right) \ge (\hat{\alpha}^2+\hat{\beta}^2) - 1 - 2\hat{\alpha} = \hat{\alpha}(\hat{\alpha}-2) + \hat{\beta}^2 - 1 \ge 0.$$

If $\hat{n}$ is even,

$$\frac{2}{m}\left(\left\lfloor\frac{n-1}{2}\right\rfloor - k\right) \ge (\hat{\alpha}^2+\hat{\beta}^2) - 2 - 2\hat{\alpha} = [\hat{\alpha}(\hat{\alpha}-2) + \hat{\beta}^2 - 1] - 1 \ge 0$$

because $[\hat{\alpha}(\hat{\alpha}-2) + \hat{\beta}^2 - 1 \ge 1$ as $(\hat{\alpha},\hat{\beta}) = (2,1)$, which gives $\hat{n} = 5$, is excluded by $\hat{n}$ even. □

Square patterns of type T appear in Cases 3 and 4 in Proposition 3.24, and these two cases are characterized by a single condition

$$\overline{\textbf{GCD-div}}\text{: } 2\gcd(\hat{k},\hat{\ell}) \text{ is not divisible by } \gcd(\hat{k}^2+\hat{\ell}^2,\hat{n}). \qquad (3.194)$$

This observation yields the following statement:

Proposition 3.31. *A bifurcating solution of type T exists if and only if* $\overline{\textbf{GCD-div}}$ *holds.*

In addition, we have the following statement for some concrete cases.

Proposition 3.32. *A bifurcating solution of type T does not exist for the cases* $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell})$, $(4\hat{\ell}, \hat{k}, \hat{\ell})$, *and* $(2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$.

Proof. First, we show that $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell})$ contradicts the condition $\overline{\textbf{GCD-div}}$ in (3.194). Let $\gcd(\hat{k}, \hat{\ell}) = \alpha$. Then, we have $\hat{n} = 4\hat{k} = 4\alpha(\hat{k}/\alpha)$. Recall that $\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1$. If $\alpha \neq 1$, then $\hat{n}, \hat{k}$, and $\hat{\ell}$ have a common divisor $\alpha \geq 2$. This contradicts $\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1$. Hence, we have $\gcd(\hat{k}, \hat{\ell}) = \alpha = 1$. Thus, we rewrite (3.194) as

$$\overline{\textbf{GCD-div}} \text{ for } (\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell})\text{: 2 is not divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, 4\hat{k}). \tag{3.195}$$

This condition is equivalent to that $\hat{k}^2 + \hat{\ell}^2$ and $4\hat{k}$ have 4 or a prime number $m \geq 3$ as a common divisor.

- For the case that $\hat{k}^2 + \hat{\ell}^2$ and $4\hat{k}$ have 4 as a common divisor, we have

$$\hat{k}^2 + \hat{\ell}^2 = 4p. \tag{3.196}$$

  Here, p is a positive integer. Using $\hat{k}^2 + \hat{\ell}^2 = (\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell}$, we have

$$(\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell} = 4p. \tag{3.197}$$

  Recall that $\gcd(\hat{k}, \hat{\ell}) = 1$. Hence, either $\hat{k}$ or $\hat{\ell}$, or both are odd. When we consider either $\hat{k}$ or $\hat{\ell}$ is odd, we see that $\hat{k} - \hat{\ell}$ is odd. Hence, $(\hat{k} - \hat{\ell})^2$ is odd. Thus, $(\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell}$ is odd. This contradicts (3.197). On the other hand, when we consider both $\hat{k}$ and $\hat{\ell}$ are odd, we see that $\hat{k} - \hat{\ell}$ is even. Hence, $(\hat{k} - \hat{\ell})^2$ is divisible by 4. Since $\hat{k}\hat{\ell}$ is odd, $2\hat{k}\hat{\ell}$ is not divisible by 4. Hence, $(\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell}$ is not divisible by 4. This contradicts (3.197).

- For the case that $\hat{k}^2 + \hat{\ell}^2$ and $4\hat{k}$ have a prime number $m \geq 3$ as a common divisor, we have

$$\hat{k}^2 + \hat{\ell}^2 = mp, \tag{3.198}$$
$$4\hat{k} = mq. \tag{3.199}$$

  Here, p and q are positive integers. Multiplying the both sides of (3.198) by q, we have

$$q(\hat{k}^2 + \hat{\ell}^2) = mpq. \tag{3.200}$$

  Multiplying the both sides of (3.199) by p, we have

$$4p\hat{k} = mpq. \tag{3.201}$$

  Combining (3.200) and (3.201), we have $q(\hat{k}^2+\hat{\ell}^2) = 4p\hat{k}$. Hence, we have $q\hat{\ell}^2 = \hat{k}(4p-q\hat{k})$. Since $\gcd(\hat{k}, \hat{\ell}) = 1$, q is divisible by $\hat{k}$. Hence, we have $q = r\hat{k}$ with some positive integer r. Substituting this into (3.199), we have $m = 4/r$. Recall that $m \geq 3$. From this, we have $r = 1$. Hence, we have $m = 4/r = 4$. Thus, we have $q = r\hat{k} = \hat{k}$. Substituting this into (3.200), we have $\hat{k}^2 + \hat{\ell}^2 = 4p$. Using $\hat{k}^2 + \hat{\ell}^2 = (\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell}$, we have

$$(\hat{k} - \hat{\ell})^2 + 2\hat{k}\hat{\ell} = 4p. \tag{3.202}$$

  This condition is equivalent to (3.197) in the above case. Hence, we have contradiction in a similar manner to the above case.

Thus, we see that $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell})$ contradicts $\overline{\textbf{GCD-div}}$. In the same way, we can see that $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{\ell}, \hat{k}, \hat{\ell})$ contradicts $\overline{\textbf{GCD-div}}$.

Next, we show that $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$ contradicts the condition $\overline{\textbf{GCD-div}}$ in (3.194). Let $\gcd(\hat{k}, \hat{\ell}) = \alpha$. Then, $\hat{n} = 2\hat{k} + 2\hat{\ell} = 2\alpha(\hat{k} + \hat{\ell})/\alpha$. Recall that $\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1$. If $\alpha \neq 1$, then $\hat{n}, \hat{k}$, and $\hat{\ell}$ have a common divisor α. This contradicts $\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1$. Hence, we have $\gcd(\hat{k}, \hat{\ell}) = \alpha = 1$. Thus, we rewrite $\overline{\textbf{GCD-div}}$ as

$$\overline{\textbf{GCD-div}} \text{ for } (\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})\text{: 2 is not divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, 2\hat{k} + 2\hat{\ell}). \tag{3.203}$$

This condition is equivalent to that $\hat{k}^2 + \hat{\ell}^2$ and $2\hat{k} + 2\hat{\ell}$ have 4 or a prime number $m \geq 3$ as a common divisor.

- For the case that $\hat{k}^2 + \hat{\ell}^2$ and $2\hat{k} + 2\hat{\ell}$ have 4 as a common divisor, we have

$$\hat{k}^2 + \hat{\ell}^2 = 4p, \tag{3.204}$$
$$2\hat{k} + 2\hat{\ell} = 4q. \tag{3.205}$$

  Here, p and q are positive integers. From (3.205), we have $\hat{k} + \hat{\ell} = 2q$. Since $\gcd(\hat{k}, \hat{\ell}) = \alpha = 1$, $\hat{k}$ and $\hat{\ell}$ are not both even. Hence, we have

$$\hat{k} = 2r + 1, \tag{3.206}$$
$$\hat{\ell} = 2s + 1. \tag{3.207}$$

  Here, r and s are positive integers. Substituting (3.206) and (3.207) into (3.204), we have $(2r + 1)^2 + (2s + 1)^2 = 4p$. Rearranging this, we have

$$p - r(r + 1) - s(s + 1) = 1/2. \tag{3.208}$$

  This equality has contradiction since $p - r(r + 1) - s(s + 1)$ is an integer.

- For the case that $\hat{k}^2 + \hat{\ell}^2$ and $2\hat{k} + 2\hat{\ell}$ have a prime number $m \geq 3$ as a common divisor, we have

$$\hat{k}^2 + \hat{\ell}^2 = mp, \tag{3.209}$$
$$2\hat{k} + 2\hat{\ell} = mq. \tag{3.210}$$

  Here, p and q are positive integers. Using $\hat{k}^2 + \hat{\ell}^2 = (\hat{k} + \hat{\ell})^2 - 2\hat{k}\hat{\ell}$, we have

$$(\hat{k} + \hat{\ell})^2 - 2\hat{k}\hat{\ell} = mp. \tag{3.211}$$

  Substituting (3.210) into (3.211), we have $q^2m^2/4 - 2\hat{k}\hat{\ell} = mp$. Rearranging this, we have

$$8\hat{k}\hat{\ell}/m = -4p + mq^2. \tag{3.212}$$

  Hence, $\hat{k}/m$ or $\hat{\ell}/m$ is an integer. When we consider $\hat{k}/m$ is an integer, we have $\hat{k} = mr$ with some positive integer r. From (3.210), we have $\hat{\ell} = m(q - r)$. Hence, $\hat{\ell}$ and $\hat{k}$ has m as a common divisor. This contradicts $\gcd(\hat{k}, \hat{\ell}) = 1$. When we consider $\hat{\ell}/m$ is an integer, we have contradiction in a similar manner.

Thus, we see that $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$ contradicts $\overline{\textbf{GCD-div}}$. □

Remark 3.14. The compatibility condition (3.104) for $\Sigma_0(\alpha, \beta)$ is satisfied as

$$n = m\,\frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}$$

with $m = \gcd(k, \ell, n)$ by (3.165) with (3.164).

□

Possible Square patterns for Several Lattice Sizes

In the previous subsections, we have investigated possible occurrences of square patterns for each of the three types V, M, and T and have enumerated all possible combinations of lattice size n and irreducible representation $(8; k, \ell)$ that can potentially engender square patterns. Compiling these results, we can capture, for each n, all square patterns that can potentially arise from critical points of multiplicity 8. The results are given in Tables 3.19 and 3.20 for several lattice sizes. The results are also incorporated in Table 3.8. Recall from Proposition 3.24 that bifurcating square patterns are associated with

$$z = \begin{cases} z^{(1)} = c(1, 1, 1, 1) & \text{for type V or type M,} \\ z^{(2)} = c(1, 1, 0, 0) & \text{for type T,} \\ z^{(3)} = c(0, 0, 1, 1) & \text{for type T.} \end{cases}$$

For $n = 5$, square patterns of type T exist for the irreducible representation $(8; k, \ell) = (8; 2, 1)$ with $\Sigma_0(\alpha, \beta) = \Sigma_0(2, 1)$ and $\Sigma_0(1, 2)$. For a composite number $n = 20$ with several divisors, square patterns of various kinds exist. Subgroups of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ expressing square patterns satisfy the inclusion relations given below.

Example 3.7. For $n = 20$, possible square patterns are of types V, M, and T. Subgroups for square patterns of type T have inclusion relations

$$\Sigma_0(2, 1) \supset \left\{ \begin{array}{ccc} \Sigma_0(1, 3) & \supset & \Sigma_0(2, 6) \\ \Sigma_0(4, 2) & \supset & \Sigma_0(8, 4) \end{array} \right\} \supset \Sigma_0(20, 0) = \langle r \rangle,$$

$$\Sigma_0(1, 2) \supset \left\{ \begin{array}{ccc} \Sigma_0(3, 1) & \supset & \Sigma_0(6, 2) \\ \Sigma_0(2, 4) & \supset & \Sigma_0(4, 8) \end{array} \right\} \supset \Sigma_0(20, 0) = \langle r \rangle,$$

and satisfy

$$\begin{aligned} &\Sigma_0(3, 1) \cap \Sigma_0(1, 3) = \Sigma_0(5, 5), \\ &\Sigma_0(4, 2) \cap \Sigma_0(2, 4) = \Sigma_0(10, 0), \\ &\Sigma_0(6, 2) \cap \Sigma_0(2, 6) = \Sigma_0(10, 10), \\ &\Sigma_0(8, 4) \cap \Sigma_0(4, 8) = \Sigma_0(20, 0) = \langle r \rangle. \end{aligned}$$

In addition, subgroups for square patterns of types V and M satisfy

$$\Sigma(1, 0) \supset \left\{ \begin{array}{ccccccccc} \Sigma(1, 1) & \supset & \Sigma(2, 0) & \supset & \Sigma(2, 2) & \supset & \Sigma(4, 0) & \supset & \Sigma(4, 4) \\ \Sigma(5, 0) & \supset & \Sigma(5, 5) & \supset & \Sigma(10, 0) & \supset & \Sigma(10, 10) & & \end{array} \right\}$$
$$\supset \Sigma(20, 0) = \langle r, s \rangle.$$

□

Table 3.19: Square patterns of types V, M, and T arising from critical points of multiplicity 8 for the $n \times n$ square lattices with $n = 5, 6, 10, 13, 17, 18$ ($\hat{D}$ is defined in (3.169)).

n	(k, ℓ) in $(8; k, \ell)$	$\hat{n}$	z	(α, β)	D	$\hat{D}$	Type
5	(2, 1)	5	$z^{(1)}$	(5, 0)	25	1	V
		5	$z^{(2)}$	(2, 1)	5	5	T
		5	$z^{(3)}$	(1, 2)	5	5	T
6	(2, 1)	6	$z^{(1)}$	(6, 0)	36	1	V
10	(4, 2)	5	$z^{(1)}$	(5, 0)	25	1	V
			$z^{(2)}$	(2, 1)	5	5	T
			$z^{(3)}$	(1, 2)	5	5	T
	(3, 1)	10	$z^{(1)}$	(5, 5)	50	2	M
			$z^{(2)}$	(3, 1)	10	10	T
			$z^{(3)}$	(1, 3)	10	10	T
	(2, 1)	10	$z^{(1)}$	(10, 0)	100	1	V
			$z^{(2)}$	(4, 2)	20	20	T
			$z^{(3)}$	(2, 4)	20	20	T
	(4, 3)	10	$z^{(1)}$	(10, 0)	100	1	V
			$z^{(2)}$	(2, 4)	20	20	T
			$z^{(3)}$	(4, 2)	20	20	T
	(3, 2), (4, 1)	10	$z^{(1)}$	(10, 0)	100	1	V
13	(3, 2), (6, 4)	13	$z^{(1)}$	(13, 0)	169	1	V
			$z^{(2)}$	(3, 2)	13	13	T
			$z^{(3)}$	(2, 3)	13	13	T
	(5, 1)	13	$z^{(1)}$	(13, 0)	169	1	V
			$z^{(2)}$	(2, 3)	13	13	T
			$z^{(3)}$	(3, 2)	13	13	T
	other (k, ℓ)'s	13	$z^{(1)}$	(13, 0)	169	1	V
17	(4, 1), (7, 6), (8, 2)	17	$z^{(1)}$	(17, 0)	17^2	1	V
			$z^{(2)}$	(4, 1)	17	17	T
			$z^{(3)}$	(1, 4)	17	17	T
	(5, 3)	17	$z^{(1)}$	(17, 0)	17^2	1	V
			$z^{(2)}$	(1, 4)	17	17	T
			$z^{(3)}$	(4, 1)	17	17	T
	other (k, ℓ)'s	17	$z^{(1)}$	(17, 0)	17^2	1	V
18	(6, 3)	6	$z^{(1)}$	(6, 0)	36	1	V
	(4, 2), (6, 2), (6, 4), (8, 2), (8, 4), (8, 6)	9	$z^{(1)}$	(9, 0)	81	1	V
	(2, 1), (3, 2), (4, 1), (4, 3), (5, 2), (5, 4), (6, 1), (6, 5) (7, 2), (7, 4), (7, 6), (8, 1), (8, 3), (8, 5), (8, 7)	18	$z^{(1)}$	(18, 0)	18^2	1	V
	(3, 1), (5, 1), (5, 3), (7, 1), (7, 3), (7, 5)	18	$z^{(1)}$	(9, 9)	162	2	M

Table 3.20: Square patterns of types V, M, and T arising from critical points of multiplicity 8 for the $n \times n$ square lattice with $n = 20, 24$ ($\hat{D}$ is defined in (3.169)).

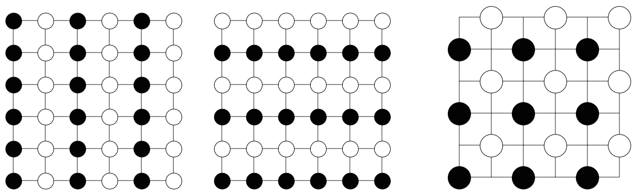
n	(k, ℓ) in $(8; k, \ell)$	$\hat{n}$	z	(α, β)	D	$\hat{D}$	Type
20	(8, 4)	5	$z^{(1)}$	(5, 0)	25	1	V
			$z^{(2)}$	(2, 1)	5	5	T
			$z^{(3)}$	(1, 2)	5	5	T
	(6, 2)	10	$z^{(1)}$	(5, 5)	50	2	M
			$z^{(2)}$	(3, 1)	10	10	T
			$z^{(3)}$	(1, 3)	10	10	T
	(4, 2)	10	$z^{(1)}$	(10, 0)	100	1	V
			$z^{(2)}$	(4, 2)	20	20	T
			$z^{(3)}$	(2, 4)	20	20	T
	(8, 6)	10	$z^{(1)}$	(10, 0)	100	1	V
			$z^{(2)}$	(2, 4)	20	20	T
			$z^{(3)}$	(4, 2)	20	20	T
	(3, 1), (9, 3)	20	$z^{(1)}$	(10, 10)	200	2	M
			$z^{(2)}$	(6, 2)	40	40	T
			$z^{(3)}$	(2, 6)	40	40	T
	(7, 1), (9, 7)	20	$z^{(1)}$	(10, 10)	200	2	M
			$z^{(2)}$	(2, 6)	40	40	T
			$z^{(3)}$	(6, 2)	40	40	T
	(4, 3), (7, 4), (8, 1), (9, 8)	20	$z^{(1)}$	(20, 0)	400	1	V
			$z^{(2)}$	(8, 4)	80	80	T
			$z^{(3)}$	(4, 8)	80	80	T
	(2, 1), (6, 3), (7, 6), (9, 2)	20	$z^{(1)}$	(20, 0)	400	1	V
			$z^{(2)}$	(4, 8)	80	80	T
			$z^{(3)}$	(8, 4)	80	80	T
	(6, 4), (8, 2)	10	$z^{(1)}$	(10, 0)	100	1	V
	(3, 2), (4, 1), (5, 2), (5, 4), (6, 1), (6, 5)	20	$z^{(1)}$	(20, 0)	400	1	V
	(7, 2), (8, 3), (8, 5), (8, 7), (9, 4), (9, 6)						
	(5, 1), (5, 3), (7, 3), (7, 5), (9, 1), (9, 5)	10	$z^{(1)}$	(10, 10)	200	2	M
24	(8, 4)	6	$z^{(1)}$	(6, 0)	36	1	V
	(6, 3), (9, 6)	8	$z^{(1)}$	(8, 0)	64	1	V
	(4, 2), (6, 4), (8, 2), (8, 6), (10, 4), (10, 8)	12	$z^{(1)}$	(12, 0)	144	1	V
	(2, 1), (3, 2), (4, 1), (4, 3), (5, 2), (5, 4), (6, 1), (6, 5)	24	$z^{(1)}$	(24, 0)	24^2	1	V
	(7, 2), (7, 4), (7, 6), (8, 1), (8, 3), (8, 5), (8, 7), (9, 2)						
	(9, 4), (9, 8), (10, 1), (10, 3), (10, 5), (10, 7), (10, 9)						
	(11, 2), (11, 4), (11, 6, (11, 8), (11, 10)						
	(9, 3)	8	$z^{(1)}$	(4, 4)	32	2	M
	(6, 2), (10, 2), (10, 6)	12	$z^{(1)}$	(6, 6)	72	2	M
	(3, 1), (5, 1), (5, 3), (7, 1), (7, 3), (7, 5), (9, 1)	24	$z^{(1)}$	(12, 12)	288	2	M
	(9, 5), (9, 7), (11, 1), (11, 3), (11, 5), (11, 7), (11, 9)						

Table 3.21: Square patterns of types V and M arising from for critical points of all kinds of multiplicity ($M = 1, 2, 4, 8$) for the $n \times n$ square lattice with $n = 18, 24$.

n	μ or (k, ℓ) in $(4; k, \ell)$ or (k, ℓ) in $(8; k, \ell)$	(α, β)	D	Type	M
18	(1; +, +, −)	(1, 1)	2	M	1
	(2; +, +)	(2, 0)	4	V	2
	(6, 0)	(3, 0)	9	V	4
	(6, 6)				
	(3, 0)	(6, 0)	36	V	
	(9, 6)				
	(2, 0), (4, 0), (8, 0)	(9, 0)	81	V	
	(2, 2), (4, 4), (8, 8)				
	(1, 0), (5, 0), (7, 0)	(18, 0)	324	V	
	(9, 2), (9, 4), (9, 8)				
	(3, 3)	(3, 3)	18	M	
	(9, 3)		18	M	
	(1, 1), (5, 5), (7, 7)	(9, 9)	162	M	
	(9, 1), (9, 5), (9, 7)		162	M	
	(6, 3)	(6, 0)	36	V	8
	(4, 2), (6, 2), (6, 4), (8, 2), (8, 4), (8, 6)	(9, 0)	81	V	
	(2, 1), (3, 2), (4, 1), (4, 3), (5, 2), (5, 4), (6, 1), (6, 5)	(18, 0)	18^2	V	
	(7, 2), (7, 4), (7, 6), (8, 1), (8, 3), (8, 5), (8, 7)				
	(3, 1), (5, 1), (5, 3), (7, 1), (7, 3), (7, 5)	(9, 9)	162	M	
24	(1; +, +, −)	(1, 1)	2	M	1
	(2; +, +)	(2, 0)	4	V	2
	(8, 0)	(3, 0)	9	V	4
	(8, 8)				
	(6, 0)	(4, 0)	16	V	
	(12, 6)				
	(4, 0)	(6, 0)	36	V	
	(12, 8)				
	(3, 0), (9, 0)	(8, 0)	64	V	
	(12, 3), (12, 9)				
	(2, 0), (10, 0)	(12, 0)	144	V	
	(12, 2), (12, 10)				
	(1, 0), (5, 0), (7, 0), (11, 0)	(24, 0)	576	V	
	(12, 1), (12, 5), (12, 7)(12, 11)				
	(6, 6)	(2, 2)	8	M	
	(4, 4)	(3, 3)	18	M	
	(12, 4)				
	(3, 3), (9, 9)	(4, 4)	32	M	
	(2, 2), (10, 10)	(6, 6)	72	M	
	(1, 1), (5, 5), (7, 7), (11, 11)	(12, 12)	288	M	
	(8, 4)	(6, 0)	36	V	8
	(6, 3), (9, 6)	(8, 0)	64	V	
	(4, 2), (6, 4), (8, 2), (8, 6), (10, 4), (10, 8)	(12, 0)	144	V	
	(2, 1), (3, 2), (4, 1), (4, 3), (5, 2), (5, 4), (6, 1), (6, 5)	(24, 0)	24^2	V	
	(7, 2), (7, 4), (7, 6), (8, 1), (8, 3), (8, 5), (8, 7), (9, 2)				
	(9, 4), (9, 8), (10, 1), (10, 3), (10, 5), (10, 7), (10, 9)				
	(11, 2), (11, 4), (11, 6), (11, 8), (11, 10)				
	(9, 3)	(4, 4)	32	M	
	(6, 2), (10, 2), (10, 6)	(6, 6)	72	M	
	(3, 1), (5, 1), (5, 3), (7, 1), (7, 3), (7, 5), (9, 1)	(12, 12)	288	M	
	(9, 5), (9, 7), (11, 1), (11, 3), (11, 5), (11, 7), (11, 9)				

Table 3.22: Square patterns of types V, M, and T arising from for critical points of all kinds of multiplicity ($M = 1, 2, 4, 8$) for the $n \times n$ square lattice with $n = 20$.

n	μ or (k,ℓ) in $(4;k,\ell)$ or (k,ℓ) in $(8;k,\ell)$	(α,β)	D	Type	M
20	(1; +, +, −)	(1, 1)	2	M	1
	(2; +, +)	(2, 0)	4	V	2
	(5, 0)	(4, 0)	16	V	4
	(10, 5)				
	(4, 0), (8, 0)	(5, 0)	25	V	
	(4, 4), (8, 8)				
	(2, 0), (6, 0)	(10, 0)	100	V	
	(10, 4), (10, 8)				
	(1, 0), (3, 0), (7, 0), (9, 0)	(20, 0)	400	V	
	(10, 1), (10, 3), (10, 7), (10, 9)				
	(5, 5)	(2, 2)	8	M	
	(2, 2), (6, 6)	(5, 5)	50	M	
	(10, 2), (10, 6)				
	(1, 1), (3, 3), (7, 7), (9, 9)	(10, 10)	200	M	
	(8, 4)	(5, 0)	25	V	8
		(2, 1)	5	T	
		(1, 2)	5	T	
	(6, 2)	(5, 5)	50	M	
		(3, 1)	10	T	
		(1, 3)	10	T	
	(4, 2)	(10, 0)	100	V	
		(4, 2)	20	T	
		(2, 4)	20	T	
	(8, 6)	(10, 0)	100	V	
		(2, 4)	20	T	
		(4, 2)	20	T	
	(3, 1), (9, 3)	(10, 10)	200	M	
		(6, 2)	40	T	
		(2, 6)	40	T	
	(7, 1), (9, 7)	(10, 10)	200	M	
		(2, 6)	40	T	
		(6, 2)	40	T	
	(4, 3), (7, 4), (8, 1), (9, 8)	(20, 0)	400	V	
		(8, 4)	80	T	
		(4, 8)	80	T	
	(2, 1), (6, 3), (7, 6), (9, 2)	(20, 0)	400	V	
		(4, 8)	80	T	
		(8, 4)	80	T	
	(6, 4), (8, 2)	(10, 0)	100	V	
	(3, 2), (4, 1), (5, 2), (5, 4), (6, 1), (6, 5)	(20, 0)	400	V	
	(7, 2), (8, 3), (8, 5), (8, 7), (9, 4), (9, 6)				
	(5, 1), (5, 3), (7, 3), (7, 5), (9, 1), (9, 5)	(10, 10)	200	M	



$\boldsymbol{q}_1^{(2;+,+)}$ $\boldsymbol{q}_2^{(2;+,+)}$ $\boldsymbol{q}_1^{(2;+,+)} + \boldsymbol{q}_2^{(2;+,+)}$

(a) Stripe patterns (b) A square pattern

Figure 3.14: Patterns of the eigenvectors $\boldsymbol{q}_1^{(2;+,+)}$ and $\boldsymbol{q}_2^{(2;+,+)}$ on the 6×6 square lattice. A black circle denotes a positive component and a white circle denotes a negative component of the associated eigenvector. The size of a circle represents the magnitude of the associated component.

In particular, possible square patterns for $n = 18, 20, 24$ for critical points of all kinds of multiplicity ($M = 1, 2, 4, 8$) are classified in Tables 3.21 and 3.22.

3.6. Stability of Bifurcating Solutions

In Section 3.5, we showed the existence of square patterns by using the equivariant branching lemma. In this section, we explain another approach of group-theoretic bifurcation analysis with bifurcation equations. As worked out in Appendix A.4, we can show the existence of bifurcating solutions by solving bifurcation equations. In addition, we can investigate the stability of bifurcating solutions by using the Jacobian matrix.

3.6.1. Illustration of Analysis

Let us consider the bifurcation equation

$$\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi}) = \boldsymbol{0} \tag{3.213}$$

in (3.108), where $\widetilde{\boldsymbol{F}} : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$ is a function, and $\widetilde{\phi} = \phi - \phi_c$ denotes the increment of ϕ. With the use of the matrix representation $T^\mu(g)$ for the associated irreducible representation μ of the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$, we have the equivariance condition

$$T^\mu(g)\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi}) = \widetilde{\boldsymbol{F}}(T^\mu(g)\boldsymbol{w}, \widetilde{\phi}), \quad g \in G \tag{3.214}$$

in (3.109).

We demonstrate, for example, group-theoretic bifurcation analysis of a critical point of multiplicity 2 associated with the irreducible representation $\mu = (2; +, +)$. The matrices $\widetilde{T}^\mu(g)$ in (3.214) for $\mu = (2; +, +)$ are represented as

$$T^{(2;+,+)}(r) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad T^{(2;+,+)}(s) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix},$$
$$T^{(2;+,+)}(p_1) = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \quad T^{(2;+,+)}(p_2) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

By the equivariance condition in (3.214), we see that $\widetilde{F}_i$ ($i = 1, 2$) in (3.213) take the form

$$\widetilde{F}_1(w_1, w_2, \widetilde{\phi}) = w_1 \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} A_{2a+1,2b}(\widetilde{\phi}) {w_1}^{2a} {w_2}^{2b}, \tag{3.215}$$

$$\widetilde{F}_2(w_1, w_2, \widetilde{\phi}) = w_2 \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} A_{2a+1,2b}(\widetilde{\phi}) {w_2}^{2a} {w_1}^{2b} \tag{3.216}$$

with coefficients $A_{2a+1,2b} \in \mathbb{R}$ (see Appendix A.4.3 for the proof). We use the column vectors $\boldsymbol{q}_1^{(2;+,+)}$ and $\boldsymbol{q}_2^{(2;+,+)}$ of $Q^{(2;+,+)}$ in (3.85) as the basis vectors of the two-dimensional space for $\boldsymbol{w} = (w_1, w_2)$. Fig. 3.14(a) depicts the spatial patterns of $\boldsymbol{q}_1^{(2;+,+)}$ and $\boldsymbol{q}_2^{(2;+,+)}$. We see that these two vectors represent stripe patterns.

Using the equations in (3.215) and (3.216), we have the following propositions for the existence and the symmetry of bifurcating solutions.

Proposition 3.33. *For a critical point of multiplicity 2 associated with $\mu = (2; +, +)$, we have the following bifurcating solutions:*

- the stripe pattern: $\boldsymbol{w}_{\text{stripe}} = (w, 0)$ ($w \in \mathbb{R}$) [Fig. 3.14(a)],
- the square pattern: $\boldsymbol{w}_{\text{sq}} = (w, w)$ ($w \in \mathbb{R}$) [Fig. 3.14(b)].

Proof. Substituting $\boldsymbol{w}_{\text{stripe}} = (w, 0)$ into (3.215), we have

$$\widetilde{F}_1(w, 0, \widetilde{\phi}) = w \sum_{a=0}^{\infty} A_{2a+1,0}(\widetilde{\phi}) w^{2a} \approx w \left\{ A'_{10}(0)\widetilde{\phi} + A_{30}(0) w^2 \right\} \tag{3.217}$$

with $A'_{10}(0) = \partial A_{10}/\partial\widetilde{\phi}(0)$. Thus, $\widetilde{F}_1(w, 0, \widetilde{\phi}) = 0$ represents the $\widetilde{\phi}$ versus w relation for $\boldsymbol{w}_{\text{stripe}}$. Substituting $\boldsymbol{w}_{\text{stripe}}$ into (3.216), we see that $\widetilde{F}_2(w, 0, \widetilde{\phi}) = 0$ is satisfied for any w. Thus, there is a bifurcating curve satisfying $\widetilde{F}_1 = \widetilde{F}_2 = 0$ for $\boldsymbol{w}_{\text{stripe}}$. Similar discussion holds for $\boldsymbol{w}_{\text{sq}}$. □

Proposition 3.34. *For a critical point of multiplicity 2 associated with $\mu = (2; +, +)$, the two bifurcating solutions $(\boldsymbol{w}, \widetilde{\phi})$ and $(-\boldsymbol{w}, \widetilde{\phi})$ are conjugate for $\boldsymbol{w} = \boldsymbol{w}_{\text{sq}}, \boldsymbol{w}_{\text{stripe}}$.*

Proof. Since $\boldsymbol{w} = (w, 0)$ and $(-w, 0)$ satisfy the same relation $\sum_{a=0}^{\infty} A_{2a+1,0}(\widetilde{\phi}) w^{2a} = 0$ (cf., (3.217)), $\widetilde{F}_1(w, 0, \widetilde{\phi})$ is an odd function in w, that is, $\widetilde{F}_1(-w, 0, \widetilde{\phi}) = -\widetilde{F}_1(w, 0, \widetilde{\phi})$. Thus, $(\boldsymbol{w}_{\text{stripe}}, \widetilde{\phi})$ and $(-\boldsymbol{w}_{\text{stripe}}, \widetilde{\phi})$ are conjugate solutions for $\widetilde{F}_1 = 0$. Similar discussion holds for $\boldsymbol{w}_{\text{sq}}$. □

We evaluate the stability of bifurcating solutions. The Jacobian matrix of $\widetilde{\boldsymbol{F}}$ becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx \begin{bmatrix} A'_{10}(0)\widetilde{\phi} + 3A_{30}(0){w_1}^2 + A_{12}(0){w_2}^2 & 2A_{12}(0) w_1 w_2 \\ 2A_{12}(0) w_1 w_2 & A'_{10}(0)\widetilde{\phi} + 3A_{30}(0){w_2}^2 + A_{12}(0){w_1}^2 \end{bmatrix} \tag{3.218}$$

with $A'_{10}(0) = \partial A_{10}/\partial\widetilde{\phi}(0)$. Evaluating the eigenvalues of $\widetilde{J}$ for each bifurcating solution and employing the assumption that the pre-bifurcation distribution is stable, we have the following proposition (see Appendix A.4.3 for the proof):

Table 3.23: Theoretically predicted bifurcating solutions for critical points with multiplicity M.

M	Bifurcating solutions ($w \in \mathbb{R}$)	Existence conditions
1	w	if n is even
2	$\boldsymbol{w}_{\rm sq} = (w, w)$	if n is even
	$\boldsymbol{w}_{\rm stripe} = (w, 0)$	if n is even
4	$\boldsymbol{w}_{\rm sq} = (w, 0, w, 0)$	Always
	$\boldsymbol{w}_{\rm stripeI} = (w, 0, 0, 0)$	Always
	$\boldsymbol{w}_{\rm stripeII} = (0, w, 0, 0)$	if $\check{n}$ is even
8	$\boldsymbol{w}_{\rm sqVM} = (w, 0, w, 0, w, 0, w, 0)$	Always
	$\boldsymbol{w}_{\rm sqT} = (w, 0, w, 0, 0, 0, 0, 0)$	if $2\,\gcd(\hat{k}, \hat{\ell})$ is not divisible by $\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n})$
	$\boldsymbol{w}_{\rm upside-downI} = (w, 0, 0, 0, w, 0, 0, 0)$	if $(\hat{k} + \hat{\ell})\gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell})$ is not divisible by $\gcd(2\hat{k}\hat{\ell}, \hat{n})$
	$\boldsymbol{w}_{\rm upside-downII} = (0, w, 0, 0, 0, w, 0, 0)$	if $\hat{n}$ is even and $(\hat{k} + \hat{\ell})\gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell})$ is not divisible by $\gcd(2\hat{k}\hat{\ell}, \hat{n})$
	$\boldsymbol{w}_{\rm stripeI} = (w, 0, 0, 0, 0, 0, 0, 0)$	if $\hat{k}^2 + \hat{\ell}^2$, $2\hat{k}\hat{\ell}$, and $\hat{k}^2 - \hat{\ell}^2$ are not divisible by $\hat{n}$
	$\boldsymbol{w}_{\rm stripeII} = (0, w, 0, 0, 0, 0, 0, 0)$	if $\hat{n}$ is even and $\hat{k}^2 + \hat{\ell}^2$, $2\hat{k}\hat{\ell}$, and $\hat{k}^2 - \hat{\ell}^2$ are not divisible by $\hat{n}$

$\check{n} = n/\gcd(k, n)$ for $M = 4$ in (3.138);
$\hat{n} = n/\gcd(k, \ell, n)$, $\hat{k} = k/\gcd(k, \ell, n)$, $\hat{\ell} = \ell/\gcd(k, \ell, n)$ for $M = 8$ in (3.164)

Proposition 3.35. *For a critical point of multiplicity 2 associated with* $\mu = (2; +, +)$*, under the assumption that all the eigenvalues of the Jacobian matrix other than those for* $\mu = (2; +, +)$ *are negative, we have the following statements in the neighborhood of the critical point.*

- *If* $A_{30}(0) < A_{12}(0) < -A_{30}(0)$ *is satisfied, the square pattern* $\boldsymbol{w}_{\rm sq}$ *is stable.*
- *If* $A_{12}(0) < A_{30}(0) < 0$ *is satisfied, the stripe pattern* $\boldsymbol{w}_{\rm stripe}$ *is stable.*
- *The two solutions* $\boldsymbol{w}_{\rm sq}$ *and* $\boldsymbol{w}_{\rm stripe}$ *are not stable simultaneously.*

Proposition 3.35 implies possible existence of stable bifurcating solutions from the uniform distribution for economic geography models on the square lattice. This makes a sharp contrast with a knowledge on S_N invariant space and a hexagonal lattice, for which all bifurcating paths are unstable in the neighborhood of the bifurcation points (Elmihirst, 2004; Ikeda et al., 2018a; Aizawa et al., 2020).

3.6.2. Summary of Theoretical Results

Similarly to the case of a critical point of multiplicity 2, we also investigate the existence and the stability of bifurcating solutions for other bifurcation points. We summarize theoretically predicted bifurcating solutions in Table 3.23. For these bifurcating solutions, we have the following propositions (see Appendices A.4.4 and A.4.5 for the proofs):

Proposition 3.36. *For a critical point of multiplicity 4, we have the following statements:*

- *For $\mu = (4; k, 0, +)$ and $\mu = (4; k, k, +)$ with $\check{n} = 3$, under the assumption that all the eigenvalues of the Jacobian matrix other than those for $\mu = (4; k, 0, +)$ and $(4; k, k, +)$ are negative, the bifurcating solutions $\boldsymbol{w}_{\mathrm{sq}}$ and $\boldsymbol{w}_{\mathrm{stripeI}}$ are always unstable in the neighborhood of the critical point. The bifurcating curve takes the form $\widetilde{\phi} \approx cw$ for some constant c.*

- *For any other cases, the stability of $\boldsymbol{w}_{\mathrm{sq}}$, $\boldsymbol{w}_{\mathrm{stripeI}}$, and $\boldsymbol{w}_{\mathrm{stripeII}}$ depends on the values of the coefficients of the power series expansion of the bifurcation equation. The bifurcating curve takes the form $\widetilde{\phi} \approx cw^2$ for some constant c.*

Proposition 3.37. *For a critical point of multiplicity 8, we have the following statements:*

- *For $\mu = (8; k, \ell)$ with $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$, under the assumption that all the eigenvalues of the Jacobian matrix other than those for $\mu = (8; k, \ell)$ are negative, the bifurcating solution $\boldsymbol{w}_{\mathrm{sqT}}$ is always unstable in the neighborhood of the critical point. The bifurcating curve takes the form $\widetilde{\phi} \approx cw$ for some constant c.*

- *For any other cases, the stability of $\boldsymbol{w}_{\mathrm{stripeI}}$, $\boldsymbol{w}_{\mathrm{stripeII}}$, $\boldsymbol{w}_{\mathrm{upside\text{-}downI}}$, $\boldsymbol{w}_{\mathrm{upside\text{-}downII}}$, $\boldsymbol{w}_{\mathrm{sqT}}$, and $\boldsymbol{w}_{\mathrm{sqVM}}$ depends on the values of the coefficients of the power series expansion of the bifurcation equation. The bifurcating curve takes the form $\widetilde{\phi} \approx cw^2$ for some constant c.*

Propositions 3.36 and 3.37 indicate that for particular lattice sizes n, several types of bifurcating solutions are always unstable in the neighborhood of the critical point. Note that these are common results for any models with the equivariance to the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

3.7. Bifurcation Behavior of Economic Geography Models

In this section, we conduct numerical bifurcation analysis for economic geography models on the square lattice. To emphasize the applicability of theoretical results in this chapter, we employ three types of economic geography models: the FO model (Forslid and Ottaviano, 2003), the Hm model (Helpman, 1998), and the PS model (Pflüger and Südekum, 2008). We demonstrate the emergence of theoretically predicted bifurcating solutions that were presented in Section 3.5 and Appendix A.4,. We search for bifurcating solution curves and investigate their stability numerically by using comparative static analysis with respect to the trade freeness, which is one of the major parameter of economic geography models and is employed here as the bifurcation parameter.

3.7.1. Group Equivariance

The FO model, the Hm model, and the PS model with the replicator dynamics on the $n \times n$ square lattice satisfies the equivariance to $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ (see Proposition 2.1 in Section 2.3.2 for the proof):

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \phi), \quad g \in G \tag{3.219}$$

for the $K(= n^2)$-dimensional permutation representation $T(g)$ of G.[9] Thus, theoretical results in this chapter is applicable to these models.

[9] The concrete form of $T(g)$ was given in Section 3.4.1.

Table 3.24: Bifurcating solutions for the 6×6 square lattice.

μ	Bifurcating solutions ($w \in \mathbb{R}$)
$(1; +, +, -)$	w
$(2; +, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, w)$, $\boldsymbol{w}_{\mathrm{stripe}} = (w, 0)$
$(4; 1, 0, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$
$(4; 2, 0, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$
$(4; 1, 1, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$
$(4; 2, 2, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$
$(4; 3, 1, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$, $\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0)$
$(4; 3, 2, +)$	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$, $\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0)$
$(8; 2, 1)$	$\boldsymbol{w}_{\mathrm{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$, $\boldsymbol{w}_{\mathrm{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$, $\boldsymbol{w}_{\mathrm{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$, $\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0, 0, 0, 0, 0)$, $\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$,

Note that the uniform distribution

$$\boldsymbol{\lambda}_{\mathrm{uniform}} = (1/K, \ldots, 1/K)^{\top} \tag{3.220}$$

on the $n \times n$ square lattice satisfies the governing equation in (2.3) for any ϕ.[10] All places have the same economic environments with the same indirect utility $v_i = \bar{v}$ $(i = 1, \ldots, n^2)$. The uniform distribution satisfies

$$T(g)\boldsymbol{\lambda}_{\mathrm{uniform}} = \boldsymbol{\lambda}_{\mathrm{uniform}}, \quad g \in G, \tag{3.221}$$

and hence this solution is G-symmetric. We investigate group-theoretic critical points on the uniform distribution in the following subsections.

3.7.2. Theoretically Predicted Bifurcating Solutions

We focus on the 6×6 square lattice that accommodates various kinds of bifurcating solutions. As a consequence of the irreducible decomposition (3.73) of the permutation representation T for this lattice, the irreducible representation μ of the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_6 \times \mathbb{Z}_6)$ to be considered in bifurcation analysis is restricted to

$$\begin{aligned} \mu = {} & (1; +, +, +), (1; +, +, -), (2; +, +), (4; 1, 0, +), (4; 2, 0, +), \\ & (4; 1, 1, +), (4; 2, 2, +), (4; 3, 1, +), (4; 3, 2, +), (8; 2, 1). \end{aligned} \tag{3.222}$$

Theoretically possible bifurcating solutions associated with μ in (3.222) are listed in Table 3.24 and depicted in Fig. 3.15. Note that for $\mu = (4; 2, 0, +)$ and $\mu = (4; 2, 2, +)$, the two solutions $\boldsymbol{w}_{\mathrm{sq}}$ and $-\boldsymbol{w}_{\mathrm{sq}}$, which have opposite signs, represent different physical behaviour. The same holds for the solutions $\boldsymbol{w}_{\mathrm{stripeI}}$ and $-\boldsymbol{w}_{\mathrm{stripeI}}$. Other bifurcating solutions with opposite signs represent the same physical behaviour.

[10] We call such distributions as invariant patterns. See Proposition 4.2 in Section 4.3.

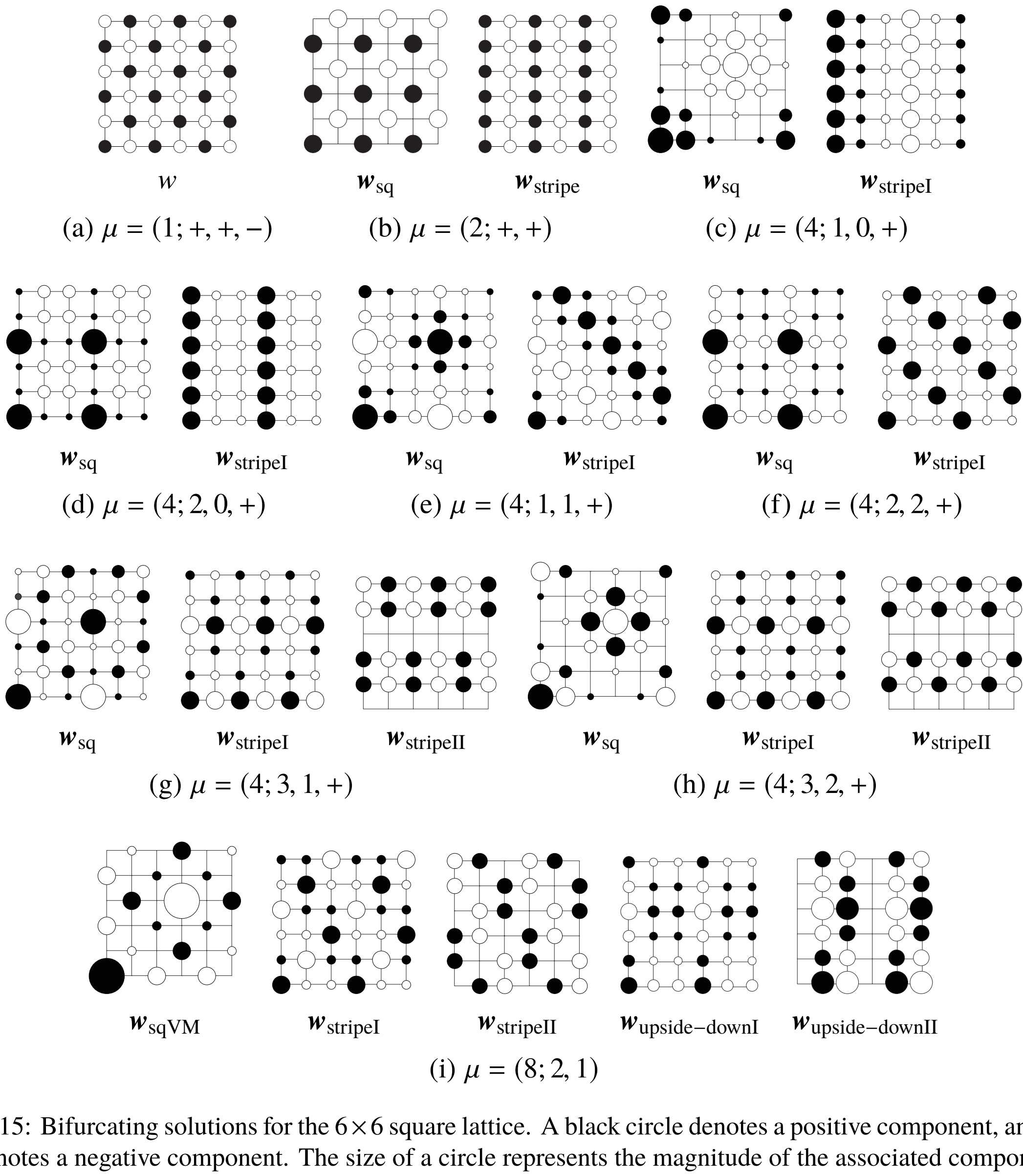


Figure 3.15: Bifurcating solutions for the 6×6 square lattice. A black circle denotes a positive component, and a white circle denotes a negative component. The size of a circle represents the magnitude of the associated component.

Remark 3.15. For the 6×6 square lattice, we have the following statements:

- For $\mu = (4; 1, 0, +), (4; 2, 0, +), (4; 1, 1, +), (4; 2, 2, +)$, the solution $\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0)$ does not exist. See Proposition A.17 in Section A.4.4. Note that the condition in Proposition A.17 is not satisfied since $\check{n}$ is odd for these cases.

- For $\mu = (8; 2, 1)$, the solution $\boldsymbol{w}_{\mathrm{sqT}} = (w, 0, w, 0, 0, 0, 0, 0)$ does not exist. See Proposition 3.32 in Section 3.5.6. This case corresponds to the case $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$. In fact, $\underline{2\gcd(\hat{k}, \hat{\ell})} = 2\gcd(2, 1) = 2$. This is divisible by $\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}) = \gcd(6, 6) = 1$. Hence, **GCD-div** in (3.194) is not satisfied.

□

3.7.3. Numerical Bifurcation Analysis

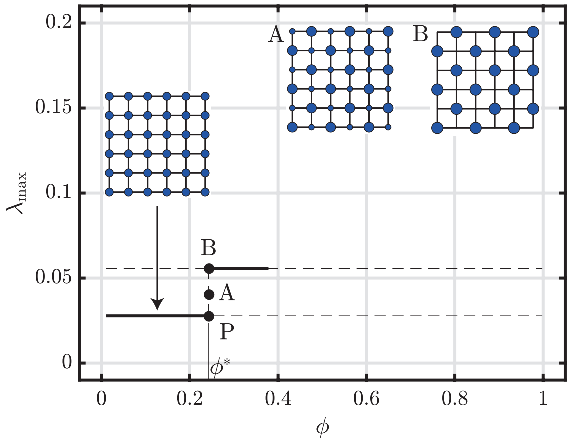
We conduct comparative static analysis for the FO model (Forslid and Ottaviano, 2003), the Hm model (Helpman, 1998), and the PS model (Pflüger and Südekum, 2008) with respect to the trade freeness $\phi \in (0, 1)$. We aim to demonstrate the applicability of theoretical results in this chapter for these models. We focus on group-theoretic critical points on the uniform distribution $\boldsymbol{\lambda}_0 = (1/36, \ldots, 1/36)^\top$ associated with the irreducible representations $\mu = (1; +, +, -)$, $\mu = (2; +, +)$, and $\mu = (4; 1, 0, +)$ in (3.222) and compute bifurcating solution curves from these points. Figures 3.16–3.18 show the results of numerical simulations, in which $\lambda_{\max} = \max(\lambda_1, \ldots, \lambda_K)$ is plotted against ϕ.

For these three models, the elasticity of substitution $\sigma \in (1, \infty)$ and the expenditure share of manufacturing goods $\mu \in (0, 1)$ are model parameters. Note that the expenditure share of housing goods $\gamma \in (0, 1)$ is another model parameter for the PS model. Bifurcating curves for other irreducible representations are presented in Appendix A.5 for the FO model.

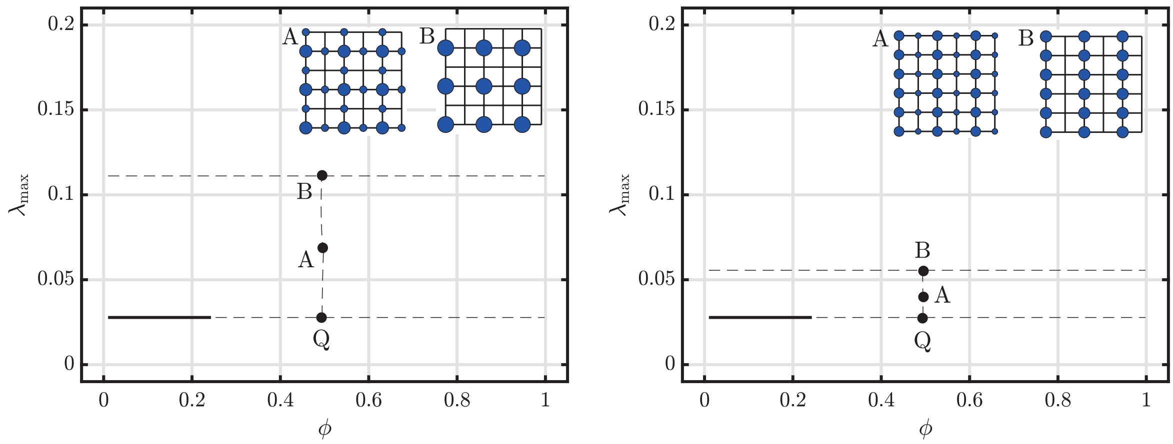
FO model

Figure 3.16 shows equilibrium curves of the FO model. Following Fujita et al. (1999b), we choose the parameter values as $\sigma = 6.0$, $\mu = 0.4$. In the early state where ϕ is small, the uniform distribution is the only stable equilibrium. When ϕ reaches ϕ^* in Fig. 3.16(a), the first bifurcation occurs at the bifurcation point P associated with $\mu = (1; +, +, -)$. A 18-centric distribution emerges and becomes stable at the point B. From the bifurcation point Q associated with $\mu = (2; +, +)$, a 9-centric distribution and a stripe one emerge simultaneously (see Fig. 3.16(b)). From the bifurcation point R associated with $\mu = (4; 1, 0, +)$, a diffused distribution and another stripe one emerge (see Fig. 3.16(c)). This diffused distribution tends to be agglomerated to a single place and arrives at the mono-centric distribution that becomes stable when ϕ is close to 1.

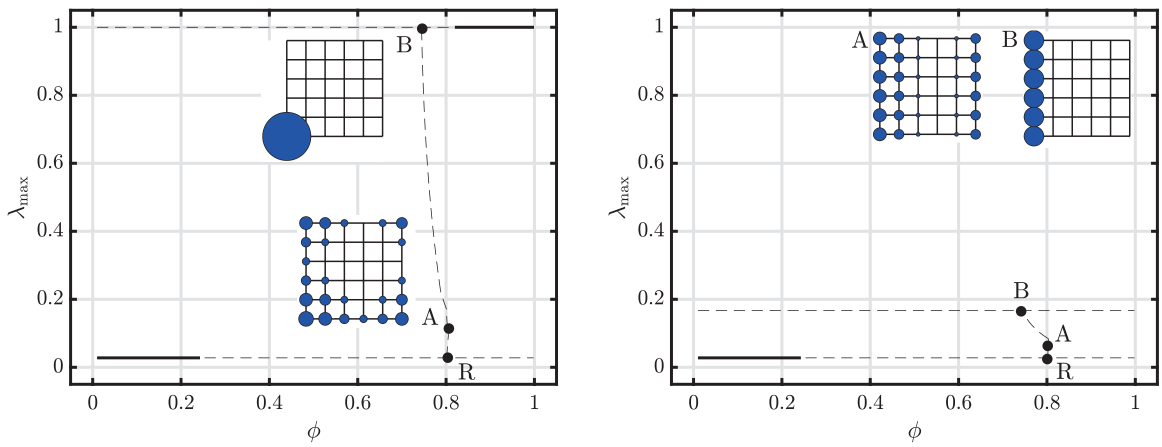
The state A of each diagram is consistent with theoretically predicted bifurcating solutions in Fig. 3.15. After the bifurcation, population tends to be agglomerated completely to places with the largest positive components of the bifurcation mode. Note that all the bifurcating solutions are unstable just after the bifurcation. Stable ones are theoretically possible and may exist for other parameter values.



(a) $\mu = (1; +, +, -)$

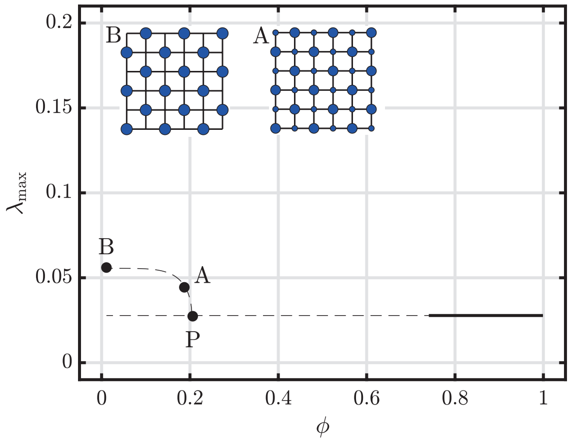


(b) $\mu = (2; +, +)$

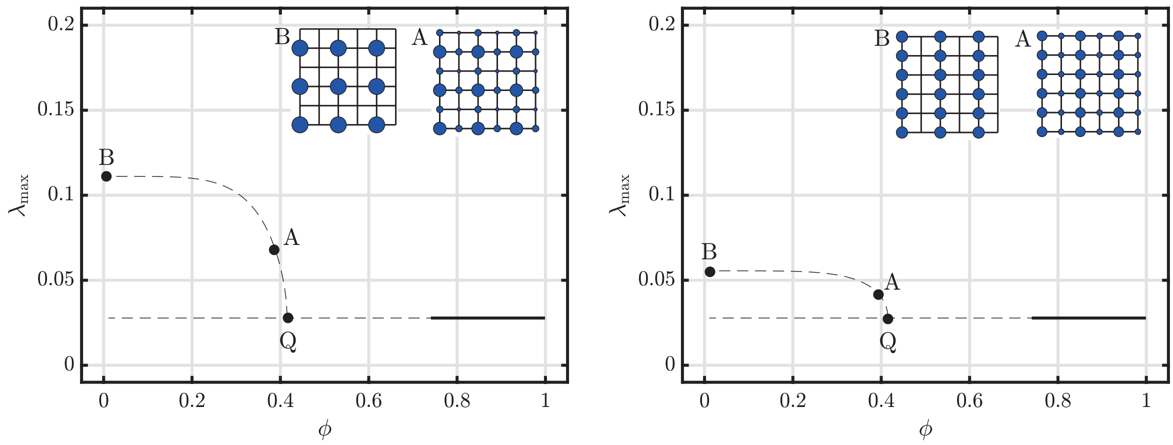


(c) $\mu = (4; 1, 0, +)$

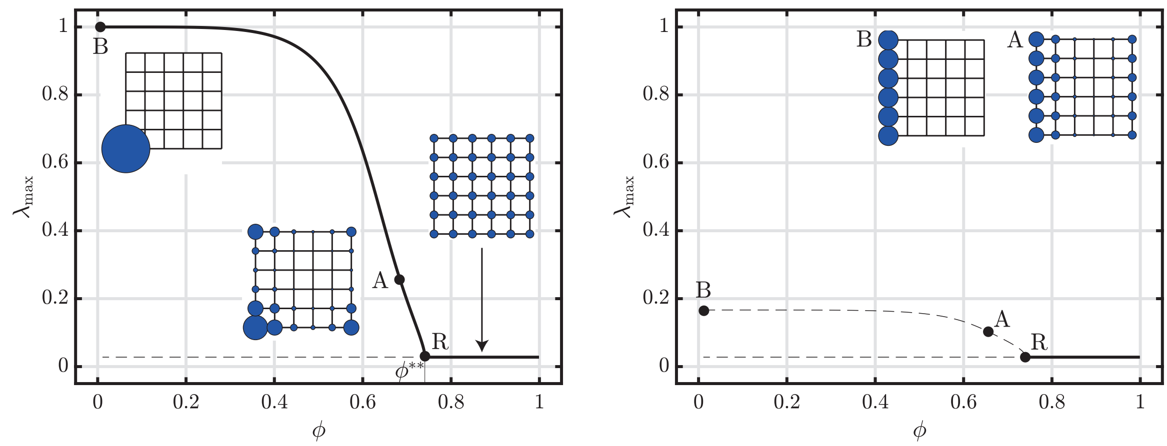
Figure 3.16: Equilibrium curves of the FO model for several irreducible representations μ. Solid curves represent stable stationary points, and dashed curves represent unstable ones.



(a) $\mu = (1; +, +, -)$



(b) $\mu = (2; +, +)$



(c) $\mu = (4; 1, 0, +)$

Figure 3.17: Equilibrium curves of the Hm model for several irreducible representations μ. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

Hm model

Figure 3.17 shows equilibrium curves of the Hm model. We choose the parameter values as $\sigma = 2.0$, $\mu = 0.8$ in order to realise bifurcation from the uniform distribution.[11] When ϕ is close to 1, the uniform distribution is stable, unlike the FO model. When ϕ decreases to ϕ^{**} in Fig. 3.17(c), the first bifurcation occurs at the bifurcation point R associated with $\mu = (4; 1, 0, +)$, and the stable mono-centric agglomeration pattern emerges. Note that this bifurcating solution is stable (subcritical bifurcation), while the others are unstable.

Similarly to the case of the FO model, we see that population tends to be agglomerated to places with the largest positive components of the bifurcation mode. Progress of stable equilibria of the Hm model as ϕ increases 0 to 1, however, is significantly different from that of the FO model. While the uniform distribution prevails for small ϕ for the FO model, the mono-centric distribution prevails for small ϕ for the Hm model.

PS model

Figure 3.18 shows equilibrium curves of the PS model. We choose the parameter values as $\sigma = 3.0$, $\mu = 0.6$, $\gamma = 0.2$ in reference to Akamatsu et al. (2021); in the early state where ϕ is small, the stable equilibrium is the uniform distribution. The first bifurcation occurs at the subcritical bifurcation point P associated with $\mu = (1; +, +, -)$ when ϕ decreases to ϕ^{***} in Fig. 3.18(a). Then, a stable 18-centric distribution emerges. The progress of stable equilibria as ϕ increases of the PS model is similar to that of the FO model.

3.8. Concluding Remarks

This chapter has tried to exhaustively find bifurcating solutions of economic geography models on an $n \times n$ square lattice by group-theoretic bifurcation analysis. We presented a complete list of typical bifurcating solutions from the uniform distribution for an arbitrary lattice size n. Possible bifurcating solutions elucidated in this chapter were square, stripe, and upside-down patterns. In numerical analysis of several economic geography models, we demonstrated the emergence of these bifurcating solutions. The stability of bifurcating solutions and the order of occurrence of particular bifurcations are found to be dependent on the models.

The main message of this chapter is not only to demonstrate the emergence of bifurcating solutions for particular models but also to develop a general theory to understand bifurcation behaviour for any economic geography model. This chapter would provide the important contribution of nonlinear mathematics to the study of economic agglomerations in spatial economics.

[11] For the Hm model, the uniform distribution is the unique equilibrium, and no bifurcation occurs when $(1-\mu)\sigma > 1$ (Redding and Sturm, 2008). For the parameter values $(\sigma, \mu) = (6.0, 0.4)$ used for the FO model, no bifurcation, accordingly, takes place for the Hm model.

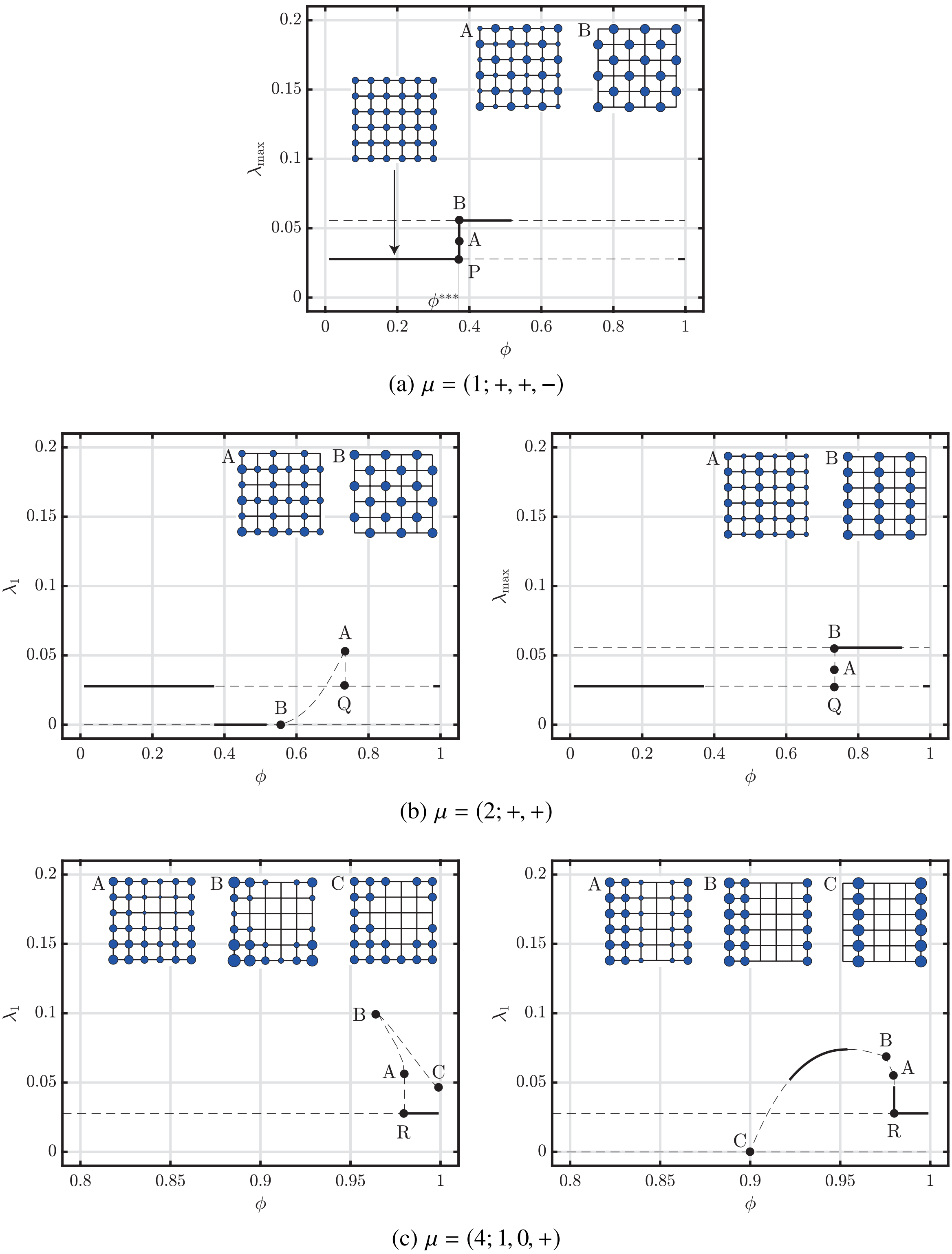


Figure 3.18: Equilibrium curves of the PS model for several irreducible representations μ. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

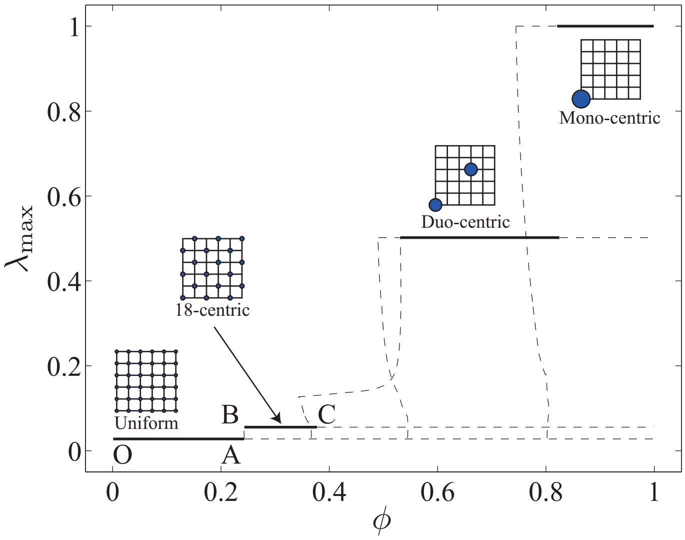


Figure 4.1: Equilibrium curves from the uniform distribution on a square lattice for the FO model ($\sigma = 6.0,\ \mu = 0.4$). Square domains denote population distributions for the associated equilibria. The horizontal axis shows the bifurcation patameter (trade freeness), and the vertical axis shows the maximum population. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

4. Invariant Patterns for the Replicator Dynamics

4.1. Introduction

Bifurcation analysis of economic geography models on lattice economies has come to be simulated in several studies (Ikeda and Murota, 2014; Ikeda et al., 2012b, 2014, 2017b, 2018a). In these analysis, we have observed a plethora of direct and further bifurcating patterns from the uniform distribution that form a complicated network of equilibrium curves. This chapter aims to elucidate the mechanism of this complicated network in the light of geometrical symmetry.

As a hint of this, we refer to an example of bifurcation diagrams for an economic geography model with the replicator dynamics on a square lattice shown in Fig. 4.1. From the curve OA of the uniform distribution, a bifurcating curve AB branches and arrives at the curve BC of an 18-centric distribution. It is to be noted that the curves OA and BC are horizontal as the population distribution remains constant along these curves when the bifurcation parameter changes. We may wonder why these curves are horizontal. The existence of such constant distributions, called invariant patterns (Ikeda et al., 2018b), has come to be acknowledged in economic geography models with the replicator dynamics, which is widely used in economics (Sandholm, 2010). The horizontal curves are connected by non-horizontal ones AB and DE to form a mesh-like structure. Moreover, we encounter, in Section 4.4, a complicated mesh-like structure with a large number of horizontal and non-horizontal curves, just like threads of warp and weft.

That said, this chapter aims to carry out a theoretical study of invariant patterns for the replicator dynamics on a square lattice. We consider an $n \times n$ square lattice that has symmetry expressed by the finite group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ with periodic boundary conditions. Exploiting the geometrical symmetry and the structure of the replicator dynamics, We obtain invariant patterns exhaustively, including the uniform, mono-centric, and poly-centric distributions. A list of invariant patterns advanced in this chapter would be of assistance in the study of economic agglomerations.

In addition, we combine invariant patterns with bifurcation mechanisms due to the geometrical symmetry. We propose the following innovative bifurcation analysis procedure to find stable equilibria:

- Obtain all invariant patterns and investigate their stability.
- Search for bifurcating equilibrium curves that connect stable invariant patterns and investigate their stability.

We apply such a procedure to the FO model (Forslid and Ottaviano, 2003) and the PS model (Pflüger and Südekum, 2008) to observe a network structure of equilibrium curves. We elucidate the connectivity between the uniform distribution and invariant patterns: Population tends to be agglomerated to places with the largest positive components of a bifurcating solution from the uniform distribution, and then the spatial distribution arrives at an invariant pattern via a bifurcating curve. We see that when two half branches at a bifurcation point are symmetric (respectively, asymmetric), they would arrive at one (respectively, two) invariant patterns.

A knowledge of invariant patterns have come to be used in analysis of economic geography models to capture a series of agglomeration patterns of economic interest (Takayama et al., 2020; Osawa et al., 2020). Such an application of invariant patterns would contribute to the study of economic geography, in which economic agglomerations are studied for several spatial platforms, including the two-places economy (Fujita et al., 1999b; Baldwin et al., 2011), a long narrow economy (Fujita and Mori, 1997; Mori, 1997; Fujita et al., 1999a), and a racetrack economy (Tabuchi and Thisse, 2011; Mossay and Picard, 2011; Akamatsu et al., 2012). Invariant patterns on a racetrack economy were observed in several studies (Castro et al., 2012; Ikeda et al., 2012a, 2019b). We use a systematic procedure proposed for a hexagonal lattice (Ikeda et al., 2019a) and obtain invariant patterns exhaustively.

This chapter is organized as follows. A general framework of economic geography models with the replicator dynamics is presented in Section 4.2. A theory of invariant patterns for the replicator dynamics is introduced in Section 4.3. Numerical stability analysis of invariant patterns on the square lattice is conducted for the FO model and PS model in Section 4.4. The bifurcation behaviour of the FO model is investigated in detail in Section 4.5 by focusing on the connectivity of bifurcating solutions from the uniform distribution to invariant patterns.

4.2. Spatial Equilibrium and the Replicator Dynamics

We employ a general framework of economic geography models with the replicator dynamics that was introduced in Section 2.1. We briefly explain a spatial equilibrium of the economy comprising K places. Mobile agents (e.g., skilled workers for the FO model) can migrate among the K places.

Let $P = \{1, \ldots, K\}$ be the set of places. Define the payoff function vector $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{\lambda}, \phi) \in \mathbb{R}^K$ as a continuous function of the spatial distribution of mobile agents $\boldsymbol{\lambda}$ ($\lambda_i \geq 0$; $i \in P$) and the trade freeness ϕ. Define a spatial equilibrium as a spatial distribution that satisfies the following

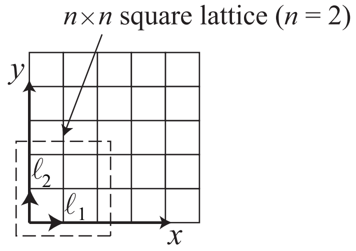


Figure 4.2: Square lattice.

conditions:

$$\begin{cases} v^* - v_i = 0 & \text{if } \lambda_i > 0, \\ v^* - v_i \leq 0 & \text{if } \lambda_i = 0, \end{cases} \tag{4.1}$$

and

$$\sum_{i \in P} \lambda_i = 1, \tag{4.2}$$

where v^* denotes the equilibrium payoff level. This condition means that there is no incentive for mobile agents to change the location choice.

We consider the replicator dynamics:

$$\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \phi), \tag{4.3}$$

where $\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = (F_i(\boldsymbol{\lambda}, \phi) \mid i \in P)$, and F_i takes the form

$$F_i(\boldsymbol{\lambda}, \phi) = \lambda_i(v_i(\boldsymbol{\lambda}, \phi) - \bar{v}(\boldsymbol{\lambda}, \phi)), \quad i \in P. \tag{4.4}$$

Here, $\bar{v} = \sum_{i \in P} \lambda_i v_i$ represents the weighted average payoff. We can restate a problem to obtain a set of stable spatial equilibria by another problem to find a set of stable stationary points of the replicator dynamics (Sandholm, 2010). A stationary point $(\boldsymbol{\lambda}, \phi)$ of the replicator dynamics is a solution to the governing equation:

$$\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{0}. \tag{4.5}$$

A stationary point is linearly stable if every eigenvalue of the Jacobian matrix $J = \partial \boldsymbol{F} / \partial \boldsymbol{\lambda}$ has a negative real part.

4.3. Theory of Invariant Patterns

In this section, we explain a theory of invariant patterns for the replicator dynamics. Consider a system of places allocated at each node of the $n \times n$ square lattice: Figure 4.2 depicts an example for $n = 2$ by the dashed lines. The $n \times n$ square lattice provides uniformly distributed $n \times n$ discrete regions ($K = n^2$), which are connected by links of the same length d forming a square mesh.

The symmetry of the $n \times n$ square lattice is described by the group

$$G = \langle r, s, p_1, p_2 \rangle = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n), \tag{4.6}$$

where $\langle \cdots \rangle$ denotes a group generated by the elements therein, and

- r: counterclockwise rotation about the origin at an angle of $\pi/2$,
- s: reflection $y \mapsto -y$,
- p_1: periodic translation along the $\boldsymbol{\ell}_1$-axis (i.e., the x-axis), and
- p_2: periodic translation along the $\boldsymbol{\ell}_2$-axis (i.e., the y-axis).

The symmetry of the $n \times n$ square lattice ensures the equivariance in the sense that

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \phi), \quad g \in G. \tag{4.7}$$

Therein, $T(g)$ ($g \in G$) denotes an orthogonal matrix representation of the group G on the K-dimensional space $\mathbb{R}^K$. The concrete form of $T(g)$ was presented in Section 3.4.

Stationary points of the replicator dynamics form solution curves $(\boldsymbol{\lambda}^*(\phi), \phi)$. In general, a spatial distribution $\boldsymbol{\lambda}^*(\phi)$ changes as the value of ϕ along a solution curve. In contrast, there can be a special solution curve $(\boldsymbol{\lambda}^*(\phi), \phi) = (\bar{\boldsymbol{\lambda}}, \phi)$ that has a constant spatial distribution $\bar{\boldsymbol{\lambda}}$ along a solution curve.[12] Such a distribution $\bar{\boldsymbol{\lambda}}$ is called an invariant pattern, and $(\bar{\boldsymbol{\lambda}}, \phi)$ is a solution for any ϕ. In contrast, a solution curve with distribution $\boldsymbol{\lambda}^*(\phi)$ that varies with ϕ is called a non-invariant pattern. Thus, stationary points are classified into

$$\begin{cases} \text{invariant pattern:} & \boldsymbol{\lambda}^* = \bar{\boldsymbol{\lambda}}, \\ \text{non-invariant pattern:} & \boldsymbol{\lambda}^* = \boldsymbol{\lambda}^*(\phi). \end{cases}$$

Rearranging the order of the components of $\boldsymbol{\lambda}^*$, we introduce $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0)$ with $\boldsymbol{\lambda}_+ = \{\lambda_i > 0 \mid i = 1, \dots, m\}$ and $\boldsymbol{\lambda}_0 = \boldsymbol{0}$ for later discussion of invariant patterns. As a candidate of invariant patterns, we consider a spatial distribution of a special form

$$(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0) = \left(\frac{1}{m}\boldsymbol{1}, \boldsymbol{0}\right), \quad 1 \le m \le K \tag{4.8}$$

with an m-dimensional vector $\boldsymbol{1} = (1, \dots, 1)^\top$. This distribution expresses equal complete agglomeration to m places and can be an invariant pattern under some symmetry conditions in the following proposition:

Proposition 4.1. *A spatial distribution $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0) = (\frac{1}{m}\boldsymbol{1}, \boldsymbol{0})$ of an economic geography model with the replicator dynamics is an invariant pattern if this distribution satisfies*

(i) $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0) = (\frac{1}{m}\boldsymbol{1}, \boldsymbol{0})$ *is invariant to some subgroup G' of G.*

(ii) *The set of points for $\boldsymbol{\lambda}_+$ belongs to an orbit of G'.*

Proof. Since the m places of $\boldsymbol{\lambda}_+$ belong to an orbit, we have $v_1 = \cdots = v_m$. Then, we have $\bar{v} = \sum_{i=1}^m \lambda_i v_i = v_i$ and $v_i - \bar{v} = 0$ ($i = 1, \dots, m$). Hence, we have $F_i(\frac{1}{m}\boldsymbol{1}, \boldsymbol{0}, \phi) = \boldsymbol{0}$ ($i = 1, \dots, m$). For $K - m$ places with no population, we have $\lambda_j = 0$ ($j = m + 1, \dots, K$). Hence, we have $F_i(\frac{1}{m}\boldsymbol{1}, \boldsymbol{0}, \phi) = \boldsymbol{0}$ ($i = m + 1, \dots, K$). This shows that $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \phi) = (\frac{1}{m}\boldsymbol{1}, \boldsymbol{0}, \phi)$ is a solution for any ϕ. Hence, $(\frac{1}{m}\boldsymbol{1}, \boldsymbol{0})$ is an invariant pattern. □

[12] Such a solution curve observed in the two-place economy (Fujita et al., 1999b).

Remark 4.1. The concept of invariant patterns is not applicable to some kind of models. We employ the replicator dynamics and admit $\boldsymbol{\lambda}$ to have zero components. Hence, it cannot be applied to models that do not take corner solutions due to the existence of the housing market such as Helpman (1998) and Allen and Arkolakis (2014) models. Also it cannot be applied to other dynamics such as the logit dynamics in (2.6). □

Spatial distributions for $m = 1, 2, K$ in (4.8) are called mono-centric, duo-centric, and uniform distribution, respectively.[13] We have the following propositions for these distributions.

Proposition 4.2. *A mono-centric distribution at any place is an invariant pattern for any economy.*

Proof. Consider $\lambda_1 = 1$ and $\lambda_i = 0$ $(i = 2, \dots, K)$. Then, we have $\bar{v} = \sum_{i=1}^{m} \lambda_i v_i = v_1$. Thus, we have $v_1 - \bar{v} = 0$. Hence, we have $F_1(1, \mathbf{0}, \phi) = \mathbf{0}$. For $K - 1$ places with no population, we have $\lambda_i = 0$. Hence, we have $F_i(1, \mathbf{0}, \phi) = \mathbf{0}$ $(i = 2, \dots, K)$. This shows that $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \phi) = (1, \mathbf{0}, \phi)$ serves as a solution for any ϕ. Hence, a mono-center at one place is an invariant pattern. □

Proposition 4.3. *The uniform and a duo-centric distribution are invariant patterns for an $n \times n$ square lattice.*

Proof. Consider two nodes (n_1, n_2) and (n_1', n_2'). Then, we have

$$r^2 {p_1}^i {p_2}^j \cdot (n_1, n_2) \equiv (-n_1 - i, -n_2 - j) \mod n.$$

Hence, for any pair of (n_1, n_2) and (n_1', n_2'), we see that

$$g \cdot (n_1, n_2) \equiv (n_1', n_2'), \quad g \cdot (n_1', n_2') \equiv (n_1, n_2) \mod n$$

by $g = r^2 {p_1}^i {p_2}^j$ with $i = -n_1 - n_1'$ and $j = -n_2 - n_2'$. By choosing $G' = \langle r^3 {p_1}^i {p_2}^j \rangle$, we see that a duo-center $(m = 2)$ at any places is an invariant pattern by Proposition 4.1. The uniform distribution can be shown as an invariant pattern by extending the proof for the duo-center. □

We search for invariant patterns on the $n \times n$ square lattice by finding a set of m nodal points and a subgroup G' that satisfy Proposition 4.1 for the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. We propose the following procedure to obtain all invariant patterns.

- Choose a set of m nodal points among a total of n^2 nodal points.
- Find elements of G that retain the set of points invariant.
- If these elements form a group and permute any two of the m nodal points, this group is chosen as G' in Proposition 4.1 to ensure that the set of points gives an invariant pattern.

In this procedure, it is convenient to note that the number m of agglomerated places is not arbitrary but depends on the lattice size n as explained in the following proposition:

Proposition 4.4. *If a spatial distribution $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0) = (\frac{1}{m}\mathbf{1}, \mathbf{0})$ is an invariant pattern on an $n \times n$ square lattice, then the number m $(1 \leq m \leq n^2)$ divides $8n^2$.*

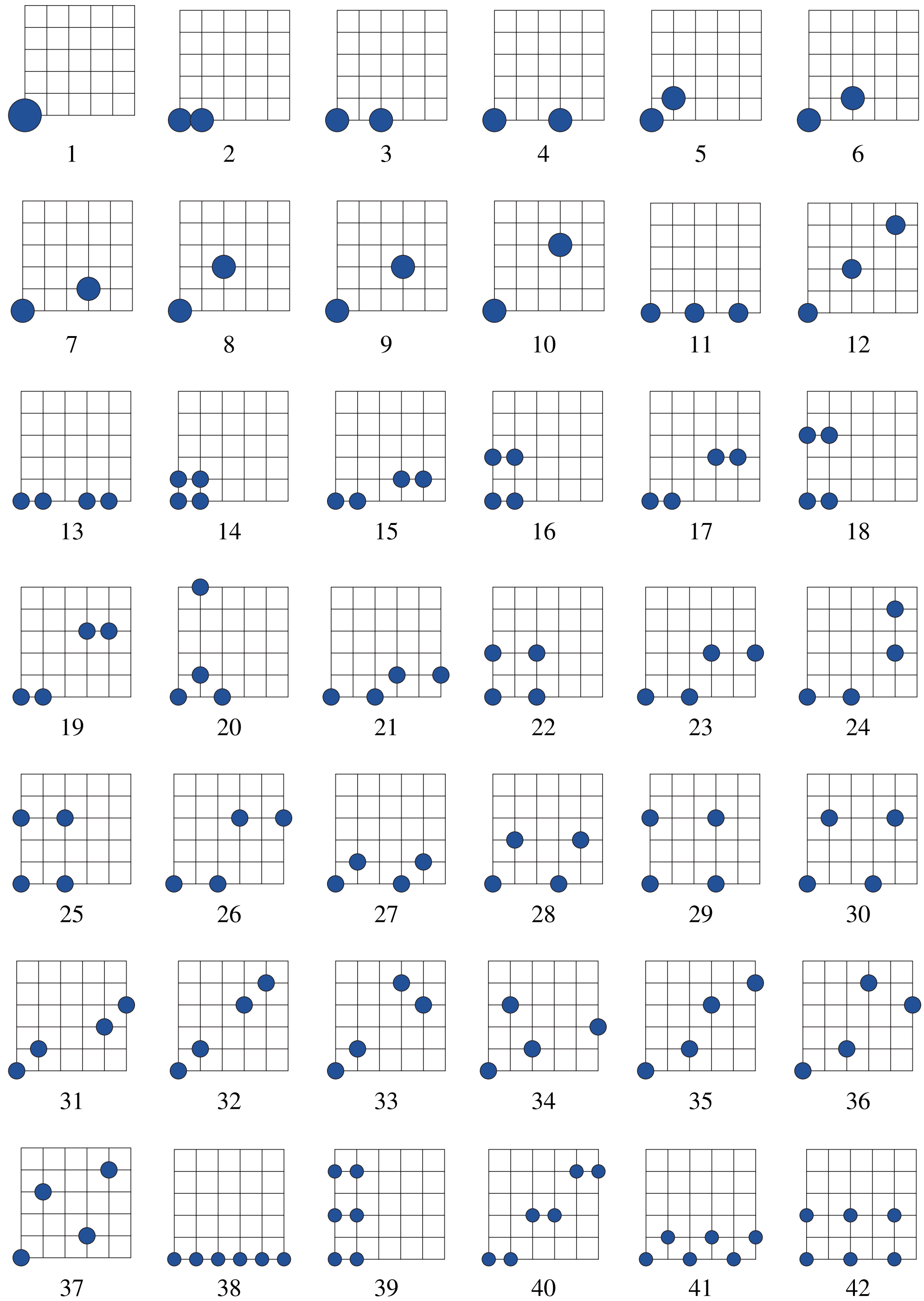


Figure 4.3: List of invariant patterns for the 6×6 square lattice. The size of a circle represents the mass of population in each place.

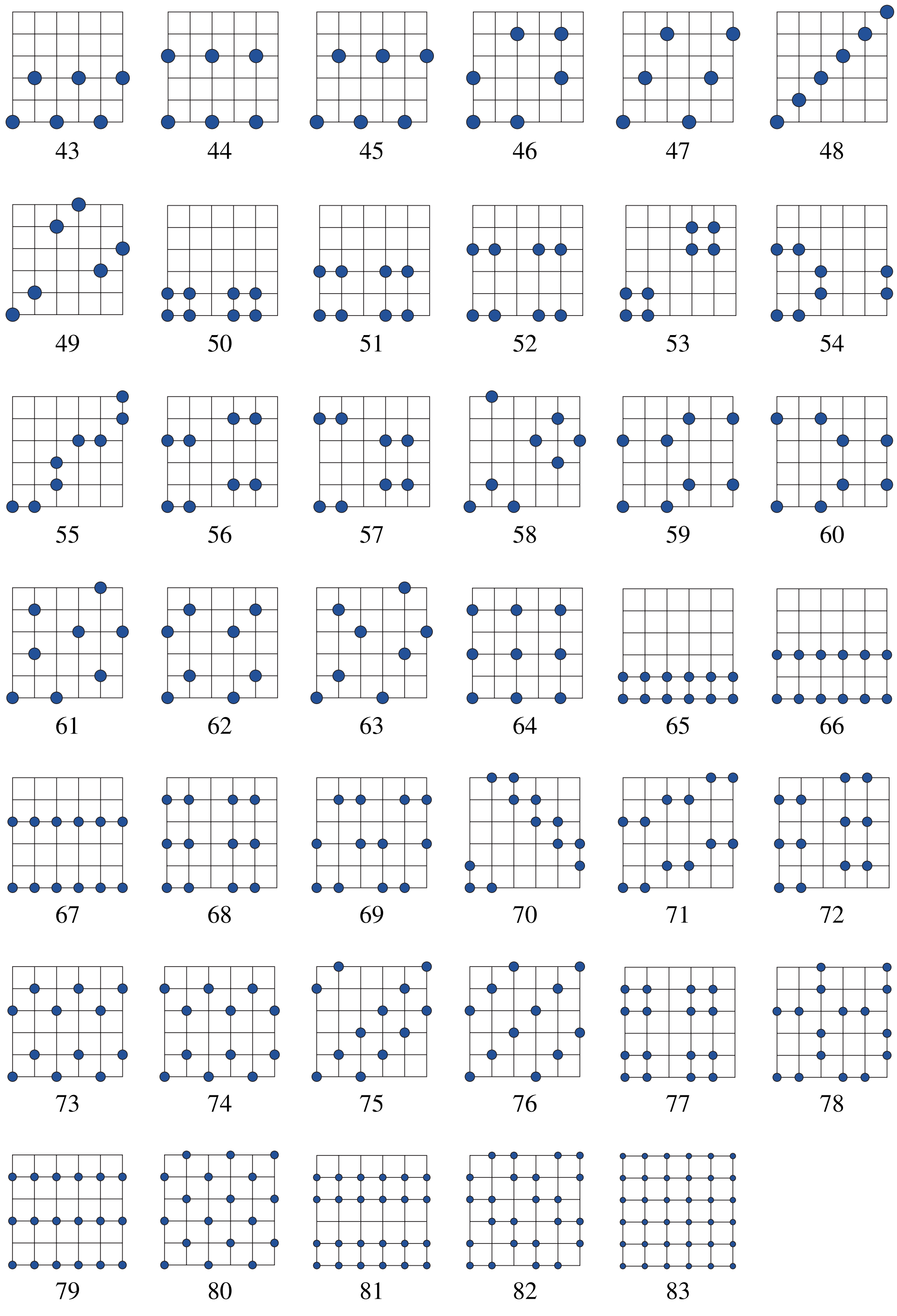


Figure 4.4: List of invariant patterns for the 6×6 square lattice. The size of a circle represents the mass of population in each place.

Proof. Since G' is a subgroup of G with $|G| = |\langle r, s, p_1, p_2 \rangle| = 8n^2$, $|G'|$ divides $8n^2$ by Lagrange's theorem. The number m of elements of an orbit divides $|G'|$ (e.g., see §3.1.2 of Kochendörfer, 1970). Hence, $8n^2$ is divisible by m. □

For example, a list of invariant patterns for $n = 6$ are depicted in Figs. 4.3 and 4.4.

4.4. Stable Invariant Patterns for Economic Geography Models

In this section, we investigate the stability of invariant patterns by numerical analysis. We employ the FO model (Forslid and Ottaviano, 2003) and the PS model (Pflüger and Südekum, 2008) as specific examples of economic geography models. Note that these models with the replicator dynamics on the $n \times n$ square lattice satisfy the equivariance to $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ (see Proposition 2.1 in Section 2.3.2 for the proof):

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \phi), \quad g \in G \tag{4.9}$$

for the $K(= n^2)$-dimensional permutation representation $T(g)$ of G.[14] Thus, the theoretical prediction of invariant patterns in Section 4.3 is applicable to these models.

We show the stability of invariant patterns on the 6×6 square lattice by comparative static analysis with respect to the bifurcation parameter (trade freeness) $\phi \in (0, 1)$ for each model. We use the same settings of the model parameters as in Section 3.7. Note that we can systematically investigate the stability of invariant patterns also for the other parameter settings.

FO model

Figure 4.5 depicts by solid lines the range of ϕ where the associated patterns are stable. Parameter values for the FO model are chosen as $\sigma = 6.0$, $\mu = 0.4$. There are as many as 22 invariant patterns that are stable for some range of ϕ. We see a tendency that when the trade freeness ϕ increases from a small value, the number of places with positive population decreases. Although most of the invariant patterns are not connected directly to the uniform distribution, some of them may be activated through secondary and further bifurcations or bifurcations from other invariant patterns.

PS model

Figure 4.6 depicts by solid lines the range of ϕ where the associated patterns are stable. Parameter values for the PS model are chosen as $\sigma = 3.0$, $\mu = 0.6$, $\gamma = 0.2$. There are as many as 8 invariant patterns that are stable for some range of ϕ. Unlike the FO model, invariant patterns with small number of agglomerated places (e.g., mono-centric and duo-centric distributions) are not stable. When ϕ is close to 0, the uniform distribution is stable. As ϕ increases, the uniform distribution loses its stability. When ϕ is close to 1, the uniform distribution becomes stable again. The 8-centric (51) and 12-centric (65) distributions also show the same feature as the uniform distribution.

[13] These three distributions are proved to be invariant patterns for the hexagonal lattice (Ikeda et al., 2019a).
[14] The concrete form of $T(g)$ was given in Section 3.4.1.

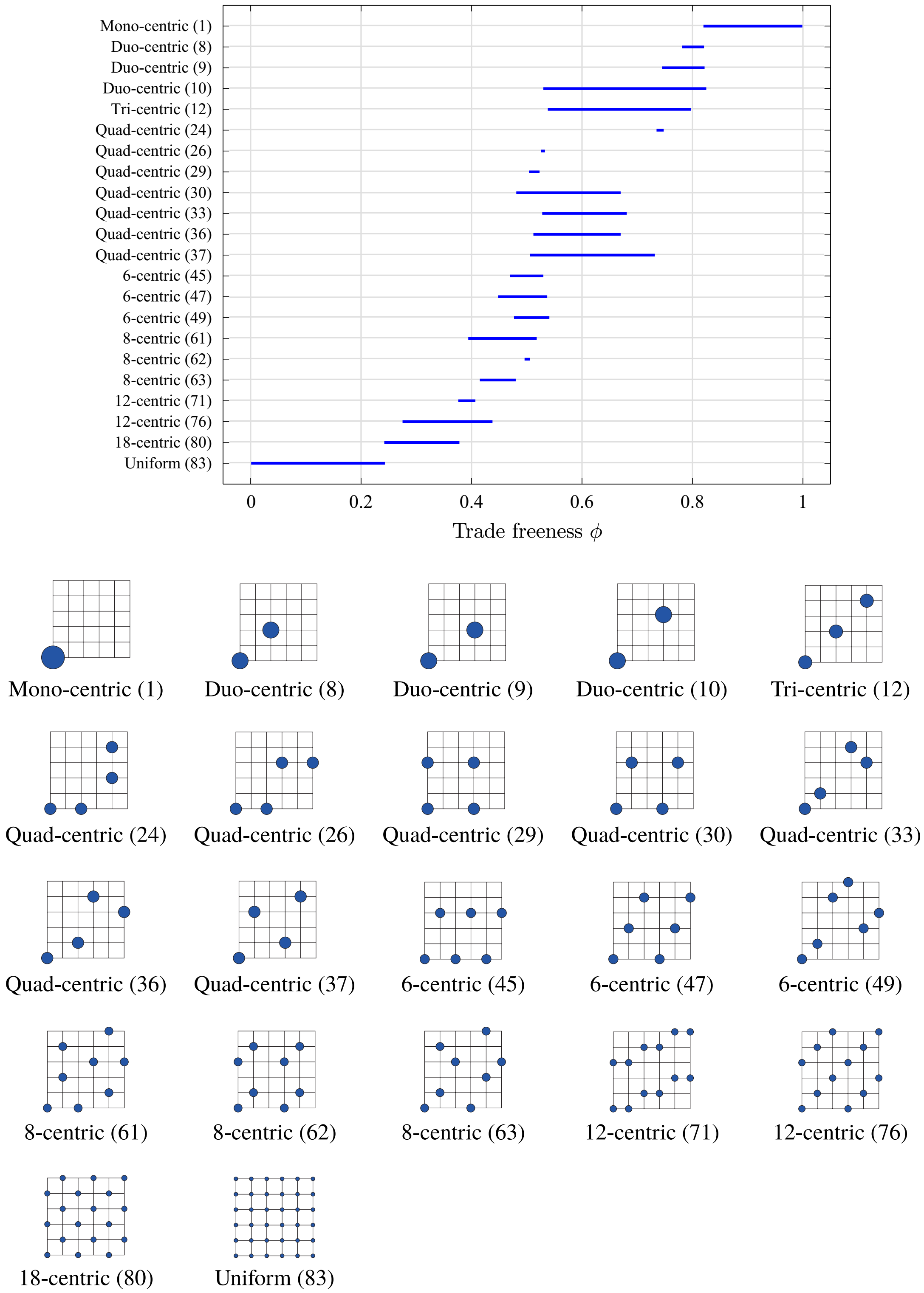


Figure 4.5: The ranges of ϕ for stable invariant patterns on the 6×6 square lattice for the FO model with $\sigma = 6.0$, $\mu = 0.4$. A number in the label of each invariant pattern corresponds to Figs. 4.3 and 4.4.

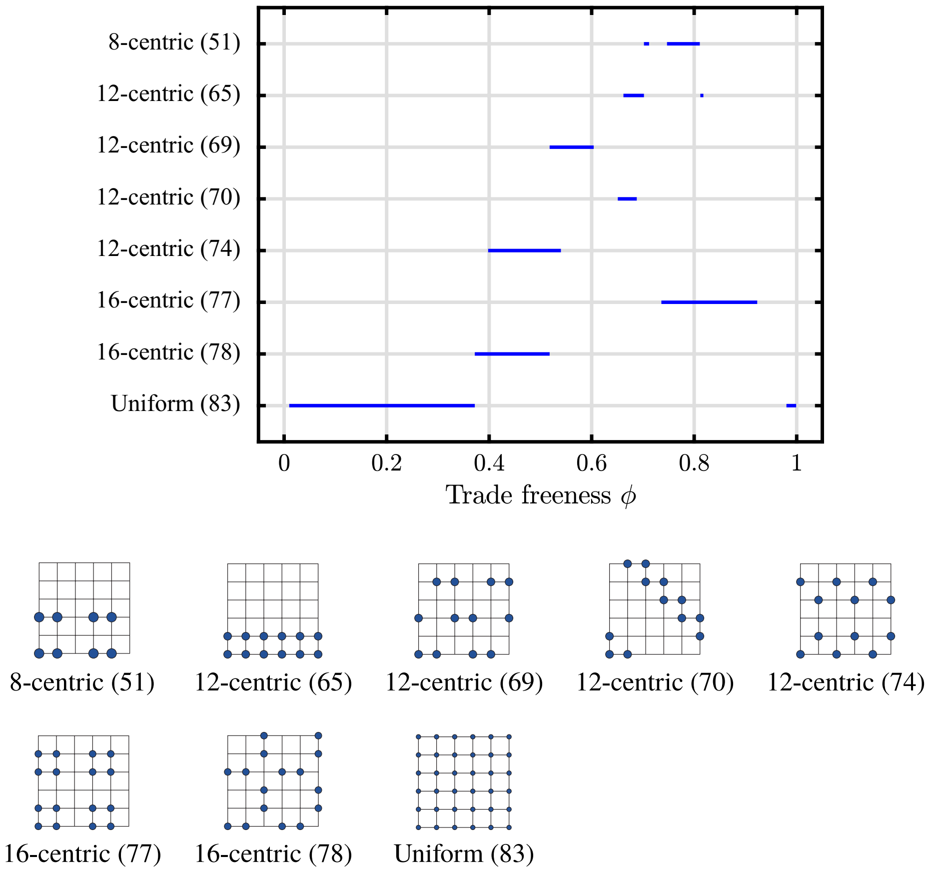


Figure 4.6: The ranges of ϕ for stable invariant patterns on the 6×6 square lattice for the PS model with $\sigma = 3.0$, $\mu = 0.6$, $\gamma = 0.2$. A number in the label of each invariant pattern corresponds to Figs. 4.3 and 4.4.

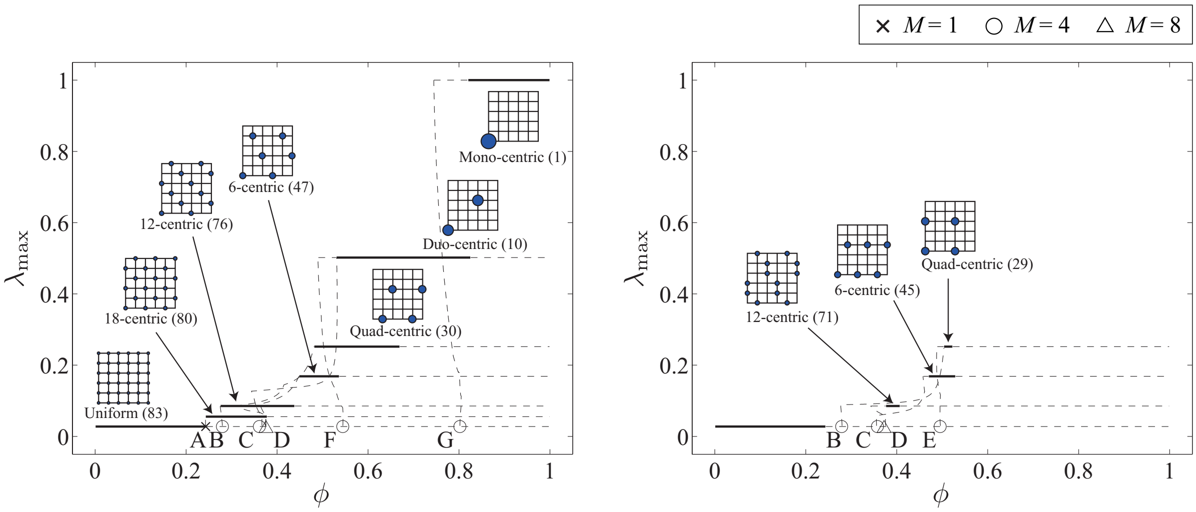


Figure 4.7: Stable invariant patterns engendered by direct bifurcation on the 6×6 square lattice for the FO model. The vertical axis shows $\lambda_{\max} = \max(\lambda_1, \ldots, \lambda_K)$. M represents the multiplicity of critical points. Solid curves represent stable equilibria, and dashed ones represent unstable ones. A number in the label of each invariant pattern corresponds to Figs. 4.3 and 4.4.

4.5. *Bifurcation Behaviour of Forslid and Ottaviano (2003) Model*

In this section, we conduct numerical bifurcation and stability analysis of the FO model (Forslid and Ottaviano, 2003) on the 6×6 square lattice, focusing on the connectivity of bifurcating solutions to invariant patterns. We would like to show the usefulness of invariant patterns for bifurcation analysis of economic geography models.

4.5.1. *Path Tracing Focusing on Invariant Patterns*

Figure 4.7 depicts the $\lambda_{\max} = \max(\lambda_1, \ldots, \lambda_K)$ versus ϕ relation of the equilibrium curves. When the trade freeness ϕ increases from 0, the uniform distribution (83) in Fig. 4.5 loses its stability at the bifurcation point A associated with $\mu = (1; +, +, -)$. Then, the bifurcating solution $\boldsymbol{q}_1^{(1;+,+,-)}$ emerges, and the solution curve connects to an invariant pattern of 18-centric distribution (80) in Fig. 4.5. During this process, population tends to be agglomerated completely to places with the largest positive components of the bifurcating solution. There is the same tendency for the solution curves from the bifurcation points B, C, D, E, F, and G associated with the irreducible representations $(4; 2, 2, +)$, $(4; 3, 1, +)$, $(8; 2, 1)$, $(4; 2, 0, +)$, $(4; 1, 1, +)$, and $(4; 1, 0, +)$, respectively.

Figure 4.7 captures stable invariant patterns engendered from the uniform distribution and solution curves for non-invariant patterns connecting the uniform distribution to these invariant patterns. So far as these solution curves are concerned, stable equilibria are always invariant patterns, and non-invariant patterns are all unstable. These unstable non-invariant patterns often regain their stability by arriving at stable invariant patterns. We have observed a mesh-like structure of the horizontal curves for stable invariant patterns and unstable non-invariant ones in Fig. 4.7. As we have seen, a knowledge of invariant patterns is useful in the understanding of the mechanism of such network-like structure of the bifurcation behaviour.

Table 4.1: Bifurcating solutions for the 6×6 square lattice

μ	Bifurcating solutions ($w \in \mathbb{R}$)
$(1; +, +, -)$	w
$(2; +, +)$	$\boldsymbol{w}_{\rm sq} = (w, w)$, $\boldsymbol{w}_{\rm stripe} = (w, 0)$
$(4; 1, 0, +)$	$\boldsymbol{w}_{\rm sq} = (w, 0, w, 0)$, $\boldsymbol{w}_{\rm stripeI} = (w, 0, 0, 0)$
$(4; 2, 0, +)$	$\boldsymbol{w}_{\rm sq} = (w, 0, w, 0)$, $\boldsymbol{w}_{\rm stripeI} = (w, 0, 0, 0)$
$(4; 1, 1, +)$	$\boldsymbol{w}_{\rm sq} = (w, 0, w, 0)$, $\boldsymbol{w}_{\rm stripeI} = (w, 0, 0, 0)$
$(4; 2, 2, +)$	$\boldsymbol{w}_{\rm sq} = (w, 0, w, 0)$, $\boldsymbol{w}_{\rm stripeI} = (w, 0, 0, 0)$
$(4; 3, 1, +)$	$\boldsymbol{w}_{\rm sq} = (w, 0, w, 0)$, $\boldsymbol{w}_{\rm stripeI} = (w, 0, 0, 0)$, $\boldsymbol{w}_{\rm stripeII} = (0, w, 0, 0)$
$(4; 3, 2, +)$	$\boldsymbol{w}_{\rm sq} = (w, 0, w, 0)$, $\boldsymbol{w}_{\rm stripeI} = (w, 0, 0, 0)$, $\boldsymbol{w}_{\rm stripeII} = (0, w, 0, 0)$
$(8; 2, 1)$	$\boldsymbol{w}_{\rm sqVM} = (w, 0, w, 0, w, 0, w, 0)$,
	$\boldsymbol{w}_{\rm upside-downI} = (w, 0, 0, 0, w, 0, 0, 0)$, $\boldsymbol{w}_{\rm upside-downII} = (0, w, 0, 0, 0, w, 0, 0)$,
	$\boldsymbol{w}_{\rm stripeI} = (w, 0, 0, 0, 0, 0, 0, 0)$, $\boldsymbol{w}_{\rm stripeII} = (0, w, 0, 0, 0, 0, 0, 0)$,

4.5.2. Connectivity of Bifurcating Solutions to Invariant Patterns

We can observe the connectivity of the uniform distribution to invariant patterns via bifurcating solutions. As a consequence of the irreducible decomposition (3.73) in Section 3.4.2 of the permutation representation T for the 6×6 square lattice, the irreducible representation μ of the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_6 \times \mathbb{Z}_6)$ to be considered in bifurcation analysis is restricted to

$$\mu = (1; +, +, +), (1; +, +, -), (2; +, +), (4; 1, 0, +), (4; 2, 0, +),$$
$$(4; 1, 1, +), (4; 2, 2, +), (4; 3, 1, +), (4; 3, 2, +), (8; 2, 1). \quad (4.10)$$

Theoretically possible bifurcating solutions associated with μ in (4.10) are listed in Table 4.1 and depicted in Fig. 4.8. Note that for $\mu = (4; 2, 0, +)$ and $\mu = (4; 2, 2, +)$, the two solutions $\boldsymbol{w}_{\rm sq}$ and $-\boldsymbol{w}_{\rm sq}$, which have opposite signs, represent different physical behaviour. The same holds for the solutions $\boldsymbol{w}_{\rm stripeI}$ and $-\boldsymbol{w}_{\rm stripeI}$. Other bifurcating solutions with opposite signs represent the same physical behaviour.

Note that the connectivity of the uniform distribution to invariant patterns via bifurcating solutions presented in Fig. 4.8. The eigenvector of a bifurcating solution at the left and the associated invariant pattern at the right connected by an arrow $\rightarrow$ that forms a pair and several are presented in Figs 4.9–4.11. Each pair displays similar geometrical patterns. In numerical bifurcation analysis of the FO model from the uniform distribution to be conducted in Section 4.5.1, we see how such connectivity arises from a bifurcation mechanism. Population in places with the positive components of bifurcating solutions tended to increase, while population in places with the negative components of bifurcating solutions tended to decrease along all bifurcating curves.

Based on this tendency, we predict that invariant patterns shown in Figs. 4.9–4.11 can be engendered from the uniform distribution as consequence of direct bifurcations. For example, a mono-center can be engendered from a critical point associated with $\boldsymbol{q}_1^{(4;1,0)} + \boldsymbol{q}_3^{(4;1,0)}$ (see the top-left of Fig. 4.9). Such connectivity is also observed for other pairs connected by the arrow $\rightarrow$.

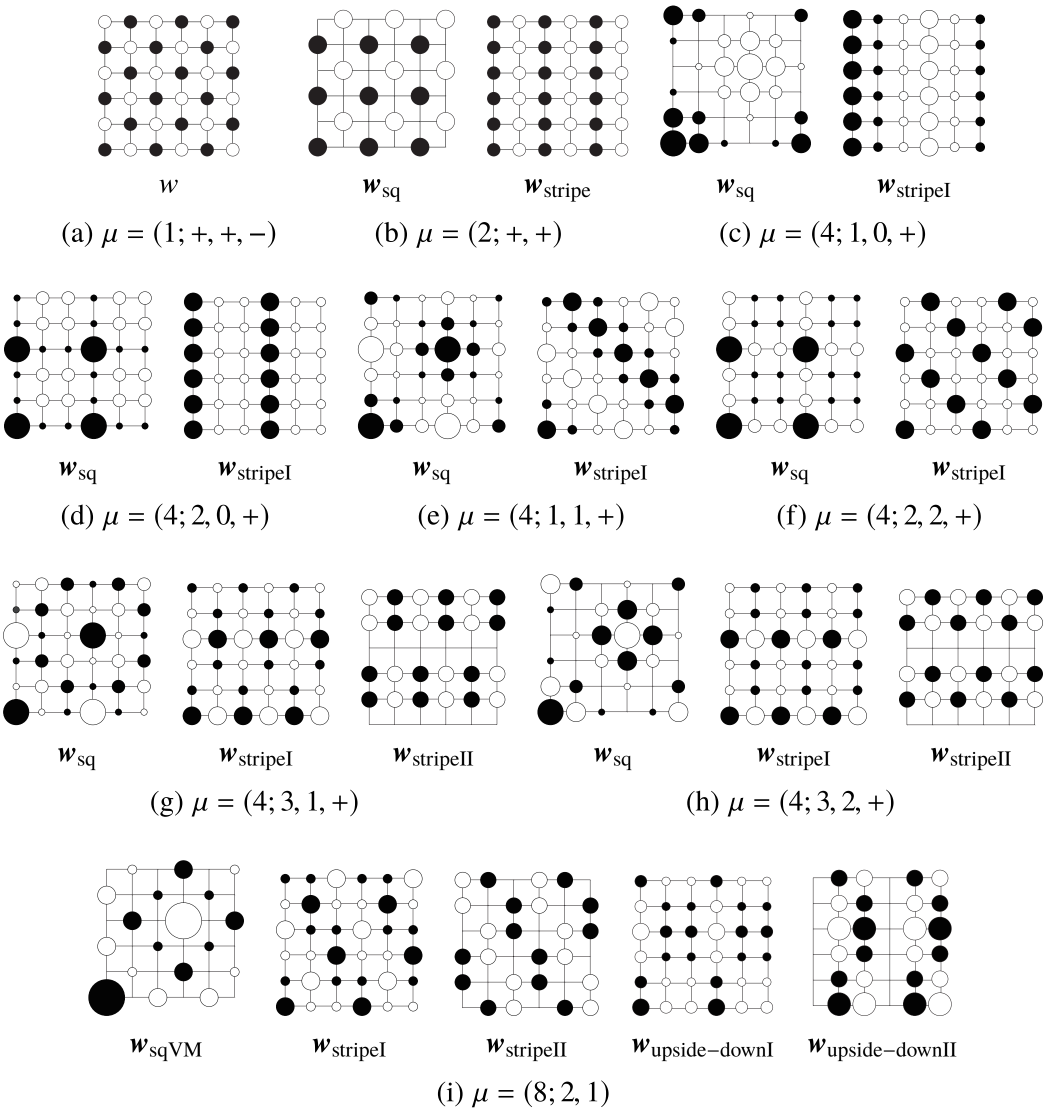


Figure 4.8: Bifurcating solutions for the 6×6 square lattice. A black circle denotes a positive component, and a white circle denotes a negative component. The size of a circle represents the magnitude of the associated component.

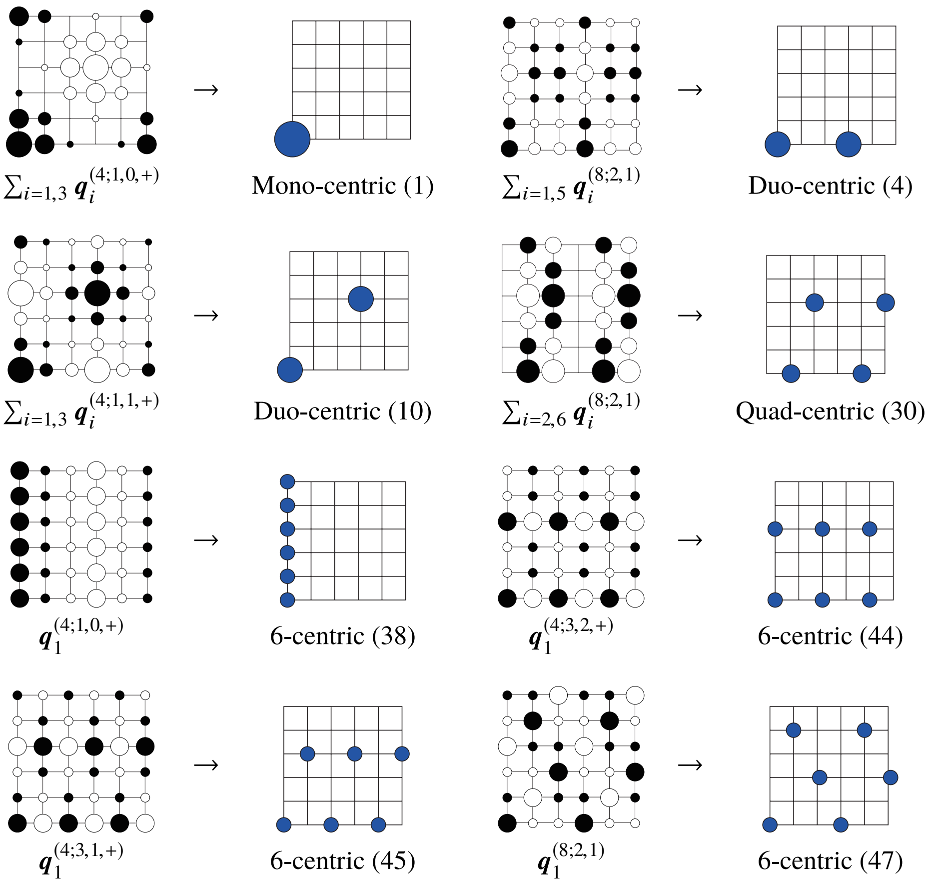


Figure 4.9: Invariant patterns that are engendered through asymmetric bifurcating solutions from the uniform distribution for the 6×6 square lattice. Figures to the left represent bifurcating solutions, and ones to the right represent corresponding invariant patterns. The number $(\cdot)$ in the parenthesis for each invariant pattern corresponds to that in Figs. 4.3 and 4.4.

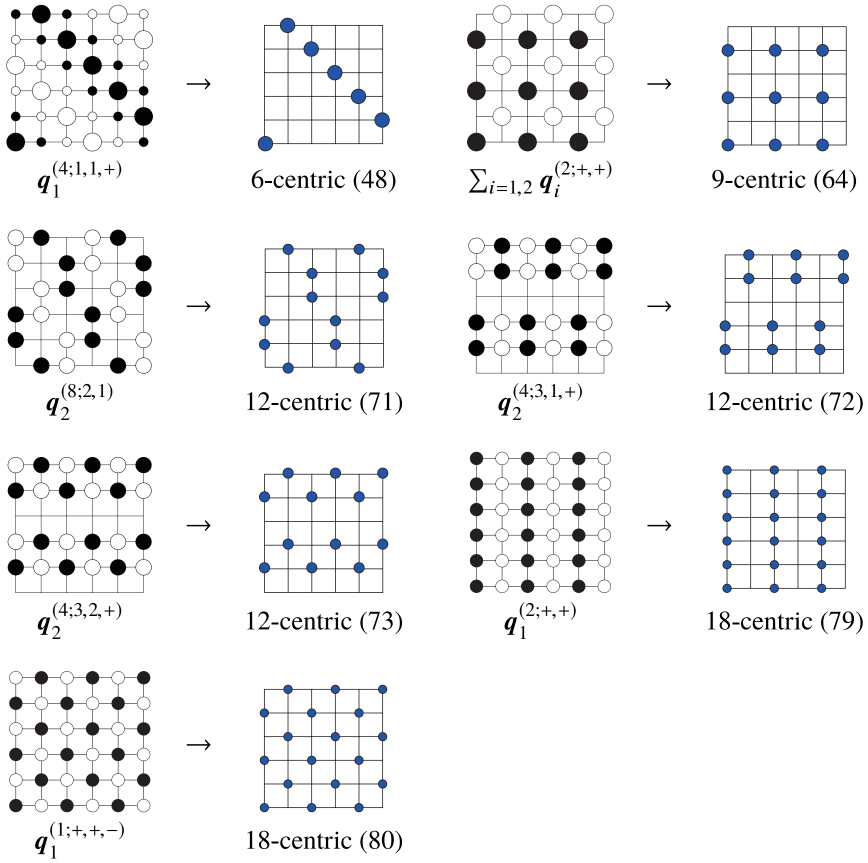


Figure 4.10: Invariant patterns that are engendered through asymmetric bifurcating solutions from the uniform distribution for the 6×6 square lattice. Figures to the left represent bifurcating solutions, and ones to the right represent corresponding invariant patterns. The number $(\cdot)$ in the parenthesis for each invariant pattern corresponds to that in Figs. 4.3 and 4.4.

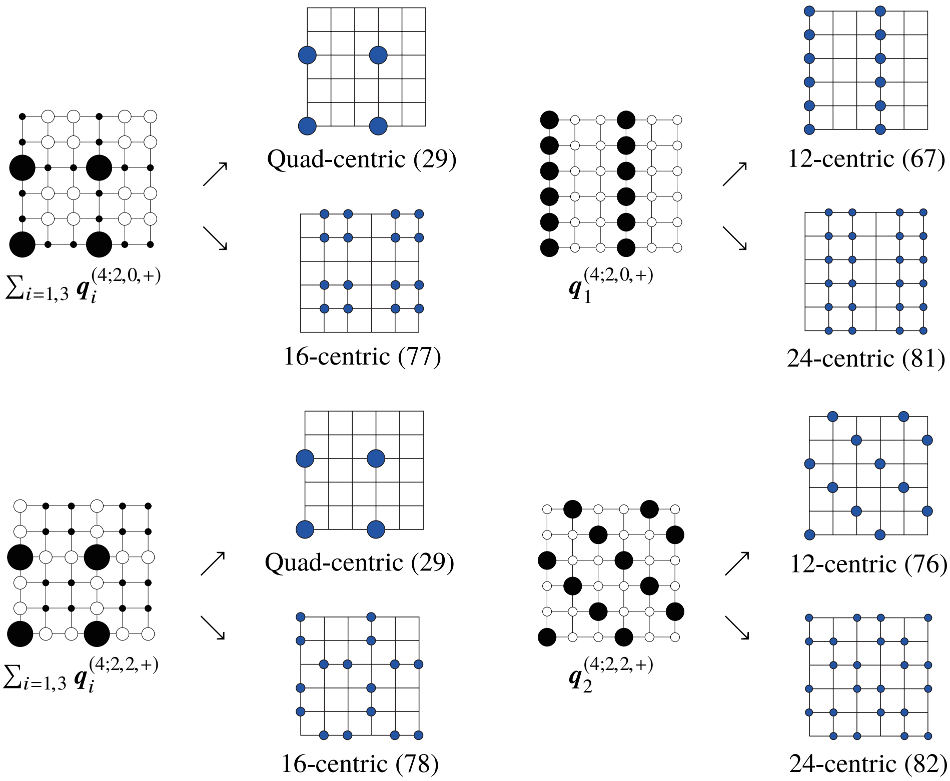


Figure 4.11: Invariant patterns that are engendered through asymmetric bifurcating solutions from the uniform distribution for the 6×6 square lattice. Figures to the left represent bifurcating solutions, and ones to the right represent corresponding invariant patterns. The number $(\cdot)$ in the parenthesis for each invariant pattern corresponds to that in Figs. 4.3 and 4.4.

This prediction is fairly in line with the bifurcation behaviour of the FO model that was investigated in Section 4.5.1 and is insightful in the understanding of spatial economic agglomerations.

A remark is on the symmetry/asymmetry of the bifurcating solutions. When the solutions in the positive and the negative directions from the bifurcation point are conjugate, these solutions can arrive at the same invariant pattern (see Figs. 4.9 and 4.10). When the two solutions are not conjugate, these solutions can arrive at two different patterns (see Fig. 4.11).

4.6. Concluding Remarks

This chapter has shown the usefulness of invariant patterns for analysis of economic geography models with the replicator dynamics. Focusing on invariant patterns, we proposed a systematic procedure to find stable equilibria of economic geography models: (i) obtaining all invariant patterns and investigate their stability, and (ii) searching for bifurcating equilibrium curves connecting stable invariant patterns. In numerical analysis of the FO model and PS model, we demonstrated the usefulness of this procedure in the elucidation of the agglomeration behaviour of economic geography models.

Invariant patterns on an $n \times n$ square lattice display characteristic geometrical patterns, including mono-centric and poly-centric distributions. Using the FO model, we showed the connectivity between such invariant patterns and bifurcating solutions via bifurcation from the uniform distribution. We demonstrated that such connectivity produces a mesh-like structure of the equilibrium curves for stable invariant patterns and unstable non-invariant ones.

The main contribution of this chapter is not only revealing the agglomeration behaviour of particular models but also proposing a general framework to understand bifurcation behaviour for any economic geography model that takes corner solutions under the replicator dynamics. Using the procedure proposed in this chapter, we can completely figure out bifurcation behaviour for any economic geography model.

5. Bifurcation Mechanism from the Mono-centric Distribution on a Square Domain

5.1. Introduction

Central place theory (Christaller, 1933; Lösch, 1940) put forward agglomeration patterns of one core city surrounded by satellite cities. In fact, such agglomeration patterns prosper worldwide. Using population data of Germany and the U.S., Ikeda et al. (2022) detected such core–periphery distributions by group-theoretic spectrum analysis.

Several studies in spatial economics dealt with the emergence of cities. The formation of satellite cities around a single large city was explored in a linear space (Mori, 1997; Fujita and Mori, 1997; Fujita et al., 1999a). A hub city formation from a central mono-center was investigated in three regions on a line segment (Ago et al., 2006). The spatial platforms in these studies, however, are restricted to be one-dimensional. To describe the mechanism of economic agglomerations in the real world, spatial platforms for economic geography models are to be extended to two-dimensional spaces or various network topologies. For example, Barbero and Zofío (2016) analyzed the agglomeration and dispersion forces of the core–periphery model with a ring and heterogeneous star network topologies.

That said, this chapter aims to elucidate a bifurcation mechanism, which can be interpreted as the formation of satellite cities, from the mono-centric distribution in a two-dimensional space for economic geography models. In search of realistic agglomeration patterns, we employ a square lattice with ordinary boundaries (cf., periodic boundaries in Chapter 3), that is, a two-dimensional square domain where discrete places are evenly distributed. Focusing on a bifurcation mechanism due to the geometrical symmetry, we present an exhaustive list of bifurcating solutions from the mono-centric distribution. The list of bifurcating solutions advanced in this chapter would be of assistance in the study of spatial economics.

In numerical analysis, we demonstrate the emergence of theoretically predicted bifurcating solutions. We use the FO model (Forslid and Ottaviano, 2003) and the PS model (Pflüger and Südekum, 2008) as specific examples of economic geography models. For each parameter value of these models, we investigate which bifurcating solution occurs from the mono-centric distribution as the trade freeness (transportation cost) changes.

This chapter is organized as follows. Basic assumptions of economic geography models with the replicator dynamics are presented in Section 5.2. A square lattice and its orbit decomposition is explained in Section 5.3. Bifurcation from the mono-centric distribution is studied in Section 5.4. Numerical analysis of economic geography models on the square lattice is conducted in Section 5.5.

5.2. Spatial Equilibrium and the Replicator Dynamics

We employ a general framework of economic geography models with the replicator dynamics that was introduced in Section 2.1. We briefly explain a spatial equilibrium of the economy comprising K places. Mobile agents (e.g., skilled workers for the FO model) can migrate among the K places.

Let $P = \{1, \ldots, K\}$ be the set of places. Define the payoff function vector $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{\lambda}, \phi) \in \mathbb{R}^K$ as a continuous function of the spatial distribution of mobile agents $\boldsymbol{\lambda}$ ($\lambda_i \geq 0$; $i \in P$) and the trade freeness ϕ. Define a spatial equilibrium as a spatial distribution that satisfies the following

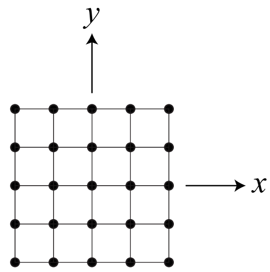


Figure 5.1: Square lattice with 25 places.

conditions:

$$\begin{cases} v^* - v_i = 0 & \text{if } \lambda_i > 0, \\ v^* - v_i \leq 0 & \text{if } \lambda_i = 0, \end{cases} \tag{5.1}$$

and

$$\sum_{i \in P} \lambda_i = 1, \tag{5.2}$$

where v^* denotes the equilibrium payoff level.

We consider the replicator dynamics:

$$\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \phi), \tag{5.3}$$

where $\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = (F_i(\boldsymbol{\lambda}, \phi) \mid i \in P)$, and F_i takes the form

$$F_i(\boldsymbol{\lambda}, \phi) = \lambda_i(v_i(\boldsymbol{\lambda}, \phi) - \bar{v}(\boldsymbol{\lambda}, \phi)), \quad i \in P. \tag{5.4}$$

Here, $\bar{v} = \sum_{i \in P} \lambda_i v_i$ represents the weighted average payoff. We can restate a problem to obtain a set of stable spatial equilibria by another problem to find a set of stable stationary points of the replicator dynamics (Sandholm, 2010). A stationary point $(\boldsymbol{\lambda}, \phi)$ of the replicator dynamics is a solution to the governing equation:

$$\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{0}. \tag{5.5}$$

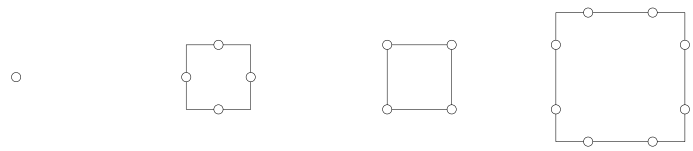
A stationary point is linearly stable if every eigenvalue of the Jacobian matrix $J = \partial \boldsymbol{F} / \partial \boldsymbol{\lambda}$ has a negative real part.

5.3. Square Lattice and Orbit Decomposition of Places

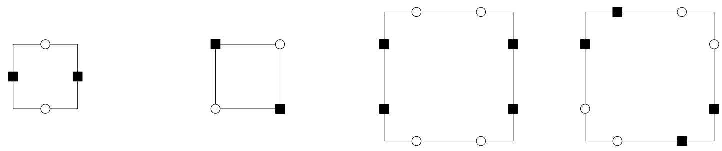
We employ a square lattice with a set of K places at the nodal points (cf., Fig. 5.1 for $K = 25$). In the description of spatial distributions in this lattice, it is essential to resort to its symmetry labeled by the dihedral group:

$$G = \mathrm{D}_4 = \{e, r, \ldots, r^3, s, sr, \ldots, sr^3\}, \tag{5.6}$$

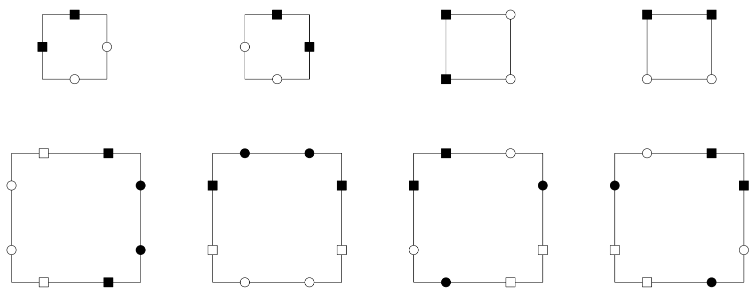
where e is the identity transformation, s is a reflection $y \mapsto -y$, and r^j is a counterclockwise rotation about the center of the square lattice by an angle of $\pi j/2$ ($j = 0, 1, 2, 3$).



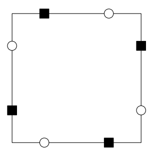
(a) D_4



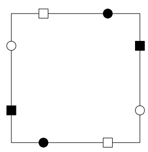
(b) D_2



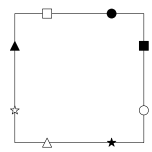
(b) D_1



(d) C_4



(d) C_2



(d) C_1

Figure 5.2: Orbit decompositions of places on a square lattice with respect to subgroups of G.

We can decompose the K places into subsets, called orbits. Each orbit has some geometrical symmetry described by a subgroup of G. Subgroups of G are given as follows:

$$\begin{aligned}
&\mathrm{D}_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\} && : \text{square symmetry},\\
&\mathrm{D}_2 = \{e, r^2, s, sr^2\} && : \text{diagonal symmetry},\\
&\mathrm{D}_1 = \{e, s\} && : \text{bilateral symmetry},\\
&\mathrm{C}_4 = \{e, r, r^2, r^3\} && : (\pi/2)\text{−rotation symmetry},\\
&\mathrm{C}_2 = \{e, r^2\} && : \pi\text{−rotation symmetry},\\
&E = \mathrm{C}_1 = \{e\} && : \text{asymmetry}.
\end{aligned}$$

The set of places P is decomposed into disjoint orbits with respect to a subgroup G' of G:

$$P = \bigcup_{l \in \mathcal{L}} P_l, \tag{5.7}$$

where P_l is an orbit, and $\mathcal{L}$ is the whole set of orbits with the symmetry labeled by G'. Orbit decompositions with respect to subgroups other than E are depicted in Fig. 5.2, while each node becomes an orbit for $G' = E$. The same symbols in Fig. 5.2 (such as ◦ or □) imply that they belong to the same orbit.

We assume that the symmetry of the square lattice ensures the equivariance with respect to the payoff function:

$$T(g)\boldsymbol{v}(\boldsymbol{\lambda}, \phi) = \boldsymbol{v}(T(g)\boldsymbol{\lambda}, \phi), \quad g \in G, \tag{5.8}$$

where $T(g)$ is a matrix representation of G that permutes place numbers. Under this assumption, we have the following lemma:

Lemma 1. *The payoff function v_i in the same orbit for a subgroup takes the same value when a spatial distribution is symmetric with respect to the subgroup.*

Proof. The spatial distribution is symmetric with respect to a subgroup G' of G, that is, $T(g)\boldsymbol{\lambda} = \boldsymbol{\lambda}$ for $g \in G'$. Hence, we have $T(g)\boldsymbol{v}(\boldsymbol{\lambda}, \phi) = \boldsymbol{v}(\boldsymbol{\lambda}, \phi)$ for $g \in G'$ by virtue of the equivariance in (5.8). This means that v_i in the same orbit is permutable. This suffices for the proof. □

5.4. Bifurcating Solutions from the Mono-centric Distribution

Let $\boldsymbol{\lambda}^{\mathrm{FA}} = (1, 0, \ldots, 0)$ be the mono-centric distribution,[15] which represents the full agglomeration to the place at the center of the square lattice. We investigate bifurcation points on the mono-centric distribution. Recall that a bifurcation occurs when the Jacobian matrix becomes singular. The following lemma provides the form of the Jacobian matrix at the mono-centric distribution.

[15] Note that the mono-centric distribution is an invariant pattern, which satisfies the governing equation in (5.5) for any ϕ (cf., Proposition 4.3 in Section 4.3).

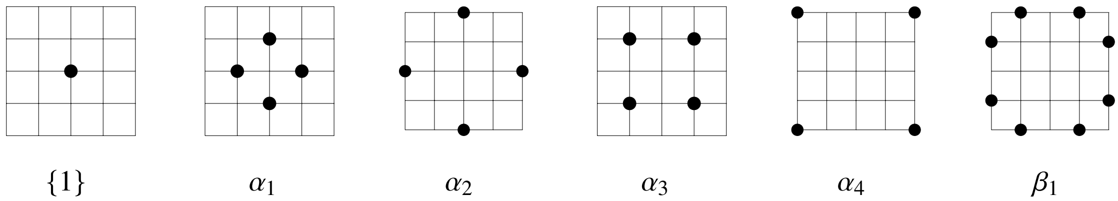


Figure 5.3: Orbits with respect to D_4 for a square lattice with 25 places.

Lemma 2. *The Jacobian matrix at the mono-centric distribution takes the following form:*

$$J(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi) = \begin{pmatrix} -v_1 & J_{+0} \\ & J_0 \end{pmatrix}, \tag{5.9}$$

where

$$J_{+0} = (-v_2, \ldots, -v_K), \quad J_0 = \mathrm{diag}(v_2 - v_1, \ldots, v_K - v_1). \tag{5.10}$$

Proof. Since the replicator dynamics takes $F_i = \lambda_i(v_i - \bar{v})$ $(i = 1, \ldots, K)$, we have

$$\frac{\partial F_i}{\partial \lambda_i} = v_i - \bar{v} + \lambda_i \left(\frac{\partial v_i}{\partial \lambda_i} - v_i - \sum_{k=1}^{K} \lambda_k \frac{\partial v_k}{\partial \lambda_i} \right), \quad i = 1, \ldots, K, \tag{5.11}$$

$$\frac{\partial F_i}{\partial \lambda_j} = \lambda_i \left(\frac{\partial v_i}{\partial \lambda_j} - v_j - \sum_{k=1}^{K} \lambda_k \frac{\partial v_k}{\partial \lambda_j} \right), \quad i, j = 1, \ldots, K, \quad j \neq i. \tag{5.12}$$

Note that $\bar{v} = \sum_{j=1}^{K} \lambda_j v_j = v_1$ at $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{FA}}$. Substituting $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{FA}}$ into (5.11) and (5.12), we have

$$\left.\frac{\partial F_i}{\partial \lambda_i}\right|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}^{\mathrm{FA}}} = \begin{cases} -v_1 & (i = 1) \\ v_i - v_1 & (i \neq 1) \end{cases}, \tag{5.13}$$

$$\left.\frac{\partial F_i}{\partial \lambda_j}\right|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}^{\mathrm{FA}}} = \begin{cases} -v_j & (i = 1) \\ 0 & (i \neq 1) \end{cases}. \tag{5.14}$$

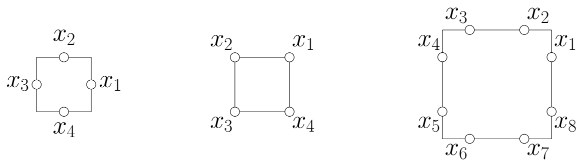
Thus, the Jacobian matrix $J = \partial \boldsymbol{F} / \partial \boldsymbol{\lambda}(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi)$ takes the form (5.9) with (5.10). □

Note that the mono-centric distribution is symmetric with respect to the group D_4:

$$T(g)\boldsymbol{\lambda}^{\mathrm{FA}} = \boldsymbol{\lambda}^{\mathrm{FA}}, \quad g \in D_4. \tag{5.15}$$

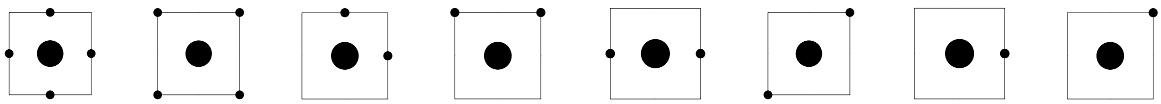
Thus, we carry out orbit decomposition with respect to D_4 in order to apply Lemma 1 to the mono-centric distribution. As s result, each orbit other than the center of the square lattice comprises four or eight places. We denote these orbits by

$$P = \{1\} \cup \alpha_1 \cup \ldots \cup \alpha_{n_1} \cup \beta_1 \cup \ldots \cup \beta_{n_2}, \tag{5.16}$$



(a) Type α_i orbit (b) Type β_i orbit

Figure 5.4: Definition of variables for each orbit.



(a) Square-I (b) Duo-I (c) Duo-II (d) Mono-I

Figure 5.5: Geometrical configurations of bifurcating solutions from the mono-centric distribution for Type α_i orbit.

where $\{1\}$ represents an orbit comprising only the place at the center, α_i ($i = 1, \ldots, n_1$) represents an orbit with square-shaped four places, and β_i ($i = 1, \ldots, n_2$) represents an orbit with eight places. For example, Fig. 5.3 depicts orbits for $K = 25$ places ($n_1 = 4, n_2 = 1$).

By Lemma 1, the payoff function v_i in the same orbit with respect to D_4 takes the same value at the mono-centric distribution. We denote such values as

$$\begin{aligned} &v_{\alpha_1}, \ldots, v_{\alpha_{n_1}} \quad \text{for } \alpha_i \ (i = 1, \ldots, n_1), \\ &v_{\beta_1}, \ldots, v_{\beta_{n_2}} \quad \text{for } \beta_i \ (i = 1, \ldots, n_2). \end{aligned} \tag{5.17}$$

Then, we have the following condition:

Lemma 3. *A bifurcating solution in the space $\sum_{j=1}^{K} \lambda_j = 1$ emerges from the mono-centric distribution if one of the following conditions is satisfied:*

$$v_{\alpha_i} - v_1 = 0 \quad \text{for some } \alpha_i \ (i = 1, \ldots, n_1), \tag{5.18}$$

$$v_{\beta_i} - v_1 = 0 \quad \text{for some } \beta_i \ (i = 1, \ldots, n_2). \tag{5.19}$$

Proof. The Jacobian matrix (5.9) becomes singular if one of the following conditions is satisfied:

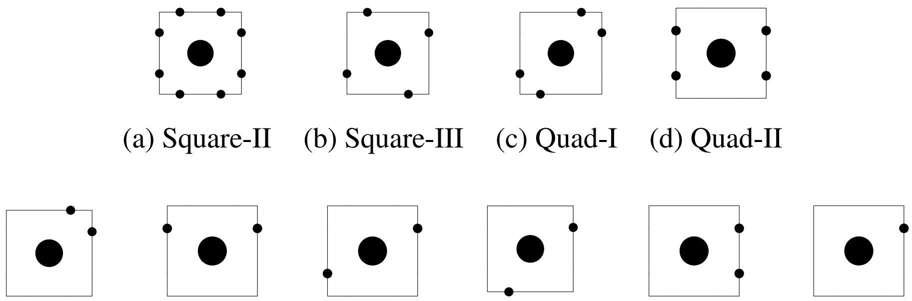
$$v_1 = 0, \tag{5.20}$$

$$v_{\alpha_i} - v_1 = 0 \quad \text{for some } \alpha_i \ (i = 1, \ldots, n_1), \tag{5.21}$$

$$v_{\beta_i} - v_1 = 0 \quad \text{for some } \beta_i \ (i = 1, \ldots, n_2). \tag{5.22}$$

For (5.20), no bifurcating solution emerges in the space $\sum_{i=1}^{K} \lambda_i = 1$ since the direction of this solution is $(1, 0, \ldots, 0)$. Thus, only (5.21) and (5.22) are the bifurcating conditions from the mono-center. □

Let ϕ_c^k be the trade freeness at $v_k - v_1 = 0$ ($k \in P - \{1\}$). In analysis of bifurcating solutions from a critical point $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_c^k)$, we employ the bifurcation equation. The bifurcation equation for Type α_i orbit takes the following form:



(a) Square-II (b) Square-III (c) Quad-I (d) Quad-II

(e) Duo-III (f) Duo-IV (g) Duo-V (h) Duo-VI (i) Duo-VII (j) Mono-II

Figure 5.6: Geometrical configurations of bifurcating solutions from the mono-centric distribution for Type β_i orbit.

Lemma 4. *For a bifurcation point associated with Type α_i orbit, the bifurcation equation is four-dimensional and is expressed as*

$$\begin{aligned}
\tilde{F}_1(x_1, x_2, x_3, x_4, \psi) &= x_1 R(x_1, x_2, x_3, x_4, \psi) = 0,\\
\tilde{F}_2(x_1, x_2, x_3, x_4, \psi) &= x_2 R(x_2, x_3, x_4, x_1, \psi) = 0,\\
\tilde{F}_3(x_1, x_2, x_3, x_4, \psi) &= x_3 R(x_3, x_4, x_1, x_2, \psi) = 0,\\
\tilde{F}_4(x_1, x_2, x_3, x_4, \psi) &= x_4 R(x_4, x_1, x_2, x_3, \psi) = 0,
\end{aligned} \tag{5.23}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_4) = \{\lambda_j \mid j \in \alpha_i\}$ *(cf.,* Fig. 5.4 (a)*), and R is a function with*

$$R(x_1, x_2, x_3, x_4, \psi) = R(x_1, x_4, x_3, x_2, \psi). \tag{5.24}$$

Proof. See Appendix B.1.1. □

Solving the bifurcation equation for α_i, we obtain the following bifurcating solutions:

Proposition 5.1. *A bifurcation point associated with Type α_i orbit has the following bifurcating solutions (cf.,* Fig. 5.5*):*

$$\boldsymbol{x} = \begin{cases} w(1,1,1,1): & \text{Square-I,}\\ w(1,1,0,0): & \text{Duo-I,}\\ w(1,0,1,0): & \text{Duo-II,}\\ w(1,0,0,0): & \text{Mono-I} \end{cases} \tag{5.25}$$

for some $w > 0$.

Proof. See Appendix B.1.2. □

The bifurcation equation for Type β_i orbit takes the following form:

Lemma 5. *For a bifurcation point associated with Type* β_i *orbit, the bifurcation equation is eight-dimensional and is expressed as*

$$\begin{aligned}
\tilde{F}_1(\boldsymbol{x}, \psi) &= x_1 R(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \psi) = 0,\\
\tilde{F}_2(\boldsymbol{x}, \psi) &= x_2 R(x_2, x_1, x_8, x_7, x_6, x_5, x_4, x_3, \psi) = 0,\\
\tilde{F}_3(\boldsymbol{x}, \psi) &= x_3 R(x_3, x_4, x_5, x_6, x_7, x_8, x_1, x_2, \psi) = 0,\\
\tilde{F}_4(\boldsymbol{x}, \psi) &= x_4 R(x_4, x_3, x_2, x_1, x_8, x_7, x_6, x_5, \psi) = 0,\\
\tilde{F}_5(\boldsymbol{x}, \psi) &= x_5 R(x_5, x_6, x_7, x_8, x_1, x_2, x_3, x_4, \psi) = 0,\\
\tilde{F}_6(\boldsymbol{x}, \psi) &= x_6 R(x_6, x_5, x_4, x_3, x_2, x_1, x_8, x_7, \psi) = 0,\\
\tilde{F}_7(\boldsymbol{x}, \psi) &= x_7 R(x_7, x_8, x_1, x_2, x_3, x_4, x_5, x_6, \psi) = 0,\\
\tilde{F}_8(\boldsymbol{x}, \psi) &= x_8 R(x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1, \psi) = 0,
\end{aligned} \tag{5.26}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_8) = \{\lambda_j \mid j \in \beta_i\}$ *(cf.,* Fig. 5.4(b)*), and* R *is a function.*

Proof. See Appendix B.2.1. □

Solving the bifurcation equation for β_i, we obtain the following bifurcating solutions:

Proposition 5.2. *A bifurcation point associated with Type* β_i *orbit has the following bifurcating solutions (cf.,* Fig. 5.6*):*

$$\boldsymbol{x} = \begin{cases}
w(1,1,1,1,1,1,1,1): & \text{Square-II},\\
w(1,0,1,0,1,0,1,0): & \text{Square-III},\\
w(1,1,0,0,1,1,0,0): & \text{Quad-I},\\
w(1,0,0,1,1,0,0,1): & \text{Quad-II},\\
w(1,1,0,0,0,0,0,0): & \text{Duo-III},\\
w(1,0,0,1,0,0,0,0): & \text{Duo-IV},\\
w(1,0,0,0,1,0,0,0): & \text{Duo-V},\\
w(1,0,0,0,0,1,0,0): & \text{Duo-VI},\\
w(1,0,0,0,0,0,0,1): & \text{Duo-VII},\\
w(1,0,0,0,0,0,0,0): & \text{Mono-II}
\end{cases} \tag{5.27}$$

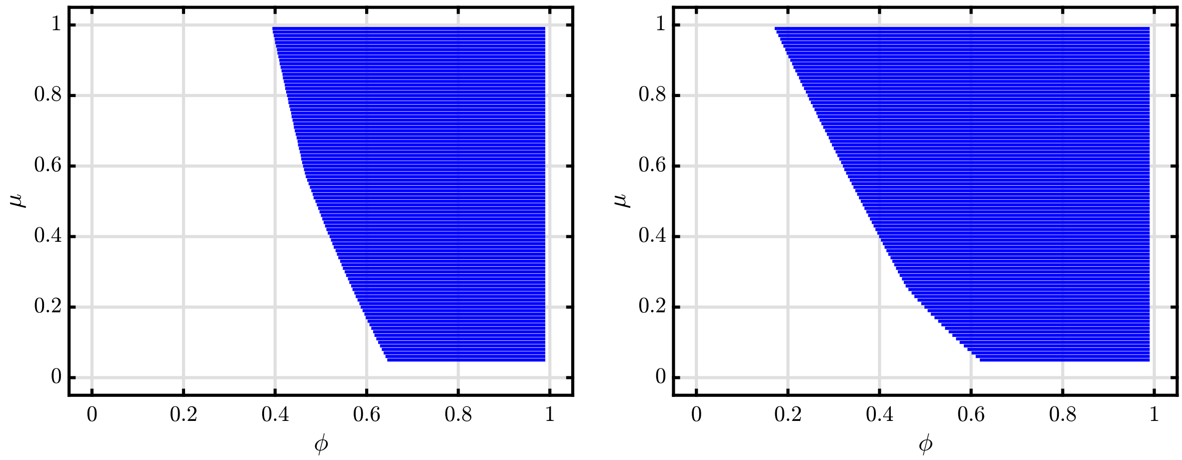
for some $w > 0$.

Proof. See Appendix B.2.2. □

Note that the stability of all the bifurcating solutions depend on cases. See Appendix B.1.3 for α_i and Appendix B.2.3 for β_i.

5.5. Bifurcation Behaviour of Economic Geography Models

Based on theoretically possible bifurcating solutions presented in Section 5.4, we conduct numerical bifurcation analysis of the FO model (Forslid and Ottaviano, 2003) and the PS model (Pflüger and Südekum, 2008) on the square lattice. For these two models, the elasticity of substitution $\sigma \in (1, \infty)$ and the expenditure share of manufacturing goods $\mu \in (0, 1)$ are model parameters. Note that the expenditure share of housing goods $\gamma \in (0, 1)$ is another model parameter for the PS model. The trade freeness $\phi \in (0, 1)$ serves as the bifurcation parameter.



(a) $\sigma = 6.0$

(b) $\sigma = 3.0$

Figure 5.7: The stability areas of the mono-centric distribution on the square lattice with 25 places for the FO model in $(\phi, \mu) \in (0, 1) \times (0, 1)$.

5.5.1. Forslid and Ottaviano (2003) Model

We conduct numerical bifurcation analysis of the FO model (Forslid and Ottaviano, 2003) on the square lattice. We investigate the influence of the model parameters σ and μ on the types of bifurcating solutions from the stable mono-centric distribution. We additionally discuss the influence of boundary conditions of the square lattice.

Bifurcation Behaviour on the Square Lattice with 25 Places

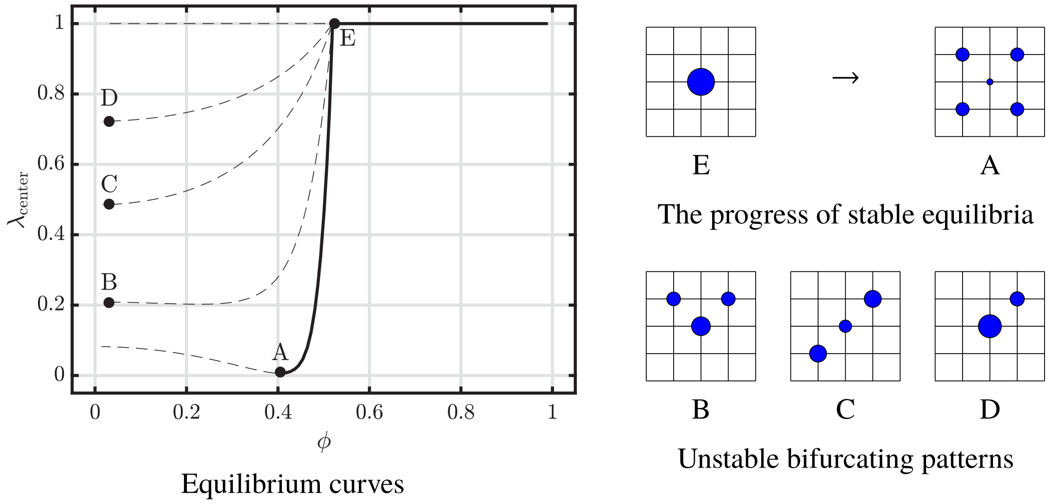
We employ the square lattice with 25 places (cf., Fig. 5.1) and demonstrate the bifurcation behaviour of the FO model. The 25 places can be decomposed into six kinds of orbits (cf., Fig. 5.3):

$$\begin{cases} \{1\}: & \text{a place at the center,} \\ \alpha_1, \ldots, \alpha_4: & \text{4 places,} \\ \beta_1: & \text{8 places.} \end{cases} \tag{5.28}$$

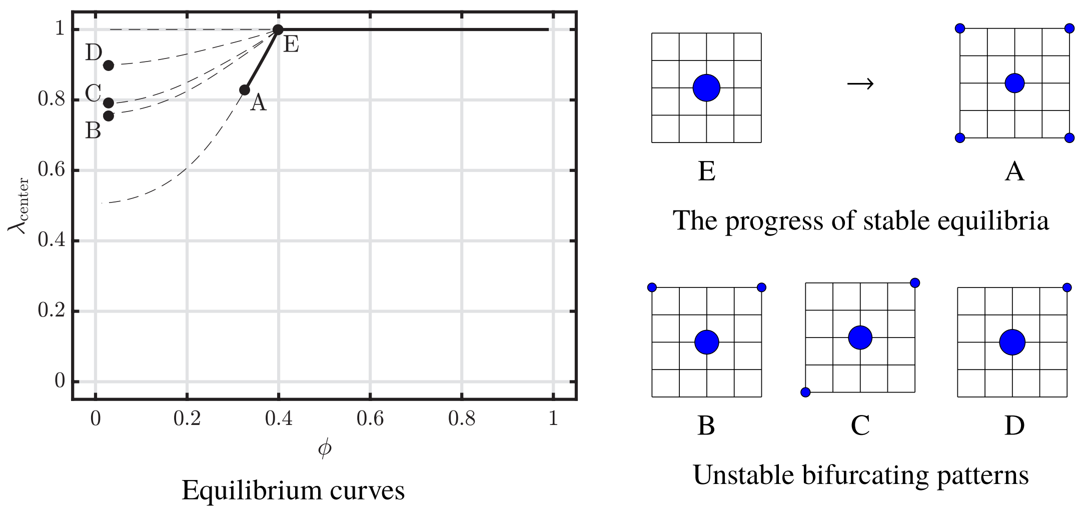
There are five kinds of bifurcation points associated with Type $\alpha_1, \ldots, \alpha_4$, and β_1 orbits, whereas the orbit $\{1\}$ is not associated with bifurcation.

We specify a bifurcation point on the stable mono-centric distribution. Figures 5.7(a) and (b) show the stability areas of the mono-centric distribution in the space of $(\phi, \mu) \in (0, 1) \times (0, 1)$ for $\sigma = 6.0$ and $\sigma = 3.0$, respectively. For each μ, the mono-centric distribution loses its stability at a bifurcation point when ϕ decreases from 1. An increase of μ expands a range of ϕ where the mono-centric distribution becomes stable. Comparing Figs. 5.7(a) and (b), we see that a decrease of σ expands the stability area of the mono-centric distribution.

We show the emergence of bifurcating solutions from the stable mono-centric distribution. Figure 5.8(a) shows equilibrium curves for $(\sigma, \mu) = (6.0, 0.4)$. When ϕ decreases from 1, a bifurcation occurs at the point E. The bifurcating path EA shows that the stable agglomeration



(a) $(\sigma, \mu) = (6.0, 0.4)$



(b) $(\sigma, \mu) = (3.0, 0.4)$

Figure 5.8: Equilibrium curves for the FO model on the square lattice with 25 places. The vertical axis shows the size of population at the center. Solid curves represent stable equilibria, and dashed ones represent unstable ones. A blue circle shows the size of population at each place.

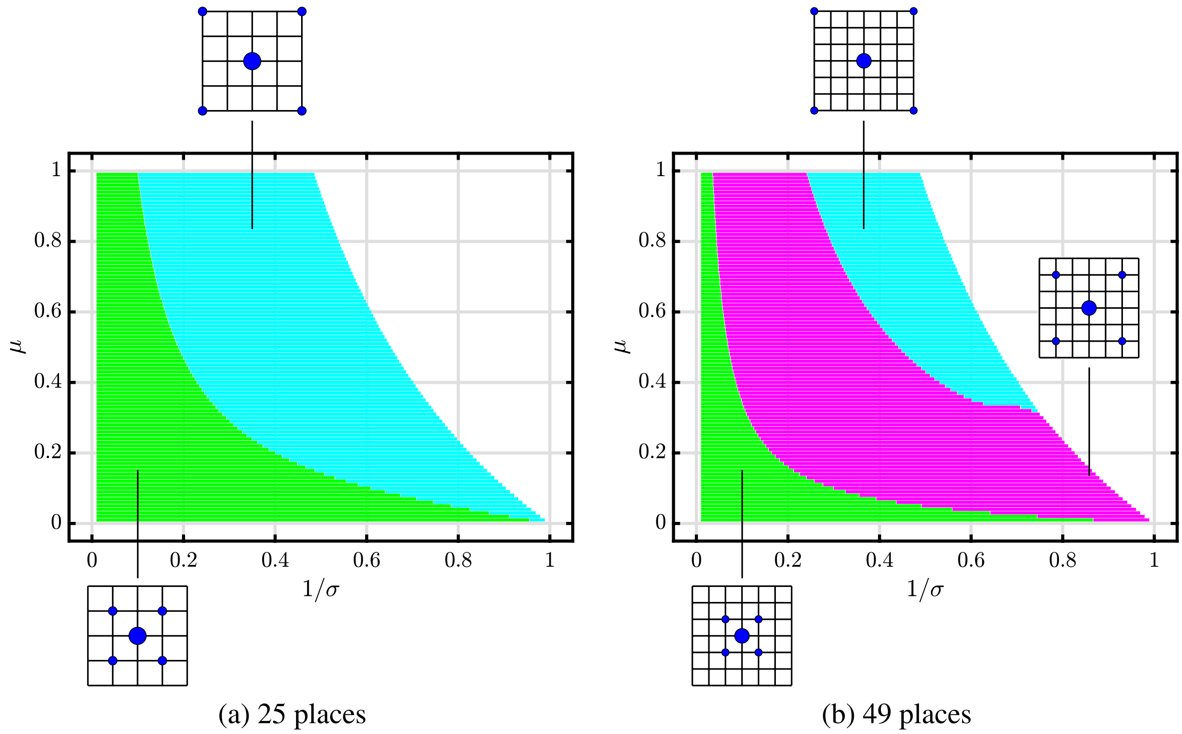


Figure 5.9: The dependence of the types of bifurcating solutions on the values of $(1/\sigma, \mu) \in (0, 1) \times (0, 1)$ for the FO model on the square lattice. A blue circle shows the size of population at each place.

pattern shifts from the mono-centric distribution to Square-I pattern for Type α_3 orbit (cf., α_3 in Fig. 5.3). The other bifurcating paths EB, EC, and ED are all unstable.

the types of bifurcating solutions from a bifurcation point on the stable mono-centric distribution varies with the model parameters σ and μ. Figure 5.8(b) depicts equilibrium curves for $(\sigma, \mu) = (3.0, 0.4)$. Similarly to those for the case of $(\sigma, \mu) = (6.0, 0.4)$, a bifurcation occurs at the point E when ϕ decreases from 1. The stable bifurcating path EA, however, represents Square-I pattern for Type α_4 orbit (cf., α_4 in Fig. 5.3).

Influence of the Model Parameters

With the results above in mind, we investigate the influence of the model parameters σ and μ on bifurcation behaviour. We use the square lattice with 25 places and that with 49 places.

Figure 5.9(a) shows the parameter dependence of the types of bifurcating solutions from the stable mono-centric distribution on the square lattice with 25 places. The areas painted using different colors express the emergence of different bifurcating solutions.[16] There are two possible bifurcating solutions. This result indicates that locations where population emerges are dependent on an agglomeration force: As an agglomeration force increases ($1/\sigma$ and μ become close to 1), population emerges at the places away from the center.

[16] For the FO model, no bifurcation occurs in the area of $\mu > \sigma - 1$. This condition is called the no-black-hole condition (Robert-Nicoud, 2005).

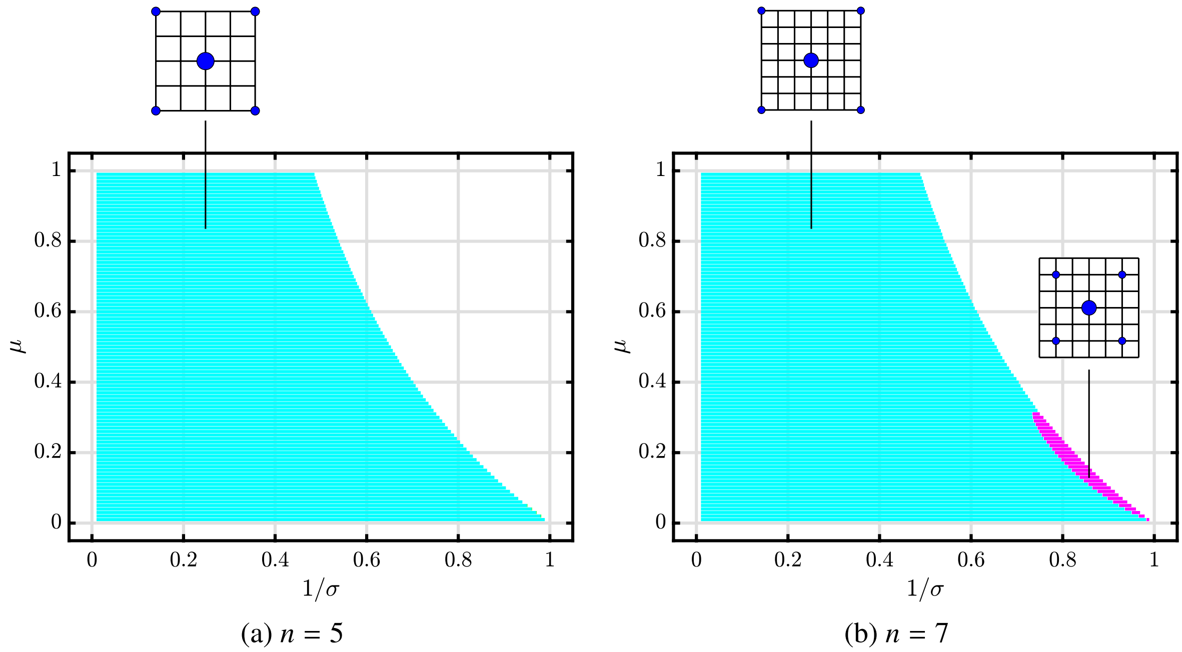


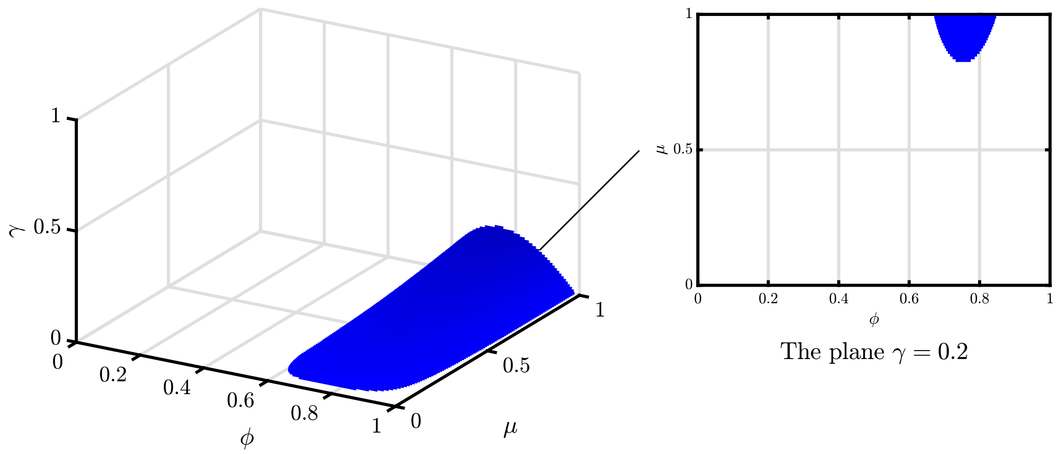
Figure 5.10: The dependence of the types of bifurcating solutions on the values of $(1/\sigma, \mu) \in (0, 1) \times (0, 1)$ for the FO model on the $n \times n$ square lattice with periodic boundaries. A blue circle shows the size of population at each place.

Figure 5.9(b) shows the parameter dependence of the types of bifurcating solutions from the stable mono-centric distribution on the square lattice with 49 places. Similarly to that for the case of 25 places, population emerges at places away from the center as σ and μ become close to 1. There are three possible bifurcating solutions. When μ or $1/\sigma$ is close to 0 (cf., the area painted using green in Fig. 5.9(b)), a bifurcating solution that can be interpreted as the formation of satellite places surrounding the center occurs.

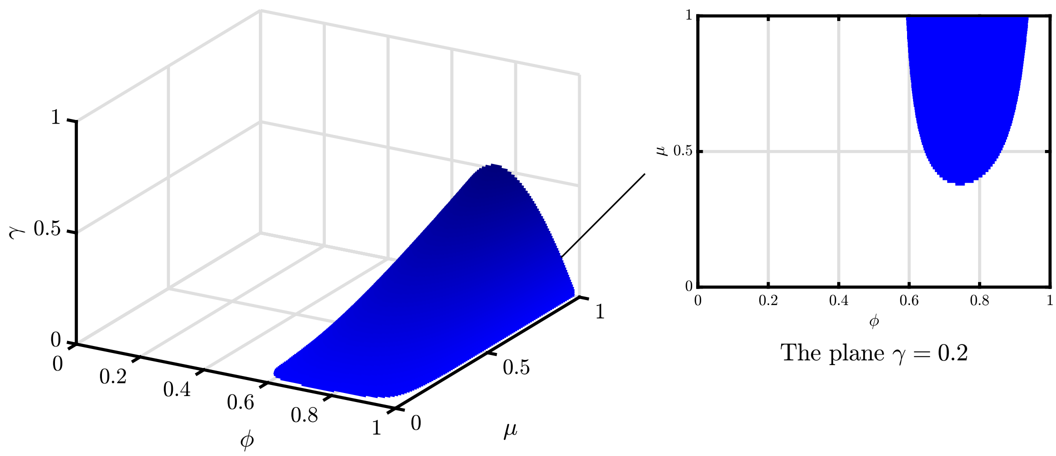
Influence of Boundary Conditions

We investigate the influence of boundary conditions for the square lattice on bifurcation behaviour. We employ the $n \times n$ square lattice ($n = 5, 7$) with periodic boundaries that was introduced in Chapter 3, while we have used the square lattice with ordinary boundaries in this chapter. Note that periodic boundaries remove the exogenous disadvantages of the places near the borders.

Figure 5.10 shows the parameter dependence of the types of bifurcating solutions from the stable mono-centric distribution on the $n \times n$ square lattice ($n = 5, 7$). For $n = 5$, population emerges at the corners for any value of σ and μ. For $n = 7$, there are two possible bifurcating solutions. For each case, bifurcating solutions that represent the formation of satellite places surrounding the center do not arise, while those can occur on the square lattice with ordinary boundaries (cf., the area painted using green in Fig. 5.9(b) for 49 places). Note that population emerges at the inside of the borders when μ is close to 0 and σ is close to 1 (cf., the area painted using magenta in Fig. 5.10(b) for $n = 7$).



(a) $\sigma = 6.0$



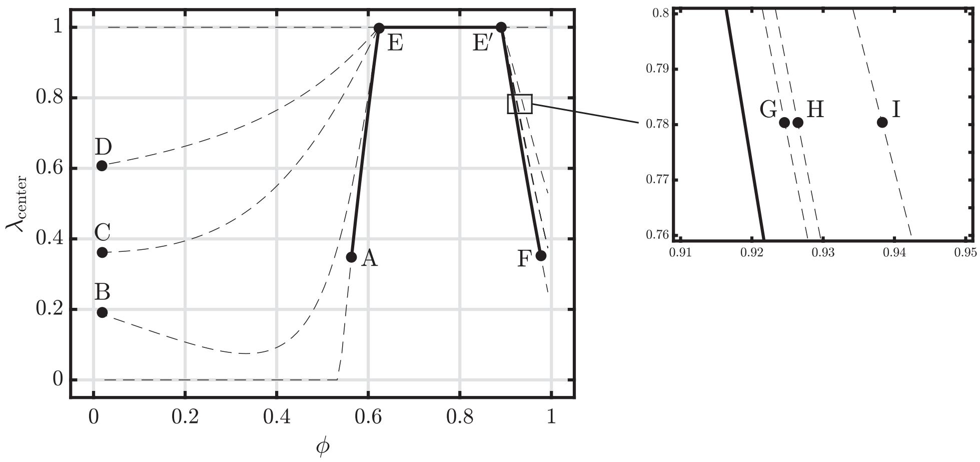
(b) $\sigma = 3.0$

Figure 5.11: The stability areas of the mono-centric distribution on the square lattice with 25 places for the PS model in $(\phi, \mu, \gamma) \in (0, 1) \times (0, 1) \times (0, 1)$.

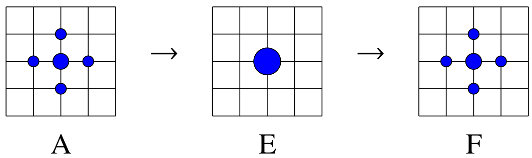
5.5.2. Pflüger and Südekum (2008) Model

We conduct numerical bifurcation analysis of the PS model (Pflüger and Südekum, 2008). We employ the square lattice with 25 places and investigate the influence of the model parameters σ, μ, and γ on the types of bifurcating solutions from the stable mono-centric distribution.

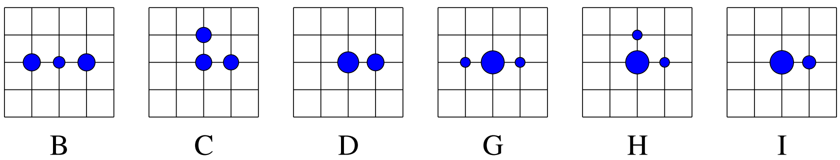
We first examine bifurcation points on the stable mono-centric distribution. Figures 5.11(a) and (b) show the stability areas of the mono-centric distribution in $(\phi, \mu, \gamma) \in (0, 1) \times (0, 1) \times (0, 1)$ for $\sigma = 6.0$ and $\sigma = 3.0$, respectively. For each case, the mono-centric distribution is unstable when ϕ is close to 0. The plane $\gamma = 0.2$ indicates that the mono-centric distribution becomes stable at a bifurcation point as ϕ increases from 0. The stable mono-centric distribution loses its stability when ϕ reaches another bifurcation point. This observation implies that the stable mono-centric distribution encounters two kinds of bifurcation points as the value of ϕ changes. Thus, the PS model can potentially describe the formation of satellite places due to an increase or decrease of ϕ.



(a) Equilibrium curves



(b) The progress of stable equilibria



(c) Unstable bifurcating patterns

Figure 5.12: Equilibrium curves for the PS model with $(\sigma, \mu, \gamma) = (3.0, 0.6, 0.2)$ on the square lattice with 25 places. The vertical axis shows the size of population at the center. Solid curves represent stable equilibria, and dashed ones represent unstable ones. A blue circle shows the size of population at each place.

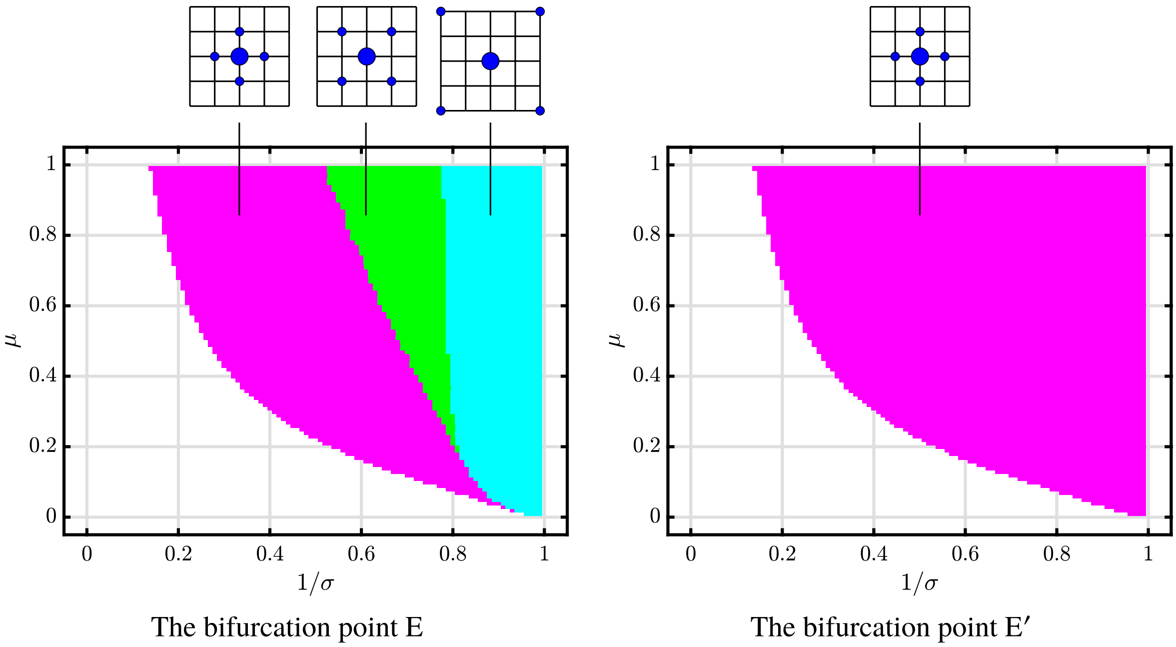
We next demonstrate the emergence of bifurcating solutions from the stable mono-centric distribution. Figure 5.12 shows equilibrium curves for $(\sigma, \mu, \gamma) = (3.0, 0.6, 0.2)$. The path AEE′F shows the progress of stable equilibria when ϕ increases from 0 to 1. The stable path AE represents Square-I pattern for Type α_1 orbit (cf., α_1 in Fig. 5.3). As ϕ increases from the point A, population surrounding the center disappears. The mono-centric distribution becomes stable at the bifurcation point E and loses its stability at another bifurcation point E′. The stable path E′A represents Square-I pattern for Type α_1 orbit again. The other bifurcating paths are all unstable.

The stable mono-centric distribution encounters two kinds of bifurcation points (cf., the points E and E′ in Fig. 5.12). Bifurcating solutions emerge from the bifurcation point E when ϕ decreases. On the other hand, bifurcating solutions emerge from the bifurcation point E′ when ϕ increases. With these results in mind, we investigate the dependence of the types of bifurcating solutions from the bifurcation points E and E′ on the model parameters. Figure 5.13(a) and (b) show the dependence of the types of bifurcating solutions on the values of the model parameters σ and μ for $\gamma = 0.2$ and $\gamma = 0.5$, respectively. For the bifurcation point E, population tends to emerge away from the center as an agglomeration force increases ($1/\sigma$ and μ become close to 1). For the bifurcation point E′, bifurcating solutions with population surrounding the center is the only possibility regardless of the value of σ and μ. Note that such a tendency is common with the case of $\gamma = 0.5$.

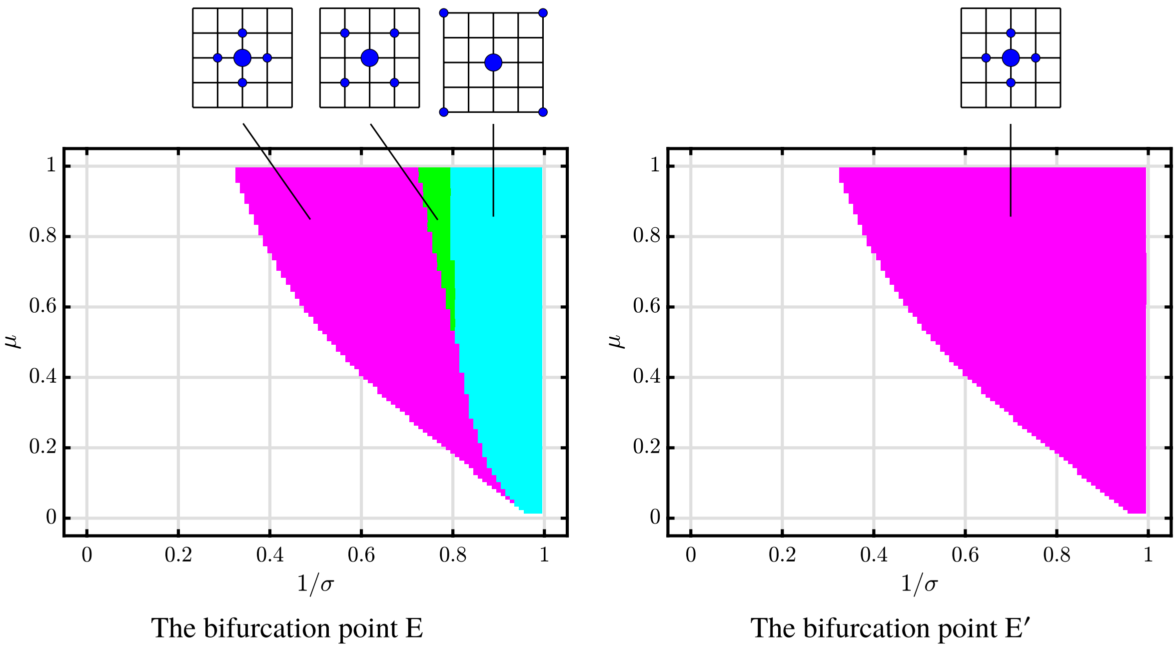
5.6. Concluding Remarks

This chapter has elucidated a bifurcation mechanism from the mono-centric distribution on a square lattice with ordinary boundaries. We derived bifurcating solutions, including spatial distributions that represent the formation of satellite places surrounding a central place, from the mono-centric distribution on a square lattice by group-theoretic bifurcation analysis. We demonstrated the emergence of theoretically predicted bifurcating solutions from the stable mono-centric distribution by numerical analysis of the FO model and the PS model. We also investigated the influence of the model parameters on the types of bifurcating solutions for these models.

The main contribution of this chapter is to propose a general theory to understand bifurcation behaviour of economic geography models from the mono-centric distribution. It is emphasized that theoretical analysis conducted in this chapter relies only on the symmetry of spatial platforms. Thus, this analysis procedure would be applicable to any economic geography model that takes the mono-centric distribution under the replicator dynamics. It is a future topic to apply such investigation to many other economic geography models.

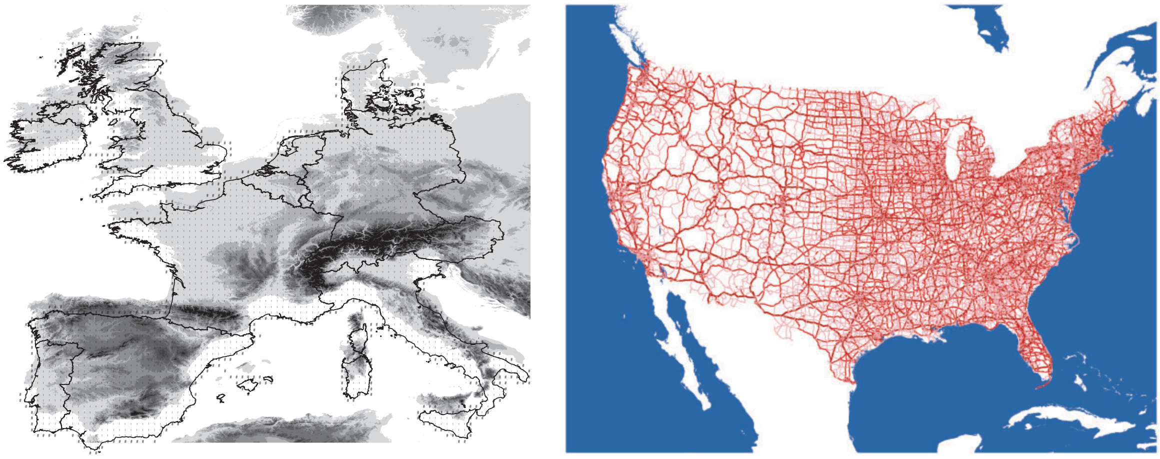


(a) $\gamma = 0.2$



(b) $\gamma = 0.5$

Figure 5.13: The dependence of the type of bifurcating solutions on the values of $(1/\sigma, \mu) \in (0, 1) \times (0, 1)$ for the PS model on the square lattice with 25 places. A blue circle shows the size of population at each place.



(a) Stelder (2005) (b) Allen and Arkolakis (2014)

Figure 6.1: A spatial platform of Europe in Stelder (2005) and that of the U.S. in Allen and Arkolakis (2014).

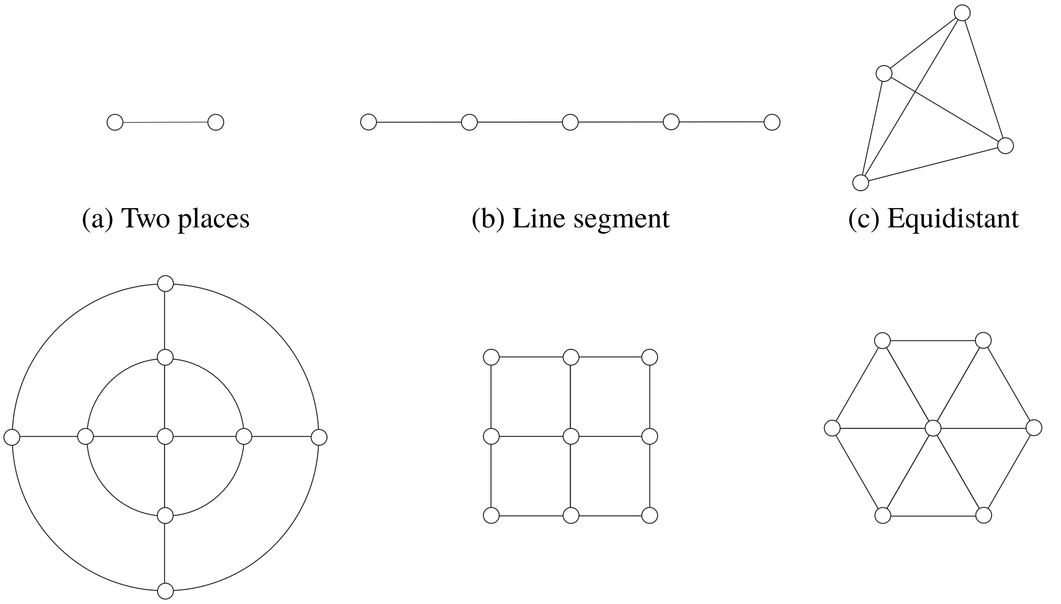
6. Interacting Local and Global Platforms for Economic Geography Models

6.1. Introduction

A hierarchical spatial structure of economic agglomerations comprising countries, cities, towns, and so on, is observed worldwide. Central place theory proposed the geometrical mechanism of the self-organization of such a hierarchical structure (Christaller, 1933; Lösch, 1940) but failed to implement microeconomic mechanisms. Krugman (1991) elucidated the microeconomic mechanism of the emergence of core and periphery places from two identical places, highlighting bifurcation as a catalyst to engender the simplest two-level hierarchy. Economic geography models mushroomed thereafter but mostly dealt with two places that is too simple to represent such a hierarchical structure. Qualitative spatial economics (Redding and Rossi-Hansberg, 2017) has been developed to deal with a realistic spatial platform with a large number of places, but does not necessarily have insightful bifurcation mechanisms.

Figure 6.1 shows a spatial platform of Europe in Stelder (2005) and that of the U.S. in Allen and Arkolakis (2014). Stelder (2005) used a grid of land points in Europe and conducted a simulation of agglomeration. Allen and Arkolakis (2014) used a geography based on the data of highway, rail, and navigable water networks in the U.S. and estimated the topography of trade costs, productivities, and amenities. Sheard (2021) studied the influence of the network of airports in the U.S on employment. Such spatial platforms with irregular and asymmetric networks can express detailed and complicated geometries but rely too heavily on numerical analysis.

This chapter aims to develop a spatial platform that can present a hierarchical structure but can still retain the insightfulness of a bifurcation mechanism. We consider a two level hierarchy of global and local systems. A global system is made up of a system of cities and expresses the geographical distribution of cites. Each city has a micro structure comprising a system of local places and has its particular population size and geography. The number of grid points in a local system is used to index the amount of mobile population of a city, and the distribution of these points to express its geographical properties.



(a) Two places (b) Line segment (c) Equidistant

(d) Radial network (e) Square lattice (f) Hexagonal lattice

Figure 6.2: Examples of spatial platforms for economic geography models. Each node represents a place to locate, and each edge indicates a transportation link.

Candidates of local and/or global spatial platforms are depicted in Fig. 6.2. The two places in (a) is most popular but does not have much spatial structure. The line segment in (b) expresses a chain of cities. The equidistant economy in (c) represents a system of cities connected each other by airplanes. The radial network in (d) can be seen in many traditional cities such as Paris. The square and hexagonal lattices in (e) and (f), respectively, are suitable in modeling densely and regularly distributed locations.

In the selection of a local spatial platform, it is to be noted that square road networks prosper worldwide. Chicago (the U.S.) and Kyoto (Japan), for example, are well-known to accommodate such square networks historically (see Fig. 6.3). Accordingly, this chapter employs a square lattice as a local spatial platform, whereas a hexagonal lattice network would be suitable in other cases. In fact, several studies of spatial economic agglomerations have been conducted on square lattices

(a) Chicago (the U.S.) (b) Kyoto (Japan)

Figure 6.3: Satellite photographs of cities provided by the Google Map displaying square road networks.

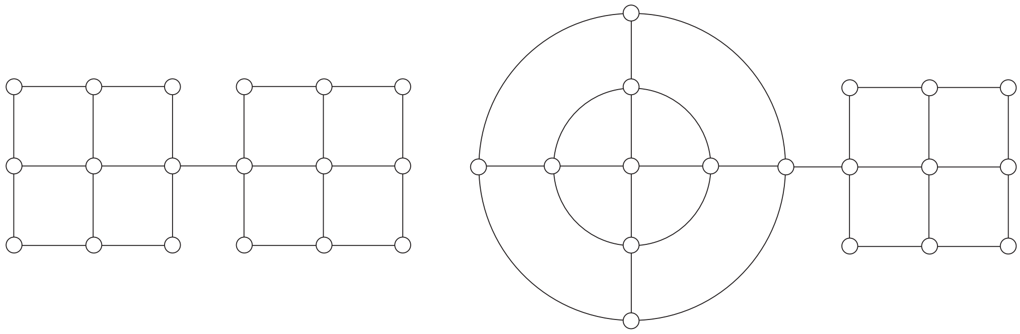


Figure 6.4: Systems comprising two local platforms connected at the borders. A group G represents the symmetry of a system.

(Clarke and Wilson, 1983, 1985; Weidlich and Haag, 1987; Munz and Weidlich, 1990; Brakman et al., 1999).

As a global spatial platform, this chapter employs an equidistant economy. This economy is welcomed as a simplifying assumption and is popular in spatial economics (Puga, 1999; Tabuchi et al., 2005; Bosker et al., 2010; Gaspar et al., 2018, 2020). Break and sustain bifurcations of an equidistant economy with arbitrary many regions were studied in Aizawa et al. (2020) extending the analysis for break bifurcation for the symmetric group S_N for N objects (Golubitsky and Stewart, 2002; Elmihirst, 2004).

We intend to investigate the bifurcation behaviour of economic geography models on local-global systems. In numerical analysis for the demonstration of the performance of local-global systems, we use the FO model (Forslid and Ottaviano, 2003) as an example.

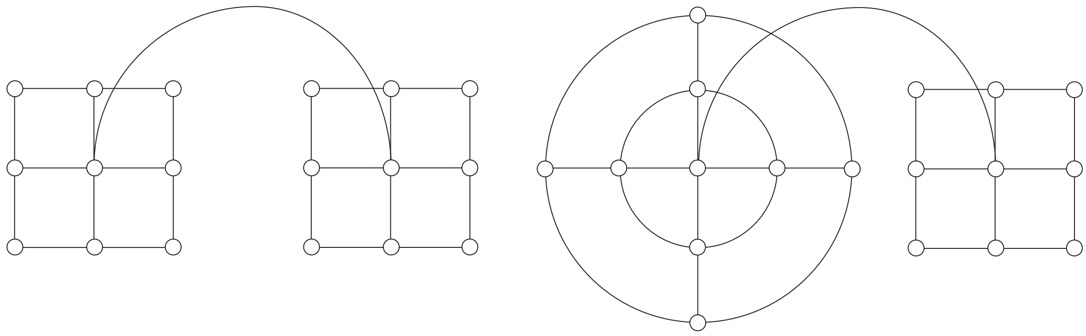
This chapter is organized as follows. The extension of spatial platforms for economic geography models is discussed in Section 6.2. A local-global system with two identical local platforms is introduced, and its symmetry is explained in Section 6.3. Systems with different local platforms are treated in Section 6.4.

6.2. Extension of Spatial Platforms for Economic Geography Models

In this section, we discuss the extension of spatial platforms, which is applicable to hub airports in the U.S. that is presented as a future target in Section 6.5, for conventional economic geography models. We explain the concept of local-global systems for realistic modeling of global transportation networks.

As a first step, we consider systems that are made up of two local platforms. The left of Fig. 6.4 depicts a system comprising two identical local platforms that are connected at the borders. The right of Fig. 6.4 depicts a system with two different ones. This kind of connections reduces the symmetry of the whole space. The symmetry of such systems is labeled by the dihedral groups $G \simeq D_2$ and $G \simeq D_1$ with simple structures. Here, D_N represents the N-dimensional dihedral group.

We next introduce hierarchical spatial platforms, called local-global systems, by connecting the centers of the local platforms as depicted in Fig. 6.5. These systems can describe economic interactions between local and global scales. Such a way of connection retains the symmetry



$G \simeq (\mathrm{D}_4 \times \mathrm{D}_4) \dotplus \mathrm{S}_2$ $\qquad$ $G \simeq \mathrm{D}_4 \times \mathrm{D}_4$

Figure 6.5: Systems comprising two local platforms connected at the centers (local-global systems). A group G represents the symmetry of a system.

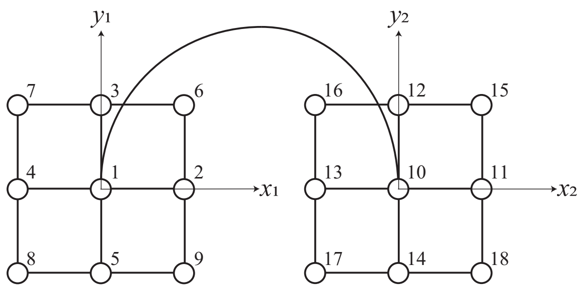


Figure 6.6: A local-global system with two identical square lattices. A number associated with each node represents the label of each place.

of each local platform and provides rich bifurcation mechanisms. The group G describing the symmetry is represented by larger groups $G \simeq (\mathrm{D}_4 \times \mathrm{D}_4) \dotplus \mathrm{S}_2$ and $G \simeq \mathrm{D}_4 \times \mathrm{D}_4$. Here, S_2 represents the two-dimensional symmetric group.

Note that radial and square lattices are suitable to represent local transportation networks in France (Paris) and Germany, respectively. We would like to employ systems comprising these two different lattices in Section 6.4 as extended examples.

6.3. Local-global System with Two Identical Local Platforms

In this section, we consider two identical local platforms. We employ the general framework of economic geography models with the replicator dynamics that was introduced in Chapter 2. As candidates of stable equilibria of the system, we obtain invariant patterns that were explained in Chapter 4.

6.3.1. Symmetry of the System

We consider the local-global system in Fig. 6.6 that is made up of two identical square lattices, which represent intra-regional (local scale) transportation networks such as local roads and railroads. The centers of the square lattices are connected by an inter-regional (global scale) transportation networks such as a high-speed train or an airplane. A set of 18 places are allocated at the nodal points.

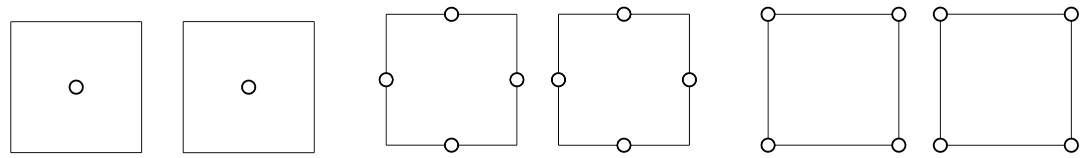


Figure 6.7: An orbit decomposition of the local-global system with two identical square lattices with respect to $G \simeq (\mathrm{D}_4 \times \mathrm{D}_4) \dotplus \mathrm{S}_2$.

The symmetry of this local-global system is described by the group

$$G = (\mathrm{G}_1 \times \mathrm{G}_2) \dotplus \mathrm{G}_3 \simeq (\mathrm{D}_4 \times \mathrm{D}_4) \dotplus \mathrm{S}_2. \tag{6.1}$$

A group G_1 is isomorphic to D_4 and is described as

$$\mathrm{G}_1 = \{e_1, r_1, {r_1}^2, {r_1}^3, s_1, s_1 r_1, s_1 {r_1}^2, s_1 {r_1}^3\}, \tag{6.2}$$

where e_1 is the identity transformation, s_1 is the reflection with respect to the x_1-axis, and ${r_1}^j$ is a counterclockwise rotation about the origin $(x_1, y_1) = (0, 0)$ by an angle of $\pi j/2$ $(j = 0, 1, 2, 3)$. A group G_2 is also isomorphic to D_4 and is described as

$$\mathrm{G}_2 = \{e_2, r_2, {r_2}^2, {r_2}^3, s_2, s_2 r_2, s_2 {r_2}^2, s_2 {r_2}^3\}, \tag{6.3}$$

where e_2 is the identity transformation, s_2 is the reflection with respect to the x_2-axis, and ${r_2}^j$ is a counterclockwise rotation about the origin $(x_2, y_2) = (0, 0)$ by an angle of $\pi j/2$ $(j = 0, 1, 2, 3)$. A group G_3 is isomorphic to S_2 and is described as

$$\mathrm{G}_3 = \{e_3, s_3\} \simeq \mathrm{S}_2, \tag{6.4}$$

where s_3 is the permutation among the coordinates $(x_1, y_1, x_2, y_2) \mapsto (x_2, y_2, x_1, y_1)$, and e_3 is the identity transformation.

A set of the nodal points is decomposed into disjoint subsets, called orbits for a subgroup of G. For example, Figure 6.7 depicts an orbit decomposition with respect to $G \simeq (\mathrm{D}_4 \times \mathrm{D}_4) \dotplus \mathrm{S}_2$. In Fig. 6.7, places belonging to the same orbit are labeled by the same symbol $\circ$.

6.3.2. Invariant Patterns

Recall the concept of invariant patterns for the replicator dynamics in Section 4.3. For the present local-global system, an identical complete agglomeration to places in the same orbit with respect to a subgroup of G becomes an invariant pattern. An orbit decomposition with respect to $G \simeq (\mathrm{D}_4 \times \mathrm{D}_4) \dotplus \mathrm{S}_2$ was shown in Fig. 6.7.

Conducting orbit decompositions with respect to all subgroups of G, we can obtain invariant patterns for this local-global system exhaustively. This local-global system has 18 invariant patterns depicted in Fig. 6.8. Through numerical stability analysis of the FO model to be conducted in Section 6.3.3, it turns out that the mono-centric distribution at $i = 1$ or 10 (at the center of a square lattice) and the duo-centric one at $i = 1$ and 10 are superior in stability.

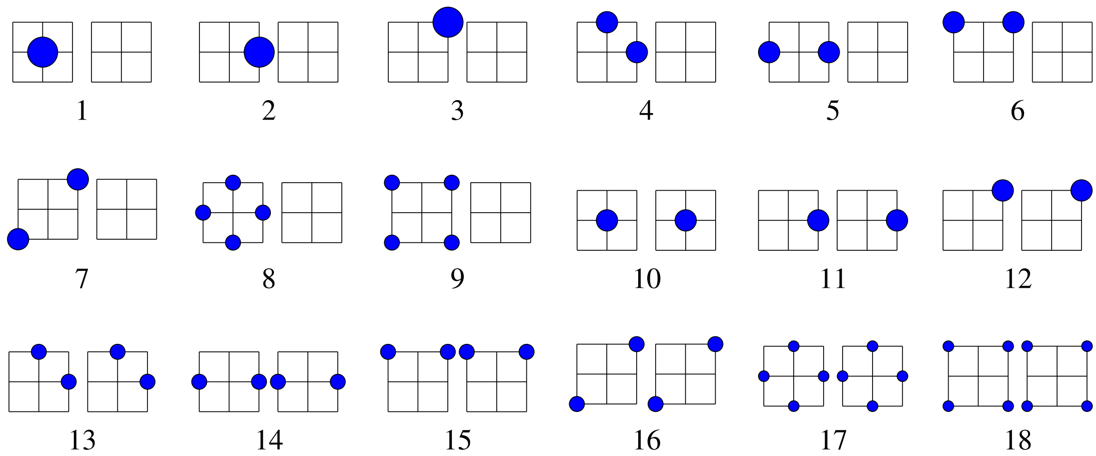


Figure 6.8: A list of invariant patterns for the local-global system with two identical square lattices. The size of a blue circle represents the size of population at each place.

6.3.3. Bifurcation Behaviour of Forslid and Ottaviano (2003) Model

We investigate by numerical analysis the bifurcation behaviour of the local-global system with two identical square lattices for the FO model (Forslid and Ottaviano, 2003). Recall a general framework of the FO model that was introduced in Section 2.3. We introduce two kinds of bifurcation parameters, ϕ_{local} and ϕ_{global}, which represent trade freeness of local and global scales, respectively. We investigate the stability of invariant patterns for the whole set $(\phi_{\mathrm{local}}, \phi_{\mathrm{global}}) \in (0,1] \times (0,1]$ of the two parameters to find stable equilibria in the space of $(\phi_{\mathrm{local}}, \phi_{\mathrm{global}})$. We obtain bifurcating solution curves emanating from the equilibrium curves of these invariant patterns to observe the transition of stable equilibria.

Basic Assumptions

We label nodal number of places as $I_1 = \{1, \ldots, 9\}$ for the square lattice at the left and $I_2 = \{10, \ldots, 18\}$ for that at the right for the present local-global system. We set the transportation cost τ_{ij} in (2.11) as follows:

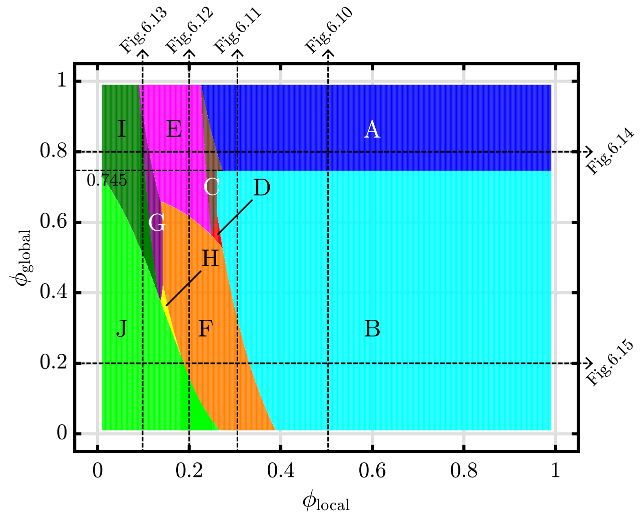
$$\tau_{ij} = \begin{cases} \exp[m(i,j)\tau_{\mathrm{local}}] & \text{for } i, j \in I_k \ (k = 1, 2), \\ \exp[(m(i,1) + m(10,j))\tau_{\mathrm{local}} + m(1,10)\tau_{\mathrm{global}}] & \text{for } i \in I_1, \ j \in I_2, \end{cases} \tag{6.5}$$

$$\tau_{ji} = \tau_{ij}. \tag{6.6}$$

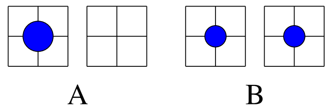
Here, τ_{local} and τ_{global} represent transportation cost parameters for intra-regional and inter-regional transportation, respectively; $m(i, j)$ denotes the shortest distance between places i and j. We choose the nominal length of the two local platforms as the unity, i.e., $m(1, 2) = 1$, and set the distance between the centers of the two local platforms to be also the unity, i.e., $m(1, 10) = 1$. Other distances $m(i, j)$'s within each lattice follow geometrically.

Define the spatial discounting factor d_{ij} as

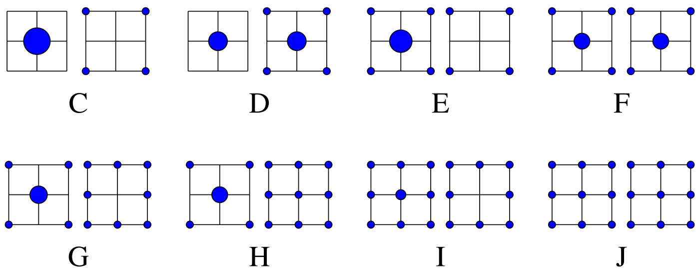
$$d_{ij} = \tau_{ij}{}^{-(\sigma-1)}. \tag{6.7}$$



(a) The stability areas of spatial distributions



(b) Stable invariant patterns



(c) Stable non-invariant patterns

Figure 6.9: The stability areas of spatial distributions that become stable for some $(\phi_{\mathrm{local}}, \phi_{\mathrm{global}})$.

Then, d_{ij} is evaluated to

$$d_{ij} = \begin{cases} {\phi_{\mathrm{local}}}^{m(i,j)} & \text{for } i, j \in I_k \ (k = 1, 2), \\ {\phi_{\mathrm{local}}}^{m(i,1)} {\phi_{\mathrm{local}}}^{m(10,j)} {\phi_{\mathrm{global}}}^{m(1,10)} & \text{for } i \in I_1, \ j \in I_2, \end{cases} \tag{6.8}$$

where

$$\phi_{\mathrm{local}} = \exp[-(\sigma - 1)\tau_{\mathrm{local}}],$$
$$\phi_{\mathrm{global}} = \exp[-(\sigma - 1)\tau_{\mathrm{global}}].$$

We use ϕ_{local} and ϕ_{global} as the bifurcation parameters.

Numerical Simulations

We conduct numerical bifurcation and stability analysis of the FO model on the present local-global system. We choose parameter values of the FO model as $(\sigma, \mu) = (6.0, 0.4)$.

Figure 6.9 shows the stability areas of spatial distributions that become stable for some $(\phi_{\mathrm{local}}, \phi_{\mathrm{global}})$. Among all the invariant patterns in Fig. 6.8, only two patterns, the mono-centric

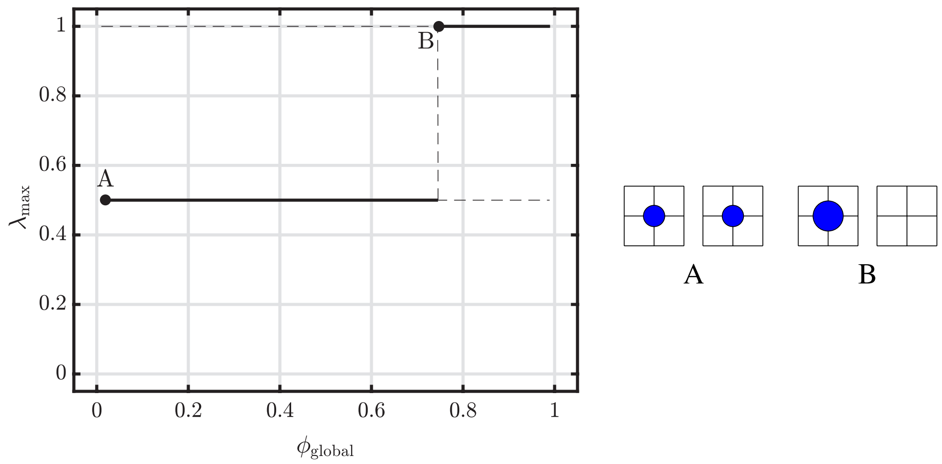


Figure 6.10: Equilibrium curves for $\phi_{\text{local}} = 0.5$. The vertical axis shows $\lambda_{\max} = \max(\lambda_1, \ldots, \lambda_{18})$. Solid curves represent stable equilibria, and dashed ones represent unstable ones.

distribution at the center of a square lattice and the duo-centric one at the centers of the square lattices, are stable for some $(\phi_{\text{local}}, \phi_{\text{global}})$. The stability areas of these two invariant patterns are disjoint. When ϕ_{local} is relatively high ($\phi_{\text{local}} > 0.389$), either the mono-centric or the duo-centric distribution is stable for any ϕ_{global}. A stable invariant pattern shifts from the duo-centric distribution to the mono-centric one at $\phi_{\text{global}} = 0.745$ as ϕ increases from 0. When ϕ_{local} is low ($\phi_{\text{local}} < 0.227$), there is no stable invariant pattern for any ϕ_{global}.

With Fig. 6.9 in mind, we first fix the local trade freeness ϕ_{local} to some particular values and investigate the transition of stable equilibria when the global trade freeness ϕ_{global} increases. Figure 6.10 shows equilibrium curves for $\phi_{\text{local}} = 0.5$. In this case, stable equilibria consist of two invariant patterns: mono-centric and duo-centric distributions. A curve of an unstable non-invariant solution connects these two invariant patterns. Such bifurcation behaviour is similar to that observed for the two-region economy (Krugman, 1991). This behaviour can be seen to prevail for $\phi_{\text{local}} > 0.5$ from Fig. 6.9.

Figure 6.11 shows equilibrium curves for $\phi_{\text{local}} = 0.3$. The equilibrium curves for $\phi_{\text{global}} > 0.355$ are similar to those for the case of $\phi_{\text{local}} = 0.5$ in Fig. 6.10. For $\phi_{\text{global}} < 0.355$, we see the emergence of satellite places at the corners of the two square lattices (cf., the state A). A bifurcation occurs from the duo-centric distribution at the point B of $\phi_{\text{global}} = 0.355$. The population agglomerates from the corners to the centers. Such agglomeration of population cannot be expressed by the two-region economy. It shows the importance of the use of square lattices as local platforms.

Figure 6.12 shows equilibrium curves for $\phi_{\text{local}} = 0.2$. In this case, no invariant pattern is stable, and some non-invariant patterns are stable. In the search of equilibrium curves, we employ a nearly uniform distribution as an initial distribution (cf., the state A). The population at the centers of the two square lattices increases along the curve ABB′ as ϕ_{global} increases. A bifurcation occurs at the point B′ of $\phi_{\text{global}} = 0.616$. Then, the population at the center of a square lattice becomes zero, while the population at the center of another square lattice increases (cf., the state C).

Figure 6.13 shows equilibrium curves for $\phi_{\text{local}} = 0.1$. Similarly to the case of $\phi_{\text{local}} = 0.2$,

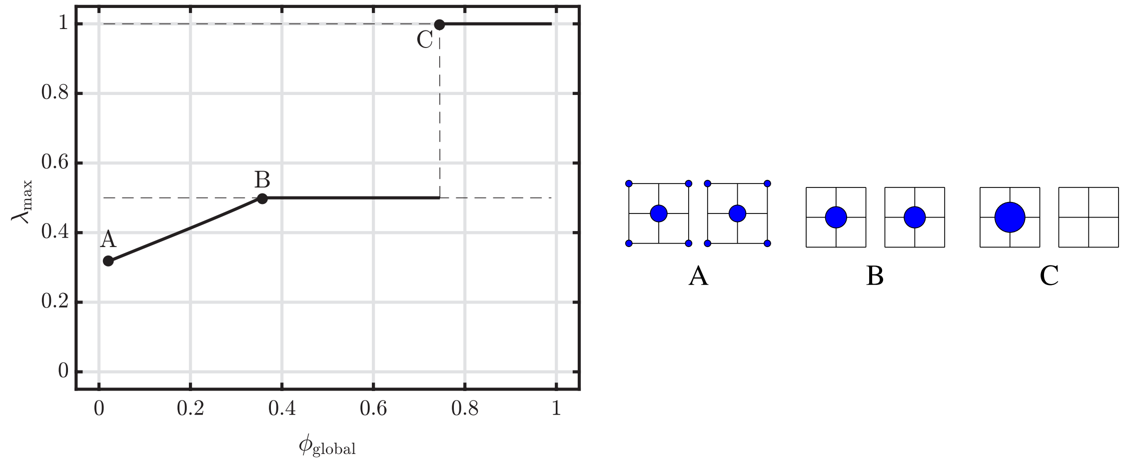


Figure 6.11: Equilibrium curves for $\phi_{\text{local}} = 0.3$. The vertical axis shows $\lambda_{\text{max}} = \max(\lambda_1, \ldots, \lambda_{18})$. Solid curves represent stable equilibria, and dashed ones represent unstable ones.

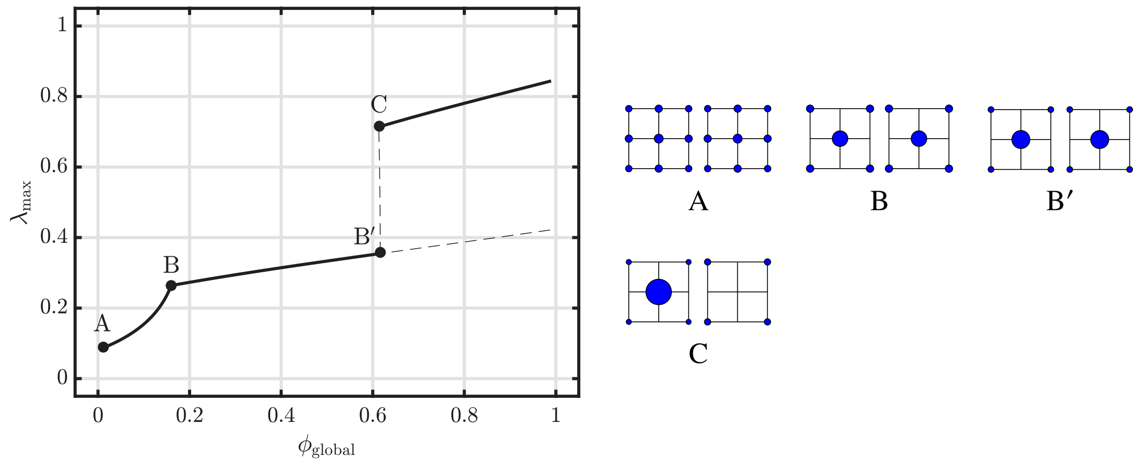


Figure 6.12: Equilibrium curves for $\phi_{\text{local}} = 0.2$. The vertical axis shows $\lambda_{\text{max}} = \max(\lambda_1, \ldots, \lambda_{18})$. Solid curves represent stable equilibria, and dashed ones represent unstable ones.

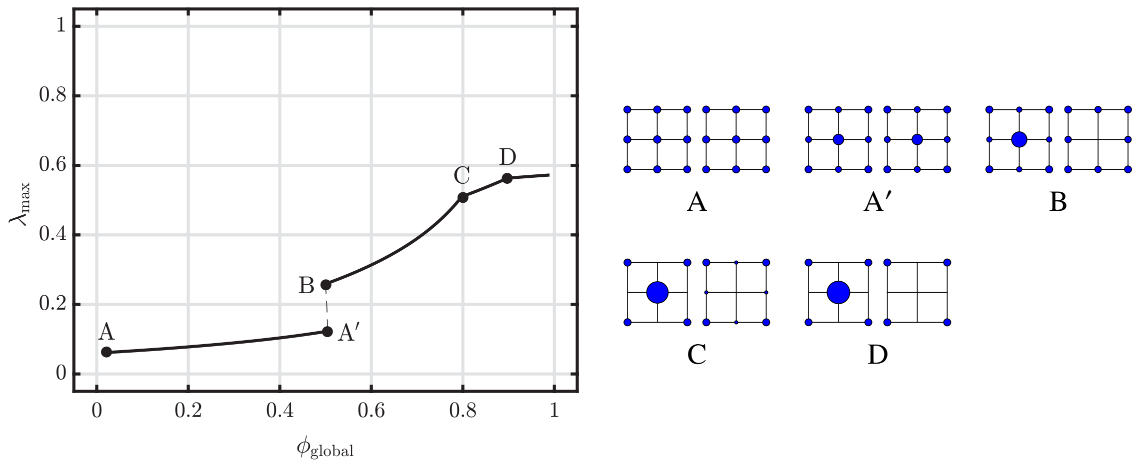


Figure 6.13: Equilibrium curves for $\phi_{\text{local}} = 0.1$. The vertical axis shows $\lambda_{\max} = \max(\lambda_1, \ldots, \lambda_{18})$. Solid curves represent stable equilibria, and dashed ones represent unstable ones.

no invariant pattern is stable. In the early state A, a nearly uniform distribution is stable. At the point A′ of $\phi_{\text{global}} = 0.504$, a bifurcation occurs. The population at the center of a square lattice becomes zero, while the population at the center of another lattice increases (cf., the state B). The number of agglomerated places decreases along the curve BCD as ϕ_{global} increases.

The results depicted in Figs. 6.10–6.13 imply that agglomeration behavior varies greatly with the value of ϕ_{local}. Note that the agglomeration behaviour on this local-global system is similar to that on the two-region economy when ϕ_{local} is high (cf., Figs. 6.10 and 6.11). When ϕ_{local} is low (cf., Figs. 6.12 and 6.13), the imbalance between the two local platforms becomes predominant due to the relative superiority of the global trade.

We next fix the global trade freeness ϕ_{global} to some particular values and examine the transition of stable equilibria when the local trade freeness ϕ_{local} increases. Figure 6.14 shows equilibrium curves for $\phi_{\text{global}} = 0.8$. For any value of ϕ_{local}, the population at the center of a square lattice is zero. The population at the center of another square lattice increases as ϕ_{local} increases. At the point E of $\phi_{\text{local}} = 0.261$, the mono-centric distribution becomes stable and remains stable thereafter.

Figure 6.15 shows equilibrium curves for $\phi_{\text{global}} = 0.2$. In contrast to the case of $\phi_{\text{global}} = 0.8$ in Fig. 6.14, the population at the two square lattices remains at some place for any value of ϕ_{local}. The population at the centers of square lattices increases as ϕ_{local} increases. At the point C of $\phi_{\text{local}} = 0.332$, the duo-centric distribution becomes stable.

The results depicted in Figs. 6.14 and 6.15 imply that the values of ϕ_{global} affect the difference of spatial distributions at each local platform. When ϕ_{global} is high, population at each square lattice shows different spatial distributions. We see that the mono-centric distribution at the center of a square lattice becomes stable. When ϕ_{global} is low, population at each square lattice shows identical spatial distributions. We see that the duo-centric distribution at the centers of the two square lattices becomes stable. It is noteworthy that a large ϕ_{global} accelerates the imbalance between the two local systems.

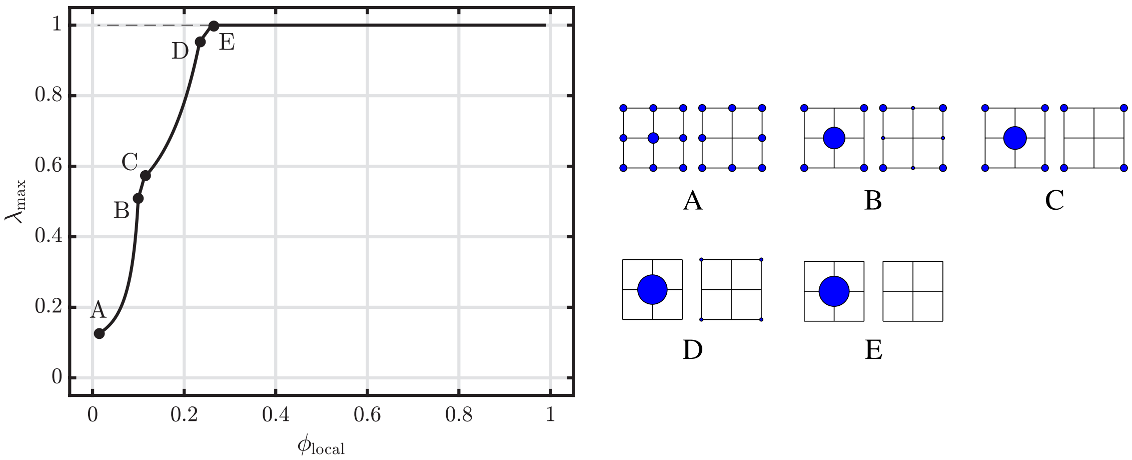


Figure 6.14: Equilibrium curves for $\phi_{\text{global}} = 0.8$. The vertical axis shows $\lambda_{\max} = \max(\lambda_1, \ldots, \lambda_{18})$. Solid curves represent stable equilibria, and dashed ones represent unstable ones.

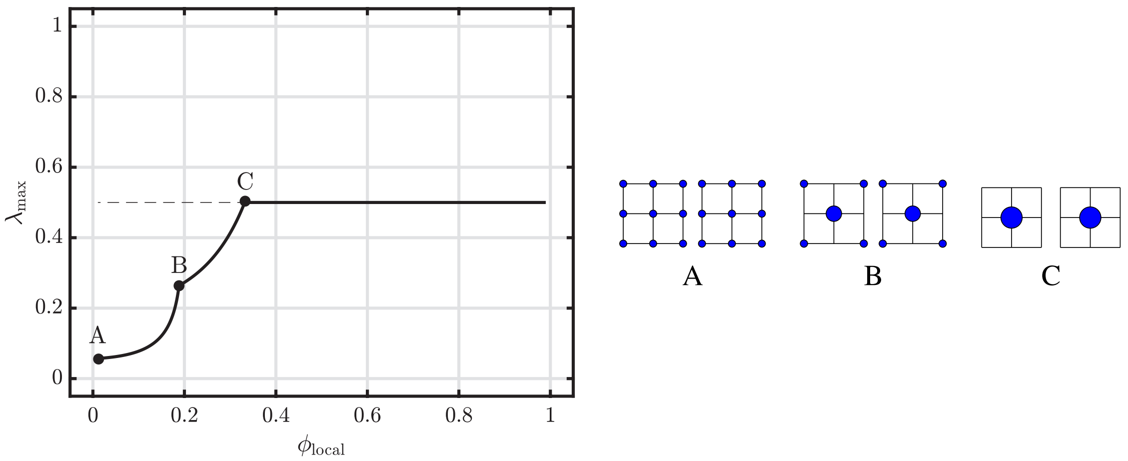


Figure 6.15: Equilibrium curves for $\phi_{\text{global}} = 0.2$. The vertical axis shows $\lambda_{\max} = \max(\lambda_1, \ldots, \lambda_{18})$. Solid curves represent stable equilibria, and dashed ones represent unstable ones.

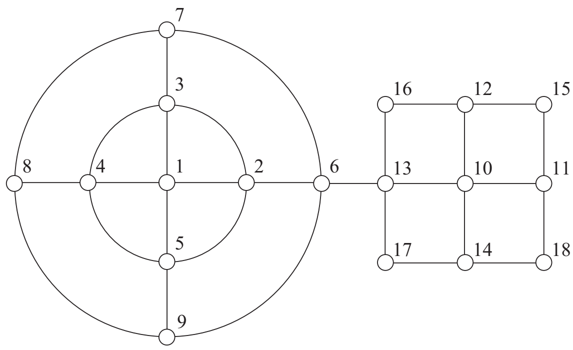


Figure 6.16: A system comprising two different local platforms connected at the borders proposed by J.-F. Thisse to K. Ikeda. A number associated with each node represents the label of each place.

6.4. Systems with Two Different Local Platforms

In this section, we consider systems comprising two different local platforms, which were discussed in Section 6.2. We show the usefulness of the bifurcation mechanism of a local-global system with reference to numerical simulations of the FO model.

6.4.1. Connection at the Borders

We introduce a spatial platform where the borders of the two local platforms are connected as depicted in Fig. 6.16. Such a simple way of connection has been used widely in spatial economics. Figure 6.17 shows equilibrium curves for the FO model on this spatial platform. Continuing from the previous section, we chose the parameter values as $\sigma = 6.0$ and $\mu = 0.4$. In the early state A with low trade freeness, the population distributes almost uniformly. As the trade freeness ϕ increases, the population agglomerates mostly at places $i = 1$ and 10, and 13 in the state J (cf., Fig. 6.16 for node numbers). Along the curve JKL, places $i = 1$ and 10 lose their population. At the end, the mono-centric distribution at $i = 13$ (the left border of the square lattice at the right) becomes stable.

6.4.2. Connection between the Centers: Local-global System

We introduce a local-global system with two different local platforms the centers of which are connected directly as depicted in Fig. 6.18. This system retains the geometrical symmetry of each local platform unlike the system with simple connection in Section 6.4.1. Thus, there exist invariant patterns and associated bifurcation mechanisms. Continuing from the previous section, we use the FO model ($\sigma = 6.0$, $\mu = 0.4$) for numerical bifurcation and stability analysis. We set the distances as $m(1, 2) = 1$, $m(1, 10) = 1$, $m(10, 11) = 1$, and the other distances follow geometrically.

Figure 6.19 depicts all invariant patterns for this local-global system. Almost all patterns correspond to those of each local platform. Note that the duo-centric distribution at $i = 1$ and 10 becomes an invariant pattern since places $i = 1$ and 10 are in the same geometrical connectivity with other places.[17]

[17] Note that Proposition 4.1 in Section 4.3 provides sufficient conditions for invariant patterns based only on their geometrical configurations although it is not obvious geometrically.

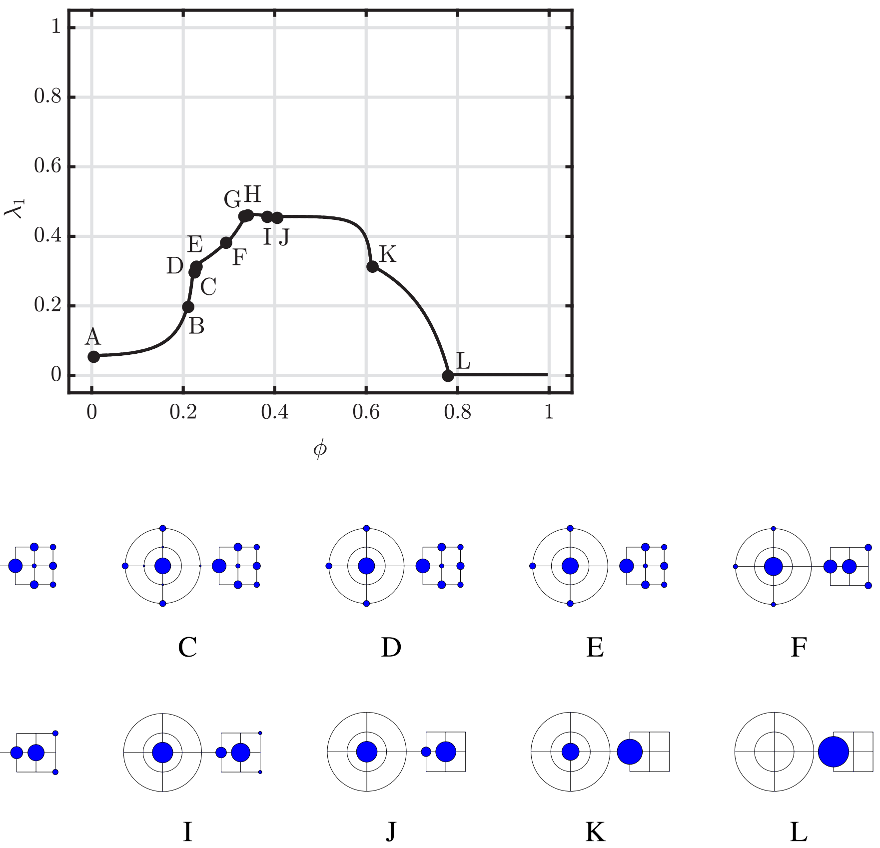


Figure 6.17: Equilibrium curves for the connection of two different local platforms at the borders. The vertical axis shows the size of population at $i = 1$ (the center of the radial network). Solid curves represent stable equilibria.

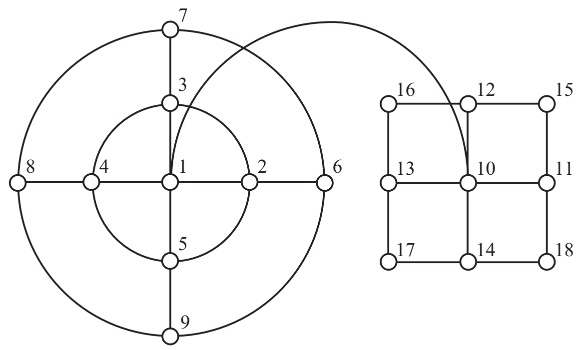


Figure 6.18: A local-global system with two different local platforms. A number associated with each node represents the label of each place.

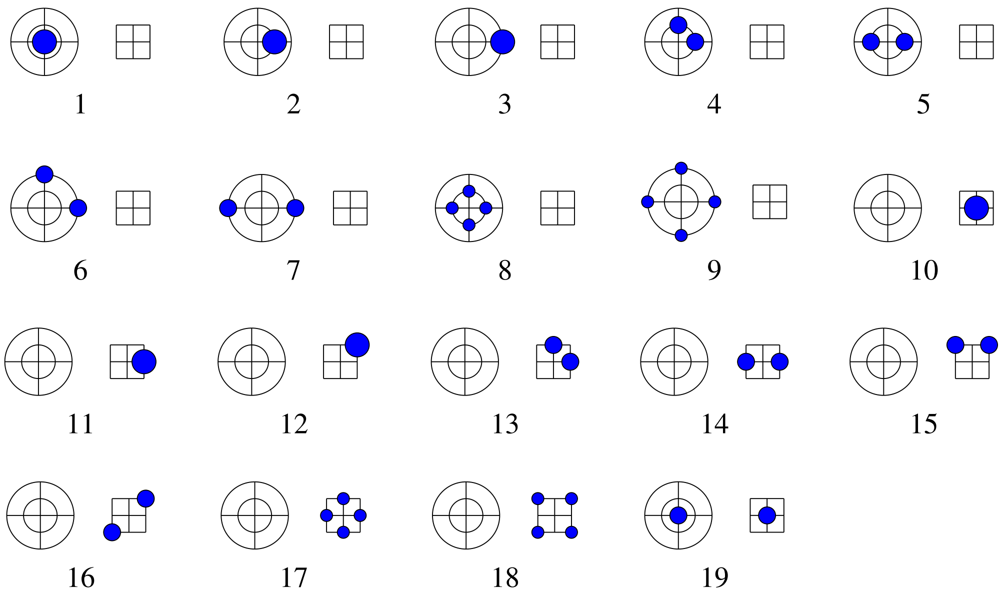


Figure 6.19: A list of invariant patterns for the local-global system with two different local platforms. The size of a blue circle represents the size of population at each place.

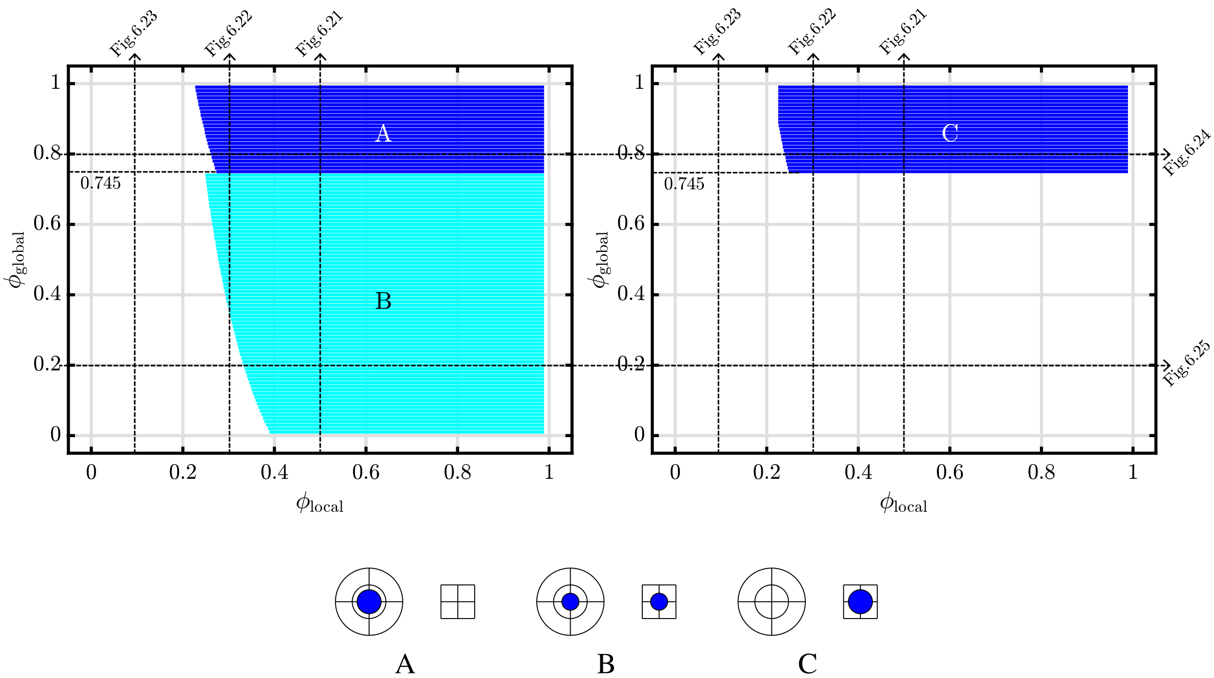


Figure 6.20: The stability areas of invariant patterns with local square symmetry that become stable for some $(\phi_{\text{local}}, \phi_{\text{global}})$.

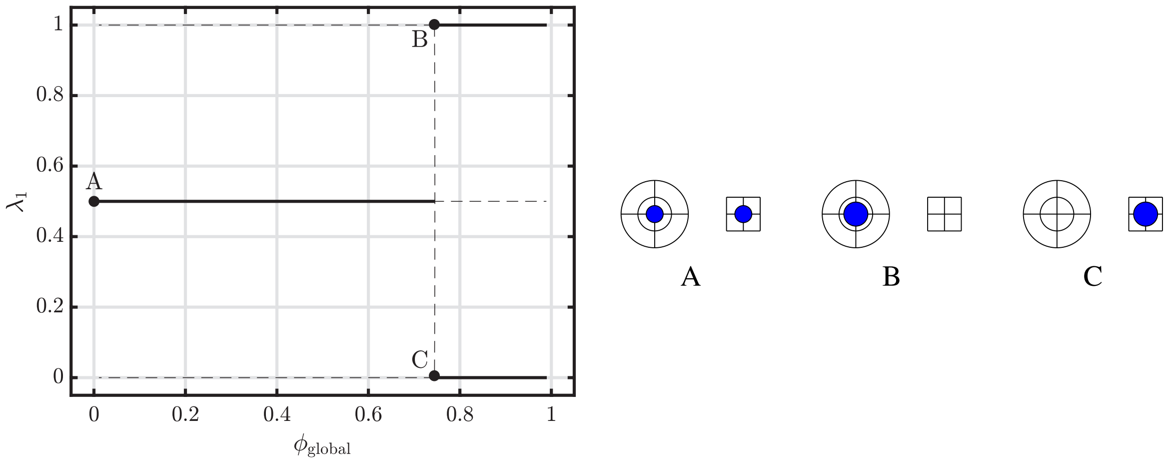


Figure 6.21: Equilibrium curves for $\phi_{\text{local}} = 0.5$. The vertical axis shows the size of population at $i = 1$ (the center of the radial network). Solid curves represent stable equilibria, and dashed ones represent unstable ones.

To find stable equilibria for this local-global system, we focus on invariant patterns in Fig. 6.19 and investigate the stability of these patterns. Among all these patterns, only three patterns, the mono-centric distribution at $i = 1$, the mono-centric one at $i = 10$, and the duo-centric one at $i = 1$ and 10, are stable for some $(\phi_{\text{local}}, \phi_{\text{global}})$. Figure 6.20 shows the stability areas of these three patterns. Note that the stability areas of the two mono-centric distributions and that of the duo-centric one are disjoint and is separated by the horizontal line at $\phi_{\text{global}} = 0.745$.

We fix the local trade freeness ϕ_{local} to some particular values to investigate the influence of the local trade freeness ϕ_{global} on the progress of stable equilibria. Figures 6.21–6.23 show equilibrium curves for $\phi_{\text{local}} = 0.5, 0.3$, and 0.1, respectively. When ϕ_{local} is high (cf., Figs. 6.21 and 6.22), the mono-centric distributions and the duo-centric one are stable for a wide range of ϕ_{global} like the two-region economy. When ϕ_{local} is low (cf., Fig. 6.23), the population agglomerates to the center of the radial network as ϕ_{global} increases; this shows the relative superiority of this center for ϕ_{local} small. Such bifurcation behaviour is quite similar to that of the system with two identical square lattices in Section 6.3.3.

We fix the global trade freeness ϕ_{global} to some particular values to examine the influence of the local trade freeness ϕ_{local} on the progress of stable equilibria. Figures 6.24 and 6.25 show equilibrium curves for $\phi_{\text{global}} = 0.8$ and 0.2, respectively. When ϕ_{global} is high (cf., Fig. 6.24), we find two kinds of stable equilibrium paths. The path ABCDE represents transition of equilibria from a spatial distribution where the center of the square lattice attracts population to the mono-centric distribution at the center of the square lattice. The path FGH shows transition of equilibria from a spatial distribution where the center of the radial network gains large population to the mono-centric distribution at the center of the radial network. When ϕ_{global} is low (cf., Fig. 6.25), population at each local platform shows similar behaviour. The duo-centric distribution at the centers of the two local platforms becomes stable for a wide range of ϕ_{local}. Similarly to the case of the system with two identical square lattices in Section 6.3.3, a large ϕ_{global} facilitates the imbalance between the two local systems.

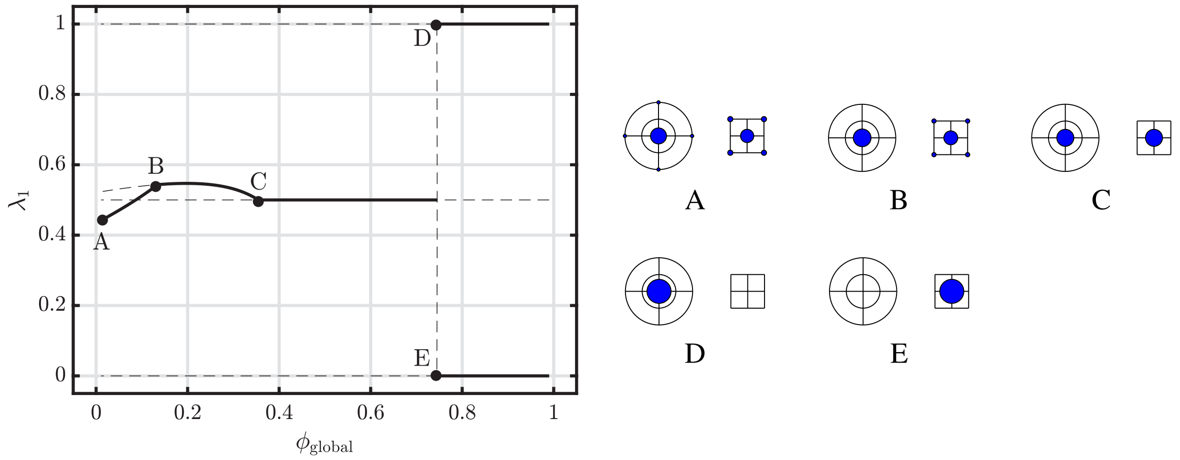


Figure 6.22: Equilibrium curves for $\phi_{\text{local}} = 0.3$. The vertical axis shows the size of population at $i = 1$ (the center of the radial network). Solid curves represent stable equilibria, and dashed ones represent unstable ones.

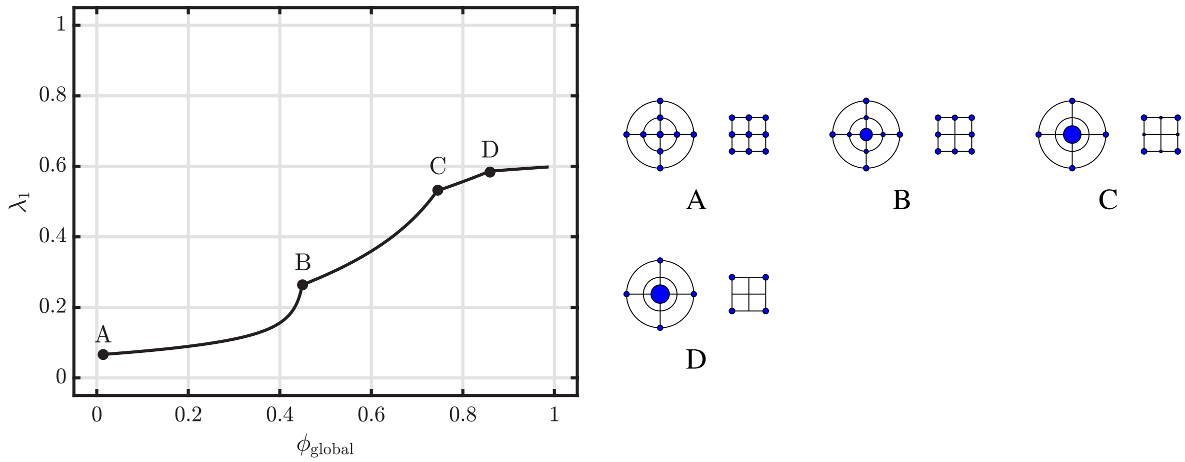


Figure 6.23: Equilibrium curves for $\phi_{\text{local}} = 0.1$. The vertical axis shows the size of population at $i = 1$ (the center of the radial network). Solid curves represent stable equilibria.

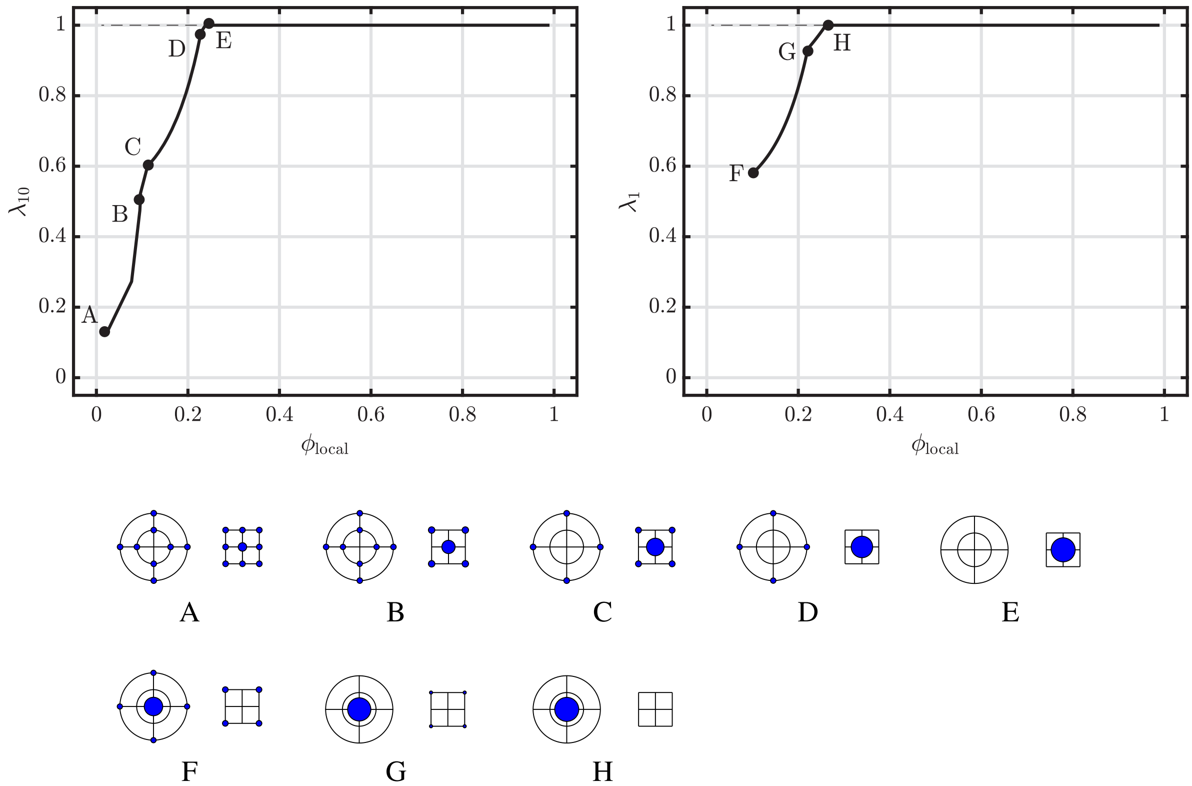


Figure 6.24: Equilibrium curves for $\phi_{\text{global}} = 0.8$. The vertical axis of a diagram to the left shows the size of population at $i = 10$ (the center of the square lattice), while that of a diagram to the right shows the size of population at $i = 1$ (the center of the radial network). Solid curves represent stable equilibria.

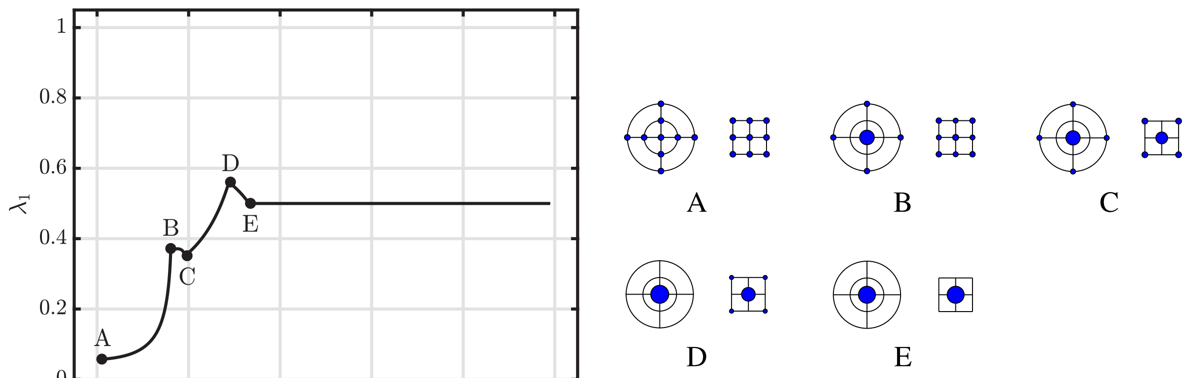


Figure 6.25: Equilibrium curves for $\phi_{\text{global}} = 0.2$. The vertical axis shows the size of population at $i = 1$ (the center of the radial network). Solid curves represent stable equilibria.

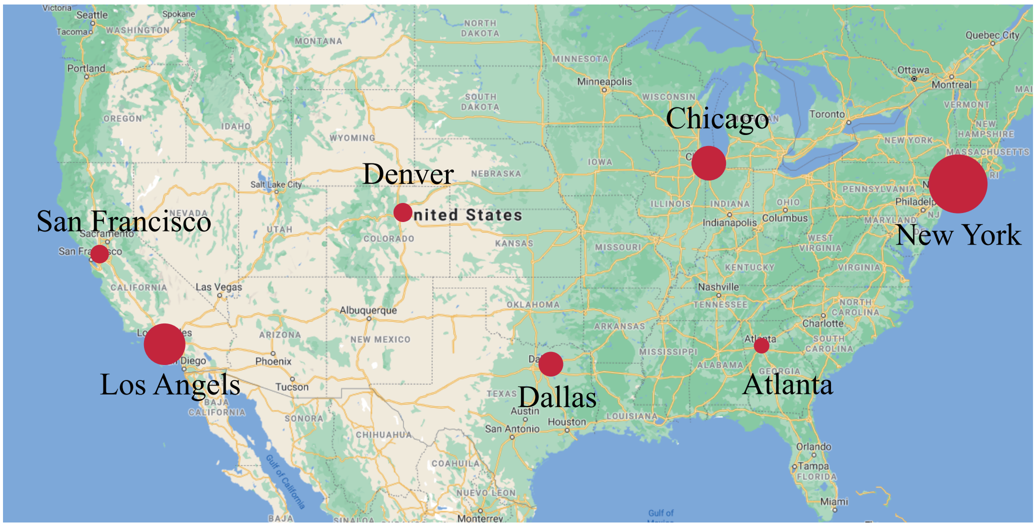


Figure 6.26: Seven largest hub cities in the U.S. (plotted on the Google Map). Population is expressed by the area of the red circles.

6.5. Concluding Remarks

This chapter has introduced local-global systems that can express a hierarchical spatial structure and can retain the insightfulness of bifurcation mechanisms. We obtained invariant patterns on these systems as candidates of stable equilibria. As a specific model for numerical bifurcation and stability analysis, we employed the FO model We considered two kinds of bifurcation parameters, the local and global trade freeness. It turns out that the mono-centric and the duo-centric invariant patterns are stable for wide ranges of the local and global trade freeness.

The main contribution of this chapter is to propose a general framework to explain economic interaction between local and global scales for any economic geography model. This chapter, however, focused on prototype local-global systems that are made up of two local platforms. It is a future topic to consider local-global systems comprising three or more local platforms. Such a research direction is essential to elucidate the mechanism of economic agglomerations on realistic global transportation networks.

Future Topic: Modeling of Hub Airports in the U.S.

We would like to target a direction of the Qualitative Spatial Economics (Redding and Rossi-Hansberg, 2017), which is based on the framework of conventional economic geography models. The spatial structures in mind, for example, are the seven largest hub cities in the U.S. in 2019 that are connected by airlines in Fig. 6.26.

We model a network of hub cities in the U.S. focusing on the seven cities that have seven largest enplaned passengers as listed in Table 6.1(a). The populations of these seven cities are listed in Table 6.1(b) and are normalized using $\lambda^* = 330,000$ as a normalizing constant.

With Table 6.1 in mind, we model the local transportation networks of these seven cities by either a two-places economy or a square lattice economy as follows:

- Square lattice economy (25 places): New York.

Table 6.1: Seven largest hub airports in the U.S. in 2019 by United States Department of Transportation (https://www.bts.gov/content/passengers-boarded-top-50-us-airports) and the population of each city in 2021 by World Population Review (https://worldpopulationreview.com/us-cities)

(a) Enplaned passengers at the seven hub airports

City	Airport	Total enplaned passengers (rank)
Atlanta	Hartsfield-Jackson Atlanta International	53,505,357 (1)
Los Angeles	Los Angeles International	42,965,731 (2)
Chicago	Chicago O'Hare International	40,887,890 (3)
Dallas	Dallas/Fort Worth International	35,785,318 (4)
Denver	Denver International	33,592,645 (5)
New York	John F. Kennedy International	31,123,436 (6)
San Francisco	San Francisco International	27,715,305 (7)

(b) The population of each city

City	Population	Normalized population	Lattice size
Atlanta	524,067	1.6	2
Los Angeles	3,983,540	12.0	3×3
Chicago	2,679,080	8.1	3×3
Dallas	1,347,120	4.1	$1 + 4$
Denver	749,103	2.3	2
New York	8,230,290	24.9	5×5
San Francisco	883,255	2.7	2

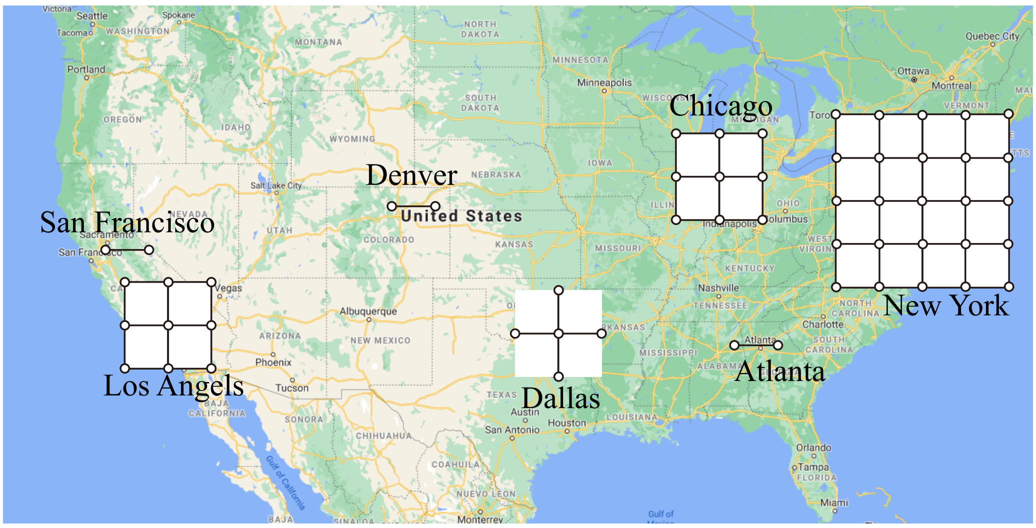


Figure 6.27: Seven largest hubs cities in the U.S. modeled by square lattices and two-places economies (plotted on the Google Map).

- Square lattice economy (9 places): Los Angels and Chicago.
- Square lattice economy (star economy with 1 + 4 places): Dallas.
- Two-places economy: San Francisco, Denver, and Atlanta.

Figure 6.27 depicts square lattices and two-places economies that represent local transportation networks in the U.S. We consider a local-global system where the centers of these local platforms (one of the two places for the two-places economy) are connected equidistantly by inter-regional transportation networks of airplanes. The symmetry of such a local-global system is described by the group

$$G \simeq \mathrm{D}_4 \times \{(\mathrm{D}_4 \times \mathrm{D}_4) \dotplus \mathrm{S}_2\} \times \mathrm{D}_4 \times \{(\mathrm{D}_2 \times \mathrm{D}_2 \times \mathrm{D}_2) \dotplus \mathrm{S}_3\}.$$

Here, S_3 represents the three-dimensional symmetric group.

7. Concluding Remarks

The present thesis developed group-theoretic methods for analyzing economic geography models on a square lattice in collaboration with nonlinear mathematics and spatial economics. Such a methodology provides an effective approach to elucidate the complicated agglomeration behaviour of economic geography models systematically in the light of bifurcation mechanisms.

Chapter 3 provided a group-theoretic bifurcation mechanism from the uniform distribution on an $n \times n$ square lattice, which has the symmetry of the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. We revealed the self-organization of square patterns as bifurcation phenomena in a system of equations modeled on the square lattice. Two different approaches, using the equivariant branching lemma and solving the bifurcation equation, were employed.

We presented in Chapter 3 a complete list of typical bifurcating solutions from the uniform distribution on the square lattice for an arbitrary lattice size n. To demonstrate the emergence of these bifurcating solutions, we conducted numerical analysis of economic geography models. For the FO model, the uniform distribution prevails for small ϕ. For the Fm model, the uniform distribution dominates for large ϕ. For the PS model, the uniform distribution becomes stable when ϕ is close to 0 or 1. All the bifurcating solutions are unstable just after the bifurcation for the FO model and the PS model, while stable bifurcating solutions occur for the Hm model.

Chapter 4 provided a theory of invariant patterns, which are one kind of stationary points of the replicator dynamics. Invariant patterns retain their spatial distribution when the value of the bifurcation parameter changes and display characteristic population distribution.

We proposed in Chapter 4 a methodology to find invariant patterns exhaustively. In view of invariant patterns, we proposed an innovative bifurcation analysis procedure to find stable equilibria: investigating the stability of invariant patterns and searching for bifurcating equilibrium curves that connect stable invariant patterns. We applied this procedure to the FO model and numerically showed the connectivity between bifurcating solutions and invariant patterns via bifurcating solutions from the uniform state. We found a mesh-like structure of the solution curves for stable invariant patterns and unstable non-invariant ones.

Chapter 5 provided a group-theoretic bifurcation mechanism from the mono-centric distribution in a two-dimensional square domain. We obtained bifurcating solutions from the mono-centric distribution by group-theoretic bifurcation analysis. We demonstrated the emergence of such bifurcating solutions by numerical analysis of economic geography models. For the FO model, the mono-centric distribution encounters a bifurcation point as ϕ decreases from 1 to 0. For the PS model, the mono-centric distribution encounters two bifurcation points as ϕ changes. When ϕ increases, a bifurcating solution that represents the emergence of satellite cities emerges. When ϕ decreases, several bifurcating solutions that represent square distributions emerge.

Chapter 6 developed a spatial platform, a local-global system, that can present a hierarchical structure but can still retain the insightfulness of bifurcation mechanisms. We employed a local-global system constructed by two identical square lattices. We introduced two kinds of bifurcating parameter, ϕ_{local} and ϕ_{global}, which represent the local trade freeness (related to transportation in a lattice) and global trade freeness (related to transportation between two lattices), respectively. We demonstrated complicated bifurcation behaviour due to the change of two bifurcation parameters by numerical analysis of the FO model.

A. Appendices for Chapter 3

A.1. *Details of Irreducible Representations*

In Section 3.3, we listed the matrix forms of the irreducible representations of the group G. In this section, we describe a systematic method using little groups to construct these irreducible representations.

A.1.1. *Characters*

We explain the characters of the irreducible representations of G, which play a vital role in the method of little groups (Appendix A.1.2).

One-Dimensional Irreducible Representations

The characters $\chi^\mu(g) = \mathrm{Tr}\, T^\mu(g)$, which are equal to $T^\mu(g)$ for one-dimensional representations, are given as follows:

g	$\chi^{(1;+,+,+)}(g)$	$\chi^{(1;+,-,+)}(g)$	$\chi^{(1;-,+,+)}(g)$	$\chi^{(1;-,-,+)}(g)$
${p_1}^i {p_2}^j$	1	1	1	1
$r {p_1}^i {p_2}^j$	1	1	−1	−1
$r^2 {p_1}^i {p_2}^j$	1	1	1	1
$r^3 {p_1}^i {p_2}^j$	1	1	−1	−1
$s r^m {p_1}^i {p_2}^j$ $(m$: even$)$	1	−1	1	−1
$(m$: odd$)$	1	−1	−1	1

(A.1)

g	$\chi^{(1;+,+,-)}(g)$	$\chi^{(1;+,-,-)}(g)$	$\chi^{(1;-,+,-)}(g)$	$\chi^{(1;-,-,-)}(g)$
${p_1}^i {p_2}^j$	$(-1)^{i+j}$	$(-1)^{i+j}$	$(-1)^{i+j}$	$(-1)^{i+j}$
$r {p_1}^i {p_2}^j$	$(-1)^{i+j}$	$(-1)^{i+j}$	$-(-1)^{i+j}$	$-(-1)^{i+j}$
$r^2 {p_1}^i {p_2}^j$	$(-1)^{i+j}$	$(-1)^{i+j}$	$(-1)^{i+j}$	$(-1)^{i+j}$
$r^3 {p_1}^i {p_2}^j$	$(-1)^{i+j}$	$(-1)^{i+j}$	$-(-1)^{i+j}$	$-(-1)^{i+j}$
$s r^m {p_1}^i {p_2}^j$ $(m$: even$)$	$(-1)^{i+j}$	$-(-1)^{i+j}$	$(-1)^{i+j}$	$-(-1)^{i+j}$
$(m$: odd$)$	$(-1)^{i+j}$	$-(-1)^{i+j}$	$-(-1)^{i+j}$	$(-1)^{i+j}$

(A.2)

where $i, j = 0, 1, \ldots, n-1$ and $m = 0, 1, 2, 3$.

Two-Dimensional Irreducible Representations

The characters $\chi^\mu(g) = \mathrm{Tr}\, T^\mu(g)$ for two-dimensional irreducible representations are given as follows. For $\mu = (2;+), (2;-)$ in (3.45) and (3.46), we have

g	$\chi^{(2;+)}(g)$	$\chi^{(2;-)}(g)$
${p_1}^i {p_2}^j$	2	$(-1)^{i+j}2$
$r {p_1}^i {p_2}^j$	0	0
$r^2 {p_1}^i {p_2}^j$	−2	$-(-1)^{i+j}2$
$r^3 {p_1}^i {p_2}^j$	0	0
$s r^m {p_1}^i {p_2}^j$	0	0

(A.3)

where $i, j = 0, 1, \ldots, n-1$ and $m = 0, 1, 2, 3$. For $\mu = (2;+,+), (2;+,-), (2;-,+), (2;-,-)$ in (3.47) and (3.48), we have

$$
\begin{array}{c|cccc}
g & \chi^{(2;+,+)}(g) & \chi^{(2;+,-)}(g) & \chi^{(2;-,+)}(g) & \chi^{(2;-,-)}(g) \\
\hline
{p_1}^i {p_2}^j & (-1)^i + (-1)^j & (-1)^i + (-1)^j & (-1)^i + (-1)^j & (-1)^i + (-1)^j \\
r {p_1}^i {p_2}^j & 0 & 0 & 0 & 0 \\
r^2 {p_1}^i {p_2}^j & (-1)^i + (-1)^j & (-1)^i + (-1)^j & -(-1)^i - (-1)^j & -(-1)^i - (-1)^j \\
r^3 {p_1}^i {p_2}^j & 0 & 0 & 0 & 0 \\
\hline
s {p_1}^i {p_2}^j & (-1)^i + (-1)^j & -(-1)^i - (-1)^j & (-1)^i - (-1)^j & -(-1)^i + (-1)^j \\
sr {p_1}^i {p_2}^j & 0 & 0 & 0 & 0 \\
sr^2 {p_1}^i {p_2}^j & (-1)^i + (-1)^j & -(-1)^i - (-1)^j & -(-1)^i + (-1)^j & (-1)^i - (-1)^j \\
sr^3 {p_1}^i {p_2}^j & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\qquad (A.4)
$$

where $i, j = 0, 1, \ldots, n-1$ and $m = 0, 1, 2, 3$.

Four-Dimensional Irreducible Representations

The characters $\chi^\mu(g) = \mathrm{Tr}\, T^\mu(g)$ for four-dimensional irreducible representations are given as follows:

$$
\begin{array}{c|ccc}
g & \chi^{(4;k,0,\sigma)}(g) & \chi^{(4;k,k,\sigma)}(g) & \chi^{(4;n/2,\ell,\sigma)}(g) \\
\hline
{p_1}^i {p_2}^j & 2[\cos(ki\theta) & 2[\cos(k(i+j)\theta) & 2[(-1)^i \cos(\ell j\theta) \\
 & +\cos(kj\theta)] & +\cos(k(i-j)\theta)] & +(-1)^j \cos(\ell i\theta)] \\
\hline
r^m {p_1}^i {p_2}^j & 0 & 0 & 0 \\
(m = 1,2,3) & & & \\
\hline
s {p_1}^i {p_2}^j & 2\sigma \cos(ki\theta) & 0 & 2\sigma(-1)^j \cos(\ell i\theta) \\
sr {p_1}^i {p_2}^j & 0 & 2\sigma \cos(k(i-j)\theta) & 0 \\
sr^2 {p_1}^i {p_2}^j & 2\sigma \cos(kj\theta) & 0 & 2\sigma(-1)^i \cos(\ell j\theta) \\
sr^3 {p_1}^i {p_2}^j & 0 & 2\sigma \cos(k(i+j)\theta) & 0 \\
\hline
\end{array}
\qquad (A.5)
$$

where $\theta = 2\pi/n$ and $i, j = 0, 1, \ldots, n-1$.

Eight-Dimensional Irreducible Representations

The characters $\chi^{(8;k,\ell)}(g) = \mathrm{Tr}\, T^{(8;k,\ell)}(g)$ are given as follows. For $g = {p_1}^i {p_2}^j$, being free from r and s, we have

$$
\chi^{(8;k,\ell)}({p_1}^i {p_2}^j) = 2\{\cos((ki+\ell j)\theta) + \cos((-\ell i + kj)\theta) + \cos((ki - \ell j)\theta) + \cos((-\ell i - kj)\theta)\}, \quad (A.6)
$$

where $\theta = 2\pi/n$ and $i, j = 0, 1, \ldots, n-1$. For other g, we have $\chi^{(8;k,\ell)}(g) = 0$.

A.1.2. Method of Little Groups

We describe a systematic method, called the method of little groups, for constructing irreducible representations of a general group with the structure of the semidirect product by an abelian group. For details about semidirect products, see Section 8.2 of Serre (1977).

Let G be a group that is a semidirect product of a group H and an abelian group A. This means that A is a normal subgroup of G, and each element $g \in G$ is represented uniquely as $g = ah$ with $a \in A$ and $h \in H$.

Since A is abelian, every irreducible representation of A over $\mathbb{C}$ is one-dimensional, and is identified with its character χ. Accordingly, the set of all irreducible representations of A over $\mathbb{C}$ can be denoted as

$$X = \{\chi^i \mid i \in R(A)\} \tag{A.7}$$

with a suitable index set $R(A)$. For $\chi \in X$ and $g \in G$, we define a function ${}^g\chi$ on A by

$${}^g\chi(a) = \chi(g^{-1}ag), \quad a \in A, \tag{A.8}$$

which is also a character of A, belonging to X. This defines an action of G on X.

With reference to the action of G on X, we classify the elements of X into orbits. It should be noted that, for $g = bh$ with $b \in A$ and $h \in H$, we have

$${}^g\chi(a) = \chi((bh)^{-1}a(bh)) = \chi(h^{-1}ah) = {}^h\chi(a), \quad a \in A, \tag{A.9}$$

in which $b^{-1}ab = a$ since A is abelian. Hence, the orbits can in fact be obtained by the action of the subgroup H on X, instead of that of G. Denote by

$$\{\chi^i \mid i \in R(A)/H\} \tag{A.10}$$

a system of representatives from the orbits, where $R(A)/H$ is an index set, or the set of "names" of the orbit. This means that

- $\chi^i \in X$ for each $i \in R(A)/H$,
- for distinct i and j in $R(A)/H$, $\chi^i \neq {}^h(\chi^j)$ for any $h \in H$, and
- for each $\chi \in X$, there exist some $i \in R(A)/H$ and $h \in H$ such that $\chi = {}^h(\chi^i)$.

For each $i \in R(A)/H$, we define

$$H^i = \{h \in H \mid {}^h(\chi^i) = \chi^i\}, \tag{A.11}$$

which is a subgroup of H associated with the orbit i, and

$$G^i = \{ah \mid a \in A,\ h \in H^i\}, \tag{A.12}$$

which is a subgroup of G, called the *little group*. Noting that each element of G^i can be represented as ah with $a \in A$ and $h \in H^i$, we define a function $\tilde{\chi}^i$ on G^i by

$$\tilde{\chi}^i(ah) = \chi^i(a), \quad a \in A,\ h \in H^i, \tag{A.13}$$

which is a one-dimensional representation (a character of degree one) of G^i.

Let T^μ be an irreducible representation of H^i over $\mathbb{C}$ indexed by $\mu \in R(H^i)$. Then the matrix-valued function $T^{(i,\mu)}$ defined on G^i of (A.12) by

$$T^{(i,\mu)}(ah) = \chi^i(a)T^\mu(h), \quad a \in A, \ h \in H^i \tag{A.14}$$

is an irreducible representation of G^i. Denote by $\tilde{T}^{(i,\mu)}$ the induced representation of G obtained from $T^{(i,\mu)}$ (see Remark A.1 below). Then $\tilde{T}^{(i,\mu)}$ is an irreducible representation of G. Moreover, all the irreducible representations of G can be obtained in this manner, and $\tilde{T}^{(i,\mu)}$'s are mutually inequivalent for different (i, μ). Thus, the irreducible representations of G are indexed by (i, μ), i.e.,

$$R(G) = \{(i, \mu) \mid i \in R(A)/H, \ \mu \in R(H^i)\} \tag{A.15}$$

and

$$\{\tilde{T}^{(i,\mu)} \mid i \in R(A)/H, \ \mu \in R(H^i)\} \tag{A.16}$$

gives a complete list of irreducible representations of G over $\mathbb{C}$.

Remark A.1. The induced representation is explained here. Let G be a group, G' be a subgroup of G, and T' be a representation of G' of dimension N'. Consider the *coset decomposition*

$$G = g_1G' + g_2G' + \cdots + g_mG', \tag{A.17}$$

where $j = 1, \ldots, m$ and $m = |G|/|G'|$. Each $g \in G$ causes a permutation of $(g_1, g_2, \ldots, g_m)$ to $(g_{\pi(1)}, g_{\pi(2)}, \ldots, g_{\pi(m)})$ according to the equation

$$gg_j = g_{\pi(j)}f_j, \quad f_j \in G' \tag{A.18}$$

for $j = 1, \ldots, m$. Note that the choice of $(g_1, g_2, \ldots, g_m)$ is not unique, but once this is fixed, f_j is uniquely determined for each g.

Define $\tilde{T}(g)$ to be an $mN' \times mN'$ matrix with rows and columns partitioned into m blocks of size N' such that the $(\pi(j), j)$-block of $\tilde{T}(g)$ equals $T'(f_j)$, whereas the (i, j)-block of $\tilde{T}(g)$ equals O if $i \neq \pi(j)$. Note that this is well-defined, since f_j and $\pi(j)$ are uniquely determined from g, and $T'(f_j)$ for $j = 1, \ldots, m$ are assumed to be given. The family of matrices $\{\tilde{T}(g) \mid g \in G\}$ is a representation of G of dimension mN', called the *induced representation*. For example, if $m = 3$, $(\pi(1), \pi(2), \pi(3)) = (2, 3, 1)$, we have

$$\tilde{T}(g) = \begin{bmatrix} & & T'(f_3) \\ T'(f_1) & & \\ & T'(f_2) & \end{bmatrix}.$$

We shall apply this construction to $T' = T^{(i,\mu)}$ on $G' = G^i$ to obtain $\tilde{T} = \tilde{T}^{(i,\mu)}$, where the dimension N' of $T^{(i,\mu)}$ is equal to that of T^μ by (A.14).

□

A.1.3. Derivation of the Little Groups

We apply the method of little groups to

$$A = \mathbb{Z}_n \times \mathbb{Z}_n = \langle p_1 \rangle \times \langle p_2 \rangle, \quad H = \mathrm{D}_4 = \langle r, s \rangle \tag{A.19}$$

to obtain the irreducible representations of $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

As the first step we determine the system of representatives (A.10) in the orbit decomposition of X. Since $A = \mathbb{Z}_n \times \mathbb{Z}_n$ is an abelian group, all the irreducible representations are one-dimensional. The set X of irreducible representations of $A = \mathbb{Z}_n \times \mathbb{Z}_n$ is indexed by

$$R(A) = \{(k, \ell) \mid 0 \le k \le n-1,\ 0 \le \ell \le n-1\}, \tag{A.20}$$

where (k, ℓ) denotes a one-dimensional representation (or character) $\chi^{(k,\ell)}$ defined by

$$\chi^{(k,\ell)}(p_1) = \omega^k, \quad \chi^{(k,\ell)}(p_2) = \omega^\ell \tag{A.21}$$

with

$$\omega = \exp(2\pi \mathrm{i}/n). \tag{A.22}$$

We extend the notation (k, ℓ) for any integers, to designate the element (k', ℓ') of $R(A)$ with $k' \equiv k \mod n$ and $\ell' \equiv \ell \mod n$.

For the orbit decomposition of X by H, we compute $h^{-1}p_1h$ and $h^{-1}p_2h$ for $h \in H$, to obtain

h	e	r	r^2	r^3	s	sr	sr^2	sr^3
$h^{-1}p_1h$	p_1	${p_2}^{-1}$	${p_1}^{-1}$	p_2	p_1	${p_2}^{-1}$	${p_1}^{-1}$	p_2
$h^{-1}p_2h$	p_2	p_1	${p_2}^{-1}$	${p_1}^{-1}$	${p_2}^{-1}$	${p_1}^{-1}$	p_2	p_1

(A.23)

For example, for $h = s$, we have $(h^{-1}p_1h, h^{-1}p_2h) = (p_1, {p_2}^{-1})$, and we see, by (A.21), that the action of s in (A.8) is given as ${}^s\chi^{(k,\ell)} = \chi^{(k,-\ell)}$, which is expressed symbolically as $(k, \ell) \Rightarrow (k, -\ell)$. In this manner, we can obtain the following orbit containing (k, ℓ):

$$\boxed{\begin{array}{ccc} (\ell, -k) & \leftarrow & (-k, -\ell) \\ \downarrow & & \uparrow \\ (k, \ell) & \rightarrow & (-\ell, k) \\ \Downarrow & & \\ (k, -\ell) & \rightarrow & (-\ell, -k) \\ \uparrow & & \downarrow \\ (\ell, k) & \leftarrow & (-k, \ell) \end{array}} \tag{A.24}$$

where "$\Downarrow$" means the action of s, and "$\rightarrow$" (or "$\leftarrow$", "$\uparrow$", "$\downarrow$") means the action of r. It should be clear that $(\ell, -k)$, for example, is understood as $(\ell \bmod n, -k \bmod n)$. The orbit (A.24) is illustrated in Fig. A.1.

The system of representatives in (A.10) in the orbit decomposition of X with respect to the action of G is given as follows. In view of Fig. A.1, it is natural to take

$$R(A)/H = \{(k, \ell) \mid 0 \le \ell \le k \le \lfloor (n-1)/2 \rfloor\}, \tag{A.25}$$

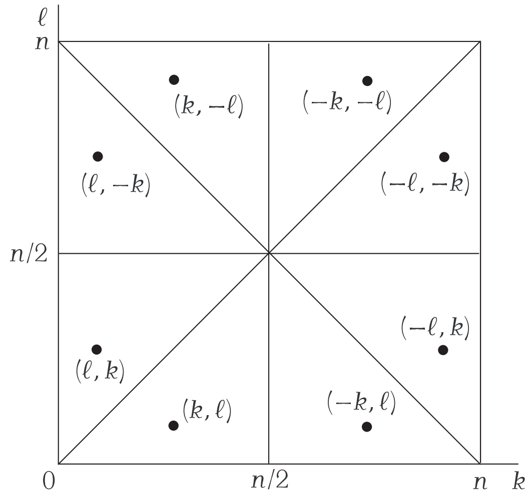


Figure A.1: Orbit of (k, ℓ) in (A.24).

which corresponds to the set of integer lattice points (k, ℓ) contained in the triangle with vertices at $(k, \ell) = (0, 0), (n/2, 0), (n/2, n/2)$, where the points on the edges of the triangle are included.

The subgroup $H^i = H^{(k,\ell)}$ in (A.11) for $i = (k, \ell)$, which is expressed as

$$H^{(k,\ell)} = \{h \in \mathrm{D}_4 \mid \chi^{(k,\ell)}(h^{-1}ah) = \chi^{(k,\ell)}(a) \text{ for all } a \in \mathbb{Z}_n \times \mathbb{Z}_n\}, \tag{A.26}$$

is obtained with reference to (A.21) and (A.23). For $h \in \mathrm{D}_4$, we have $h \in H^{(k,\ell)}$ if and only if

$$\chi^{(k,\ell)}(hp_1h^{-1}) = \chi^{(k,\ell)}(p_1), \quad \chi^{(k,\ell)}(hp_2h^{-1}) = \chi^{(k,\ell)}(p_2). \tag{A.27}$$

For $(k, \ell) = (0, 0)$, for example, this condition is satisfied by all $h \in \mathrm{D}_4$, and hence $H^{(0,0)} = \langle r, s\rangle$. In this manner, we obtain

$$H^{(k,\ell)} = \begin{cases} \langle r, s\rangle & \text{for } (k,\ell) = (0,0), \\ \langle r, s\rangle & \text{for } (k,\ell) = (n/2, n/2) \quad (n\text{: even}), \\ \langle r^2, s\rangle & \text{for } (k,\ell) = (n/2, 0) \quad (n\text{: even}), \\ \{e, s\} & \text{for } (k,\ell) = (k, 0) \quad (1 \le k \le \lfloor \frac{n-1}{2} \rfloor), \\ \{e, sr^3\} & \text{for } (k,\ell) = (k, k) \quad (1 \le k \le \lfloor \frac{n-1}{2} \rfloor), \\ \{e, sr^2\} & \text{for } (k,\ell) = (n/2, \ell) \quad (n\text{: even},\ 1 \le \ell \le \lfloor \frac{n-1}{2} \rfloor), \\ \{e\} & \text{for } (k,\ell) \quad (1 \le \ell \le k-1,\ 2 \le k \le \lfloor \frac{n-1}{2} \rfloor). \end{cases} \tag{A.28}$$

The little group $G^i = G^{(k,\ell)}$ in (A.12) for $i = (k, \ell)$ is obtained as the semidirect product of $H^{(k,\ell)}$ by $A = \langle p_1, p_2\rangle$.

Example A.1. The system of representatives $R(A)/H$ and the associated subgroups $H^{(k,\ell)}$ in (A.28) for $n = 3, 4, 7, 8, 9$ are given as follows:

$n = 3$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r,s\rangle$
$(1,0)$	$\{e,s\}$
$(1,1)$	$\{e,sr^3\}$

$n = 4$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r,s\rangle$
$(2,2)$	$\langle r,s\rangle$
$(2,0)$	$\langle r^2,s\rangle$
$(1,0)$	$\{e,s\}$
$(1,1)$	$\{e,sr^3\}$
$(2,1)$	$\{e,sr^2\}$

$n = 7$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r,s\rangle$
$(1,0),(2,0),(3,0)$	$\{e,s\}$
$(1,1),(2,2),(3,3)$	$\{e,sr^3\}$
$(2,1),(3,1),(3,2)$	$\{e\}$

$n = 8$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r,s\rangle$
$(4,4)$	$\langle r,s\rangle$
$(4,0)$	$\langle r^2,s\rangle$
$(1,0),(2,0),(3,0)$	$\{e,s\}$
$(1,1),(2,2),(3,3)$	$\{e,sr^3\}$
$(4,1),(4,2),(4,3)$	$\{e,sr^2\}$
$(2,1),(3,1),(3,2)$	$\{e\}$

$n = 9$	
(k,ℓ)	$H^{(k,\ell)}$
$(0,0)$	$\langle r,s\rangle$
$(1,0),(2,0),(3,0),(4,0)$	$\{e,s\}$
$(1,1),(2,2),(3,3),(4,4)$	$\{e,sr^3\}$
$(2,1),(3,1),(3,2)$	$\{e\}$
$(4,1),(4,2),(4,3)$	$\{e\}$

□

A.1.4. Construction of the Irreducible Representations

The procedure for constructing irreducible representations of $G = \langle r,s,p_1,p_2\rangle$ using orbit decomposition and little groups is summarized as follows.

For each $(k,\ell) \in R(A)/H$, we have the associated subgroup $H^{(k,\ell)}$ in (A.28). Let T^μ be an irreducible representation of $H^{(k,\ell)}$ indexed by $\mu \in R(H^{(k,\ell)})$, and define $T^{(k,\ell,\mu)}$ by

$$T^{(k,\ell,\mu)}({p_1}^i {p_2}^j h) = \chi^{(k,\ell)}({p_1}^i {p_2}^j) T^\mu(h) = \omega^{ki+\ell j} T^\mu(h), \quad 0 \le i,j \le n-1,\ h \in H^{(k,\ell)}, \tag{A.29}$$

which is an irreducible representation of the little group $G^{(k,\ell)}$.

The coset decomposition (A.17) takes the form of

$$G = g_1 G^{(k,\ell)} + g_2 G^{(k,\ell)} + \cdots + g_m G^{(k,\ell)} \tag{A.30}$$

with $m = |G|/|G^{(k,\ell)}| = |\mathrm{D}_4|/|H^{(k,\ell)}| = 8/|H^{(k,\ell)}|$. Since $G^{(k,\ell)} \supseteq \langle p_1,p_2\rangle$, we may assume that $g_j \in \langle r,s\rangle$ for $j = 1,\ldots,m$ and $g_1 = e$.

The induced representation $\tilde{T}^{(k,\ell,\mu)}(g)$ is determined by its values at $g = p_1,p_2,r,s$ that generate the group G. Hence, it suffices to consider $g = p_1,p_2,r,s$ in the equation (A.18):

$$g g_j = g_{\pi(j)} f_j, \tag{A.31}$$

where $\pi(j)$ and $f_j \in G^{(k,\ell)}$ are to be found for $j = 1,\ldots,m$. The induced representation $\tilde{T}^{(k,\ell,\mu)}$ is an irreducible representation of dimension $mN^\mu = 8N^\mu/|H^{(k,\ell)}|$ over $\mathbb{C}$, where N^μ denotes the dimension of T^μ.

Table A.1: Induced irreducible representations of $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

(k,ℓ)	$H^{(k,\ell)}$	m	Induced irreducible representations
$(0,0)$	$\langle r,s\rangle$	1	$(1;+,+,+), (1;+,-,+), (1;-,+,+), (1;-,-,+), (2;+)$
$(n/2,n/2)$	$\langle r,s\rangle$	1	$(1;+,+,-), (1;+,-,-), (1;-,+,-), (1;-,-,-), (2;-)$
$(n/2,0)$	$\langle r^2,s\rangle$	2	$(2;+,+), (2;+,-), (2;-,+), (2;-,-)$
$(k,0)$	$\{e,s\}$	4	$(4;k,0,+), (4;k,0,-)$
(k,k)	$\{e,sr^3\}$	4	$(4;k,k,+), (4;k,k,-)$
$(n/2,\ell)$	$\{e,sr^2\}$	4	$(4;n/2,\ell,+), (4;n/2,\ell,-)$
(k,ℓ)	$\{e\}$	8	$(8;k,\ell)$

$(k,\ell) = (n/2,n/2)$ and $(n/2,0)$ exist if n is even;
$(k,0)$ with $1 \le k \le \lfloor \frac{n-1}{2} \rfloor$ in (3.49);
(k,k) with $1 \le k \le \lfloor \frac{n-1}{2} \rfloor$ in (3.50);
$(n/2,\ell)$ with $1 \le \ell \le \lfloor \frac{n-1}{2} \rfloor$ in (3.51);
(k,ℓ) with $1 \le \ell \le k-1$, $2 \le k \le \lfloor (n-1)/2 \rfloor$ in (3.60)

According to the general theory, $\tilde{T}^{(k,\ell,\mu)}$ obtained in this manner is not a representation over $\mathbb{R}$ but over $\mathbb{C}$, as is evident from the fact that ω appearing in (A.29) is a complex number defined by (A.22). Fortunately, however, all irreducible representations thus obtained are representable over $\mathbb{R}$. We can thus determine a complete list of irreducible representations over $\mathbb{R}$ of the group $G = D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. Table A.1 is a summary of the derivations below.

Case of $(k,\ell) = (0,0)$

For $(k,\ell) = (0,0)$, $\chi^{(k,\ell)}$ is the unit representation by (A.21), and therefore

$$H^{(k,\ell)} = \langle r,s\rangle = D_4,$$

as is shown in (A.28). D_4 has four one-dimensional irreducible representations $(+,+,+)$, $(+,-,+)$, $(-,+,+)$, $(-,-,+)$, and one two-dimensional irreducible representation $(2;+)$ (e.g., see Kim, 1999; Kettle, 2007).

Since $G^{(k,\ell)} = G$, the coset decomposition (A.30) is trivial with $m = 1$ and $g_1 = e$, and the equation (A.31) reads $g \cdot g_1 = g_1 \cdot g$ for every $g \in G$. For each μ, the induced representation $\tilde{T}^{(0,0,\mu)}(g)$ for $g = {p_1}^i {p_2}^j h$ with $h \in D_4$ is given by (A.29) as

$$\tilde{T}^{(0,0,\mu)}(g) = \tilde{T}^{(0,0,\mu)}({p_1}^i {p_2}^j h) = \chi^{(0,0)}({p_1}^i {p_2}^j) T^{\mu}(h) = T^{\mu}(h).$$

With this result, we have the one-dimensional irreducible representations $(1;+,+,+)$, $(1;+,-,+)$, $(1;-,+,+)$, $(1;-,-,+)$, and the two-dimensional irreducible representation $(2;+)$ as the irreducible representations for the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

Case of $(k,\ell) = (n/2,n/2)$

In this case, $\chi = \chi^{(k,\ell)} = \chi^{(n/2,n/2)}$ is given by (A.21) as $\chi(p_1) = \chi(p_2) = \omega^{n/2} = -1$. For $(k,\ell) = (n/2,n/2)$, we have

$$H^{(k,\ell)} = \langle r,s\rangle = D_4,$$

as is shown in (A.28). Hence we have the one-dimensional irreducible representations $(1;+,+,-)$, $(1;+,-,-)$, $(1;-,+,-)$, $(1;-,-,-)$, and the two-dimensional irreducible representation $(2;-)$.

Case of $(k,\ell) = (n/2, 0)$

The case of $(k,\ell) = (n/2, 0)$ occurs when n is even. In this case, $\chi = \chi^{(k,\ell)} = \chi^{(n/2,0)}$ is given by (A.21) as $\chi(p_1) = -1$ and $\chi(p_2) = 1$, and therefore

$$H^{(k,\ell)} = \{e, r^2, s, sr^2\} = \langle r^2, s\rangle \simeq \mathrm{D}_2,$$

as is shown in (A.28). This group has four one-dimensional irreducible representations, say, $\mu = (\sigma_r, \sigma_s) = (+,+), (+,-), (-,+), (-,-)$ defined by

$$T^\mu(r^2) = \sigma_r = \pm 1, \quad T^\mu(s) = \sigma_s = \pm 1.$$

Since $G^{(k,\ell)} = \langle r^2, s, p_1, p_2\rangle$, the coset decomposition in (A.30) is given by

$$G = g_1 G^{(k,\ell)} + g_2 G^{(k,\ell)} = e \cdot \langle r^2, s, p_1, p_2\rangle + r \cdot \langle r^2, s, p_1, p_2\rangle$$

with $m = 2$, $g_1 = e$ and $g_2 = r$. The equation (A.31) for $g = p_1, p_2, r, s$ reads as follows:

$p_1 \cdot g_j = g_{\pi(j)} \cdot f_j$	$p_2 \cdot g_j = g_{\pi(j)} \cdot f_j$	$r \cdot g_j = g_{\pi(j)} \cdot f_j$	$s \cdot g_j = g_{\pi(j)} \cdot f_j$
$p_1 \cdot e = e \cdot p_1$	$p_2 \cdot e = e \cdot p_2$	$r \cdot e = r \cdot e$	$s \cdot e = e \cdot s$
$p_1 \cdot r = r \cdot {p_2}^{-1}$	$p_2 \cdot r = r \cdot p_1$	$r \cdot r = e \cdot r^2$	$s \cdot r = r \cdot sr^2$

For the one-dimensional representation $\mu = (\sigma)$ with $\sigma \in \{+,-\}$, the induced representation $\tilde{T} = \tilde{T}^{(n/2,0,\mu)}$ is given by

$$\tilde{T}(p_1) = \begin{bmatrix} \chi(p_1)T^\mu(e) & \\ & \chi({p_2}^{-1})T^\mu(e) \end{bmatrix} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix},$$

$$\tilde{T}(p_2) = \begin{bmatrix} \chi(p_2)T^\mu(e) & \\ & \chi(p_1)T^\mu(e) \end{bmatrix} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix},$$

$$\tilde{T}(r) = \begin{bmatrix} & \chi(e)T^\mu(r^2) \\ \chi(e)T^\mu(e) & \end{bmatrix} = \begin{bmatrix} & \sigma_r \\ 1 & \end{bmatrix},$$

$$\tilde{T}(s) = \begin{bmatrix} \chi(e)T^\mu(s) & \\ & \chi(e)T^\mu(sr^2) \end{bmatrix} = \sigma_s \begin{bmatrix} 1 & \\ & \sigma_r \end{bmatrix},$$

where (A.29) is used and the nonzero blocks here are determined with reference to $\pi(j)$ and f_j computed above (see Remark A.1).

Case of $(k,\ell) = (k, 0), (k, k), (n/2, \ell)$

For $(k,\ell) = (k, 0)$ in (3.49), we have $\chi^{(k,\ell)}(p_1) = \omega^k$ and $\chi^{(k,\ell)}(p_2) = 1$ by (A.21), and therefore

$$H^{(k,\ell)} = \{e, s\},$$

as is shown in (A.28). For $(k,\ell) = (k, k)$ in (3.50), we have $\chi^{(k,\ell)}(p_1) = \chi^{(k,\ell)}(p_2) = \omega^k$, and therefore

$$H^{(k,\ell)} = \{e, sr^3\}.$$

For $(k,\ell) = (n/2,\ell)$ in (3.51), we have $\chi^{(k,\ell)}(p_1) = -1$ and $\chi^{(k,\ell)}(p_2) = \omega^{\ell}$, and therefore

$$H^{(k,\ell)} = \{e, sr^2\}.$$

Let $h_0 = s$ for $(k,\ell) = (k,0)$, $h_0 = sr^3$ for $(k,\ell) = (k,k)$, and $h_0 = sr^2$ for $(k,\ell) = (n/2,\ell)$. In either case $H^{(k,\ell)} = \{e, h_0\}$ is isomorphic to D_1 and has two one-dimensional irreducible representations, say, $\mu = \mu_1, \mu_2$ defined by

$$T^{\mu_1}(h_0) = 1, \quad T^{\mu_2}(h_0) = -1.$$

That is, $T^{\mu}(h_0) = \sigma^{\mu}$ with $\sigma^{\mu_1} = 1$ and $\sigma^{\mu_2} = -1$. The notation is summarized as follows:

(k,ℓ)	$H^{(k,\ell)}$	h_0	$T^{\mu_1}(h_0)$	$T^{\mu_2}(h_0)$
$(k,0)$	$\{e,s\}$	sr	1	-1
(k,k)	$\{e,sr^3\}$	sr^3	1	-1
$(n/2,\ell)$	$\{e,sr^2\}$	sr^2	1	-1

The coset decomposition in (A.30) is given by $G^{(k,\ell)} = \langle h_0, p_1, p_2 \rangle$, $m = 4$, and $g_j = r^{j-1}$ for $j = 1,\ldots,4$. The equation (A.31) for $g = p_1, p_2, r, s$ reads as follows (see (A.23) for p_1 and p_2):

$p_1 \cdot g_j = g_{\pi(j)} \cdot f_j$	$p_2 \cdot g_j = g_{\pi(j)} \cdot f_j$	$r \cdot g_j = g_{\pi(j)} \cdot f_j$
$p_1 \cdot e = e \cdot p_1$	$p_2 \cdot e = e \cdot p_2$	$r \cdot e = r \cdot e$
$p_1 \cdot r = r \cdot {p_2}^{-1}$	$p_2 \cdot r = r \cdot p_1$	$r \cdot r = r^2 \cdot e$
$p_1 \cdot r^2 = r^2 \cdot {p_1}^{-1}$	$p_2 \cdot r^2 = r^2 \cdot {p_2}^{-1}$	$r \cdot r^2 = r^3 \cdot e$
$p_1 \cdot r^3 = r^3 \cdot p_2$	$p_2 \cdot r^3 = r^3 \cdot p_1$	$r \cdot r^3 = e \cdot e$

$s \cdot g_j = g_{\pi(j)} \cdot f_j$		
$(k,0)$	(k,k)	$(n/2,\ell)$
$s \cdot e = e \cdot s$	$s \cdot e = r^3 \cdot sr^3$	$s \cdot e = r^2 \cdot sr^2$
$s \cdot r = r^3 \cdot s$	$s \cdot r = r^2 \cdot sr^3$	$s \cdot r = r \cdot sr^2$
$s \cdot r^2 = r^2 \cdot s$	$s \cdot r^2 = r \cdot sr^3$	$s \cdot r^2 = e \cdot sr^2$
$s \cdot r^3 = r \cdot s$	$s \cdot r^3 = e \cdot sr^3$	$s \cdot r^3 = r^3 \cdot sr^2$

For $(k,\ell) = (k,0), (k,k), (n/2,\ell)$ and $\mu = \mu_1, \mu_2$, the induced representation $\tilde{T}^{(k,\ell,\mu)}$ is given, with $\omega = \exp(2\pi\,\mathrm{i}/n)$, by

$$\tilde{T}^{(k,\ell,\mu)}(p_1) = \mathrm{diag}(\chi(p_1), \chi({p_2}^{-1}), \chi({p_1}^{-1}), \chi(p_2)) = \mathrm{diag}(\omega^{k}, \omega^{-\ell}, \omega^{-k}, \omega^{\ell}),$$
$$\tilde{T}^{(k,\ell,\mu)}(p_2) = \mathrm{diag}(\chi(p_2), \chi(p_1), \chi({p_2}^{-1}), \chi({p_1}^{-1})) = \mathrm{diag}(\omega^{\ell}, \omega^{k}, \omega^{-\ell}, \omega^{-k}),$$

$$\tilde{T}^{(k,\ell,\mu)}(r) = T^{\mu}(e) \begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix} = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix},$$

and

$$\tilde{T}^{(k,0,\mu)}(s) = T^{\mu}(s)\begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{bmatrix},$$

$$\tilde{T}^{(k,k,\mu)}(s) = T^{\mu}(sr^3)\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} = \sigma^{\mu}\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix},$$

$$\tilde{T}^{(n/2,\ell,\mu)}(s) = T^{\mu}(sr^2)\begin{bmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix} = \sigma^{\mu}\begin{bmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix}.$$

The above representation over $\mathbb{C}$ can be transformed to a real representation. By permuting the rows and columns as $(1, 3, 2, 4)$, we obtain

$$\hat{T}^{(k,\ell,\mu)}(p_1) = \left[\begin{array}{cc|cc} \omega^{k} & & & \\ & \omega^{-k} & & \\ \hline & & \omega^{-\ell} & \\ & & & \omega^{\ell} \end{array}\right], \quad \hat{T}^{(k,\ell,\mu)}(p_2) = \left[\begin{array}{cc|cc} \omega^{\ell} & & & \\ & \omega^{-\ell} & & \\ \hline & & \omega^{k} & \\ & & & \omega^{-k} \end{array}\right], \quad \hat{T}^{(k,\ell,\mu)}(r) = \left[\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline 1 & & & \\ & 1 & & \end{array}\right],$$

$$\hat{T}^{(k,0,\mu)}(s) = \sigma^{\mu}\left[\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right], \quad \hat{T}^{(k,k,\mu)}(s) = \sigma^{\mu}\left[\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right], \quad \hat{T}^{(n/2,\ell,\mu)}(s) = \sigma^{\mu}\left[\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right].$$

It is apparent that these representations are equivalent, respectively, to the four-dimensional real irreducible representations $(4; k, 0, \sigma)$ and $(4; k, k, \sigma)$ with $\sigma = \sigma^{\mu}$.

Case of General (k, ℓ)

For (k, ℓ) in (3.60), $\chi = \chi^{(k,\ell)}$ is given by (A.21), and $H^{(k,\ell)} = \{e\}$. The unit representation μ is the only irreducible representation of $H^{(k,\ell)}$.

The coset decomposition in (A.30) is given by $G^{(k,\ell)} = \langle p_1, p_2 \rangle$, $m = 8$, and

$$g_1 = e,\ g_2 = r,\ g_3 = r^2,\ g_4 = r^3,\ g_5 = s,\ g_6 = sr,\ g_7 = sr^2,\ g_8 = sr^3.$$

The equation (A.31) for $g = p_1, p_2, r, s$ reads as follows:

$p_1 \cdot g_j = g_{\pi(j)} \cdot f_j$	$p_2 \cdot g_j = g_{\pi(j)} \cdot f_j$	$r \cdot g_j = g_{\pi(j)} \cdot f_j$	$s \cdot g_j = g_{\pi(j)} \cdot f_j$
$p_1 \cdot e = e \cdot p_1$	$p_2 \cdot e = e \cdot p_2$	$r \cdot e = r \cdot e$	$s \cdot e = s \cdot e$
$p_1 \cdot r = r \cdot {p_2}^{-1}$	$p_2 \cdot r = r \cdot p_1$	$r \cdot r = r^2 \cdot e$	$s \cdot r = sr \cdot e$
$p_1 \cdot r^2 = r^2 \cdot {p_1}^{-1}$	$p_2 \cdot r^2 = r^2 \cdot {p_2}^{-1}$	$r \cdot r^2 = r^3 \cdot e$	$s \cdot r^2 = sr^2 \cdot e$
$p_1 \cdot r^3 = r^3 \cdot p_2$	$p_2 \cdot r^3 = r^3 \cdot {p_1}^{-1}$	$r \cdot r^3 = e \cdot e$	$s \cdot r^3 = sr^3 \cdot e$
$p_1 \cdot s = s \cdot p_1$	$p_2 \cdot s = s \cdot {p_2}^{-1}$	$r \cdot s = sr^3 \cdot e$	$s \cdot s = e \cdot e$
$p_1 \cdot sr = sr \cdot {p_2}^{-1}$	$p_2 \cdot sr = sr \cdot {p_1}^{-1}$	$r \cdot sr = s \cdot e$	$s \cdot sr = r \cdot e$
$p_1 \cdot sr^2 = sr^2 \cdot {p_1}^{-1}$	$p_2 \cdot sr^2 = sr^2 \cdot p_2$	$r \cdot sr^2 = sr \cdot e$	$s \cdot sr^2 = r^2 \cdot e$
$p_1 \cdot sr^3 = sr^3 \cdot p_2$	$p_2 \cdot sr^3 = sr^3 \cdot p_1$	$r \cdot sr^3 = sr^2 \cdot e$	$s \cdot sr^3 = r^3 \cdot e$

The induced representation $\tilde{T} = \tilde{T}^{(k,\ell,\mu)}$, of dimension 8, is given in terms of $\omega = \exp(2\pi \mathrm{i}/n)$ as follows:

$$\begin{aligned}
\tilde{T}(p_1) &= \mathrm{diag}(\omega^k, \omega^{-\ell}, \omega^{-k}, \omega^{\ell}, \omega^k, \omega^{-\ell}, \omega^{-k}, \omega^{\ell}),\\
\tilde{T}(p_2) &= \mathrm{diag}(\omega^{\ell}, \omega^k, \omega^{-\ell}, \omega^{-k}, \omega^{-\ell}, \omega^{-k}, \omega^{\ell}, \omega^k),\\
\tilde{T}(r) &= \begin{bmatrix} C & O \\ O & C^\top \end{bmatrix}, \quad \tilde{T}(s) = \begin{bmatrix} O & I \\ I & O \end{bmatrix}
\end{aligned}$$

with

$$C = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix}, \quad I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

The above representation over $\mathbb{C}$ can be transformed to a real representation. By permuting the rows and columns as $(1, 3, 2, 4, 5, 7, 6, 8)$, we obtain

$$\hat{T}(p_1) = \begin{bmatrix} \Omega_1 & \\ & \Omega_1 \end{bmatrix}, \quad \hat{T}(p_2) = \begin{bmatrix} \Omega_2 & \\ & \Omega_3 \end{bmatrix}, \quad \hat{T}(r) = \begin{bmatrix} D & \\ & D^\top \end{bmatrix}, \quad \hat{T}(s) = \begin{bmatrix} & I \\ I & \end{bmatrix}$$

with

$$\Omega_1 = \left[\begin{array}{cc|cc} \omega^k & & & \\ & \omega^{-k} & & \\ \hline & & \omega^{-\ell} & \\ & & & \omega^{\ell} \end{array}\right], \quad \Omega_2 = \left[\begin{array}{cc|cc} \omega^{\ell} & & & \\ & \omega^{-\ell} & & \\ \hline & & \omega^{k} & \\ & & & \omega^{-k} \end{array}\right],$$

$$\Omega_3 = \left[\begin{array}{cc|cc} \omega^{-\ell} & & & \\ & \omega^{\ell} & & \\ \hline & & \omega^{-k} & \\ & & & \omega^{k} \end{array}\right], \quad D = \left[\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline 1 & & & \\ & 1 & & \end{array}\right].$$

This representation is easily seen to be equivalent to the eight-dimensional real irreducible representation $(8; k, \ell)$.

A.2. Construction of the Function Φ

A systematic construction procedure of the function Φ in (3.182) is given here.

Basic Facts about Integer Matrices

We present here some basic facts about integer matrices[18] that are used in the construction of the correspondence Φ and in the proofs in Appendix A.3.

A square integer matrix U is called *unimodular* if its determinant is equal to ± 1; U is unimodular if and only if its inverse U^{-1} exists and is an integer matrix. For an integer matrix A, the kth *determinantal divisor*, denoted $d_k(A)$, is the greatest common divisor of all $k \times k$ minors (subdeterminants) of A. By convention we put $d_0(A) = 1$.

The first theorem states that every integer matrix can be brought to the *Smith normal form* by a bilateral unimodular transformation.

Theorem A.1. *Let A be an $m \times n$ integer matrix. There exist unimodular matrices U and V such that*

$$UAV = \left[\begin{array}{ccc|c} \alpha_1 & & 0 & \\ & \ddots & & O_{r,n-r} \\ 0 & & \alpha_r & \\ \hline & O_{m-r,r} & & O_{m-r,n-r} \end{array}\right], \tag{A.32}$$

where $r = \mathrm{rank}\, A$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$ are positive integers with the divisibility property:[19]

$$\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_r.$$

Such integers $\alpha_1, \alpha_2, \ldots, \alpha_r$ are uniquely determined by A, and are expressed as

$$\alpha_k = \frac{d_k(A)}{d_{k-1}(A)}, \qquad k = 1, \ldots, r,$$

in terms of the determinantal divisors $d_1(A), d_2(A), \ldots, d_r(A)$ of A.

The second theorem gives a solvability criterion for a system of linear equations in unknown integer vectors.

Theorem A.2. *Let A be an $m \times n$ integer matrix and $\boldsymbol{b}$ an m-dimensional integer vector. The following two conditions* (a) *and* (b) *are equivalent.*

(a) *The system of equations $A\boldsymbol{x} = \boldsymbol{b}$ admits an integer solution $\boldsymbol{x}$.*
(b) *Two matrices A and $[A \mid \boldsymbol{b}]$ share the same determinantal divisors, i.e.,* $\mathrm{rank}\, A = \mathrm{rank}\, [A \mid \boldsymbol{b}]$ *and $d_k(A) = d_k([A \mid \boldsymbol{b}])$ for all k.*

As a corollary of Theorem A.2 we can obtain the following facts.

[18] See Schrijver (1998) for more details on integer matrices.
[19] Notation "$a \mid b$" means that a divides b, that is, b is a multiple of a.

Proposition A.1. *Let $a_1, \dots, a_n$ be integers.*

(i) $\gcd(a_1, \dots, a_n) = 1$ *if and only if there exist some integers $x_1, \dots, x_n$ such that $a_1 x_1 + \cdots + a_n x_n = 1$.*

(ii) *An integer b is divisible by* $\gcd(a_1, \dots, a_n)$ *if and only if there exist some integers $x_1, \dots, x_n$ such that $a_1 x_1 + \cdots + a_n x_n = b$.*

The third theorem is a kind of duality theorem, which is sometimes referred to as the *integer analogue of the Farkas lemma*.

Theorem A.3. *Let A be an $m \times n$ integer matrix and $\boldsymbol{b}$ an m-dimensional integer vector. The following two conditions* (a) *and* (b) *are equivalent.*

(a) *The system of equations $A\boldsymbol{x} = \boldsymbol{b}$ admits an integer solution $\boldsymbol{x}$.*
(b) *We have "$\boldsymbol{y}^\top A \in \mathbb{Z}^n \Longrightarrow \boldsymbol{y}^\top \boldsymbol{b} \in \mathbb{Z}$" for any m-dimensional vector $\boldsymbol{y}$.*

Construction of Φ via the Smith Normal Form

The correspondence $\Phi : (k, \ell) \mapsto (\alpha, \beta)$ can be constructed with the aid of the Smith normal form. Recall notations

$$\hat{k} = \frac{k}{\gcd(k, \ell, n)}, \quad \hat{\ell} = \frac{\ell}{\gcd(k, \ell, n)}, \quad \hat{n} = \frac{n}{\gcd(k, \ell, n)}$$

in (3.164), for which

$$\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1. \tag{A.33}$$

By the definition of the correspondence Φ of (3.182) in Proposition 3.21, we have

$$\mathcal{A}(k, \ell, n) = \mathcal{L}(\alpha, \beta) \ \text{ for } \ (\alpha, \beta) = \Phi(k, \ell, n), \tag{A.34}$$

where

$$\mathcal{A}(k, \ell, n) = \{(a, b) \in \mathbb{Z}^2 \mid \hat{k}a + \hat{\ell}b \equiv 0, \ \hat{\ell}a - \hat{k}b \equiv 0 \quad \text{mod } \hat{n}\}, \tag{A.35}$$

$$\mathcal{L}(\alpha, \beta) = \{(a, b) \in \mathbb{Z}^2 \mid (a, b) = n_1(\alpha, \beta) + n_2(-\beta, \alpha), \ n_1, n_2 \in \mathbb{Z}\}. \tag{A.36}$$

The condition in the definition of $\mathcal{A}(k, \ell, n)$ can be rewritten in a matrix form as

$$\begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{mod } \hat{n}. \tag{A.37}$$

We define matrices K and A as

$$K = \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix}, \quad A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \tag{A.38}$$

which play the key role in our analysis. Note that

$$\mathcal{L}(\alpha, \beta) = \left\{(a, b) \mid \begin{bmatrix} a \\ b \end{bmatrix} = A \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}; \ n_1, n_2 \in \mathbb{Z}\right\} \tag{A.39}$$

by (A.36).

The condition for $\mathcal{A}(k, \ell, n)$ in (A.37) is equivalent to the existence of integers p and q such that

$$\left[\begin{array}{cc|cc} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{array}\right] \begin{bmatrix} a \\ b \\ p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A.40}$$

Since the determinantal divisors d_1 and d_2 of this 2×4 coefficient matrix are

$$\begin{aligned} d_1 &= \gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1, \\ d_2 &= \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{k}\hat{n}, \hat{\ell}\hat{n}, \hat{n}^2) = \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n} \gcd(\hat{k}, \hat{\ell}, \hat{n})) \\ &= \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}), \end{aligned}$$

the Smith normal form of that matrix is given (see Theorem A.1) as

$$U \left[\begin{array}{cc|cc} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{array}\right] V = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \end{array}\right], \tag{A.41}$$

where U and V are unimodular matrices and

$$\kappa = \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}). \tag{A.42}$$

The 4×4 matrix V for the Smith normal form in (A.41) affords an explicit representation of the correspondence Φ that is defined rather implicitly by the relationship in (A.34). As stated in the following proposition, the correspondence $(\alpha, \beta) = \Phi(k, \ell, n)$ is encoded in the upper-right block of a suitably chosen matrix V. Partition the matrix V into 2×2 submatrices as

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix},$$

and recall the matrix A in (A.38) that is parameterized by (α, β).

Proposition A.2. *We can take V such that $V_{12} = A$ for some (α, β) with $\alpha > \beta \geq 0$. Then $\Phi(k, \ell, n) = (\alpha, \beta)$.*

Proof. Putting

$$\boldsymbol{a} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \boldsymbol{p} = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = V^{-1} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{p} \end{bmatrix}$$

and using (A.41), we can rewrite (A.40) as

$$U[K \mid -\hat{n}I]V \cdot V^{-1} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{p} \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \end{array}\right] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{0}.$$

This shows that $\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{y}$ is free. Therefore, the solutions of (A.40) are given as

$$\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{p} \end{bmatrix} = V \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y} \in \mathbb{Z}^2.$$

This means, by (A.34), that

$$\mathcal{L}(\alpha,\beta) = \{\boldsymbol{a} = (a,b)^\top \mid \boldsymbol{a} = V_{12}\boldsymbol{y},\ \boldsymbol{y} \in \mathbb{Z}^2\}.$$

By comparing this with (A.39), we see that the column vectors of V_{12} and those of A are both basis vectors of the same lattice. As is well-known, this implies that the matrices V_{12} and A are related as $V_{12}W = A$ for some unimodular matrix W. Therefore,

$$\tilde{V} = V \begin{bmatrix} I & O \\ O & W \end{bmatrix} = \begin{bmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ \tilde{V}_{21} & \tilde{V}_{22} \end{bmatrix}$$

is also a valid choice for the Smith normal form (A.41), with the property that $\tilde{V}_{12} = A$. □

In what follows we assume $V_{12} = A$, i.e.,

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} V_{11} & A \\ V_{21} & V_{22} \end{bmatrix}. \tag{A.43}$$

Remark A.2. In Remark 3.7. we indicated a simpler construction of Φ that works when $\hat{n}/(\hat{k}^2+\hat{\ell}^2)$ is an integer. This simpler construction can also be understood in the framework of the general method here. Let U and V_{11} be some unimodular matrices that transform the matrix K in (A.38) to its Smith normal form: $UKV_{11} = \mathrm{diag}\,(1,\kappa)$. By choosing

$$V_{12} = \frac{\hat{n}}{\hat{k}^2+\hat{\ell}^2}\begin{bmatrix} \hat{k} & -\hat{\ell} \\ \hat{\ell} & \hat{k} \end{bmatrix}, \quad V_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_{22} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

in (A.43), we obtain a unimodular matrix V since $|\det V| = |\det V_{11}| \cdot |\det V_{22}| = 1$. Then we have (A.41), and therefore $(\alpha,\beta) = \Phi(k,\ell,n)$ is obtained from the first column of V_{12}, i.e., $(\alpha,\beta) = m(\hat{k},\hat{\ell})$ with $m = \hat{n}/(\hat{k}^2+\hat{\ell}^2)$.

□

The use of the Smith normal form is demonstrated below when $\hat{n}/(\hat{k}^2+\hat{\ell}^2)$ is not an integer, whereas when $\hat{n}/(\hat{k}^2+\hat{\ell}^2)$ is an integer, the simpler method of construction in Remark 3.7 is used.

The example is a case with a solution of type V and without one of type T.

Example A.2. [Case 1 of Proposition 3.23] For $(k,\ell,n) = (2m,m,6m)$ with $m \geq 1$, we have $(\hat{k},\hat{\ell},\hat{n}) = (2,1,6)$, $\hat{k}^2+\hat{\ell}^2 = 5$, and $\kappa = \gcd(5,6) = 1$. The transformation to the Smith normal form in (A.41) is given as

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \left[\begin{array}{cc|cc} 2 & 1 & -6 & 0 \\ 1 & -2 & 0 & -6 \end{array}\right] \left[\begin{array}{cc|cc} 2 & 1 & 6 & 0 \\ 1 & -2 & 0 & 6 \\ \hline 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

This shows $\mathcal{A}(2m,m,6m) = \mathcal{L}(6,0)$, i.e., $\Phi(2m,m,6m) = (6,0) = (\alpha,\beta)$. We have $\alpha = \hat{n} = 6$ and $(\alpha',\beta') = (6,0)$ by (3.178). This is a case of $(\alpha,\beta) = (\alpha',\beta')$, and we have

$$\Sigma_0(\alpha,\beta) = \Sigma_0(\alpha',\beta') = \Sigma_0(\alpha,\beta) \cap \Sigma_0(\alpha',\beta') = \Sigma_0(6,0).$$

When $m = 1$, $\Sigma_0(6,0)$ reduces to $\langle r \rangle$. We have $(\hat{\alpha},\hat{\beta}) = (1,0)$, $\hat{D} = 1 \notin 2\mathbb{Z}$, $\gcd(\hat{k}-\hat{\ell},\hat{n}) = \gcd(1,6) = 1 \notin 2\mathbb{Z}$, and **GCD-div** since $2\gcd(\hat{k},\hat{\ell}) = 2\gcd(2,1) = 2$ is divisible by $\kappa = 1$.

□

A.3. Proofs of Propositions 3.15, 3.17, and 3.18

In this section we establish a series of propositions, which together serve as the proofs of Propositions 3.15, 3.17, and 3.18 presented in Section 3.5.6.

We first focus on Proposition 3.18.

Proposition A.3.

(i) $\gcd(\hat{\alpha}+\hat{\beta}, \hat{\alpha}-\hat{\beta}) \in \{1, 2\}$.

(ii) $\gcd(\hat{\alpha}+\hat{\beta}, \hat{\alpha}-\hat{\beta}) = 2 \iff \hat{D} \in 2\mathbb{Z}$.

(iii) $\gcd(\hat{\alpha}+\hat{\beta}, \hat{\alpha}-\hat{\beta}) = 1 \iff \hat{D} \notin 2\mathbb{Z}$.

Proof. (i) Since $\gcd(\hat{\alpha}, \hat{\beta}) = 1$, Proposition A.1(i) implies the existence of integers x and y such that $x\hat{\alpha} + y\hat{\beta} = 1$. For $p = x + y$, $q = x - y$, we have

$$p(\hat{\alpha}+\hat{\beta}) + q(\hat{\alpha}-\hat{\beta}) = 2(x\hat{\alpha} + y\hat{\beta}) = 2.$$

Then Proposition A.1(ii) shows that 2 is divisible by $\gcd(\hat{\alpha}+\hat{\beta}, \hat{\alpha}-\hat{\beta})$, which is equivalent to the statement of (i) of this proposition.

(ii) We have $\{1, 2\} \ni \gcd(\hat{\alpha}+\hat{\beta}, \hat{\alpha}-\hat{\beta}) = \gcd(\hat{\alpha}+\hat{\beta}, 2\hat{\alpha})$. Therefore, $\gcd(\hat{\alpha}+\hat{\beta}, \hat{\alpha}-\hat{\beta}) = 2$ if and only if $\hat{\alpha}+\hat{\beta} \in 2\mathbb{Z}$. Finally we note a simple identity $\hat{D} = (\hat{\alpha}+\hat{\beta})^2 - 2\hat{\alpha}\hat{\beta}$ to see that $\hat{\alpha}+\hat{\beta} \in 2\mathbb{Z}$ if and only if $\hat{D} \in 2\mathbb{Z}$.

(iii) This is obvious from (i) and (ii) above. □

Proposition A.4.

$$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\beta, \alpha) = \begin{cases} \Sigma_0(\alpha'', 0) & \text{if } \hat{D} \notin 2\mathbb{Z}, \\ \Sigma_0(\beta'', \beta'') & \text{if } \hat{D} \in 2\mathbb{Z} \end{cases} \tag{A.44}$$

with

$$\alpha'' = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}, \quad \beta'' = \frac{D(\alpha, \beta)}{2\gcd(\alpha, \beta)}. \tag{A.45}$$

Proof. First note that $\Sigma_0(\alpha, \beta) \cap \Sigma_0(\beta, \alpha)$ is the subgroup generated by r and $p_1^a p_2^b$ for $(a, b) \in \mathcal{L}(\alpha, \beta) \cap \mathcal{L}(\beta, \alpha)$. In considering $\mathcal{L}(\alpha, \beta)$ of (A.36), it is convenient to have $\mathcal{H}(\alpha, \beta)$ of (3.96) in mind, as it has a natural correspondence with $\mathcal{L}(\alpha, \beta)$. The set $\mathcal{H}(\alpha, \beta) \cap \mathcal{H}(\beta, \alpha)$ is a square sublattice with the reflection symmetry with respect to the x-axis, and hence it can be represented as $\mathcal{H}(\alpha'', 0)$ or $\mathcal{H}(\beta'', \beta'')$ for some α'' or β''. Such α'' is determined as the minimum α'' satisfying $\mathcal{L}(\alpha'', 0) \subseteq \mathcal{L}(\alpha, \beta)$, and β'' as the minimum β'' satisfying $\mathcal{L}(\beta'', \beta'') \subseteq \mathcal{L}(\alpha, \beta)$. Then $\mathcal{L}(\alpha, \beta) \cap \mathcal{L}(\beta, \alpha)$ coincides with the larger of $\mathcal{L}(\alpha'', 0)$ and $\mathcal{L}(\beta'', \beta'')$.

The parameter α'' is determined as follows. The inclusion $\mathcal{L}(\alpha'', 0) \subseteq \mathcal{L}(\alpha, \beta)$ holds if and only if integers n_1 and n_2 exist such that

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \alpha'' \\ 0 \end{bmatrix}.$$

By the solvability criterion using determinantal divisors given in Theorem A.2, this holds if and only if

$$d_1\left(\left[\begin{array}{cc|c} \alpha & -\beta & \alpha'' \\ \beta & \alpha & 0 \end{array}\right]\right) \text{ equals } d_1\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}\right) = \gcd(\alpha, \beta),$$
$$d_2\left(\left[\begin{array}{cc|c} \alpha & -\beta & \alpha'' \\ \beta & \alpha & 0 \end{array}\right]\right) \text{ equals } d_2\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}\right) = D(\alpha, \beta).$$

The former condition is equivalent to α'' being a multiple of $\gcd(\alpha, \beta)$, and the latter to α'' being a multiple of $D(\alpha, \beta)/\gcd(\alpha, \beta)$. Hence we have $\alpha'' = D(\alpha, \beta)/\gcd(\alpha, \beta)$, which is a multiple of $\gcd(\alpha, \beta)$ since $D(\alpha, \beta)/\gcd(\alpha, \beta) = \hat{D} \gcd(\alpha, \beta)$.

The parameter β'' is determined as follows. The inclusion $\mathcal{L}(\beta'', \beta'') \subseteq \mathcal{L}(\alpha, \beta)$ holds if and only if integers n_1 and n_2 exist such that

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \beta'' \\ \beta'' \end{bmatrix}.$$

Again by Theorem A.2, this holds if and only if

$$d_1\left(\left[\begin{array}{cc|c} \alpha & -\beta & \beta'' \\ \beta & \alpha & \beta'' \end{array}\right]\right) \text{ equals } d_1\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}\right) = \gcd(\alpha, \beta),$$
$$d_2\left(\left[\begin{array}{cc|c} \alpha & -\beta & \beta'' \\ \beta & \alpha & \beta'' \end{array}\right]\right) \text{ equals } d_2\left(\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}\right) = D(\alpha, \beta).$$

The former condition is equivalent to β'' being a multiple of $\gcd(\alpha, \beta)$, and the latter to β'' being a multiple of

$$\frac{D(\alpha, \beta)}{\gcd(\alpha + \beta, \alpha - \beta)} = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta) \gcd(\hat{\alpha} + \hat{\beta}, \hat{\alpha} - \hat{\beta})}.$$

Then by Proposition A.3, we obtain

$$\beta'' = \begin{cases} D(\alpha, \beta)/\gcd(\alpha, \beta) & \text{if } \hat{D} \notin 2\mathbb{Z}, \\ D(\alpha, \beta)/(2\gcd(\alpha, \beta)) & \text{if } \hat{D} \in 2\mathbb{Z}. \end{cases}$$

We have $\mathcal{L}(\alpha'', 0) \supset \mathcal{L}(\beta'', \beta'')$ (with $\beta'' = \alpha''$) if $\hat{D} \notin 2\mathbb{Z}$, and $\mathcal{L}(\beta'', \beta'') \supset \mathcal{L}(\alpha'', 0)$ (with $\beta'' = \alpha''/2$) if $\hat{D} \in 2\mathbb{Z}$. This completes the proof. □

We next focus on Proposition 3.15(i). With this aim in mind, we rephrase (A.44) in Proposition A.4 in terms of (k, ℓ) instead of (α, β).

Proposition A.5.

(i) $\gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell}, \hat{n}) \in \{1, 2\}$.

(ii) $\gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell}, \hat{n}) = 2 \iff \gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}$.

(iii) $\gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell}, \hat{n}) = 1 \iff \gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}$.

Proof. (i) Since $\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1$, Proposition A.1(i) implies the existence of integers a, b, and c such that $a\hat{k} + b\hat{\ell} + c\hat{n} = 1$. For $p = a + b$, $q = a - b$, $r = 2c$, we have

$$p(\hat{k} + \hat{\ell}) + q(\hat{k} - \hat{\ell}) + r\hat{n} = 2(a\hat{k} + b\hat{\ell} + c\hat{n}) = 2.$$

Then Proposition A.1(ii) shows that 2 is divisible by $\gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell}, \hat{n})$, which is equivalent to the claim in (i).

(ii) We have $\{1, 2\} \ni \gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell}, \hat{n}) = \gcd(\hat{k} - \hat{\ell}, 2\hat{\ell}, \hat{n})$. Hence follows the claim.

(iii) This is obvious from (i) and (ii) above. □

Proposition A.6.

$$\Sigma_0(\alpha, \beta) \cap \Sigma_0(\beta, \alpha) = \begin{cases} \Sigma_0(\hat{n}, 0) & \text{if } \gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}, \\ \Sigma_0(\hat{n}/2, \hat{n}/2) & \text{if } \gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}. \end{cases} \tag{A.46}$$

Proof. Recall the notation $\mathcal{A}(k, \ell, n)$ in (A.35). By the same argument as in the proof of Proposition A.4, we compute the minimum α'' satisfying $(\alpha'', 0) \in \mathcal{A}(k, \ell, n)$ and the minimum β'' satisfying $(\beta'', \beta'') \in \mathcal{A}(k, \ell, n)$. Then $\mathcal{L}(\alpha, \beta) \cap \mathcal{L}(\beta, \alpha)$ coincides with the larger of $\mathcal{L}(\alpha'', 0)$ and $\mathcal{L}(\beta'', \beta'')$.

By the definition of $\mathcal{A}(k, \ell, n)$ in (A.35) we have $(\alpha'', 0) \in \mathcal{A}(k, \ell, n)$ if and only if

$$\hat{k}\alpha'' \equiv 0, \quad \hat{\ell}\alpha'' \equiv 0 \quad \mod \hat{n}.$$

Since $\gcd(\hat{k}, \hat{\ell}, \hat{n}) = 1$, the smallest α'' satisfying this condition is given by $\alpha'' = \hat{n}$. As for β'', we have $(\beta'', \beta'') \in \mathcal{A}(k, \ell, n)$ if and only if

$$(\hat{k} + \hat{\ell})\beta'' \equiv 0, \quad (\hat{k} - \hat{\ell})\beta'' \equiv 0 \quad \mod \hat{n}.$$

The smallest β'' satisfying this condition is given by

$$\beta'' = \frac{\hat{n}}{\gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell}, \hat{n})} = \begin{cases} \hat{n} & \text{if } \gcd(\hat{k} - \hat{\ell}, \hat{n}) \notin 2\mathbb{Z}, \\ \hat{n}/2 & \text{if } \gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z}, \end{cases}$$

where Proposition A.5 is used. We finally note $\mathcal{L}(\hat{n}, \hat{n}) \subset \mathcal{L}(\hat{n}, 0)$ and $\mathcal{L}(\hat{n}, 0) \subset \mathcal{L}(\hat{n}/2, \hat{n}/2)$ if $\hat{n} \in 2\mathbb{Z}$. This completes the proof. □

Proposition A.7.

(i) $\gcd(\hat{k} - \hat{\ell}, \hat{n}) \in 2\mathbb{Z} \iff \hat{D} \in 2\mathbb{Z}$.

(ii)

$$\hat{n} = \frac{D(\alpha, \beta)}{\gcd(\alpha, \beta)}. \tag{A.47}$$

Proof. This follows from a comparison of Proposition A.4 with Proposition A.6. □

We next focus on Proposition 3.15(ii).

Proposition A.8.

$$\frac{\hat{n}}{\gcd(\hat{k}^2+\hat{\ell}^2,\hat{n})} = \gcd(\alpha,\beta). \tag{A.48}$$

Proof. We rely on the representation of Φ given in Proposition A.2 in terms of the transformation matrix V in the Smith normal form of $[K \mid -\hat{n}I]$ in (A.41) with (A.38). Let

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

be the inverse of the matrix V in (A.41). We have $|\det V| = 1$ since V is unimodular. By a well-known formula in linear algebra and $V_{12} = A$ in (A.43), we have

$$|\det W_{12}| = |\det V_{12}|/|\det V| = |\det A| = D(\alpha,\beta). \tag{A.49}$$

On the other hand, it follows from (A.41) with $V = W^{-1}$ that

$$U \left[\begin{array}{cc|cc} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \end{array}\right] \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

This implies

$$-\hat{n}U = \begin{bmatrix} 1 & 0 \\ 0 & \kappa \end{bmatrix} W_{12},$$

which shows

$$\hat{n}^2 = \kappa \, |\det W_{12}| \tag{A.50}$$

since $|\det U| = 1$.

Combining (A.49) and (A.50) with the expression (A.42) of κ, we obtain

$$\hat{n}^2 = \kappa \, D(\alpha,\beta) = \gcd(\hat{k}^2+\hat{\ell}^2,\hat{n}) \cdot D(\alpha,\beta).$$

By eliminating $D(\alpha,\beta)$ using (A.47), we obtain (A.48). □

Propositions A.9–A.12 below are concerned with the symmetry of $\mathcal{A}(k,\ell,n)$ of (A.35), or that of $\Sigma_0(\alpha,\beta)$. Interestingly, such symmetry consideration leads to the proof of Proposition 3.17 of duality nature.

Proposition A.9. *The four conditions* (a), (b), (c), *and* (d) *below are equivalent.*

(a) $(u_1,u_2) \in \mathbb{Z}^2$ *exists such that*

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix} \equiv \begin{bmatrix} \hat{\ell} & \hat{k} \end{bmatrix} \mod \hat{n}. \tag{A.51}$$

(b) *An integer matrix U exists such that*

$$U \begin{bmatrix} \hat{k} & \hat{\ell} \\ \hat{\ell} & -\hat{k} \end{bmatrix} \equiv \begin{bmatrix} \hat{\ell} & \hat{k} \\ \hat{k} & -\hat{\ell} \end{bmatrix} \mod \hat{n}. \tag{A.52}$$

(c) $\gcd(\hat{k}^2-\hat{\ell}^2,\ 2\hat{k}\hat{\ell})$ is divisible by $\gcd(\hat{k}^2+\hat{\ell}^2,\hat{n})$.

(d) **GCD-div** *in* (3.167):

$$2\gcd(\hat{k},\hat{\ell}) \text{ is divisible by } \gcd(\hat{k}^2+\hat{\ell}^2,\hat{n}).$$

Proof. First, we show (a) ⇔ (b). For $(u_1, u_2) \in \mathbb{Z}^2$ satisfying (A.51), the matrix $U = \begin{bmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{bmatrix}$ is an integer matrix that satisfies (A.52). This shows (a) ⇒ (b), whereas (b) ⇒ (a) is obvious.

Next, we show (a) ⇔ (c). The condition (a) is equivalent to the existence of integers u_1, u_2, p, and q that satisfy

$$\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \\ q \end{bmatrix} = \begin{bmatrix} \hat{\ell} \\ \hat{k} \end{bmatrix}.$$

By the solvability criterion using determinantal divisors given in Theorem A.2, this holds if and only if

$$d_1\left(\left[\begin{array}{cccc|c} \hat{k} & \hat{\ell} & -\hat{n} & 0 & \hat{\ell} \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} & \hat{k} \end{array}\right]\right) \quad \text{equals} \quad d_1\left(\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix}\right) = 1,$$

$$d_2\left(\left[\begin{array}{cccc|c} \hat{k} & \hat{\ell} & -\hat{n} & 0 & \hat{\ell} \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} & \hat{k} \end{array}\right]\right) \quad \text{equals} \quad d_2\left(\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix}\right).$$

The former condition imposes nothing and the latter reduces to (c). We have thus shown (a) ⇔ (c).

Finally, we show (c) ⇔ (d). Since $\hat{k}^2 + \hat{\ell}^2$ is a multiple of $\kappa = \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n})$, $\hat{k}^2 - \hat{\ell}^2$ is divisible by κ if and only if $(\hat{k}^2 - \hat{\ell}^2) + (\hat{k}^2 + \hat{\ell}^2) = 2\hat{k}^2$ is divisible by κ. Therefore, $\gcd(\hat{k}^2 - \hat{\ell}^2,\ 2\hat{k}\hat{\ell})$ is divisible by κ if and only if $\gcd(2\hat{k}^2,\ 2\hat{k}\hat{\ell}) = 2\hat{k}\gcd(\hat{k}, \hat{\ell})$ is divisible by κ. Since $\gcd(\hat{k}, \hat{n}) = 1$, $\gcd(\hat{k}^2 - \hat{\ell}^2,\ 2\hat{k}\hat{\ell})$ is divisible by κ if and only if $2\gcd(\hat{k}, \hat{\ell})$ is divisible by κ. □

Proposition A.10. *The following two conditions are equivalent.*

(a) $\mathcal{A}(k, \ell, n) = \mathcal{A}(\ell, k, n)$.

(b) $(a, b) \in \mathcal{A}(k, \ell, n) \Longrightarrow (b, a) \in \mathcal{A}(k, \ell, n)$.

Proof. The defining equations in (A.35) for $\mathcal{A}(k, \ell, n)$ are invariant under the change of variables $(a, b, k, \ell) \mapsto (b, a, \ell, k)$, and therefore, $\mathcal{A}(\ell, k, n) = \{(b, a) \mid (a, b) \in \mathcal{A}(k, \ell, n)\}$. This shows the equivalence of (a) and (b). □

Proposition A.11. *The following two conditions are equivalent.*

(a) $\mathcal{A}(k, \ell, n) = \mathcal{A}(\ell, k, n)$.

(b) *An integer matrix U exists such that* (A.52) *holds.*

Proof. Although the claim is intuitively obvious from symmetry, we provide here a rigorous proof on the basis of Theorem A.3 (the integer analogue of the Farkas lemma).

As in the proof of Proposition A.9, the condition (b) is equivalent to the existence of integer tuples (u_1, u_2, p, q) and (u'_1, u'_2, p', q') such that

$$\begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \\ q \end{bmatrix} = \begin{bmatrix} \hat{\ell} \\ \hat{k} \end{bmatrix}, \quad \begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ p' \\ q' \end{bmatrix} = \begin{bmatrix} \hat{k} \\ -\hat{\ell} \end{bmatrix}.$$

By Theorem A.3, the existence of such (u_1, u_2, p, q) is equivalent to the following condition:

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} \hat{k} & \hat{\ell} & -\hat{n} & 0 \\ \hat{\ell} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} \in \mathbb{Z}^4 \quad \Longrightarrow \quad \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} \hat{\ell} \\ \hat{k} \end{bmatrix} \in \mathbb{Z},$$

which can be rewritten as

$$[\hat{k}y_1 + \hat{\ell}y_2,\ \hat{\ell}y_1 - \hat{k}y_2,\ -\hat{n}y_1,\ -\hat{n}y_2\,] \in \mathbb{Z}^4 \quad \Longrightarrow \quad \hat{\ell}y_1 + \hat{k}y_2 \in \mathbb{Z}.$$

Integrality condition for the third and fourth components allows us to put $y_1 = a/\hat{n}$ and $y_2 = b/\hat{n}$ with integers a and b. Then we can rewrite the above as

$$\hat{k}a + \hat{\ell}b \equiv 0, \quad \hat{\ell}a - \hat{k}b \equiv 0 \mod \hat{n} \quad \Longrightarrow \quad \hat{\ell}a + \hat{k}b \equiv 0 \mod \hat{n}.$$

Similarly, the existence of (u_1', u_2', p', q') above is equivalent to the following condition:

$$\hat{k}a + \hat{\ell}b \equiv 0, \quad \hat{\ell}a - \hat{k}b \equiv 0 \mod \hat{n} \quad \Longrightarrow \quad \hat{k}a - \hat{\ell}b \equiv 0 \mod \hat{n}.$$

The above two conditions together are nothing but the statement that $(a, b) \in \mathcal{A}(k, \ell, n)$ implies $(b, a) \in \mathcal{A}(k, \ell, n)$, which is equivalent to (a) by Proposition A.10. □

Proposition A.12. *Let* $(\alpha, \beta) = \Phi(k, \ell, n)$.
(i) $\Sigma_0(\alpha, \beta) = \Sigma_0(\beta, \alpha) \iff \beta = 0$ *or* $\alpha = \beta$.
(ii) $\Sigma_0(\alpha, \beta) = \Sigma_0(\beta, \alpha) \iff$ **GCD-div** *in* (3.167).

Proof. (i) is obvious, and (ii) follows from Propositions A.9 and A.11. Note that $\Sigma_0(\alpha, \beta)$ is the subgroup generated by r and $p_1{}^a p_2{}^b$ for $(a, b) \in \mathcal{A}(k, \ell, n)$. □

A.4. Details of Stability Analysis

We introduced the $n \times n$ square lattice as a two-dimensional discretized space and presented the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ labeling the symmetry of this lattice in Section 3.2. We obtained the irreducible decomposition of the permutation representation of the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in Sections 3.3 and 3.4 to identify the irreducible representations. We presented the equivariant branching lemma in Section 3.5 as a pertinent and sufficient means to show the existence of the square patterns for each irreducible representation.

In this section, we advance bifurcation analysis by solving bifurcation equations as a more informative means to investigate the properties of bifurcating solutions for each irreducible representations. We derive the expanded forms of bifurcation equations by exploiting the symmetry of the square lattice. We evaluate the stability of bifurcating solutions and present stability conditions.

This section is organized as follows. Fundamentals of analysis are summarized in Appendix A.4.1. Bifurcation points of multiplicity 1, 2, 4, and 8 are studied in Appendices A.4.2–A.4.5, respectively.

Table A.2: Irreducible representations of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ to be considered in bifurcation analysis.

$n \setminus M$	1	2	4	8
$2m$	$(1;+,+,+), (1;+,+,-)$	$(2;+,+)$	$(4;k,0;+), (4;k,k;+), (4;n/2,\ell,+)$	$(8;k,\ell)$
$2m-1$	$(1;+,+,+)$		$(4;k,0;+), (4;k,k;+)$	$(8;k,\ell)$

$(4;k,0;+), (4;k,k;+)$ with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$;
$(4;n/2,\ell;+)$ with $1 \leq \ell \leq \lfloor (n-1)/2 \rfloor$;
$(8;k,\ell)$ with $1 \leq \ell \leq k-1,\ 2 \leq k \leq \lfloor (n-1)/2 \rfloor$

A.4.1. Analysis Procedure Solving Bifurcation Equations

Let us consider the governing equation

$$\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{0} \tag{A.53}$$

in (2.3) endowed with the equivariance to the group $G = \mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ as

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \phi), \quad g \in G \tag{A.54}$$

in (2.7). Recall that ϕ is the bifurcation parameter, $\boldsymbol{\lambda} \in \mathbb{R}^K$ is an $K = n^2$ dimensional independent variable vector expressing a distribution of mobile population, $\boldsymbol{F} : \mathbb{R}^K \times \mathbb{R} \to \mathbb{R}^K$ is a nonlinear function, and T is the K-dimensional permutation representation of the group G. The Jacobian matrix of $\boldsymbol{F}$ is an $K \times K$ matrix expressed as

$$J(\boldsymbol{\lambda}, \phi) = \left(\frac{\partial F_i}{\partial \lambda_j} \,\middle|\, i, j = 1, \ldots, K \right). \tag{A.55}$$

Let $(\boldsymbol{\lambda}_{\mathrm{c}}, \phi_{\mathrm{c}})$ be a critical point of multiplicity M (≥ 1), at which the Jacobian matrix of $\boldsymbol{F}$ has a rank deficiency M. The critical point $(\boldsymbol{\lambda}_{\mathrm{c}}, \phi_{\mathrm{c}})$ is assumed to be G-symmetric in the sense of

$$T(g)\boldsymbol{\lambda}_{\mathrm{c}} = \boldsymbol{\lambda}_{\mathrm{c}}, \quad g \in G. \tag{A.56}$$

Moreover, it is assumed to be group-theoretic, which means, by definition, that the M-dimensional kernel space of the Jacobian matrix at $(\boldsymbol{\lambda}_{\mathrm{c}}, \phi_{\mathrm{c}})$ is irreducible with respect to the representation T. The critical point $(\boldsymbol{\lambda}_{\mathrm{c}}, \phi_{\mathrm{c}})$ is associated with one of the irreducible representations μ of G in Table A.2. The multiplicity M corresponds to the dimension of μ, and a matrix representation for μ is denoted by $T^{\mu}(g)$.

By the Liapunov–Schmidt reduction with symmetry (Sattinger, 1979; Chow and Hale, 1982; Golubitsky et al., 1988), the full system of the governing equation in (A.53) is reduced, in the neighborhood of the critical point $(\boldsymbol{\lambda}_{\mathrm{c}}, \phi_{\mathrm{c}})$, to a system of M equations

$$\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi}) = \boldsymbol{0} \tag{A.57}$$

in $\boldsymbol{w} \in \mathrm{Ker}(J_c)$, where $\widetilde{\boldsymbol{F}} \colon \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$ is a function, $\widetilde{\phi} = \phi - \phi_{\mathrm{c}}$ denotes the increment of ϕ, and $\mathrm{Ker}(J_c)$ is the kernel space of $J(\boldsymbol{\lambda}_{\mathrm{c}}, \phi_{\mathrm{c}})$. We define variables $\boldsymbol{w} = (w_1, \ldots, w_M)^\top$ in the bifurcation equation in (A.57) by using the column vectors of $Q^\mu = [\boldsymbol{q}_1^\mu, \ldots, \boldsymbol{q}_M^\mu]$ in Section 3.4.3 that span $\mathrm{Ker}(J_c)$.

In the reduction process, the equivariance in (A.54) of the full system is inherited by the reduced system in (A.57). With the use of the matrix representation $T^\mu(g)$ for the associated irreducible representation μ, the equivariance of $\widetilde{\boldsymbol{F}}$ can be expressed as

$$T^\mu(g)\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\phi}) = \widetilde{\boldsymbol{F}}(T^\mu(g)\boldsymbol{w}, \widetilde{\phi}), \quad g \in G. \tag{A.58}$$

The reduced equation (A.57) can possibly admit multiple solutions $\boldsymbol{w} = \boldsymbol{w}(\widetilde{\phi})$ with $\boldsymbol{w}(0) = \boldsymbol{0}$, since $(\boldsymbol{w}, \widetilde{\phi}) = (\boldsymbol{0}, 0)$ is a singular point of (A.57). This gives rise to bifurcation. Each $\boldsymbol{w}$ uniquely determines a solution $\boldsymbol{\lambda}$ to the full system (A.53).

Group-theoretic bifurcation analysis to investigate the stability of a bifurcating solution for a critical point proceeds as follows:

- Specify an irreducible representation μ of $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ in Table A.2.
- Obtain the expanded form of the bifurcation equation by exploiting the symmetry.
- Obtain a bifurcating solution by using the equivariant branching lemma (Cicogna, 1981; Vanderbauwhede, 1982; Golubitsky et al., 1988) or solving the bifurcation equation.
- Obtain the Jacobian matrix of $\widetilde{\boldsymbol{F}}$.
- Substitute the bifurcating solution into the Jacobian matrix and evaluate the eigenvalues to determine their stability as

$$\begin{cases} \text{linearly stable:} & \text{every eigenvalue has a negative real part,} \\ \text{linearly unstable:} & \text{at least one eigenvalue has a positive real part.} \end{cases}$$

We showed the existence of square patterns by using the equivariant branching lemma in Section 3.5. Additionally, in this section, we show the existence of some other bifurcating solutions by solving bifurcation equations. Theoretically predicted bifurcating solutions are summarized in Table A.3. Stability analysis for these solutions is also conducted in this section.

A.4.2. Bifurcation Point of Multiplicity 1

We consider a critical point associated with the one-dimensional irreducible representation $\mu = (1; +, +, -)$ of the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. The action in $(1; +, +, -)$ on a variable $w \in \mathbb{R}$ can be expressed as

$$r, s : \ w \mapsto w, \quad p_1, p_2 : \ w \mapsto -w. \tag{A.59}$$

This case is nothing but pitchfork bifurcation and is well-known.

The bifurcation equation for a critical point of multiplicity 1 is a one-dimensional equation over $\mathbb{R}$ as

$$\widetilde{F}(w, \widetilde{\phi}) = 0, \tag{A.60}$$

Table A.3: Theoretically predicted bifurcating solutions for critical points with multiplicity M.

M	Bifurcating solutions ($w \in \mathbb{R}$)	Existence conditions
1	w	if n is even
2	$\boldsymbol{w}_{\mathrm{sq}} = (w, w)$	if n is even
	$\boldsymbol{w}_{\mathrm{stripe}} = (w, 0)$	if n is even
4	$\boldsymbol{w}_{\mathrm{sq}} = (w, 0, w, 0)$	Always
	$\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0)$	Always
	$\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0)$	if $\check{n}$ is even
8	$\boldsymbol{w}_{\mathrm{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$	Always
	$\boldsymbol{w}_{\mathrm{sqT}} = (w, 0, w, 0, 0, 0, 0, 0)$	if $2\gcd(\hat{k}, \hat{\ell})$ is not divisible by $\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n})$
	$\boldsymbol{w}_{\mathrm{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$	if $(\hat{k} + \hat{\ell})\gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell})$ is not divisible by $\gcd(2\hat{k}\hat{\ell}, \hat{n})$
	$\boldsymbol{w}_{\mathrm{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$	if $\hat{n}$ is even and $(\hat{k} + \hat{\ell})\gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell})$ is not divisible by $\gcd(2\hat{k}\hat{\ell}, \hat{n})$
	$\boldsymbol{w}_{\mathrm{stripeI}} = (w, 0, 0, 0, 0, 0, 0, 0)$	if $\hat{k}^2 + \hat{\ell}^2$, $2\hat{k}\hat{\ell}$, and $\hat{k}^2 - \hat{\ell}^2$ are not divisible by $\hat{n}$
	$\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$	if $\hat{n}$ is even and $\hat{k}^2 + \hat{\ell}^2$, $2\hat{k}\hat{\ell}$, and $\hat{k}^2 - \hat{\ell}^2$ are not divisible by $\hat{n}$

$\check{n} = n/\gcd(k, n)$ for $M = 4$ in (A.92);

$\hat{n} = n/\gcd(k, \ell, n)$, $\hat{k} = k/\gcd(k, \ell, n)$, $\hat{\ell} = \ell/\gcd(k, \ell, n)$ for $M = 8$ in (A.228) where $(w, \widetilde{\phi}) = (0, 0)$ is assumed to correspond to the critical point. We expand $\widetilde{F}$ into a power series as

$$\widetilde{F}(w, \widetilde{\phi}) = \sum_{a=0} A_a(\widetilde{\phi}) w^a \tag{A.61}$$

with coefficients $A_a(\widetilde{\phi}) \in \mathbb{R}$. Since $(w, \widetilde{\phi}) = (0, 0)$ corresponds to the critical point, we have

$$A_0(0) = 0, \quad A_1(0) = 0.$$

Hence, we have

$$A_1(\widetilde{\phi}) \approx A_1'(0)\widetilde{\phi}.$$

for $A_1'(0)$, which is generically nonzero.[20]

The equivariance of the bifurcation equation to the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the actions of the four elements r, s, p_1, and p_2 generating this group. Hence, the equivariance condition in (A.58) of the bifurcation equation is written for (A.61) as

$$r, s : \widetilde{F}(w, \widetilde{\phi}) = \widetilde{F}(w, \widetilde{\phi}), \tag{A.62}$$

$$p_1, p_2 : -\widetilde{F}(w, \widetilde{\phi}) = \widetilde{F}(-w, \widetilde{\phi}). \tag{A.63}$$

From the equivariance condition (A.63), we have

$$\sum_{a=0} (-A_a(\widetilde{\phi})) w^a = \sum_{a=0} A_a(\widetilde{\phi})(-w)^a.$$

This condition implies $(-1)^a = -1$, that is,

$$a = 2b + 1, \quad b \in \mathbb{Z}_+,$$

where $\mathbb{Z}_+$ represents the set of nonnegative integers. Hence, (A.61) is restricted to

$$\widetilde{F}(w, \widetilde{\phi}) = w \sum_{b=0} A_{2b+1}(\widetilde{\phi}) w^{2b}. \tag{A.64}$$

The form of (A.64) implies that $\widetilde{F}(w, \widetilde{\phi}) = 0$ has the trivial solution and a bifurcating solution. Note that $\widetilde{F}(w, \widetilde{\phi})$ is an odd function in w. Thus, $(w, \widetilde{\phi})$ and $(-w, \widetilde{\phi})$ are conjugate solutions for $\widetilde{F} = 0$. We hereafter call the two solutions that are conjugate as symmetric bifurcating solutions and those that are not as asymmetric ones.

We evaluate the stability of the bifurcating solution by considering the asymptotic form of the bifurcation equation. The asymptotic form of the bifurcation equation in (A.64) becomes

$$\widetilde{F}(w, \widetilde{\phi}) \approx w(A_1'(0)\widetilde{\phi} + A_3(0) w^2), \tag{A.65}$$

[20] Notation $A_1'(0)$ means the derivative of $A_1(\widetilde{\phi})$ with respect to $\widetilde{\phi}$, evaluated at $\widetilde{\phi} = 0$. Generically we have $A_1'(0) \neq 0$ since the group symmetry imposes no condition.

and the Jacobian of $\widetilde{F}$ becomes

$$\widetilde{J}(w, \widetilde{\phi}) = \frac{\partial \widetilde{F}}{\partial w} \approx A_1'(0)\widetilde{\phi} + 3A_3(0)w^2. \tag{A.66}$$

Solving $\widetilde{F} = 0$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{sq} \approx -w^2 \frac{A_3(0)}{A_1'(0)}.$$

Substituting $\widetilde{\phi}_{sq}$ into (A.66), we have

$$\widetilde{J}(w, \widetilde{\phi}_{sq}) \approx 2w^2 A_3(0). \tag{A.67}$$

Hence, the stability of the bifurcating solution in the neighborhood of the critical point depends on the sign of $A_3(0)$, that is,

$$\begin{cases} A_3(0) < 0: & \text{stable,} \\ A_3(0) > 0: & \text{unstable.} \end{cases}$$

A.4.3. Bifurcation Point of Multiplicity 2

We consider a critical point associated with the two-dimensional irreducible representation $\mu = (2; +, +)$ of the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$. The action in $(2; +, +)$ on a two-dimensional vector $(w_1, w_2) \in \mathbb{R}^2$ can be expressed as

$$r : \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} w_2 \\ w_1 \end{bmatrix}, \quad s : \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{A.68}$$

$$p_1 : \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} -w_1 \\ w_2 \end{bmatrix}, \quad p_2 : \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} w_1 \\ -w_2 \end{bmatrix}. \tag{A.69}$$

The bifurcation equation for a critical point of multiplicity 2 is a two-dimensional equation in $\boldsymbol{w} = (w_1, w_2) \in \mathbb{R}^2$ expressed as

$$\widetilde{F}_i(\boldsymbol{w}, \widetilde{\phi}) = 0, \quad i = 1, 2, \tag{A.70}$$

where $(w_1, w_2, \widetilde{\phi}) = (0, 0, 0)$ is assumed to correspond to the critical point. Accordingly, the Jacobian matrix of $\widetilde{\boldsymbol{F}}$ is an 2×2 matrix expressed as

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) = \left(\frac{\partial \widetilde{F}_i}{\partial w_j} \,\middle|\, i, j = 1, \ldots, 2 \right). \tag{A.71}$$

We expand $\widetilde{F}_1$ into a power series as

$$\widetilde{F}_1(w_1, w_2, \widetilde{\phi}) = \sum_{a=0} \sum_{b=0} A_{ab}(\widetilde{\phi}) {w_1}^a {w_2}^b \tag{A.72}$$

with coefficients $A_{ab}(\widetilde{\phi}) \in \mathbb{R}$. Since $(w_1, w_2, \widetilde{\phi}) = (0, 0, 0)$ corresponds to the critical point, we have

$$A_{00}(0) = 0, \quad A_{10}(0) = A_{01}(0) = 0.$$

Since $A'_{10}(0)$ is generically nonzero, we have

$$A_{10}(\widetilde{\phi}) \approx A'_{10}(0)\widetilde{\phi}.$$

The equivariance of the bifurcation equation to the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the actions of the four elements r, s, p_1, and p_2 generating this group. Hence, the equivariance condition (A.58) of the bifurcation equation is written for (A.70) as

$$r: \quad \widetilde{F}_2(w_1, w_2) = \widetilde{F}_1(w_2, w_1), \tag{A.73}$$

$$\widetilde{F}_1(w_1, w_2) = \widetilde{F}_2(w_2, w_1), \tag{A.74}$$

$$s: \quad \widetilde{F}_1(w_1, w_2) = \widetilde{F}_1(w_1, w_2), \tag{A.75}$$

$$\widetilde{F}_2(w_1, w_2) = \widetilde{F}_2(w_1, w_2), \tag{A.76}$$

$$p_1: \; -\widetilde{F}_1(w_1, w_2) = \widetilde{F}_1(-w_1, w_2), \tag{A.77}$$

$$\widetilde{F}_2(w_1, w_2) = \widetilde{F}_2(-w_1, w_2), \tag{A.78}$$

$$p_2: \quad \widetilde{F}_1(w_1, w_2) = \widetilde{F}_1(w_1, -w_2), \tag{A.79}$$

$$-\widetilde{F}_2(w_1, w_2) = \widetilde{F}_2(w_1, -w_2). \tag{A.80}$$

From the equivariance condition (A.77) or (A.80), we have

$$\sum_{a=0}\sum_{b=0}(-A_{ab}(\widetilde{\phi}))w_1{}^a w_2{}^b = \sum_{a=0}\sum_{b=0} A_{ab}(\widetilde{\phi})(-w_1)^a w_2{}^b.$$

From the equivariance condition (A.78) or (A.79), we have

$$\sum_{a=0}\sum_{b=0} A_{ab}(\widetilde{\phi})w_1{}^a w_2{}^b = \sum_{a=0}\sum_{b=0} A_{ab}(\widetilde{\phi})w_1{}^a(-w_2)^b.$$

These conditions imply that a is odd, and b is even. Thus,

$$a = 2c + 1, \quad c \in \mathbb{Z}_+,$$
$$b = 2d, \quad d \in \mathbb{Z}_+.$$

where $\mathbb{Z}_+$ represents the set of nonnegative integers. Hence, $\widetilde{F}_i$ $(i = 1, 2)$ is restricted to

$$\widetilde{F}_1(w_1, w_2, \widetilde{\phi}) = w_1 \sum_{c=0}\sum_{d=0} A_{2c+1,2d}(\widetilde{\phi})w_1{}^{2c}w_2{}^{2d}. \tag{A.81}$$

$$\widetilde{F}_2(w_1, w_2, \widetilde{\phi}) = w_2 \sum_{c=0}\sum_{d=0} A_{2c+1,2d}(\widetilde{\phi})w_2{}^{2c}w_1{}^{2d}. \tag{A.82}$$

Therein, $\widetilde{F}_2$ is obtained by (A.73).

We have the following propositions on the existence and the symmetry of bifurcating solutions by solving the bifurcation equation.

Proposition A.13. *For a critical point of multiplicity 2 associated with* $\mu = (2; +, +)$*, we have the following bifurcating solutions:*

- the stripe pattern $\boldsymbol{w}_{\text{stripe}} = (w, 0)$ $(w \in \mathbb{R})$,
- the square pattern $\boldsymbol{w}_{\text{sq}} = (w, w)$ $(w \in \mathbb{R})$.

Proof. Substituting $\boldsymbol{w}_{\text{stripe}} = (w, 0)$ into (A.81), we have

$$\widetilde{F}_1(w, 0, \widetilde{\phi}) = w \sum_{a=0}^{\infty} A_{2a+1,0}(\widetilde{\phi}) w^{2a} \approx w \left\{ A'_{10}(0)\widetilde{\phi} + A_{30}(0) w^2 \right\} \tag{A.83}$$

with $A'_{10}(0) = \partial A_{10}/\partial \widetilde{\phi}(0)$. Thus, $\widetilde{F}_1(w, 0, \widetilde{\phi}) = 0$ represents $\widetilde{\phi}$ versus w relation for $\boldsymbol{w}_{\text{stripe}}$. Substituting $\boldsymbol{w}_{\text{stripe}}$ into (A.82), we have $\widetilde{F}_2(w, 0, \widetilde{\phi}) = 0$. Thus, there is a bifurcating curve satisfying $\widetilde{F}_1 = \widetilde{F}_2 = 0$ for $\boldsymbol{w}_{\text{stripe}}$. Similar discussion holds for $\boldsymbol{w}_{\text{sq}}$. □

Proposition A.14. *For a critical point of multiplicity 2 associated with* $\mu = (2; +, +)$*, the two bifurcating solutions* $(\boldsymbol{w}, \widetilde{\phi})$ *and* $(-\boldsymbol{w}, \widetilde{\phi})$ *are conjugate for* $\boldsymbol{w} = \boldsymbol{w}_{\text{sq}}, \boldsymbol{w}_{\text{stripe}}$.

Proof. Since $\boldsymbol{w}_{\text{stripe}} = (w, 0)$ and $-\boldsymbol{w}_{\text{stripe}} = (-w, 0)$ satisfy the same relation (cf., (A.83))

$$\sum_{a=0}^{\infty} A_{2a+1,0}(\widetilde{\phi}) w^{2a} = 0,$$

$\widetilde{F}_1(w, 0, \widetilde{\phi})$ is an odd function in w, that is,

$$\widetilde{F}_1(-w, 0, \widetilde{\phi}) = -\widetilde{F}_1(w, 0, \widetilde{\phi}).$$

Thus, $(\boldsymbol{w}_{\text{stripe}}, \widetilde{\phi})$ and $(-\boldsymbol{w}_{\text{stripe}}, \widetilde{\phi})$ are conjugate solutions for $\widetilde{F}_1 = 0$. Similar discussion holds for $(\boldsymbol{w}_{\text{sq}}, \widetilde{\phi})$ and $(-\boldsymbol{w}_{\text{sq}}, \widetilde{\phi})$. □

We evaluate the stability of the bifurcating solutions by considering the asymptotic form of the bifurcation equation. The asymptotic form of the bifurcation equation becomes

$$\widetilde{F}_1(w_1, w_2, \widetilde{\phi}) \approx w_1 (A'_{10}(0)\widetilde{\phi} + A_{30}(0) w_1{}^2 + A_{12}(0) w_2{}^2), \tag{A.84}$$

$$\widetilde{F}_2(w_1, w_2, \widetilde{\phi}) \approx w_2 (A'_{10}(0)\widetilde{\phi} + A_{30}(0) w_2{}^2 + A_{12}(0) w_1{}^2), \tag{A.85}$$

and the Jacobian matrix of $\widetilde{\boldsymbol{F}}$ in (A.71) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx \begin{bmatrix} A'_{10}(0)\widetilde{\phi} + 3A_{30}(0) w_1{}^2 + A_{12}(0) w_2{}^2 & 2A_{12}(0) w_1 w_2 \\ 2A_{12}(0) w_1 w_2 & A'_{10}(0)\widetilde{\phi} + 3A_{30}(0) w_2{}^2 + A_{12}(0) w_1{}^2 \end{bmatrix}. \tag{A.86}$$

Substituting $\boldsymbol{w}_{\text{sq}} = (w, w)$ into (A.84) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sq}} \approx -w^2 \frac{A_{30}(0) + A_{12}(0)}{A'_{10}(0)}.$$

Evaluating the Jacobian matrix (A.86) at $(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}}) \approx 2w^2 \begin{bmatrix} A_{30}(0) & A_{12}(0) \\ A_{12}(0) & A_{30}(0) \end{bmatrix}. \tag{A.87}$$

The eigenvalues of $\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$ are given by

$$\lambda_1, \lambda_2 \approx 2w^2(A_{30}(0) \pm A_{12}(0)).$$

Hence, the sign of the eigenvalues depends on the values of the coefficients $A_{30}(0)$ and $A_{12}(0)$.

Substituting $\boldsymbol{w}_{\mathrm{stripe}} = (w, 0)$ into (A.84) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripe}} \approx -w^2 \frac{A_{30}(0)}{A'_{10}(0)}.$$

Evaluating the Jacobian matrix (A.86) at $(\boldsymbol{w}_{\mathrm{stripe}}, \widetilde{\phi}_{\mathrm{stripe}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripe}}, \widetilde{\phi}_{\mathrm{stripe}}) \approx w^2 \begin{bmatrix} 2A_{30}(0) & 0 \\ 0 & -A_{30}(0) + A_{12}(0) \end{bmatrix}. \tag{A.88}$$

The eigenvalues of $\widetilde{J}(\boldsymbol{w}_{\mathrm{stripe}}, \widetilde{\phi}_{\mathrm{stripe}})$ are given by

$$\begin{aligned} \lambda_1 &\approx 2w^2 A_{30}(0), \\ \lambda_2 &\approx 2w^2(A_{12}(0) - A_{30}(0)). \end{aligned}$$

Hence, the sign of the eigenvalues depends on the values of the coefficients $A_{30}(0)$ and $A_{12}(0)$.

To sum up, we have the following proposition:

Proposition A.15. *For a critical point of multiplicity 2 associated with $\mu = (2; +, +)$, suppose that all eigenvalues of $J(\boldsymbol{\lambda}_{\mathrm{c}}, \phi)$ other than those for $\mu = (2; +, +)$ are negative. Then, we have the following statements on the stability in the neighborhood of the critical point.*

(i) *If $A_{30}(0) < A_{12}(0) < -A_{30}(0)$ are satisfied, the square pattern $\boldsymbol{w}_{\mathrm{sq}}$ is stable.*

(ii) *If $A_{12}(0) < A_{30}(0) < 0$ are satisfied, the stripe pattern $\boldsymbol{w}_{\mathrm{stripe}}$ is stable.*

(iii) *The two solutions $\boldsymbol{w}_{\mathrm{sq}}$ and $\boldsymbol{w}_{\mathrm{stripe}}$ are not stable simultaneously.*

Proof. The first and second statements are obtained by assuming that all the eigenvalues of the Jacobian matrix at each bifurcating solution are negative. The last statement are obtained by the fact that $A_{30}(0) < A_{12}(0)$ and $A_{12}(0) < A_{30}(0)$ cannot be satisfied simultaneously. □

A.4.4. Bifurcation Point of Multiplicity 4

We consider a critical point associated with the four-dimensional irreducible representations μ of the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$:

$$(4; k, 0, +) \ \text{with} \ 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{A.89}$$

$$(4; k, k, +) \ \text{with} \ 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{A.90}$$

$$(4; n/2, \ell, +) \ \text{with} \ 1 \le \ell \le \frac{n}{2} - 1, \tag{A.91}$$

where $n \ge 3$ and $(4; n/2, \ell, +)$ exists when n is even. For $(4; k, 0, +)$ and $(4; k, k, +)$, we use the following notations:

$$\check{n} = \frac{n}{\gcd(k, n)}, \quad \check{k} = \frac{k}{\gcd(k, n)}. \tag{A.92}$$

For $(4; n/2, \ell, +)$, we use the following notations:

$$\tilde{n} = \frac{n}{\gcd(\ell, n)}, \quad \tilde{\ell} = \frac{\ell}{\gcd(\ell, n)}. \tag{A.93}$$

The action in $(4; k, 0, +)$ on a four-dimensional vector $(w_1, \dots, w_4) \in \mathbb{R}^4$ can be expressed for a two-dimensional vector (z_1, z_2) with complex variables $z_j = w_{2j-1} + \mathrm{i} w_{2j}$ $(j = 1, 2)$ as (cf., (3.132))

$$r : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z_2} \\ z_1 \end{bmatrix}, \quad s : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \overline{z_2} \end{bmatrix}, \tag{A.94}$$

$$p_1 : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ z_2 \end{bmatrix}, \quad p_2 : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ \omega^k z_2 \end{bmatrix} \tag{A.95}$$

with $\omega = \exp(\mathrm{i}2\pi/n)$. The action in $(4; k, k, +)$ can be expressed as (cf., (3.133))

$$r : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z_2} \\ z_1 \end{bmatrix}, \quad s : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z_2} \\ \overline{z_1} \end{bmatrix}, \tag{A.96}$$

$$p_1 : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^{-k} z_2 \end{bmatrix}, \quad p_2 : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^k z_2 \end{bmatrix}, \tag{A.97}$$

and the action in $(4; n/2, \ell, +)$ can be expressed as (cf., (3.134))

$$r : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z_2} \\ z_1 \end{bmatrix}, \quad s : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix}, \tag{A.98}$$

$$p_1 : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} -z_1 \\ \omega^{-\ell} z_2 \end{bmatrix}, \quad p_2 : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega^{\ell} z_1 \\ -z_2 \end{bmatrix}. \tag{A.99}$$

Derivation of Bifurcation Equation

The bifurcation equation for a critical point of multiplicity 4 is a four-dimensional equation in $\boldsymbol{w} = (w_1, \ldots, w_4) \in \mathbb{R}^4$ expressed as

$$\widetilde{F}_i(\boldsymbol{w}, \widetilde{\phi}) = 0, \quad i = 1, \ldots, 4, \tag{A.100}$$

where $(w_1, \ldots, w_4, \widetilde{\phi}) = (0, \ldots, 0, 0)$ is assumed to correspond to the critical point. Accordingly, the Jacobian matrix of $\widetilde{\boldsymbol{F}}$ is a 4×4 matrix expressed as

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) = \left(\frac{\partial \widetilde{F}_i}{\partial w_j} \,\middle|\, i, j = 1, \ldots, 4 \right). \tag{A.101}$$

The bifurcation equation (A.100) can be represented as a 2-dimensional equation in complex variables $z_j = w_{2j-1} + \mathrm{i} w_{2j}$ $(j = 1, 2)$ as

$$F_i(z_1, z_2, \widetilde{\phi}) = 0, \quad i = 1, 2, \tag{A.102}$$

where $(z_1, z_2, \widetilde{\phi}) = (0, 0, 0)$ corresponds to the critical point, and there are the following relationship:

$$F_1(z_1, z_2, \widetilde{\phi}) = \widetilde{F}_1 + \mathrm{i}\widetilde{F}_2, \tag{A.103}$$

$$F_2(z_1, z_2, \widetilde{\phi}) = \widetilde{F}_3 + \mathrm{i}\widetilde{F}_4. \tag{A.104}$$

We expand F_1 into a power series as

$$F_1(z_1, z_2, \widetilde{\phi}) = \sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b \,{\overline{z}_1}^C \,{\overline{z}_2}^d \tag{A.105}$$

with coefficients $A_{abcd}(\widetilde{\phi})$. Since $(z_1, z_2, \widetilde{\phi}) = (0, 0, 0)$ corresponds to the critical point, we have

$$A_{0000}(0) = 0, \quad A_{1000}(0) = A_{0100}(0) = A_{0010}(0) = A_{0001}(0) = 0.$$

In addition, since $a_1 = A'_{1000}(0)$ is generically nonzero, we have

$$A_{1000}(\widetilde{\phi}) \approx a_1 \widetilde{\phi}.$$

The equivariance of the bifurcation equation to the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the actions of the four elements r, s, p_1, and p_2 generating this group. The equivariance condition for $(4; k, 0, +)$ is written as

$$r: \quad \overline{F_2(z_1, z_2)} = F_1(\overline{z}_2, z_1), \tag{A.106}$$

$$F_1(z_1, z_2) = F_2(\overline{z}_2, z_1), \tag{A.107}$$

$$s: \quad F_1(z_1, z_2) = F_1(z_1, \overline{z}_2), \tag{A.108}$$

$$\overline{F_2(z_1, z_2)} = F_2(z_1, \overline{z}_2), \tag{A.109}$$

$$p_1: \quad \omega^k F_1(z_1, z_2) = F_1(\omega^k z_1, z_2), \tag{A.110}$$

$$F_2(z_1, z_2) = F_2(\omega^k z_1, z_2), \tag{A.111}$$

$$p_2: \quad F_1(z_1, z_2) = F_1(z_1, \omega^k z_2), \tag{A.112}$$

$$\omega^k F_2(z_1, z_2) = F_2(z_1, \omega^k z_2) \tag{A.113}$$

with $\omega = \exp(\mathrm{i}2\pi/n)$. The equivariance condition for $(4; k, k, +)$ is written as

$$r: \quad \overline{F_2(z_1, z_2)} = F_1(\overline{z}_2, z_1), \tag{A.114}$$

$$F_1(z_1, z_2) = F_2(\overline{z}_2, z_1), \tag{A.115}$$

$$s: \quad \overline{F_2(z_1, z_2)} = F_1(\overline{z}_2, \overline{z}_1), \tag{A.116}$$

$$\overline{F_1(z_1, z_2)} = F_2(\overline{z}_2, \overline{z}_1), \tag{A.117}$$

$$p_1: \quad \omega^k F_1(z_1, z_2) = F_1(\omega^k z_1, \omega^{-k} z_2), \tag{A.118}$$

$$\omega^{-k} F_2(z_1, z_2) = F_2(\omega^k z_1, \omega^{-k} z_2), \tag{A.119}$$

$$p_2: \quad \omega^k F_1(z_1, z_2) = F_1(\omega^k z_1, \omega^k z_2), \tag{A.120}$$

$$\omega^k F_2(z_1, z_2) = F_2(\omega^k z_1, \omega^k z_2). \tag{A.121}$$

The equivariance condition for $(4; n/2, \ell, +)$ is written as

$$r: \quad \overline{F_2(z_1, z_2)} = F_1(\overline{z}_2, z_1), \tag{A.122}$$

$$F_1(z_1, z_2) = F_2(\overline{z}_2, z_1), \tag{A.123}$$

$$s: \quad \overline{F_1(z_1, z_2)} = F_1(\overline{z}_1, z_2), \tag{A.124}$$

$$F_2(z_1, z_2) = F_2(\overline{z}_1, z_2), \tag{A.125}$$

$$p_1: \quad -F_1(z_1, z_2) = F_1(-z_1, \omega^{-\ell} z_2), \tag{A.126}$$

$$\omega^{-\ell} F_2(z_1, z_2) = F_2(-z_1, \omega^{-\ell} z_2), \tag{A.127}$$

$$p_2: \quad \omega^{\ell} F_1(z_1, z_2) = F_1(\omega^{\ell} z_1, -z_2), \tag{A.128}$$

$$-F_2(z_1, z_2) = F_2(\omega^{\ell} z_1, -z_2). \tag{A.129}$$

The equivariance condition with respect to r is equivalent to

$$F_2(z_1, z_2) = F_1(z_2, \overline{z}_1), \tag{A.130}$$

$$F_1(z_1, z_2) = \overline{F_1(\overline{z}_1, \overline{z}_2)} \tag{A.131}$$

for each irreducible representation. Hence, we can obtain F_2 from F_1 by the condition (A.130) and see that

$$A_{abcd}(\widetilde{\phi}) \in \mathbb{R} \tag{A.132}$$

by the condition (A.131).

The equivariance condition with respect to s is equivalent to $F_1(z_1, z_2) = F_1(z_1, \overline{z}_2)$ in (A.108), which gives

$$A_{abcd}(\widetilde{\phi}) = A_{adcb}(\widetilde{\phi}) \tag{A.133}$$

for each irreducible representation as explained below. For $(4; k, 0, +)$, the condition (A.108) applies. For $(4; k, k, +)$, substituting (A.114) into (A.116), we have $F_1(\overline{z}_2, z_1) = F_1(\overline{z}_2, \overline{z}_1)$. This condition is equivalent to $F_1(z_1, z_2) = F_1(z_1, \overline{z}_2)$. For $(4; n/2, \ell, +)$, the condition (A.124) gives $F_1(z_1, z_2) = \overline{F_1(\overline{z}_1, z_2)}$. Using (A.132), we have $\overline{F_1(\overline{z}_1, z_2)} = F_1(z_1, \overline{z}_2)$. Thus, we have $F_1(z_1, z_2) = F_1(z_1, \overline{z}_2)$.

For $(4; k, 0, +)$, the equivariance condition with respect to p_1 and p_2 is expressed as follows. The equivariance condition (A.110) for p_1 is expressed as

$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} \omega^k A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b {\bar{z}_1}^C {\bar{z}_2}^d = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^k z_1)^a {z_2}^b (\omega^{-k}\bar{z}_1)^C {\bar{z}_2}^d,$$

which implies

$$\omega^{k(a-c-1)} = \exp\left[\frac{\mathrm{i}2\pi}{n} k(a-c-1)\right] = 1. \tag{A.134}$$

The equivariance condition (A.112) for p_2 is expressed as

$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b {\bar{z}_1}^C {\bar{z}_2}^d = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) {z_1}^a (\omega^k z_2)^b {\bar{z}_1}^C (\omega^{-k}\bar{z}_2)^d,$$

which implies

$$\omega^{k(b-d)} = \exp\left[\frac{\mathrm{i}2\pi}{n} k(b-d)\right] = 1. \tag{A.135}$$

Using (A.130), we rewrite the remaining equivariance conditions (A.111) and (A.113) as

$$F_1(z_2, \bar{z}_1) = F_1(z_2, \omega^{-k}\bar{z}_1),$$
$$\omega^k F_1(z_2, \bar{z}_1) = F_1(\omega^k z_2, \bar{z}_1),$$

which are expressed as

$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) {z_2}^a {\bar{z}_1}^b {\bar{z}_2}^C {z_1}^d = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) {z_2}^a (\omega^{-k}\bar{z}_1)^b {\bar{z}_2}^C (\omega^k z_1)^d,$$
$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} \omega^k A_{abcd}(\widetilde{\phi}) {z_2}^a {\bar{z}_1}^b {\bar{z}_2}^C {z_1}^d = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^k z_2)^a {\bar{z}_1}^b (\omega^{-k}\bar{z}_2)^C {z_1}^d.$$

Each of these conditions leads to the same result as (A.135) and (A.134), respectively. To sum up, from (A.134) and (A.135), we have the following conditions for $(4; k, 0, +)$:

$$k(a-c-1) \equiv 0 \mod n,$$
$$k(b-d) \equiv 0 \mod n.$$

Using (A.92), we rewrite these conditions as

$$\check{k}(a-c-1) \equiv 0 \mod \check{n},$$
$$\check{k}(b-d) \equiv 0 \mod \check{n},$$

which are equivalent to the following condition:

$$a = c + p\check{n} + 1, \quad b = d + q\check{n} \quad (p, q \in \mathbb{Z}). \tag{A.136}$$

Then, F_1 in (A.105) becomes

$$F_1(z_1, z_2, \widetilde{\phi}) = \sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p\in\mathbb{Z},\ c+p\check{n}+1\geq 0}\ \sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{c+p\check{n}+1,d+q\check{n},cd}(\widetilde{\phi}) {z_1}^{c+p\check{n}+1} {z_2}^{d+q\check{n}}\, {\bar{z}_1}^{C}\, {\bar{z}_2}^{d}. \tag{A.137}$$

Note that $a = 0$ and $c = 0$ are not satisfied simultaneously in (A.136):

$$a = 0 \Rightarrow c = -p\check{n} - 1 \neq 0, \quad c = 0 \Rightarrow a = p\check{n} + 1 \neq 0.$$

Thus, F_1 in (A.137) becomes

$$\begin{aligned} F_1(z_1, z_2, \widetilde{\phi}) = z_1 \sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{p\in\mathbb{Z},\ c+p\check{n}+1>0}\ \sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{c+p\check{n}+1,d+q\check{n},cd}(\widetilde{\phi}) {z_1}^{c+p\check{n}} {z_2}^{d+q\check{n}}\, {\bar{z}_1}^{C}\, {\bar{z}_2}^{d} \\ + \bar{z}_1 \sum_{d=0}^{\infty}\sum_{p=1}^{\infty}\ \sum_{q\in\mathbb{Z},\ d+q\check{n}\geq 0} A_{0,d+q\check{n},p\check{n}-1,d}(\widetilde{\phi}) {z_2}^{d+q\check{n}}\, {\bar{z}_1}^{p\check{n}-2}\, {\bar{z}_2}^{d}. \end{aligned} \tag{A.138}$$

For $(4; k, k, +)$, the equivariance condition (A.118) is expressed as

$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} \omega^k A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b\, {\bar{z}_1}^C\, {\bar{z}_2}^d = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^k z_1)^a (\omega^{-k} z_2)^b (\omega^{-k}\, \bar{z}_1)^C (\omega^k\, \bar{z}_2)^d,$$

which implies

$$\omega^{k(a-b-c+d-1)} = \exp\left[\frac{\mathrm{i}2\pi}{n} k(a-b-c+d-1)\right] = 1. \tag{A.139}$$

The equivariance condition (A.120) is expressed as

$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} \omega^k A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b\, {\bar{z}_1}^C\, {\bar{z}_2}^d = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^k z_1)^a (\omega^{k} z_2)^b (\omega^{-k}\, \bar{z}_1)^C (\omega^{-k}\, \bar{z}_2)^d,$$

which implies

$$\omega^{k(a+b-c-d-1)} = \exp\left[\frac{\mathrm{i}2\pi}{n} k(a+b-c-d-1)\right] = 1. \tag{A.140}$$

Using (A.130), we rewrite the remaining equivariance conditions (A.119) and (A.121) as

$$\begin{aligned} \omega^{-k} F_1(z_2, \bar{z}_1) &= F_1(\omega^{-k} z_2, \omega^{-k}\, \bar{z}_1), \\ \omega^{k} F_1(z_2, \bar{z}_1) &= F_1(\omega^{k} z_2, \omega^{-k}\, \bar{z}_1), \end{aligned}$$

which are expressed as

$$\begin{aligned} &\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} \omega^{-k} A_{abcd}(\widetilde{\phi}) {z_2}^a\, {\bar{z}_1}^b\, {\bar{z}_2}^C {z_1}^d \\ &\qquad = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^{-k} z_2)^a (\omega^{-k}\, \bar{z}_1)^b (\omega^{k}\, \bar{z}_2)^C (\omega^{k} z_1)^d, \\ &\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} \omega^{k} A_{abcd}(\widetilde{\phi}) {z_2}^a\, {\bar{z}_1}^b\, {\bar{z}_2}^C\, {z_1}^d \\ &\qquad = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0} A_{abcd}(\widetilde{\phi}) (\omega^{k} z_2)^a (\omega^{-k}\, \bar{z}_1)^b (\omega^{-k}\, \bar{z}_2)^C (\omega^{k} z_1)^d. \end{aligned}$$

Each of these conditions leads to the same result as (A.140) and (A.139), respectively. To sum up, from (A.139) and (A.140), we have the following conditions for $(4; k, k, +)$:

$$k(a-b-c+d-1) \equiv 0 \mod n,$$
$$k(a+b-c-d-1) \equiv 0 \mod n.$$

We rewrite these conditions as

$$\check{k}(a-b-c+d-1) \equiv 0 \mod \check{n},$$
$$\check{k}(a+b-c-d-1) \equiv 0 \mod \check{n},$$

which are equivalent to the following condition:

$$a-b-c+d-1 = v\check{n}, \quad a+b-c-d-1 = w\check{n} \quad (v, w \in \mathbb{Z}).$$

Adding and subtracting the two equations from each other, we have

$$2(a-c-1) = (v+w)\check{n}, \quad 2(b-d) = (w-v)\check{n}.$$

This condition is equivalent to

$$a = c + (v+w)\check{n}/2 + 1, \quad b = d + (w-v)\check{n}/2. \tag{A.141}$$

Since the indices a, b, c, and d are integers, we have the following condition ($p, q \in \mathbb{Z}$):

$$\begin{cases} v+w = p, \quad w-v = 2q-p & \text{if } \check{n} \text{ is even,} \\ v+w = 2p, \quad w-v = 2(q-p) & \text{if } \check{n} \text{ is odd.} \end{cases} \tag{A.142}$$

Note that for $\check{n}$ odd, we can replace $q-p$ as q ($q \in \mathbb{Z}$). From (A.141) and (A.142), we have the following condition:

$$\begin{cases} a = c + p\check{n}/2 + 1, \quad b = d + (2q-p)\check{n}/2 & \text{if } \check{n} \text{ is even,} \\ a = c + p\check{n} + 1, \quad b = d + q\check{n} & \text{if } \check{n} \text{ is odd.} \end{cases} \tag{A.143}$$

Note that for both cases in (A.143), $a = 0$ and $c = 0$ are not satisfied simultaneously:

$$\begin{cases} a = 0 \Rightarrow c = -p\check{n}/2 - 1 \neq 0, \quad c = 0 \Rightarrow a = p\check{n}/2 + 1 \neq 0 & \text{if } \check{n} \text{ is even,} \\ a = 0 \Rightarrow c = -p\check{n} - 1 \neq 0, \quad c = 0 \Rightarrow a = p\check{n} + 1 \neq 0 & \text{if } \check{n} \text{ is odd.} \end{cases}$$

If $\check{n}$ is even, F_1 in (A.105) becomes

$$\begin{aligned} F_1(z_1, z_2, \widetilde{\phi}) = z_1 \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{p,q \in \mathbb{Z},\ c+p\frac{\check{n}}{2}+1>0,\ d+(2q-p)\frac{\check{n}}{2} \geq 0} A_{c+p\frac{\check{n}}{2}+1, d+(2q-p)\frac{\check{n}}{2}, cd}(\widetilde{\phi}) {z_1}^{c+p\frac{\check{n}}{2}} {z_2}^{d+(2q-p)\frac{\check{n}}{2}} {\overline{z}_1}^{C} {\overline{z}_2}^{d} \\ + \overline{z}_1 \sum_{d=0}^{\infty} \sum_{p=1}^{\infty} \sum_{q \in \mathbb{Z},\ d+(2q+p)\frac{\check{n}}{2} \geq 0} A_{0, d+(2q+p)\frac{\check{n}}{2}, p\frac{\check{n}}{2}-1, d}(\widetilde{\phi}) {z_2}^{d+(2q+p)\frac{\check{n}}{2}} {\overline{z}_1}^{p\frac{\check{n}}{2}-2} {\overline{z}_2}^{d}. \end{aligned} \tag{A.144}$$

If $\check{n}$ is odd, F_1 in (A.105) becomes

$$F_1(z_1, z_2, \widetilde{\phi}) = z_1 \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{p \in \mathbb{Z},\ c+p\check{n}+1>0} \sum_{q \in \mathbb{Z},\ d+q\check{n} \geq 0} A_{c+p\check{n}+1, d+q\check{n}, cd}(\widetilde{\phi}) {z_1}^{c+p\check{n}} {z_2}^{d+q\check{n}} {\overline{z}_1}^{C} {\overline{z}_2}^{d}$$
$$+ \overline{z}_1 \sum_{d=0}^{\infty} \sum_{p=1}^{\infty} \sum_{q \in \mathbb{Z},\ d+q\check{n} \geq 0} A_{0, d+q\check{n}, p\check{n}-1, d}(\widetilde{\phi}) {z_2}^{d+q\check{n}} {\overline{z}_1}^{p\check{n}-2} {\overline{z}_2}^{d}. \tag{A.145}$$

For $(4; n/2, \ell, +)$, the equivariance condition (A.126) is expressed as

$$\sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} (-A_{abcd}(\widetilde{\phi})) {z_1}^{a} {z_2}^{b} {\overline{z}_1}^{C} {\overline{z}_2}^{d} = \sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} A_{abcd}(\widetilde{\phi})(-z_1)^{a} (\omega^{-\ell} z_2)^{b} (-\overline{z}_1)^{C} (\omega^{\ell}\, \overline{z}_2)^{d},$$

which implies

$$-1 = (-1)^{a+c} \omega^{\ell(d-b)}.$$

We rewrite this condition as

$$\exp\left[\frac{\mathrm{i}2\pi}{n} \left\{\frac{n}{2}(a+c) + \ell(d-b)\right\}\right] = -1. \tag{A.146}$$

Therein, we used

$$(-1)^{a+c} = \exp\left[\frac{\mathrm{i}\pi}{n}(a+c)\right] \quad (a, c \in \mathbb{Z}_+),$$

where $\mathbb{Z}_+$ represents the set of nonnegative integers. The equivariance condition (A.128) is expressed as

$$\sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} (\omega^{\ell} A_{abcd}(\widetilde{\phi})) {z_1}^{a} {z_2}^{b} {\overline{z}_1}^{C} {\overline{z}_2}^{d} = \sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} A_{abcd}(\widetilde{\phi})(\omega^{\ell} z_1)^{a} (-z_2)^{b} (\omega^{-\ell}\, \overline{z}_1)^{C} (-\overline{z}_2)^{d},$$

which implies

$$\omega^{\ell} = (-1)^{b+d} \omega^{\ell(a-c)}.$$

We rewrite this condition as

$$\exp\left[\frac{\mathrm{i}2\pi}{n} \left\{\frac{n}{2}(b+d) + \ell(a-c-1)\right\}\right] = 1. \tag{A.147}$$

Therein, we used

$$(-1)^{b+d} = \exp\left[\frac{\mathrm{i}\pi}{n}(b+d)\right] \quad (b, d \in \mathbb{Z}_+).$$

Using (A.130), we rewrite the remaining equivariance conditions (A.127) and (A.129) as

$$\omega^{-\ell} F_1(z_2, \overline{z}_1) = F_1(\omega^{-\ell} z_2, -\overline{z}_1),$$
$$-F_1(z_2, \overline{z}_1) = F_1(-z_2, \omega^{-\ell}\, \overline{z}_1),$$

which are expressed as

$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0}\omega^{-\ell}A_{abcd}(\widetilde{\phi}){z_2}^a\,{\overline{z}_1}^b\,{\overline{z}_2}^C{z_1}^d = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0}A_{abcd}(\widetilde{\phi})(\omega^{-\ell}z_2)^a(-\overline{z}_1)^b(\omega^{\ell}\,\overline{z}_2)^C(-z_1)^d,$$
$$\sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0}(-A_{abcd}(\widetilde{\phi})){z_2}^a\,{\overline{z}_1}^b\,{\overline{z}_2}^C\,{z_1}^d = \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0}A_{abcd}(\widetilde{\phi})(-z_2)^a(\omega^{-\ell}\,\overline{z}_1)^b(-\overline{z}_2)^C(\omega^{\ell}z_1)^d.$$

Each of these conditions leads to the same result as (A.147) and (A.146), respectively.

To sum up, from (A.146) and (A.147), we have the following conditions for $(4; n/2, \ell, +)$:

$$\frac{n}{2}(a+c-1)+\ell(d-b)\equiv 0 \mod n,$$
$$\frac{n}{2}(b+d)+\ell(a-c-1)\equiv 0 \mod n.$$

We rewrite these conditions as

$$\tilde{n}(a+c-1)+2\tilde{\ell}(d-b)\equiv 0 \mod 2\tilde{n},$$
$$\tilde{n}(b+d)+2\tilde{\ell}(a-c-1)\equiv 0 \mod 2\tilde{n},$$

which are equivalent to the following condition:

$$\tilde{n}(a+c-1)+2\tilde{\ell}(d-b)=2p\tilde{n},\quad \tilde{n}(b+d)+2\tilde{\ell}(a-c-1)=2q\tilde{n}\quad (p,q\in\mathbb{Z}). \tag{A.148}$$

We investigate this condition dependent on the parity of $\tilde{n}$.

When $\tilde{n}$ is even, the condition (A.148) is equivalent to

$$(a+c-1-2p)\tilde{n}/2=(b-d)\tilde{\ell},\quad (a-c-1)\tilde{\ell}=-(b+d-2q)\tilde{n}/2.$$

Since $\tilde{\ell}$ and $\tilde{n}$ are coprime, we have the following conditions ($v, w\in\mathbb{Z}$):

$$b-d=v\tilde{n}/2,\quad b+d-2q=w\tilde{\ell}, \tag{A.149}$$
$$a+c-1-2p=v\tilde{\ell},\quad a-c-1=-w\tilde{n}/2. \tag{A.150}$$

Adding and subtracting the two equations in (A.149) from each other, we have

$$2(b-q)=v\tilde{n}/2+w\tilde{\ell},\quad 2(d-q)=-v\tilde{n}/2+w\tilde{\ell}.$$

This condition is equivalent to

$$\begin{bmatrix} b\\ d\end{bmatrix}=q\begin{bmatrix}1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix} v\tilde{n}/2+w\tilde{\ell}\\ -v\tilde{n}/2+w\tilde{\ell}\end{bmatrix}. \tag{A.151}$$

Since the indices b and d in (A.151) are integers, we have

$$v\tilde{n}/2+w\tilde{\ell}\in 2\mathbb{Z}. \tag{A.152}$$

Note that if the condition (A.152) is satisfied, then $-v\tilde{n}/2 + w\tilde{\ell} \in 2\mathbb{Z}$ is also satisfied. Adding and subtracting the two equations in (A.150) from each other, we have

$$2(a-1-p) = v\tilde{\ell} - w\tilde{n}/2, \quad 2(c-p) = v\tilde{\ell} + w\tilde{n}/2.$$

This condition is equivalent to

$$\begin{bmatrix} a \\ c \end{bmatrix} = p \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} v\tilde{\ell} - w\tilde{n}/2 \\ v\tilde{\ell} + w\tilde{n}/2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{A.153}$$

Since the indices a and c in (A.153) are integers, we have

$$v\tilde{\ell} + w\tilde{n}/2 \in 2\mathbb{Z}. \tag{A.154}$$

Note that if the condition (A.154) is satisfied, then $v\tilde{\ell} - w\tilde{n}/2 \in 2\mathbb{Z}$ is also satisfied. Since $\tilde{\ell}$ and $\tilde{n}$ are coprime, $\tilde{\ell}$ is odd. Thus, the conditions (A.152) and (A.154) are equivalent to the following condition ($t, u, t', u' \in \mathbb{Z}$):

$$\begin{cases} (v,w) = (2t, 2u) & \text{if } \tilde{n}/2 \text{ is even,} \\ (v,w) = (2t,2u), (2t'+1, 2u'+1) & \text{if } \tilde{n}/2 \text{ is odd.} \end{cases} \tag{A.155}$$

If $\tilde{n}/2$ is even, the indices a, b, c, and d take the form

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = p \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \\ -\tilde{n}/2 \end{bmatrix} + u \begin{bmatrix} -\tilde{n}/2 \\ \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{A.156}$$

Note that $a = 0$ and $c = 0$ are not satisfied simultaneously:

$$a = 0 \Rightarrow c = u\tilde{n} - 1 \neq 0, \quad c = 0 \Rightarrow a = -u\tilde{n} + 1 \neq 0.$$

With this result, we define disjoint sets U and V as

$$U = \{(p,q,t,u) \in \mathbb{Z}^4 \mid a > 0,\ b \geq 0,\ c \geq 0,\ d \geq 0\},$$
$$V = \{(p,q,t,u) \in \mathbb{Z}^4 \mid a = 0,\ b \geq 0,\ c > 0,\ d \geq 0\},$$

which satisfy $U \cup V = \phi$ and are rewritten as

$$U = \left\{ (p,q,t,u) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{l} p + t\tilde{\ell} - u\tilde{n}/2 + 1 > 0 \\ q + t\tilde{n}/2 + u\tilde{\ell} \geq 0 \\ p + t\tilde{\ell} + u\tilde{n}/2 \geq 0 \\ q - t\tilde{n}/2 + u\tilde{\ell} \geq 0 \end{array} \right\}, \tag{A.157}$$

$$V = \left\{ (p,q,t,u) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{l} p + t\tilde{\ell} - u\tilde{n}/2 + 1 = 0 \\ q + t\tilde{n}/2 + u\tilde{\ell} \geq 0 \\ u\tilde{n} - 1 > 0 \\ q - t\tilde{n}/2 + u\tilde{\ell} \geq 0 \end{array} \right\}. \tag{A.158}$$

Then, F_1 in (A.105) becomes

$$F_1(z_1, z_2, \widetilde{\phi}) = z_1 \sum_{(p,q,t,u)\in U} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+1,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},p+t\tilde{\ell}+u\frac{\tilde{n}}{2},q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) {z_1}^{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}} {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}} {\overline{z}_1}^{p+t\tilde{\ell}+u\frac{\tilde{n}}{2}} {\overline{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}$$
$$+ \overline{z}_1 \sum_{(p,q,t,u)\in V} A_{0,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},u\tilde{n}-1,q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}} {\overline{z}_1}^{u\tilde{n}-2} {\overline{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}. \tag{A.159}$$

If $\tilde{n}/2$ is odd, the indices a, b, c, and d take the form

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = p \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \\ -\tilde{n}/2 \end{bmatrix} + u \begin{bmatrix} -\tilde{n}/2 \\ \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{A.160}$$
$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = p' \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + q' \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t' \begin{bmatrix} \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \\ -\tilde{n}/2 \end{bmatrix} + u' \begin{bmatrix} -\tilde{n}/2 \\ \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \tilde{\ell}-\tilde{n}/2 \\ \tilde{\ell}+\tilde{n}/2 \\ \tilde{\ell}+\tilde{n}/2 \\ \tilde{\ell}-\tilde{n}/2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{A.161}$$

The first relation (A.160) is nothing but (A.156). Note that (A.160) and (A.161) take different vectors. In fact, assuming (A.160) = (A.161), we have

$$(p'-p) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (q'-q) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + (t'-t) \begin{bmatrix} \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \\ -\tilde{n}/2 \end{bmatrix} + (u'-u) \begin{bmatrix} -\tilde{n}/2 \\ \tilde{\ell} \\ \tilde{n}/2 \\ \tilde{\ell} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \tilde{\ell}-\tilde{n}/2 \\ \tilde{\ell}+\tilde{n}/2 \\ \tilde{\ell}+\tilde{n}/2 \\ \tilde{\ell}-\tilde{n}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Substituting the first equation into the third equation, we have $(u'-u+1/2)\tilde{n} = 0$. This is a contradiction since $u'-u \in \mathbb{Z}$. In addition, note that $a = 0$ and $c = 0$ are not satisfied simultaneously:

$$\begin{cases} a = 0 \Rightarrow c = u\tilde{n} - 1 \neq 0, \quad c = 0 \Rightarrow a = -u\tilde{n} + 1 \neq 0 & \text{for (A.160)}, \\ a = 0 \Rightarrow c = (2u'+1)\check{n}/2 - 1 \neq 0, \quad c = 0 \Rightarrow a = -(2u'+1)\check{n}/2 + 1 \neq 0 & \text{for (A.161)}. \end{cases}$$

With this result, we can define four disjoint sets U and V in (A.157) and (A.158) and U' and V' from (A.161) as

$$U' = \left\{ (p,q,t,u) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{l} p + t\tilde{\ell} - u\tilde{n}/2 + (\tilde{\ell}-\tilde{n}/2)/2 + 1 > 0 \\ q + t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell}+\tilde{n}/2)/2 \geq 0 \\ p + t\tilde{\ell} + u\tilde{n}/2 + (\tilde{\ell}+\tilde{n}/2)/2 \geq 0 \\ q - t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell}-\tilde{n}/2)/2 \geq 0 \end{array} \right\}, \tag{A.162}$$

$$V' = \left\{ (p,q,t,u) \in \mathbb{Z}^4 \;\middle|\; \begin{array}{l} p + t\tilde{\ell} - u\tilde{n}/2 + (\tilde{\ell}-\tilde{n}/2)/2 + 1 = 0 \\ q + t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell}+\tilde{n}/2)/2 \geq 0 \\ (2u+1)\tilde{n}/2 - 1 > 0 \\ q - t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell}-\tilde{n}/2)/2 \geq 0 \end{array} \right\}. \tag{A.163}$$

Then, F_1 in (A.105) becomes

$$\begin{aligned}
F_1(z_1, z_2, \widetilde{\phi}) = z_1 &\sum_{(p,q,t,u)\in U} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+1,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},p+t\tilde{\ell}+u\frac{\tilde{n}}{2},q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) {z_1}^{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}} {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}} {\overline{z}_1}^{p+t\tilde{\ell}+u\frac{\tilde{n}}{2}} {\overline{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}} \\
&+ \overline{z}_1 \sum_{(p,q,t,u)\in V} A_{0,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},u\tilde{n}-1,q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}} {\overline{z}_1}^{u\tilde{n}-2} {\overline{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}} \\
&+ z_1 \sum_{(p,q,t,u)\in U'} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})+1,q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}),p+t\tilde{\ell}+u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}),q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})}(\widetilde{\phi}) \\
&\quad \times {z_1}^{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})} {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2})} {\overline{z}_1}^{p+t\tilde{\ell}+u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2})} {\overline{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})} \\
&+ \overline{z}_1 \sum_{(p,q,t,u)\in V'} A_{0,q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}),(2u+1)\frac{\tilde{n}}{2}-1,q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})}(\widetilde{\phi}) \\
&\quad \times {z_2}^{q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2})} {\overline{z}_1}^{(2u+1)\frac{\tilde{n}}{2}-2} {\overline{z}_2}^{q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})}.
\end{aligned} \tag{A.164}$$

When $\tilde{n}$ is odd, the condition (A.148) is rewritten as

$$(a + c - 1 - 2p)\tilde{n} = 2\tilde{\ell}(b - d), \quad 2\tilde{\ell}(a - c - 1) = -(b + d - 2q)\tilde{n}.$$

Since $2\tilde{\ell}$ and $\tilde{n}$ are coprime, we have the following conditions ($v, w \in \mathbb{Z}$):

$$b - d = v\tilde{n}, \quad b + d - 2q = 2w\tilde{\ell}, \tag{A.165}$$

$$a + c - 1 - 2p = 2v\tilde{\ell}, \quad a - c - 1 = -w\tilde{n}. \tag{A.166}$$

Adding and subtracting the two equations in (A.165) from each other, we have

$$2(b - q) = v\tilde{n} + 2w\tilde{\ell}, \quad 2(d - q) = -v\tilde{n} + 2w\tilde{\ell}.$$

This condition is equivalent to

$$\begin{bmatrix} b \\ d \end{bmatrix} = q \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} v \begin{bmatrix} \tilde{n} \\ -\tilde{n} \end{bmatrix} + w \begin{bmatrix} \tilde{\ell} \\ \tilde{\ell} \end{bmatrix}. \tag{A.167}$$

Since the indices b and d in (A.167) are integers, and $\tilde{n}$ is odd, we have $v \in 2\mathbb{Z}$. Therefore, we replace v as $2t$ ($t \in \mathbb{Z}$). Adding and subtracting the two equations in (A.166) from each other, we have

$$2(a - 1 - p) = 2v\tilde{\ell} - w\tilde{n}, \quad 2(c - p) = 2v\tilde{\ell} + w\tilde{n}.$$

This condition is equivalent to

$$\begin{bmatrix} a \\ c \end{bmatrix} = p \begin{bmatrix} 1 \\ 1 \end{bmatrix} + v \begin{bmatrix} \tilde{\ell} \\ \tilde{\ell} \end{bmatrix} + \frac{1}{2} w \begin{bmatrix} -\tilde{n} \\ \tilde{n} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{A.168}$$

Since the indices a and c in (A.168) are integers, and $\tilde{n}$ is odd, we have $w \in 2\mathbb{Z}$. Therefore, we replace w as $2u$ ($u \in \mathbb{Z}$). To sum up, we have

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = p \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2\tilde{\ell} \\ \tilde{n} \\ 2\tilde{\ell} \\ -\tilde{n} \end{bmatrix} + u \begin{bmatrix} -\tilde{n} \\ 2\tilde{\ell} \\ \tilde{n} \\ 2\tilde{\ell} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{A.169}$$

Note that $a = 0$ and $c = 0$ are not satisfied simultaneously:

$$a = 0 \Rightarrow c = 2u\tilde{n} - 1 \neq 0, \quad c = 0 \Rightarrow a = -u\tilde{n} + 1 \neq 0.$$

Similarly to the case that $\tilde{n}$ is even, we define sets U and V as

$$U = \left\{ (p, q, t, u) \in \mathbb{Z}^4 \;\middle|\; \begin{matrix} p + 2t\tilde{\ell} - u\tilde{n} + 1 > 0 \\ q + t\tilde{n} + 2u\tilde{\ell} \geq 0 \\ p + 2t\tilde{\ell} + u\tilde{n} \geq 0 \\ q - t\tilde{n} + 2u\tilde{\ell} \geq 0 \end{matrix} \right\}, \tag{A.170}$$

$$V = \left\{ (p, q, t, u) \in \mathbb{Z}^4 \;\middle|\; \begin{matrix} p + 2t\tilde{\ell} - u\tilde{n} + 1 = 0 \\ q + t\tilde{n} + 2u\tilde{\ell} \geq 0 \\ 2u\tilde{n} - 1 > 0 \\ q - t\tilde{n} + 2u\tilde{\ell} \geq 0 \end{matrix} \right\}. \tag{A.171}$$

Then, F_1 in (A.105) becomes

$$\begin{aligned} F_1(z_1, z_2, \widetilde{\phi}) = z_1 & \sum_{(p,q,t,u) \in U} A_{p+2t\tilde{\ell}-u\tilde{n}+1, q+t\tilde{n}+2u\tilde{\ell}, p+2t\tilde{\ell}+u\tilde{n}, q-t\tilde{n}+2u\tilde{\ell}}(\widetilde{\phi}) z_1{}^{p+2t\tilde{\ell}-u\tilde{n}} z_2{}^{q+t\tilde{n}+2u\tilde{\ell}} \,\overline{z}_1{}^{p+2t\tilde{\ell}+u\tilde{n}} \,\overline{z}_2{}^{q-t\tilde{n}+2u\tilde{\ell}} \\ & + \overline{z}_1 \sum_{(p,q,t,u) \in V} A_{0, q+t\tilde{n}+2u\tilde{\ell}, 2u\tilde{n}-1, q-t\tilde{n}+2u\tilde{\ell}}(\widetilde{\phi}) z_2{}^{q+t\tilde{n}+2u\tilde{\ell}} \,\overline{z}_1{}^{2u\tilde{n}-2} \,\overline{z}_2{}^{q-t\tilde{n}+2u\tilde{\ell}}. \end{aligned} \tag{A.172}$$

Symmetry of Square Patterns

For the irreducible representations $\mu = (4; k, 0, +), (4; k, k, +), (4; n/2, \ell, +)$, a system of the bifurcation equations $F_1 = F_2 = 0$ has a bifurcating solution, which represent the square pattern: $(z_1, z_2) = (w, w)$ ($w \in \mathbb{R}$). In Section 3.5.5, we showed the existence of this bifurcating solution by using the equivariant branching lemma (see Propositions 3.9–3.11). In this section, we discuss the symmetry of this bifurcating solution.

Consider $\mu = (4; k, 0, +)$. Substituting the square pattern $(z_1, z_2) = (w, w)$ into (A.138), we have

$$\begin{aligned} F_1(w, w, \widetilde{\phi}) &= w \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{p \in \mathbb{Z},\ c+p\check{n}+1>0} \ \sum_{q \in \mathbb{Z},\ d+q\check{n} \geq 0} A_{c+p\check{n}+1, d+q\check{n}, cd}(\widetilde{\phi}) w^{2(c+d)+(p+q)\check{n}} \\ &\quad + w \sum_{d=0}^{\infty} \sum_{p=1}^{\infty} \sum_{q \in \mathbb{Z},\ d+q\check{n} \geq 0} A_{0, d+q\check{n}, p\check{n}-1, d}(\widetilde{\phi}) w^{2d+(p+q)\check{n}-2} \\ &\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0) + A_{2010}(0)) w^2 + A_{00,\check{n}-1,0}(0) w^{\check{n}-2} \right\}. \end{aligned}$$

If $\check{n}$ is even, then $F_1(w, w, \widetilde{\phi})$ becomes an odd function in w, and hence the two bifurcating solutions $(w, w, \widetilde{\phi})$ and $(-w, -w, \widetilde{\phi})$ are conjugate. If $\check{n}$ is odd, the two solutions are not conjugate.

Consider $\mu = (4; k, k, +)$ with $\check{n}$ even. Substituting the square pattern $(z_1, z_2) = (w, w)$ into (A.144), we have

$$\begin{aligned} F_1(w, w, \widetilde{\phi}) &= w \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{p,q \in \mathbb{Z},\ c+p\frac{\check{n}}{2}+1>0,\ d+(2q-p)\frac{\check{n}}{2} \geq 0} A_{c+p\frac{\check{n}}{2}+1, d+(2q-p)\frac{\check{n}}{2}, cd}(\widetilde{\phi}) w^{2(c+d)+q\check{n}} \\ &\quad + w \sum_{d=0}^{\infty} \sum_{p=1}^{\infty} \sum_{q \in \mathbb{Z},\ d+(2q+p)\frac{\check{n}}{2} \geq 0} A_{0, d+(2q+p)\frac{\check{n}}{2}, p\frac{\check{n}}{2}-1, d}(\widetilde{\phi}) w^{2d+q\check{n}-2} \\ &\approx w\{A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0) + A_{2010}(0))w^2 \\ &\qquad + (A_{00,\check{n}-1,0}(0) + A_{0,\frac{\check{n}}{2},\frac{\check{n}}{2}-1,0}(0) + A_{00,\frac{\check{n}}{2}-1,\frac{\check{n}}{2}}(0))w^{\check{n}-2}\}. \end{aligned}$$

Since $\check{n}$ is even, $F_1(w, w, \widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w, w, \widetilde{\phi})$ and $(-w, -w, \widetilde{\phi})$ are conjugate.

Consider $\mu = (4; k, k, +)$ with $\check{n}$ odd. Substituting the square pattern $(z_1, z_2) = (w, w)$ into (A.145), we have

$$\begin{aligned} F_1(w, w, \widetilde{\phi}) &= w \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{p \in \mathbb{Z},\ c+p\check{n}+1>0} \sum_{q \in \mathbb{Z},\ d+q\check{n} \geq 0} A_{c+p\check{n}+1, d+q\check{n}, cd}(\widetilde{\phi}) w^{2(c+d)+(p+q)\check{n}} \\ &\quad + w \sum_{d=0}^{\infty} \sum_{p=1}^{\infty} \sum_{q \in \mathbb{Z},\ d+q\check{n} \geq 0} A_{0, d+q\check{n}, p\check{n}-1, d}(\widetilde{\phi}) w^{2d+(p+q)\check{n}-2} \\ &\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0) + A_{2010}(0))w^2 + A_{00,\check{n}-1,0}(0) w^{\check{n}-2} \right\}. \end{aligned}$$

Since $\check{n}$ is odd, $F_1(w, w, \widetilde{\phi})$ is not an odd function in w. Hence, the two bifurcating solutions $(w, w, \widetilde{\phi})$ and $(-w, -w, \widetilde{\phi})$ are not conjugate.

Consider $\mu = (4; n/2, \ell, +)$ with $\tilde{n}/2$ even. Substituting the square pattern $(z_1, z_2) = (w, w)$ into (A.159), we have

$$\begin{aligned} F_1(w, w, \widetilde{\phi}) &= w \sum_{(p,q,t,u) \in U} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+1, q+t\frac{\tilde{n}}{2}+u\tilde{\ell}, p+t\tilde{\ell}+u\frac{\tilde{n}}{2}, q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) w^{2(p+q)+2(t+u)\tilde{\ell}} \\ &\quad + w \sum_{(p,q,t,u) \in V} A_{0, q+t\frac{\tilde{n}}{2}+u\tilde{\ell}, u\tilde{n}-1, q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi}) w^{2q+2u(\tilde{\ell}+\frac{\tilde{n}}{2})-2} \\ &\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0) + A_{2010}(0))w^2 + A_{00,\tilde{n}-1,0}(0) w^{\tilde{n}-2} \right\}. \end{aligned}$$

Then, $F_1(w, w, \widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w, w, \widetilde{\phi})$ and $(-w, -w, \widetilde{\phi})$ are conjugate.

Consider $\mu = (4; n/2, \ell, +)$ with $\tilde{n}/2$ odd. Substituting the square pattern $(z_1, z_2) = (w, w)$ into

(A.164), we have

$$F_1(z_1,z_2,\widetilde{\phi}) = w \sum_{(p,q,t,u)\in U_1} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+1,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},p+t\tilde{\ell}+u\frac{\tilde{n}}{2},q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi})w^{2(p+q)+2(t+u)\tilde{\ell}}$$
$$+ w \sum_{(p,q,t,u)\in V_1} A_{0,q+t\frac{\tilde{n}}{2}+u\tilde{\ell},u\tilde{n}-1,q-t\frac{\tilde{n}}{2}+u\tilde{\ell}}(\widetilde{\phi})w^{2q+2u(\tilde{\ell}+\frac{\tilde{n}}{2})-2}$$
$$+ w \sum_{(p,q,t,u)\in U_2} A_{p+t\tilde{\ell}-u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})+1,q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}),p+t\tilde{\ell}+u\frac{\tilde{n}}{2}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}),q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})}(\widetilde{\phi})$$
$$\times w^{2(p+q)+2(t+u+1)\tilde{\ell}}$$
$$+ w \sum_{(p,q,t,u)\in V_2} A_{0,q+t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}+\frac{\tilde{n}}{2}),(2u+1)\frac{\tilde{n}}{2}-1,q-t\frac{\tilde{n}}{2}+u\tilde{\ell}+\frac{1}{2}(\tilde{\ell}-\frac{\tilde{n}}{2})}(\widetilde{\phi})w^{2q+(2u+1)(\tilde{\ell}+\frac{\tilde{n}}{2})-2}$$
$$\approx w\left\{A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0)+A_{2010}(0))w^2 + A_{00,\tilde{n}-1,0}(0)w^{\tilde{n}-2}\right\}.$$

Since $\tilde{\ell}+\tilde{n}/2$ is even, $F_1(w,w,\widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w,w,\widetilde{\phi})$ and $(-w,-w,\widetilde{\phi})$ are conjugate.

Consider $\mu = (4;n/2,\ell,+)$ with $\tilde{n}$ odd. Substituting the square pattern $(z_1,z_2)=(w,w)$ into (A.172), we have

$$F_1(w,w,\widetilde{\phi}) = w \sum_{(p,q,t,u)\in U} A_{p+2t\tilde{\ell}-u\tilde{n}+1,q+t\tilde{n}+2u\tilde{\ell},p+2t\tilde{\ell}+u\tilde{n},q-t\tilde{n}+2u\tilde{\ell}}(\widetilde{\phi})w^{2(p+q)+4(t+u)\tilde{\ell}}$$
$$+ w \sum_{(p,q,t,u)\in V} A_{0,q+t\tilde{n}+2u\tilde{\ell},2u\tilde{n}-1,q-t\tilde{n}+2u\tilde{\ell}}(\widetilde{\phi})w^{2q+2u(2\tilde{\ell}+\tilde{n})-2}$$
$$\approx w\left\{A'_{1000}(0)\widetilde{\phi} + (A_{1101}(0)+A_{2010}(0))w^2\right\}.$$

Then, $F_1(w,w,\widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w,w,\widetilde{\phi})$ and $(-w,-w,\widetilde{\phi})$ are conjugate.

To sum up, we have the following proposition on the symmetry of the square pattern.

Proposition A.16. *For a critical point of multiplicity 4, the two bifurcating solutions* $(w,w,\widetilde{\phi})$ *and* $(-w,-w,\widetilde{\phi})$ $(w\in\mathbb{R})$ *are conjugate for the following cases:*

- $\mu = (4;k,0,+),(4;k,k,+)$ *with* $\check{n} = n/\gcd(n,k)$ *even,*
- $\mu = (4;n/2,\ell,+)$ *for any* $\tilde{n} = n/\gcd(n,\ell)$*,*

and are not conjugate for $\mu = (4;k,0,+),(4;k,k,+)$ *with* $\check{n}$ *odd.*

Existence and Symmetry of Stripe Patterns

In this section, we would like to show the existence and the symmetry of two types of stripe patterns, which are represented as

Type I stripe pattern: $(z_1,z_2)=(w,0)$ $(w\in\mathbb{R})$,
Type II stripe pattern: $(z_1,z_2)=(\mathrm{i}w,0)$ $(w\in\mathbb{R})$.

Consider $\mu = (4; k, 0, +)$. Substituting Type I stripe pattern $(z_1, z_2) = (w, 0)$ into (A.138), we have

$$F_1(w, 0, \widetilde{\phi}) = w \sum_{c=0}^{\infty} \sum_{p\in\mathbb{Z},\ c+p\check{n}+1>0} A_{c+p\check{n}+1,0c0}(\widetilde{\phi}) w^{2c+p\check{n}} + w \sum_{p=1}^{\infty} A_{00,p\check{n}-1,0}(\widetilde{\phi}) w^{p\check{n}-2}$$
$$\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0)w^2 + A_{00,\check{n}-1,0}(0)w^{\check{n}-2} \right\}.$$

Thus, $F_1(w, 0, \widetilde{\phi}) = 0$ has the trivial solution $w = 0$ and a bifurcating solution. From (A.130), we have $F_2(w, 0) = F_1(0, w)$, and hence we have $F_1 = F_2 = 0$ for $(z_1, z_2) = (w, 0)$. If $\check{n}$ is even, then $F_1(w, 0, \widetilde{\phi})$ becomes an odd function in w, and hence the two bifurcating solutions $(w, 0, \widetilde{\phi})$ and $(-w, 0, \widetilde{\phi})$ are conjugate. If $\check{n}$ is odd, the two solutions are not conjugate. Next, substituting Type II stripe pattern $(z_1, z_2) = (\mathrm{i}w, 0)$ into (A.138), we have

$$F_1(\mathrm{i}w, 0, \widetilde{\phi}) = \mathrm{i}w \sum_{c=0}^{\infty} \sum_{p\in\mathbb{Z},\ c+p\check{n}+1>0} A_{c+p\check{n}+1,0c0}(\widetilde{\phi}) \mathrm{i}^{p\check{n}} w^{2c+p\check{n}} - \mathrm{i}w \sum_{p=1}^{\infty} A_{00,p\check{n}-1,0}(\widetilde{\phi})(-\mathrm{i})^{p\check{n}-2} w^{p\check{n}-2}$$
$$\approx \mathrm{i}w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0)w^2 - A_{00,\check{n}-1,0}(0)(-\mathrm{i})^{\check{n}-2}w^{\check{n}-2} \right\}.$$

If $\check{n}$ is even ($\mathrm{i}^{p\check{n}}$ and $(-\mathrm{i})^{p\check{n}-2}$ are real), then $F_1(\mathrm{i}w, 0, \widetilde{\phi}) = 0$ has the trivial solution $(w = 0)$ and a bifurcating solution, and a discussion similar to that for Type I stripe pattern holds.

Consider $\mu = (4; k, k, +)$ with $\check{n}$ even. Substituting Type I stripe pattern $(z_1, z_2) = (w, 0)$ into (A.144), we have

$$F_1(w, 0, \widetilde{\phi}) = w \sum_{c=0}^{\infty} \sum_{q\in\mathbb{Z},\ c+q\check{n}+1>0} A_{c+q\check{n}+1,0c0}(\widetilde{\phi}) w^{2c+q\check{n}} + w \sum_{p=1}^{\infty} A_{00,q\check{n}-1,0}(\widetilde{\phi}) w^{q\check{n}-2}$$
$$\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0)w^2 + A_{00,\check{n}-1,0}(0)w^{\check{n}-2} \right\}.$$

Thus, $F_1(w, 0, \widetilde{\phi}) = 0$ has the trivial solution $(w = 0)$ and a bifurcating solution. From (A.130), we have $F_2(w, 0) = F_1(0, w)$, and hence we have $F_1 = F_2 = 0$ for $(z_1, z_2) = (w, 0)$. Since $\check{n}$ is even, $F_1(w, 0, \widetilde{\phi})$ is an odd function in w, and hence the two bifurcating solutions $(w, 0, \widetilde{\phi})$ and $(-w, 0, \widetilde{\phi})$ are conjugate. Next, substituting Type II stripe pattern $(z_1, z_2) = (\mathrm{i}w, 0)$ into (A.144), we have

$$F_1(\mathrm{i}w, 0, \widetilde{\phi}) = \mathrm{i}w \sum_{c=0}^{\infty} \sum_{q\in\mathbb{Z},\ c+q\check{n}+1>0} A_{c+q\check{n}+1,0c0}(\widetilde{\phi}) \mathrm{i}^{q\check{n}} w^{2c+q\check{n}} - \mathrm{i}w \sum_{q=1}^{\infty} A_{00,q\check{n}-1,0}(\widetilde{\phi})(-\mathrm{i})^{q\check{n}-2} w^{q\check{n}-2}$$
$$\approx \mathrm{i}w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0)w^2 - A_{00,\check{n}-1,0}(0)(-\mathrm{i})^{\check{n}-2}w^{\check{n}-2} \right\}.$$

Since $\check{n}$ is even ($\mathrm{i}^{q\check{n}}$ and $(-\mathrm{i})^{q\check{n}-2}$ are real), $F_1(\mathrm{i}w, 0, \widetilde{\phi}) = 0$ has the trivial solution $(w = 0)$ and a bifurcating solution, and a discussion similar to that for Type I stripe pattern holds.

Consider $\mu = (4; k, k, +)$ with $\check{n}$ odd. Substituting Type I stripe pattern $(z_1, z_2) = (w, 0)$ into (A.145), we have

$$F_1(w, 0, \widetilde{\phi}) = w \sum_{c=0}^{\infty} \sum_{p\in\mathbb{Z},\ c+p\check{n}+1>0} A_{c+p\check{n}+1,0c0}(\widetilde{\phi}) w^{2c+p\check{n}} + w \sum_{p=1}^{\infty} A_{00,p\check{n}-1,0}(\widetilde{\phi}) w^{p\check{n}-2}$$
$$\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0)w^2 + A_{00,\check{n}-1,0}(0)w^{\check{n}-2} \right\}.$$

Thus, $F_1(w, 0, \widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution. From (A.130), we have $F_2(w, 0) = F_1(0, w)$, and hence we have $F_1 = F_2 = 0$ for $(z_1, z_2) = (w, 0)$. Since $\check{n}$ is odd, $F_1(w, 0, \widetilde{\phi})$ is not an odd function in w, and hence the two bifurcating solutions $(w, 0, \widetilde{\phi})$ and $(-w, 0, \widetilde{\phi})$ are not conjugate. Next, substituting Type II stripe pattern $(z_1, z_2) = (\mathrm{i}w, 0)$ into (A.145), we have

$$F_1(\mathrm{i}w, 0, \widetilde{\phi}) = \mathrm{i}w \sum_{c=0}^{\infty} \sum_{p\in\mathbb{Z},\ c+p\check{n}+1>0} A_{c+p\check{n}+1,0c0}(\widetilde{\phi})\mathrm{i}^{p\check{n}} w^{2c+p\check{n}} - \mathrm{i}w \sum_{p=1}^{\infty} A_{00,p\check{n}-1,0}(\widetilde{\phi})(-\mathrm{i})^{p\check{n}-2} w^{p\check{n}-2}.$$

Since $\check{n}$ is odd ($\mathrm{i}^{p\check{n}}$ and $(-\mathrm{i})^{p\check{n}-2}$ can be imaginary), $F_1(\mathrm{i}w, 0, \widetilde{\phi}) = 0$ cannot be solved for $\widetilde{\phi}$.

Consider $\mu = (4; n/2, \ell, +)$ with $\tilde{n}/2$ even. In (A.156), we have

$$\begin{cases} b = q + t\tilde{n}/2 + u\tilde{\ell} = 0 \\ d = q - t\tilde{n}/2 + u\tilde{\ell} = 0 \end{cases} \Rightarrow \begin{cases} q = -u\tilde{\ell} \\ t = 0 \end{cases}.$$

Thus, we have

$$F_1(z_1, 0, \widetilde{\phi}) = z_1 \sum_{p,u\in\mathbb{Z},\ p-u\frac{\tilde{n}}{2}+1>0,\ p+u\frac{\tilde{n}}{2}\geq 0} A_{p-u\frac{\tilde{n}}{2}+1,0,p+u\frac{\tilde{n}}{2},0}(\widetilde{\phi}) {z_1}^{p-u\frac{\tilde{n}}{2}}\, {\overline{z}_1}^{p+u\frac{\tilde{n}}{2}} + \overline{z}_1 \sum_{u=1}^{\infty} A_{00,u\tilde{n}-1,0}(\widetilde{\phi}){\overline{z}_1}^{u\tilde{n}-2}. \tag{A.173}$$

Substituting Type I stripe pattern $(z_1, z_2) = (w, 0)$ into (A.173), we have

$$\begin{aligned} F_1(w, 0, \widetilde{\phi}) &= w \sum_{p,u\in\mathbb{Z},\ p-u\frac{\tilde{n}}{2}+1>0,\ p+u\frac{\tilde{n}}{2}\geq 0} A_{p-u\frac{\tilde{n}}{2}+1,0,p+u\frac{\tilde{n}}{2},0}(\widetilde{\phi}) w^{2p} + w \sum_{u=1}^{\infty} A_{00,u\tilde{n}-1,0}(\widetilde{\phi}) w^{u\tilde{n}-2} \\ &\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0) w^2 + A_{00,\tilde{n}-1,0}(0) w^{\tilde{n}-2} \right\}. \end{aligned}$$

Thus, $F_1(w, 0, \widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution. From (A.130), we have $F_2(w, 0) = F_1(0, w)$. Thus, we have $F_1 = F_2 = 0$ for $(z_1, z_2) = (w, 0)$. Since $\tilde{n}$ is even, $F_1(w, 0, \widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w, 0, \widetilde{\phi})$ and $(-w, 0, \widetilde{\phi})$ are conjugate. Next, substituting Type II stripe pattern $(z_1, z_2) = (\mathrm{i}w, 0)$ into (A.173), we have

$$\begin{aligned} &F_1(\mathrm{i}w, 0, \widetilde{\phi}) \\ &= \mathrm{i}w \sum_{p,u\in\mathbb{Z},\ p-u\frac{\tilde{n}}{2}+1>0,\ p+u\frac{\tilde{n}}{2}\geq 0} A_{p-u\frac{\tilde{n}}{2}+1,0,p+u\frac{\tilde{n}}{2},0}(\widetilde{\phi})(-1)^{p+u\frac{\tilde{n}}{2}} \mathrm{i}^{2p} w^{2p} - \mathrm{i}w \sum_{u=1}^{\infty} A_{00,u\tilde{n}-1,0}(\widetilde{\phi}) \mathrm{i}^{u\tilde{n}-2} w^{u\tilde{n}-2} \\ &\approx \mathrm{i}w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0) w^2 - A_{00,\tilde{n}-1,0}(0)\mathrm{i}^{\tilde{n}-2} w^{\tilde{n}-2} \right\}. \end{aligned}$$

Since $\tilde{n}$ is even ($\mathrm{i}^{u\tilde{n}-2}$ is real), $F_1(\mathrm{i}w, 0, \widetilde{\phi}) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution. Then, a discussion similar to that for Type I stripe pattern holds.

Consider $\mu = (4; n/2, \ell, +)$ with $\tilde{n}/2$ odd. In (A.161), we have

$$\begin{cases} b = q + t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell} + \tilde{n}/2)/2 = 0 \\ d = q - t\tilde{n}/2 + u\tilde{\ell} + (\tilde{\ell} - \tilde{n}/2)/2 = 0 \end{cases} \Rightarrow 2q + (2u+1)\tilde{\ell} = 0.$$

Since $\tilde{\ell}$ is odd, this relation is a contradiction. Hence, $b = 0$ and $d = 0$ are not satisfied simultaneously. In (A.160), we have

$$\begin{cases} b = q + u\tilde{\ell} + t\tilde{n}/2 = 0 \\ d = q + u\tilde{\ell} - t\tilde{n}/2 = 0 \end{cases} \Rightarrow \begin{cases} q = -u\tilde{\ell} \\ t = 0 \end{cases}.$$

To sum up, we have

$$F_1(z_1, 0, \widetilde{\phi}) = z_1 \sum_{p,u\in\mathbb{Z},\ p-u\frac{\tilde{n}}{2}+1>0,\ p+u\frac{\tilde{n}}{2}\geq 0} A_{p-u\frac{\tilde{n}}{2}+1,0,p+u\frac{\tilde{n}}{2},0}(\widetilde{\phi}) {z_1}^{p-u\frac{\tilde{n}}{2}}\ {\overline{z}_1}^{p+u\frac{\tilde{n}}{2}} + \overline{z}_1 \sum_{u=1}^{\infty} A_{00,u\tilde{n}-1,0}(\widetilde{\phi}) {\overline{z}_1}^{u\tilde{n}-2}.$$

Then, a discussion similar to that for $\mu = (4; n/2, \ell, +)$ with $(\tilde{\ell}, \tilde{n}/2) = (\text{odd}, \text{even})$ holds.

Consider $\mu = (4; n/2, \ell, +)$ with $\tilde{n}$ odd. In (A.169), we have

$$\begin{cases} b = q + t\tilde{n} + 2u\tilde{\ell} = 0 \\ d = q - t\tilde{n} + 2u\tilde{\ell} = 0 \end{cases} \Rightarrow \begin{cases} q = -2u\tilde{\ell} \\ t = 0 \end{cases}.$$

Thus, we have

$$F_1(z_1, 0, \widetilde{\phi}) = z_1 \sum_{p,u\in\mathbb{Z},\ p-u\tilde{n}+1>0,\ p+u\tilde{n}\geq 0} A_{p-u\tilde{n}+1,0,p+u\tilde{n},0}(\widetilde{\phi}) {z_1}^{p-u\tilde{n}}\ {\overline{z}_1}^{p+u\tilde{n}} + \overline{z}_1 \sum_{u=1}^{\infty} A_{00,2u\tilde{n}-1,0}(\widetilde{\phi}) {\overline{z}_1}^{2(u\tilde{n}-1)}. \tag{A.174}$$

Substituting Type I stripe pattern $(z_1, z_2) = (w, 0)$ into (A.174), we have

$$\begin{aligned} F_1(w, 0, \widetilde{\phi}) &= w \sum_{p,u\in\mathbb{Z},\ p-u\tilde{n}+1>0,\ p+u\tilde{n}\geq 0} A_{p-u\tilde{n}+1,0,p+u\tilde{n},0}(\widetilde{\phi}) w^{2p} + w \sum_{u=1}^{\infty} A_{00,2u\tilde{n}-1,0}(\widetilde{\phi}) w^{2(u\tilde{n}-1)} \\ &\approx w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0) w^2 \right\}. \end{aligned}$$

Thus, $F_1(w, 0, \widetilde{\phi}) = 0$ has the trivial solution $(w = 0)$ and a bifurcating solution. From (A.130), we have $F_2(w, 0) = F_1(0, w)$. Thus, we have $F_1 = F_2 = 0$ for $(z_1, z_2) = (w, 0)$. We see that $F_1(w, 0, \widetilde{\phi})$ is an odd function in w. Hence, the two bifurcating solutions $(w, 0, \widetilde{\phi})$ and $(-w, 0, \widetilde{\phi})$ are conjugate. Next, substituting Type II stripe pattern $(z_1, z_2) = (\mathrm{i}w, 0)$ into (A.174), we have

$$\begin{aligned} &F_1(w, 0, \widetilde{\phi}) \\ &= \mathrm{i}w \sum_{p,u\in\mathbb{Z},\ p-u\tilde{n}+1>0,\ p+u\tilde{n}\geq 0} A_{p-u\tilde{n}+1,0,p+u\tilde{n},0}(\widetilde{\phi}) (-1)^{p+u\tilde{n}} \mathrm{i}^{2p} w^{2p} - \mathrm{i}w \sum_{u=1}^{\infty} A_{00,2u\tilde{n}-1,0}(\widetilde{\phi}) \mathrm{i}^{2(u\tilde{n}-1)} w^{2(u\tilde{n}-1)} \\ &\approx \mathrm{i}w \left\{ A'_{1000}(0)\widetilde{\phi} + A_{2010}(0) w^2 \right\}. \end{aligned}$$

Since the indices of i are real, $F_1(\mathrm{i}w, 0, \widetilde{\phi}) = 0$ has the trivial solution $(w = 0)$ and a bifurcating solution. Then, a discussion similar to that for Type I stripe pattern holds.

To sum up, we have the following propositions on the existence and the symmetry of the stripe patterns.

Proposition A.17. *For a critical point of multiplicity 4, the stripe patterns* $\boldsymbol{z} = (w, 0)$, $(\mathrm{i}w, 0)$ $(w \in \mathbb{R})$ *exist for the following cases:*

- $\mu = (4; k, 0, +), (4; k, k, +)$ *of Type I for any* $\check{n} = n/\gcd(n, k)$ *and Type II with* $\check{n}$ *even,*
- $\mu = (4; n/2, \ell, +)$ *of Type I and Type II for any* $\tilde{n} = n/\gcd(n, \ell)$.

Proposition A.18. *For a critical point of multiplicity 4, the two bifurcating solutions* $(\boldsymbol{z}, \widetilde{\phi})$ *and* $(-\boldsymbol{z}, \widetilde{\phi})$ *are conjugate for* $\boldsymbol{z} = (w, 0)$, $(\mathrm{i}w, 0)$ $(w \in \mathbb{R})$ *for the following cases:*

- $\mu = (4; k, 0, +), (4; k, k, +)$ *with* $\check{n} = n/\gcd(n, k)$ *even,*
- $\mu = (4; n/2, \ell, +)$ *for any* $\tilde{n} = n/\gcd(n, \ell)$,

and are not conjugate for $\boldsymbol{z} = (w, 0)$ *for* $\mu = (4; k, 0, +), (4; k, k, +)$ *with* $\check{n}$ *odd.*

Stability of bifurcating solutions

In Section 3.5.5, we found square patterns for a critical point of multiplicity 4 by using the equivariant branching lemma. In the previous subsections, we showed two kinds of stripe patterns by solving the bifurcation equation. These bifurcating solutions are represented for the bifurcation equation in real variables in (A.100) as follows ($w \in \mathbb{R}$):

$$\begin{aligned}\boldsymbol{w}_{\mathrm{sq}} &= (w, 0, w, 0),\\ \boldsymbol{w}_{\mathrm{stripeI}} &= (w, 0, 0, 0),\\ \boldsymbol{w}_{\mathrm{stripeII}} &= (0, w, 0, 0).\end{aligned}$$

We would like to evaluate the stability of these bifurcating solutions.

We denote by S the set of nonnegative indices (a, b, c, d) as

$$S = \begin{cases} \{(a, b, c, d) \in \mathbb{Z}_+^4 \mid \text{(A.136)}\} & \text{for } \mu = (4; k, 0, +),\\ \{(a, b, c, d) \in \mathbb{Z}_+^4 \mid \text{(A.143)}\} & \text{for } \mu = (4; k, k, +),\\ \{(a, b, c, d) \in \mathbb{Z}_+^4 \mid \text{(A.148)}\} & \text{for } \mu = (4; n/2, \ell, +),\end{cases} \tag{A.175}$$

where $\mathbb{Z}_+^4$ represents the set of nonnegative integers in $\mathbb{Z}^4$. Note that (a, b, c, d) must belong to S when $A_{abcd}(\widetilde{\phi}) \neq 0$. Hence, we replace the power series (A.105) with

$$F_1(z_1, z_2, \widetilde{\phi}) = \sum_S A_{abcd}(\widetilde{\phi}) {z_1}^a {z_2}^b\, \overline{z}_1{}^c\, \overline{z}_2{}^d. \tag{A.176}$$

To obtain the asymptotic form of the bifurcation equation and the Jacobian matrix, we elucidate the elements of S in (A.175) and specify the form of the power series in (A.176). In other words, we investigate which coefficient $A_{abcd}(\widetilde{\phi})$ becomes nonzero in (A.176). We focus on the coefficients of linear terms, quadratic terms, and cubic terms, which play a vital role as leading terms in (A.176). For this purpose, we take $(a, b, c, d) \in \mathbb{Z}_+^4$ with $a + b + \cdots + h \leq 3$ exhaustively and investigate whether it belongs to S or not. For $(4; k, 0, +)$, $(4; k, k, +)$, and $(4; n/2, \ell, +)$, we can see

$$(1, 0, 0, 0), (1, 1, 0, 1), (2, 0, 1, 0) \in S.$$

In addition, for some specific cases, we can see

$$\begin{array}{ll} (0,0,2,0) \in S & \text{for } (4; k, 0, +) \text{ with } \check{n} = 3, \\ (0,0,3,0) \in S & \text{for } (4; k, 0, +) \text{ with } \check{n} = 4, \\ (0,0,2,0) \in S & \text{for } (4; k, k, +) \text{ with } \check{n} = 3, \\ (0,0,3,0), (0,2,1,0), (0,0,1,2) \in S & \text{for } (4; k, k, +) \text{ with } \check{n} = 4, \\ (0,0,3,0) \in S & \text{for } (4; n/2, \ell, +) \text{ with } \tilde{n} = 4. \end{array}$$

Based on the above results, F_i $(i = 1, 2)$ in (A.102) is restricted to the form of

$$F_i = a_1 \widetilde{\phi} z_i + F_i^C + (\text{other terms}), \quad i = 1, 2, \tag{A.177}$$

where

$$F_1^C = a_2 z_1 z_2 \overline{z}_2 + a_3 {z_1}^2 \overline{z}_1, \tag{A.178}$$
$$F_2^C = a_2 z_2 \overline{z}_1 z_1 + a_3 {z_2}^2 \overline{z}_2 \tag{A.179}$$

with the following notations:

$$a_1 = A'_{1000}(0), \quad a_2 = A_{1101}(0), \quad a_3 = A_{2010}(0). \tag{A.180}$$

Therein, F_2 is obtained by (A.130). The form of "(other terms)" depends on the type of the irreducible representations in (A.175). Accordingly, $\widetilde{F}_i$ $(i = 1, \ldots, 4)$ in (A.100) is restricted to the form of

$$\widetilde{F}_i = a_1 \widetilde{\phi} w_i + \widetilde{F}_i^C + (\text{other terms}), \quad i = 1, \ldots, 4 \tag{A.181}$$

with

$$\widetilde{F}_1^C = a_2 w_1 ({w_3}^2 + {w_4}^2) + a_3 w_1 ({w_1}^2 + {w_2}^2), \tag{A.182}$$
$$\widetilde{F}_2^C = a_2 w_2 ({w_3}^2 + {w_4}^2) + a_3 w_2 ({w_1}^2 + {w_2}^2), \tag{A.183}$$
$$\widetilde{F}_3^C = a_2 w_3 ({w_1}^2 + {w_2}^2) + a_3 w_3 ({w_3}^2 + {w_4}^2), \tag{A.184}$$
$$\widetilde{F}_4^C = a_2 w_4 ({w_1}^2 + {w_2}^2) + a_3 w_4 ({w_3}^2 + {w_4}^2). \tag{A.185}$$

In (A.177), F_i^C corresponds to cubic terms, and the form of "(other terms)" varies with the irreducible representations. For the case $(4; k, 0, +)$ with $\check{n} = 3$, we have quadratic terms as leading terms. For any other cases, we have cubic terms as leading terms that vary with the irreducible representations. From this point of view, we can classify the form of the bifurcation equation as shown in Table A.4 for each irreducible representation.

As mentioned earlier, the form of "(other terms)" in (A.181) depends on the type μ of the irreducible representations in (A.175). Therefore, we checked all the possible cases numerically and classified each case by the form of leading terms. All the possible cases and stability conditions for the bifurcating solutions are summarized in Table A.5. The main finding of this section is as follows:

Proposition A.19. *For a critical point of multiplicity 4, we have the following statements:*

Table A.4: Nonzero coefficients of leading terms which belong to "other terms" in (A.177).

μ	Cases	Nonzero coefficients
$(4; k, 0, +)$	General $\check{n}$	None
	$\check{n} = 3$	$A_{0020}(0)$
	$\check{n} = 4$	$A_{0030}(0)$
$(4; k, k, +)$	General $\check{n}$	None
	$\check{n} = 3$	$A_{0020}(0)$
	$\check{n} = 4$	$A_{0030}(0),\ A_{0210}(0),\ A_{0012}(0)$
$(4; n/2, \ell, +)$	General $\tilde{n}$	None
	$\tilde{n} = 4$	$A_{0030}(0)$

$\check{n} = n/\gcd(k, n)$ in (A.92); $\tilde{n} = n/\gcd(\ell, n)$ in (A.93)

- *For $\mu = (4; k, 0, +)$ and $\mu = (4; k, k, +)$ with $\check{n} = 3$, the bifurcating solutions $\boldsymbol{w}_{\rm sq}$ and $\boldsymbol{w}_{\rm stripeI}$ are always unstable in the neighborhood of the critical point, and the bifurcating curve takes the form $\widetilde{\phi} \approx cw$ for some constant c.*
- *For any other cases, the stability of the bifurcating solutions $\boldsymbol{w}_{\rm sq}$, $\boldsymbol{w}_{\rm stripeI}$, and $\boldsymbol{w}_{\rm stripeII}$ depends on the values of the coefficients of the power series expansion of the bifurcation equation in (A.176), and the bifurcating curve takes the form $\widetilde{\phi} \approx cw^2$ for some constant c.*

To show these results, we derive the asymptotic form of the bifurcation equation for each case and conduct stability analysis for the bifurcating solutions in the remainder of this section.

Case 1: General $(4; k, 0, +)$

For general $(4; k, 0, +)$, we have

$$(1, 0, 0, 0), (1, 1, 0, 1), (2, 0, 1, 0) \in S,$$

and the other elements of S correspond to higher order terms. Then, the asymptotic form of F_i $(i = 1, 2)$ in (A.177) becomes

$$F_1(z_1, z_2, \widetilde{\phi}) \approx a_1\widetilde{\phi}z_1 + F_1^C, \tag{A.186}$$
$$F_2(z_1, z_2, \widetilde{\phi}) \approx a_1\widetilde{\phi}z_2 + F_2^C, \tag{A.187}$$

where F_i^C $(i = 1, 2)$ is given in (A.178) and (A.179). By (A.103) and (A.104), the asymptotic form of $\widetilde{F}_i$ $(i = 1, \ldots, 4)$ in (A.100) becomes

$$\widetilde{F}_1 \approx a_1\widetilde{\phi}w_1 + \widetilde{F}_1^C, \tag{A.188}$$
$$\widetilde{F}_2 \approx a_1\widetilde{\phi}w_2 + \widetilde{F}_1^C, \tag{A.189}$$
$$\widetilde{F}_3 \approx a_1\widetilde{\phi}w_3 + \widetilde{F}_1^C, \tag{A.190}$$
$$\widetilde{F}_4 \approx a_1\widetilde{\phi}w_4 + \widetilde{F}_1^C, \tag{A.191}$$

Table A.5: Stability conditions of bifurcating solutions for group-theoretic critical points with multiplicity 4.

μ	Cases	Solutions	Stability conditions
$(4; k, 0, +)$	$\check{n} = 3$	$\boldsymbol{w}_{\rm sq}$	Always unstable
		$\boldsymbol{w}_{\rm stripeI}$	Always unstable
		$\boldsymbol{w}_{\rm stripeII}$	Does not exist
	$\check{n} = 4$	$\boldsymbol{w}_{\rm sq}$	$a_3 < -a_5 < 0, \ a_3 + a_5 < -\lvert a_2 \rvert$
		$\boldsymbol{w}_{\rm stripeI}$	$a_3 < -a_5 < 0, \ a_2 < a_3 + a_5$
		$\boldsymbol{w}_{\rm stripeII}$	$a_3 < -a_5 < 0, \ a_2 < a_3 + a_5$
$(4; k, k, +)$	$\check{n} = 3$	$\boldsymbol{w}_{\rm sq}$	Always unstable
		$\boldsymbol{w}_{\rm stripeI}$	Always unstable
		$\boldsymbol{w}_{\rm stripeII}$	Does not exist
	$\check{n} = 4$	$\boldsymbol{w}_{\rm sq}$	$a_5 + a_6 > 0, \ a_3 + a_5 < -\lvert a_2 + 2a_6 \rvert$
		$\boldsymbol{w}_{\rm stripeI}$	$a_3 < -a_5 < 0, \ -2\lvert a_6 \rvert < a_3 + a_5$
		$\boldsymbol{w}_{\rm stripeII}$	$a_3 < -a_5 < 0, \ -2\lvert a_6 \rvert < a_3 + a_5$
$(4; n/2, \ell, +)$	$\tilde{n} = 4$	$\boldsymbol{w}_{\rm sq}$	$a_3 < -a_5 < 0, \ a_3 + a_5 < -\lvert a_2 \rvert$
		$\boldsymbol{w}_{\rm stripeI}$	$a_3 < -a_5 < 0, \ a_2 < a_3 + a_5$
		$\boldsymbol{w}_{\rm stripeII}$	$a_3 < -a_5 < 0, \ a_2 < a_3 + a_5$

μ	Cases	Solutions	Stability conditions (necessary condition)
$(4; k, 0, +)$	General $\check{n}$	$\boldsymbol{w}_{\rm sq}$	$a_3 < -\lvert a_2 \rvert$
		$\boldsymbol{w}_{\rm stripeI}$	$a_2 < a_3 < 0$
		$\boldsymbol{w}_{\rm stripeII}$	$a_2 < a_3 < 0$ if $\check{n}$ is even Does not exist if $\check{n}$ is odd
$(4; k, k, +)$	General $\check{n}$	$\boldsymbol{w}_{\rm sq}$	$a_3 < -\lvert a_2 \rvert$
		$\boldsymbol{w}_{\rm stripeI}$	$a_2 < a_3 < 0$
		$\boldsymbol{w}_{\rm stripeII}$	$a_2 < a_3 < 0$ if $\check{n}$ is even Does not exist if $\check{n}$ is odd
$(4; n/2, \ell, +)$	General $\tilde{n}$	$\boldsymbol{w}_{\rm sq}$	$a_3 < -\lvert a_2 \rvert$
		$\boldsymbol{w}_{\rm stripeI}$	$a_2 < a_3 < 0$
		$\boldsymbol{w}_{\rm stripeII}$	$a_2 < a_3 < 0$

$\check{n} = n/\gcd(k, n)$ in (A.92); $\tilde{n} = n/\gcd(\ell, n)$ in (A.93);

$a_2 = A_{1101}(0)$, $a_3 = A_{2010}(0)$, $a_4 = A_{0020}(0)$, $a_5 = A_{0030}(0)$, $a_6 = A_{0210}(0)$ in (A.176) where $\widetilde{F}_i^C$ ($i = 1, \ldots, 4$) is given in (A.182) – (A.185). Hence, the asymptotic form of the Jacobian matrix in (A.101) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1 \widetilde{\phi} I_4 + B_C \tag{A.192}$$

with

$$B_C = a_2 B_2 + a_3 B_3, \tag{A.193}$$

$$B_2 = \begin{bmatrix} w_3{}^2 + w_4{}^2 & 0 & 2w_1w_3 & 2w_1w_4 \\ 0 & w_3{}^2 + w_4{}^2 & 2w_2w_3 & 2w_2w_4 \\ 2w_1w_3 & 2w_2w_3 & w_1{}^2 + w_2{}^2 & 0 \\ 2w_1w_4 & 2w_2w_4 & 0 & w_1{}^2 + w_2{}^2 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 3w_1{}^2 + w_2{}^2 & 2w_1w_2 & 0 & 0 \\ 2w_1w_2 & w_1{}^2 + 3w_2{}^2 & 0 & 0 \\ 0 & 0 & 3w_3{}^2 + w_4{}^2 & 2w_3w_4 \\ 0 & 0 & 2w_3w_4 & w_3{}^2 + 3w_4{}^2 \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\mathrm{sq}}$ into (A.188) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{sq}} \approx -\frac{a_2 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.192) at $(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}}) \approx 2w^2 \begin{bmatrix} a_3 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 \\ a_2 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + O(w^3). \tag{A.194}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$ are given by

$$\lambda_1, \lambda_2 \approx 2(a_3 \pm a_2)w^2,$$
$$\lambda_3 \approx O(w^3) \quad \text{(repeated twice)}.$$

A necessary condition where $\boldsymbol{w}_{\mathrm{sq}}$ is stable is $a_3 < -|a_2|$. A more rigorous stability condition relies on the concrete form of the terms of $O(w^3)$ for λ_3. Thus, the stability of $\boldsymbol{w}_{\mathrm{sq}}$ depends on the values of a_2 and a_3.

Substituting $\boldsymbol{w}_{\mathrm{stripeI}}$ into (A.188) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripeI}} \approx -\frac{a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.192) at $(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}}) \approx w^2 \begin{bmatrix} 2a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 - a_3 & 0 \\ 0 & 0 & 0 & a_2 - a_3 \end{bmatrix} + O(w^3). \tag{A.195}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$ are given by the diagonal components, i.e.,

$$\begin{aligned}
\lambda_1 &\approx 2a_3 w^2, \\
\lambda_2 &\approx O(w^3), \\
\lambda_3 &\approx (a_2 - a_3) w^2 \quad \text{(repeated twice)}.
\end{aligned}$$

Necessary conditions where $\boldsymbol{w}_{\text{stripeI}}$ is stable are $a_2 < a_3 < 0$. A more rigorous stability condition relies on the concrete form of the terms of $O(w^3)$ for λ_2. Thus, the stability of $\boldsymbol{w}_{\text{stripeI}}$ depends on the values of a_2 and a_3.

Substituting $\boldsymbol{w}_{\text{stripeII}}$ into (A.189) and solving $\widetilde{F}_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeII}} \approx -\frac{a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.192) at $(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}}) \approx w^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2a_3 & 0 & 0 \\ 0 & 0 & a_2 - a_3 & 0 \\ 0 & 0 & 0 & a_2 - a_3 \end{bmatrix} + O(w^3). \tag{A.196}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$.

Case 2: $(4; k, 0, +)$ *with* $\check{n} = 3$

For the case $(4; k, 0, +)$ with $\check{n} = 3$, we have

$$(0, 0, 2, 0) \in S$$

as well as

$$(1, 0, 0, 0), (1, 1, 0, 1), (2, 0, 1, 0) \in S.$$

Then, the asymptotic form of F_i $(i = 1, 2)$ in (A.177) becomes

$$F_1(z_1, z_2, \widetilde{\phi}) \approx a_1 \widetilde{\phi} z_1 + a_4 \overline{z_1}^2 + F_1^C, \tag{A.197}$$

$$F_2(z_1, z_2, \widetilde{\phi}) \approx a_1 \widetilde{\phi} z_2 + a_4 \overline{z_2}^2 + F_2^C \tag{A.198}$$

with $a_4 = A_{0020}(0)$, where F_i^C $(i = 1, 2)$ is given in (A.178) and (A.179). By (A.103) and (A.104), the asymptotic form of $\widetilde{F}_i$ $(i = 1, \ldots, 4)$ in (A.100) becomes

$$\widetilde{F}_1 \approx a_1 \widetilde{\phi} w_1 + a_4 ({w_1}^2 - {w_2}^2) + \widetilde{F}_1^C, \tag{A.199}$$

$$\widetilde{F}_2 \approx a_1 \widetilde{\phi} w_2 - 2a_4 w_1 w_2 + \widetilde{F}_2^C, \tag{A.200}$$

$$\widetilde{F}_3 \approx a_1 \widetilde{\phi} w_3 + a_4 ({w_3}^2 - {w_4}^2) + \widetilde{F}_3^C, \tag{A.201}$$

$$\widetilde{F}_4 \approx a_1 \widetilde{\phi} w_4 - 2a_4 w_3 w_4 + \widetilde{F}_4^C, \tag{A.202}$$

where $\widetilde{F}_i^C$ $(i = 1, \ldots, 4)$ is given in (A.182)–(A.185). Hence, the asymptotic form of the Jacobian matrix in (A.101) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1 \widetilde{\phi} I_4 + a_4 B_4 + B_C \tag{A.203}$$

with

$$B_4 = 2 \begin{bmatrix} w_1 & -w_2 & 0 & 0 \\ -w_2 & -w_1 & 0 & 0 \\ 0 & 0 & w_3 & -w_4 \\ 0 & 0 & -w_4 & -w_3 \end{bmatrix},$$

where B_C is given in (A.193).

Substituting $\boldsymbol{w}_{\text{sq}}$ into (A.199) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sq}} \approx -\frac{a_4}{a_1} w.$$

Evaluating the Jacobian matrix (A.203) at $(\boldsymbol{w}_{\text{sq}}, \widetilde{\phi}_{\text{sq}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sq}}, \widetilde{\phi}_{\text{sq}}) \approx a_4 w \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}. \tag{A.204}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sq}}, \widetilde{\phi}_{\text{sq}})$ are given by the diagonal components, i.e., $a_4 w$ (repeated twice) and $-3a_4 w$ (repeated twice). Since the eigenvalues $a_4 w$ and $-3a_4 w$ have opposite signs, the bifurcating solution $\boldsymbol{w}_{\text{sq}}$ is always unstable.

Substituting $\boldsymbol{w}_{\text{stripeI}}$ into (A.199) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeI}} \approx -\frac{a_4}{a_1} w.$$

Evaluating the Jacobian matrix (A.203) at $(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}}) \approx a_4 w \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{A.205}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$ are given by the diagonal components, i.e., $a_4 w$, $-3a_4 w$ and $-a_4 w$ (repeated twice). Since the eigenvalues $a_4 w$ and $-3a_4 w$ have opposite signs, the bifurcating solution $\boldsymbol{w}_{\text{stripeI}}$ is always unstable.

Remark A.3. Since $\check{n}$ is odd, $\boldsymbol{w}_{\text{stripeII}}$ does not exist for the case $(4; k, 0, +)$ with $\check{n} = 3$. See Proposition A.17.

□

Case 3: $(4;k,0,+)$ *with* $\check{n}=4$

For the case $(4;k,0,+)$ with $\check{n}=4$, we have

$$(0,0,3,0)\in S$$

as well as

$$(1,0,0,0),(1,1,0,1),(2,0,1,0)\in S.$$

Then, the asymptotic form of F_i $(i=1,2)$ in (A.177) becomes

$$F_1(z_1,z_2,\widetilde{\phi})\approx a_1\widetilde{\phi}z_1+a_5\overline{z}_1{}^3+F_1^C, \tag{A.206}$$
$$F_2(z_1,z_2,\widetilde{\phi})\approx a_1\widetilde{\phi}z_2+a_5\overline{z}_2{}^3+F_2^C \tag{A.207}$$

with $a_5=A_{0030}(0)$, where F_i^C $(i=1,2)$ is given in (A.178) and (A.179). By (A.103) and (A.104), the asymptotic form of $\widetilde{F}_i$ $(i=1,\dots,4)$ in (A.100) becomes

$$\widetilde{F}_1\approx a_1\widetilde{\phi}w_1+a_5w_1(w_1{}^2-3w_2{}^2)+\widetilde{F}_1^C, \tag{A.208}$$
$$\widetilde{F}_2\approx a_1\widetilde{\phi}w_2+a_5w_2(w_2{}^2-3w_1{}^2)+\widetilde{F}_2^C, \tag{A.209}$$
$$\widetilde{F}_3\approx a_1\widetilde{\phi}w_3+a_5w_3(w_3{}^2-3w_4{}^2)+\widetilde{F}_3^C, \tag{A.210}$$
$$\widetilde{F}_4\approx a_1\widetilde{\phi}w_4+a_5w_4(w_4{}^2-3w_3{}^2)+\widetilde{F}_4^C, \tag{A.211}$$

where $\widetilde{F}_i^C$ $(i=1,\dots,4)$ is given in (A.182) – (A.185). Hence, the asymptotic form of the Jacobian matrix in (A.101) becomes

$$\widetilde{J}(\boldsymbol{w},\widetilde{\phi})\approx a_1\widetilde{\phi}I_4+a_5B_5+B_C \tag{A.212}$$

with

$$B_5=3\begin{bmatrix} w_1{}^2-w_2{}^2 & -2w_1w_2 & 0 & 0\\ -2w_1w_2 & w_2{}^2-w_1{}^2 & 0 & 0\\ 0 & 0 & w_3{}^2-w_4{}^2 & -2w_3w_4\\ 0 & 0 & -2w_3w_4 & w_4{}^2-w_3{}^2 \end{bmatrix}, \tag{A.213}$$

where B_C is given in (A.193).

Substituting $\boldsymbol{w}_{\mathrm{sq}}$ into (A.208) and solving $\widetilde{F}_1=0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi}=\widetilde{\phi}_{\mathrm{sq}}\approx -\frac{a_5+a_2+a_3}{a_1}w^2.$$

Evaluating the Jacobian matrix (A.212) at $(\boldsymbol{w}_{\mathrm{sq}},\widetilde{\phi}_{\mathrm{sq}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}},\widetilde{\phi}_{\mathrm{sq}})\approx 2w^2\begin{bmatrix} a_5+a_3 & 0 & a_2 & 0\\ 0 & -2a_5 & 0 & 0\\ a_2 & 0 & a_5+a_3 & 0\\ 0 & 0 & 0 & -2a_5 \end{bmatrix}. \tag{A.214}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$ are given by

$$\lambda_1, \lambda_2 \approx 2(a_5 + a_3 \pm a_2)w^2,$$
$$\lambda_3 \approx -4a_5 w^2 \quad \text{(repeated twice)}.$$

If $a_3 < -a_5 < 0$ and $a_5 + a_3 < -|a_2|$ are satisfied, $\boldsymbol{w}_{\mathrm{sq}}$ is stable. Otherwise, $\boldsymbol{w}_{\mathrm{sq}}$ is unstable. Thus, the stability of $\boldsymbol{w}_{\mathrm{sq}}$ depends on the values of a_2, a_3 and a_5.

Substituting $\boldsymbol{w}_{\mathrm{stripeI}}$ into (A.208) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripeI}} \approx -\frac{a_5 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.212) at $(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}}) \approx w^2 \begin{bmatrix} 2(a_5 + a_3) & 0 & 0 & 0 \\ 0 & -4a_5 & 0 & 0 \\ 0 & 0 & -a_5 + a_2 - a_3 & 0 \\ 0 & 0 & 0 & -a_5 + a_2 - a_3 \end{bmatrix}. \tag{A.215}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}})$ are given by the diagonal components, i.e.,

$$\lambda_1 \approx 2(a_5 + a_3)w^2,$$
$$\lambda_2 \approx -4a_5 w^2,$$
$$\lambda_3 \approx -(a_5 - a_2 + a_3)w^2 \quad \text{(repeated twice)}.$$

If $a_3 < -a_5 < 0$ and $a_2 < a_5 + a_3$ are satisfied, $\boldsymbol{w}_{\mathrm{stripeI}}$ is stable. Thus, the stability of $\boldsymbol{w}_{\mathrm{stripeI}}$ depends on the values of a_2, a_3 and a_5.

Substituting $\boldsymbol{w}_{\mathrm{stripeII}}$ into (A.209) and solving $\widetilde{F}_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripeII}} \approx -\frac{a_5 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.212) at $(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}}) \approx w^2 \begin{bmatrix} -4a_5 & 0 & 0 & 0 \\ 0 & 2(a_5 + a_3) & 0 & 0 \\ 0 & 0 & -a_5 + a_2 - a_3 & 0 \\ 0 & 0 & 0 & -a_5 + a_2 - a_3 \end{bmatrix}. \tag{A.216}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\mathrm{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\mathrm{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\mathrm{stripeI}}$.

Case 4: General $(4; k, k, +)$

For general $(4; k, k, +)$, we have

$$(1, 0, 0, 0), (1, 1, 0, 1), (2, 0, 1, 0) \in S,$$

and the other elements of S correspond to higher order terms. Then, the asymptotic form of F_1 in (A.182) is equivalent to that for the case 1: General $(4; k, 0, +)$. Hence, a discussion similar to that for the case 1 holds.

Case 5: $(4; k, k, +)$ *with* $\check{n} = 3$

For the case $(4; k, k, +)$ with $\check{n} = 3$, we have

$$(0, 0, 2, 0) \in S$$

as well as

$$(1, 0, 0, 0), (1, 1, 0, 1), (2, 0, 1, 0) \in S.$$

Then, the asymptotic form of F_1 in (A.182) is equivalent to that for the case 2: $(4; k, 0, +)$ with $\check{n} = 3$. Hence, a discussion similar to that for the case 2 holds, that is, $\boldsymbol{w}_{\mathrm{sq}}$ and $\boldsymbol{w}_{\mathrm{stripeI}}$ are always unstable. Since $\check{n}$ is odd, $\boldsymbol{w}_{\mathrm{stripeII}}$ does not exist for this case (see Proposition A.17).

Case 6: $(4; k, k, +)$ *with* $\check{n} = 4$

For the case $(4; k, k, +)$ with $\check{n} = 4$, we have

$$(0, 0, 3, 0), (0, 2, 1, 0), (0, 0, 1, 2) \in S$$

as well as

$$(1, 0, 0, 0), (1, 1, 0, 1), (2, 0, 1, 0) \in S.$$

From the condition (A.133), we have $A_{0210}(0) = A_{0012}(0)$. Then, the asymptotic form of F_i $(i = 1, 2)$ in (A.177) becomes

$$F_1(z_1, z_2, \widetilde{\phi}) \approx a_1 \widetilde{\phi} z_1 + a_5 \overline{z}_1{}^3 + a_6 \overline{z}_1 (z_2{}^2 + \overline{z}_2{}^2) + F_1^C, \tag{A.217}$$

$$F_2(z_1, z_2, \widetilde{\phi}) \approx a_1 \widetilde{\phi} z_2 + a_5 \overline{z}_2{}^3 + a_6 \overline{z}_2 (\overline{z}_1{}^2 + z_1{}^2) + F_2^C \tag{A.218}$$

with $a_6 = A_{0210}(0)$, where F_i^C $(i = 1, 2)$ is given in (A.178) and (A.179). By (A.103) and (A.104), the asymptotic form of $\widetilde{F}_i$ $(i = 1, \ldots, 4)$ in (A.100) becomes

$$\widetilde{F}_1 \approx a_1 \widetilde{\phi} w_1 + a_5 w_1 (w_1{}^2 - 3w_2{}^2) + 2a_6 w_1 (w_3{}^2 - w_4{}^2) + \widetilde{F}_1^C, \tag{A.219}$$

$$\widetilde{F}_2 \approx a_1 \widetilde{\phi} w_2 + a_5 w_2 (w_2{}^2 - 3w_1{}^2) + 2a_6 w_2 (w_4{}^2 - w_3{}^2) + \widetilde{F}_2^C, \tag{A.220}$$

$$\widetilde{F}_3 \approx a_1 \widetilde{\phi} w_3 + a_5 w_3 (w_3{}^2 - 3w_4{}^2) + 2a_6 w_3 (w_1{}^2 - w_2{}^2) + \widetilde{F}_3^C, \tag{A.221}$$

$$\widetilde{F}_4 \approx a_1 \widetilde{\phi} w_4 + a_5 w_4 (w_4{}^2 - 3w_3{}^2) + 2a_6 w_4 (w_2{}^2 - w_1{}^2) + \widetilde{F}_4^C, \tag{A.222}$$

where $\widetilde{F}_i^C$ $(i = 1, \ldots, 4)$ is given in (A.182) – (A.185). Hence, the asymptotic form of the Jacobian matrix in (A.101) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1 \widetilde{\phi} I_4 + a_5 B_5 + a_6 B_6 + B_C \tag{A.223}$$

with

$$B_5 = 3 \begin{bmatrix} w_1{}^2 - w_2{}^2 & -2w_1 w_2 & 0 & 0 \\ -2w_1 w_2 & w_2{}^2 - w_1{}^2 & 0 & 0 \\ 0 & 0 & w_3{}^2 - w_4{}^2 & -2w_3 w_4 \\ 0 & 0 & -2w_3 w_4 & w_4{}^2 - w_3{}^2 \end{bmatrix},$$

$$B_6 = 2\begin{bmatrix} {w_3}^2 - {w_4}^2 & 0 & 2w_1w_3 & -2w_1w_4 \\ 0 & {w_4}^2 - {w_3}^2 & -2w_2w_3 & 2w_2w_4 \\ 2w_1w_3 & -2w_2w_3 & {w_1}^2 - {w_2}^2 & 0 \\ -2w_1w_4 & 2w_2w_4 & 0 & {w_2}^2 - {w_1}^2 \end{bmatrix},$$

where B_5 and B_C are given in (A.213) and (A.193).

Substituting $\boldsymbol{w}_{\mathrm{sq}}$ into (A.219) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{sq}} \approx -\frac{a_2 + a_3 + a_5 + 2a_6}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.223) at $(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}}) \approx 2w^2 \begin{bmatrix} a_5 + a_3 & 0 & 2a_6 + a_2 & 0 \\ 0 & -2(a_5 + a_6) & 0 & 0 \\ 2a_6 + a_2 & 0 & a_5 + a_3 & 0 \\ 0 & 0 & 0 & -2(a_5 + a_6) \end{bmatrix}. \tag{A.224}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{sq}}, \widetilde{\phi}_{\mathrm{sq}})$ are given by

$$\lambda_1, \lambda_2 \approx 2\{(a_5 + a_3) \pm (2a_6 + a_2)\} w^2,$$
$$\lambda_3 \approx -4(a_5 + a_6) w^2 \quad \text{(repeated twice)}.$$

If $a_5 + a_6 > 0$ and $a_5 + a_3 < -|2a_6 + a_2|$ are satisfied, $\boldsymbol{w}_{\mathrm{sq}}$ is stable. Otherwise, $\boldsymbol{w}_{\mathrm{sq}}$ is unstable. Thus, the stability of $\boldsymbol{w}_{\mathrm{sq}}$ depends on the values of a_2, a_3, a_5 and a_6.

Substituting $\boldsymbol{w}_{\mathrm{stripeI}}$ into (A.219) and solving $\widetilde{F}_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripeI}} \approx -\frac{a_5 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.223) at $(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}}) \approx -w^2 \begin{bmatrix} -2(a_5 + a_3) & 0 & 0 & 0 \\ 0 & 4a_5 & 0 & 0 \\ 0 & 0 & a_5 - 2a_6 + a_3 & 0 \\ 0 & 0 & 0 & a_5 + 2a_6 + a_3 \end{bmatrix}. \tag{A.225}$$

Then, the eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}})$ are given by the diagonal components, i.e.,

$$\lambda_1 \approx 2(a_5 + a_3) w^2,$$
$$\lambda_2 \approx -4a_5 w^2,$$
$$\lambda_3, \lambda_4 \approx -(a_5 + a_3 \pm 2a_6) w^2,$$

If $a_3 < -a_5 < 0$ and $-2|a_6| < a_5 + a_3$ are satisfied, $\boldsymbol{w}_{\mathrm{stripeI}}$ is stable. Otherwise, $\boldsymbol{w}_{\mathrm{stripeI}}$ is unstable. Thus, the stability of $\boldsymbol{w}_{\mathrm{stripeI}}$ depends on the values of a_3, a_5 and a_6.

Substituting $\boldsymbol{w}_{\text{stripeII}}$ into (A.220) and solving $\widetilde{F}_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeII}} \approx -\frac{a_5 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.223) at $(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}}) \approx -w^2 \begin{bmatrix} 4a_5 & 0 & 0 & 0 \\ 0 & -2(a_5 + a_3) & 0 & 0 \\ 0 & 0 & a_5 - 2a_6 + a_3 & 0 \\ 0 & 0 & 0 & a_5 + 2a_6 + a_3 \end{bmatrix}. \tag{A.226}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{stripeII}}$ is equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$.

Case 7: General $(4; n/2, \ell, +)$

For general $(4; n/2, \ell, +)$, we have

$$(1, 0, 0, 0), (1, 1, 0, 1), (2, 0, 1, 0) \in S,$$

and the other elements of S correspond to higher order terms. Then, the asymptotic form of F_1 in (A.182) is equivalent to that for the case 1: General $(4; k, 0, +)$. Hence, a discussion similar to that for the case 1 holds.

Case 8: $(4; n/2, \ell, +)$ *with* $\tilde{n} = 4$

For the case $(4; n/2, \ell, +)$ with $\tilde{n} = 4$, we have

$$(0, 0, 3, 0) \in S$$

as well as

$$(1, 0, 0, 0), (1, 1, 0, 1), (2, 0, 1, 0) \in S.$$

Then, the asymptotic form of F_1 in (A.182) is equivalent to that for the case 3: $(4; k, 0, +)$ with $\check{n} = 4$. Hence, a discussion similar to that for the case 3 holds.

A.4.5. Bifurcation Point of Multiplicity 8

We consider a critical point associated with eight-dimensional irreducible representations μ of the group $\mathrm{D}_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$:

$$(8; k, \ell) \ \text{ with } \ 1 \leq \ell \leq k - 1, \ 2 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{A.227}$$

where $n \geq 5$. For $(8; k, \ell)$, we use the following notations:

$$\hat{k} = \frac{k}{\gcd(k, \ell, n)}, \quad \hat{\ell} = \frac{\ell}{\gcd(k, \ell, n)}, \quad \hat{n} = \frac{n}{\gcd(k, \ell, n)}. \tag{A.228}$$

The action in $(8; k, \ell)$ on an eight-dimensional vector $(w_1, \dots, w_8) \in \mathbb{R}^8$ can be expressed for a four-dimensional vector $(z_1, \dots, z_4)$ with complex variables $z_j = w_{2j-1} + \mathrm{i}w_{2j}$ $(j = 1, \dots, 4)$ as (cf., (3.159) and (3.160))

$$r : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_2 \\ z_1 \\ z_4 \\ \overline{z}_3 \end{bmatrix}, \quad s : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} z_3 \\ z_4 \\ z_1 \\ z_2 \end{bmatrix}, \tag{A.229}$$

$$p_1 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k z_1 \\ \omega^{-\ell} z_2 \\ \omega^k z_3 \\ \omega^{-\ell} z_4 \end{bmatrix}, \quad p_2 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \omega^\ell z_1 \\ \omega^k z_2 \\ \omega^{-\ell} z_3 \\ \omega^{-k} z_4 \end{bmatrix} \tag{A.230}$$

with $\omega = \exp(\mathrm{i}2\pi/n)$.

Derivation of Bifurcation Equations

The bifurcation equation for a critical point of multiplicity 8 is an eight-dimensional equation in $\boldsymbol{w} = (w_1, \dots, w_8) \in \mathbb{R}^8$ expressed as

$$\widetilde{F}_i(\boldsymbol{w}, \widetilde{\phi}) = 0, \quad i = 1, \dots, 8, \tag{A.231}$$

where $(w_1, \dots, w_8, \widetilde{\phi}) = (0, \dots, 0, 0)$ is assumed to correspond to the critical point. Accordingly, the Jacobian matrix of $\widetilde{\boldsymbol{F}}$ is an 8×8 matrix expressed as

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) = \left(F \frac{\partial \widetilde{F}_i}{\partial w_j} \,\middle|\, i, j = 1, \dots, 8 \right). \tag{A.232}$$

The bifurcation equation (A.231) can be expressed as a four-dimensional equation in complex variables $z_j = w_{2j-1} + \mathrm{i}w_{2j}$ $(j = 1, \dots, 4)$ as

$$F_i(z_1, z_2, z_3, z_4, \widetilde{\phi}) = 0, \quad i = 1, \dots, 4, \tag{A.233}$$

where $(z_1, \dots, z_4, \widetilde{\phi}) = (0, \dots, 0, 0)$ corresponds to the critical point. There are the following relationships between (A.231) and (A.233):

$$F_1(z_1, z_2, z_3, z_4, \widetilde{\phi}) = \widetilde{F}_1 + \mathrm{i}\widetilde{F}_2, \tag{A.234}$$

$$F_2(z_1, z_2, z_3, z_4, \widetilde{\phi}) = \widetilde{F}_3 + \mathrm{i}\widetilde{F}_4, \tag{A.235}$$

$$F_3(z_1, z_2, z_3, z_4, \widetilde{\phi}) = \widetilde{F}_5 + \mathrm{i}\widetilde{F}_6, \tag{A.236}$$

$$F_4(z_1, z_2, z_3, z_4, \widetilde{\phi}) = \widetilde{F}_7 + \mathrm{i}\widetilde{F}_8. \tag{A.237}$$

We expand F_1 into a power series as

$$F_1(z_1, z_2, z_3, z_4, \widetilde{\phi}) = \sum_{a=0} \sum_{b=0} \sum_{c=0} \sum_{d=0} \sum_{e=0} \sum_{f=0} \sum_{g=0} \sum_{h=0} A_{abcdefgh}(\widetilde{\phi}) {z_1}^a {z_2}^b {z_3}^c {z_4}^d \, {\overline{z}_1}^e \, {\overline{z}_2}^f \, {\overline{z}_3}^g \, {\overline{z}_4}^h. \tag{A.238}$$

Since $(z_1, z_2, z_3, z_4, \widetilde{\phi}) = (0, 0, 0, 0, 0)$ corresponds to the critical point, we have

$$A_{00000000}(0) = 0, \quad A_{10000000}(0) = A_{01000000}(0) = \cdots = A_{00000001}(0) = 0.$$

Since $a_1 = A'_{10000000}(0)$ is generically nonzero, we have

$$A_{10000000}(\widetilde{\phi}) \approx a_1 \widetilde{\phi}.$$

The equivariance of the bifurcation equation to the group $D_4 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the actions of the four elements r, s, p_1, and p_2 generating this group. The equivariance condition for $(8; k, \ell)$ is written as

$$\begin{aligned}
r: \quad & \overline{F_2(z_1, z_2, z_3, z_4)} = F_1(\overline{z}_2, z_1, z_4, \overline{z}_3), && \text{(A.239)}\\
& F_1(z_1, z_2, z_3, z_4) = F_2(\overline{z}_2, z_1, z_4, \overline{z}_3), && \text{(A.240)}\\
& F_4(z_1, z_2, z_3, z_4) = F_3(\overline{z}_2, z_1, z_4, \overline{z}_3), && \text{(A.241)}\\
& \overline{F_3(z_1, z_2, z_3, z_4)} = F_4(\overline{z}_2, z_1, z_4, \overline{z}_3), && \text{(A.242)}\\
s: \quad & F_3(z_1, z_2, z_3, z_4) = F_1(z_3, z_4, z_1, z_2), && \text{(A.243)}\\
& F_4(z_1, z_2, z_3, z_4) = F_2(z_3, z_4, z_1, z_2), && \text{(A.244)}\\
& F_1(z_1, z_2, z_3, z_4) = F_3(z_3, z_4, z_1, z_2), && \text{(A.245)}\\
& F_2(z_1, z_2, z_3, z_4) = F_4(z_3, z_4, z_1, z_2), && \text{(A.246)}\\
p_1: \quad & \omega^k F_1(z_1, z_2, z_3, z_4) = F_1(\omega^k z_1, \omega^{-\ell} z_2, \omega^k z_3, \omega^{-\ell} z_4), && \text{(A.247)}\\
& \omega^{-\ell} F_2(z_1, z_2, z_3, z_4) = F_2(\omega^k z_1, \omega^{-\ell} z_2, \omega^k z_3, \omega^{-\ell} z_4), && \text{(A.248)}\\
& \omega^k F_3(z_1, z_2, z_3, z_4) = F_3(\omega^k z_1, \omega^{-\ell} z_2, \omega^k z_3, \omega^{-\ell} z_4), && \text{(A.249)}\\
& \omega^{-\ell} F_4(z_1, z_2, z_3, z_4) = F_4(\omega^k z_1, \omega^{-\ell} z_2, \omega^k z_3, \omega^{-\ell} z_4), && \text{(A.250)}\\
p_2: \quad & \omega^\ell F_1(z_1, z_2, z_3, z_4) = F_1(\omega^\ell z_1, \omega^k z_2, \omega^{-\ell} z_3, \omega^{-k} z_4), && \text{(A.251)}\\
& \omega^k F_2(z_1, z_2, z_3, z_4) = F_2(\omega^\ell z_1, \omega^k z_2, \omega^{-\ell} z_3, \omega^{-k} z_4), && \text{(A.252)}\\
& \omega^{-\ell} F_3(z_1, z_2, z_3, z_4) = F_3(\omega^\ell z_1, \omega^k z_2, \omega^{-\ell} z_3, \omega^{-k} z_4), && \text{(A.253)}\\
& \omega^{-k} F_4(z_1, z_2, z_3, z_4) = F_4(\omega^\ell z_1, \omega^k z_2, \omega^{-\ell} z_3, \omega^{-k} z_4). && \text{(A.254)}
\end{aligned}$$

The equivariance conditions with respect to r and s are expressed as follows. The equivariance condition (A.240) for r implies

$$F_2(z_1, z_2, z_3, z_4) = F_1(z_2, \overline{z}_1, \overline{z}_4, z_3). \quad \text{(A.255)}$$

The equivariance condition (A.243) and (A.244) for s implies

$$\begin{aligned}
F_3(z_1, z_2, z_3, z_4) &= F_1(z_3, z_4, z_1, z_2), && \text{(A.256)}\\
F_4(z_1, z_2, z_3, z_4) &= F_2(z_3, z_4, z_1, z_2). && \text{(A.257)}
\end{aligned}$$

Combining (A.255) and (A.257), we have

$$F_4(z_1, z_2, z_3, z_4) = F_1(z_4, \overline{z}_3, \overline{z}_2, z_1). \tag{A.258}$$

Hence, we obtain F_2, F_3 and F_4 from F_1 by using (A.255), (A.256) and (A.258). Combining (A.239) and (A.255), we have

$$\overline{F_1(\overline{z}_2, z_1, \overline{z}_4, z_3)} = F_1(z_2, \overline{z}_1, z_4, \overline{z}_3). \tag{A.259}$$

Hence, we have

$$A_{ab\cdots h}(\widetilde{\phi}) \in \mathbb{R}. \tag{A.260}$$

It is ensured that the equivariance conditions (A.239) – (A.246) are satisfied by (A.255), (A.256), (A.258), and (A.259).

The equivariance conditions with respect to p_1 and p_2 are expressed as follows. The equivariance condition (A.247) for p_1 is expressed as

$$\begin{aligned}
&\sum_{a=0}\sum_{b=0}\cdots\sum_{h=0} \omega^k A_{ab\cdots h}(\widetilde{\phi}) {z_1}^a {z_2}^b {z_3}^c {z_4}^d\, {\overline{z}_1}^e\, {\overline{z}_2}^f\, {\overline{z}_3}^g\, {\overline{z}_4}^h \\
&= \sum_{a=0}\sum_{b=0}\cdots\sum_{h=0} \omega^{k(a+c-e-g)-\ell(b+d-f-h)} A_{ab\cdots h}(\widetilde{\phi}) {z_1}^a {z_2}^b {z_3}^c {z_4}^d\, {\overline{z}_1}^e\, {\overline{z}_2}^f\, {\overline{z}_3}^g\, {\overline{z}_4}^h,
\end{aligned}$$

which implies

$$\omega^{k(a+c-e-g-1)-\ell(b+d-f-h)} = \exp\left[\frac{2\pi \mathrm{i}}{n}\{k(a+c-e-g-1)-\ell(b+d-f-h)\}\right] = 1. \tag{A.261}$$

The equivariance condition (A.251) for p_2 is expressed as

$$\begin{aligned}
&\sum_{a=0}\sum_{b=0}\cdots\sum_{h=0} \omega^\ell A_{ab\cdots h}(\widetilde{\phi}) {z_1}^a {z_2}^b {z_3}^c {z_4}^d\, {\overline{z}_1}^e\, {\overline{z}_2}^f\, {\overline{z}_3}^g\, {\overline{z}_4}^h \\
&= \sum_{a=0}\sum_{b=0}\cdots\sum_{h=0} \omega^{k(b-d-f+h)+\ell(a-c-e+g)} A_{ab\cdots h}(\widetilde{\phi}) {z_1}^a {z_2}^b {z_3}^c {z_4}^d\, {\overline{z}_1}^e\, {\overline{z}_2}^f\, {\overline{z}_3}^g\, {\overline{z}_4}^h,
\end{aligned}$$

which implies

$$\omega^{k(b-d-f+h)+\ell(a-c-e+g-1)} = \exp\left[\frac{2\pi \mathrm{i}}{n}\{k(b-d-f+h)+\ell(a-c-e+g-1)\}\right] = 1. \tag{A.262}$$

Using (A.255), (A.256), or (A.258), we rewrite the equivariance conditions (A.248)–(A.250) for p_1 as follows:

$$\omega^{-\ell} F_1(z_2, \overline{z}_1, \overline{z}_4, z_3) = F_1(\omega^{-\ell} z_2, \omega^{-k}\, \overline{z}_1, \omega^{\ell}\, \overline{z}_4, \omega^k z_3), \tag{A.263}$$

$$\omega^k F_1(z_3, z_4, z_1, z_2) = F_1(\omega^k z_3, \omega^{-\ell} z_4, \omega^k z_1, \omega^{-\ell} z_2), \tag{A.264}$$

$$\omega^{-\ell} F_1(z_4, \overline{z}_3, \overline{z}_2, z_1) = F_1(\omega^{-\ell} z_4, \omega^{-k}\, \overline{z}_3, \omega^{\ell}\, \overline{z}_2, \omega^k z_1). \tag{A.265}$$

Similarly, we rewrite the equivariance conditions (A.252)–(A.254) for p_2 as follows:

$$\omega^k F_1(z_2, \bar{z}_1, \bar{z}_4, z_3) = F_1(\omega^k z_2, \omega^{-\ell}\,\bar{z}_1, \omega^k\,\bar{z}_4, \omega^{-\ell} z_3), \tag{A.266}$$
$$\omega^{-\ell} F_1(z_3, z_4, z_1, z_2) = F_1(\omega^{-\ell} z_3, \omega^{-k} z_4, \omega^{\ell} z_1, \omega^k z_2), \tag{A.267}$$
$$\omega^{-k} F_1(z_4, \bar{z}_3, \bar{z}_2, z_1) = F_1(\omega^{-k} z_4, \omega^{\ell}\,\bar{z}_3, \omega^{-k}\,\bar{z}_2, \omega^{\ell} z_1). \tag{A.268}$$

The equivariance conditions (A.263), (A.265), and (A.267) lead to the same result as (A.262). The equivariance conditions (A.264), (A.266), and a complex conjugate of (A.268) lead to the same result as (A.261). To sum up, we have the following conditions for $(8; k, \ell)$:

$$k(a + c - e - g - 1) - \ell(b + d - f - h) \equiv 0 \mod n,$$
$$k(b - d - f + h) + \ell(a - c - e + g - 1) \equiv 0 \mod n,$$

which are equivalent to

$$\hat{k}(a + c - e - g - 1) - \hat{\ell}(b + d - f - h) \equiv 0 \mod \hat{n}, \tag{A.269}$$
$$\hat{k}(b - d - f + h) + \hat{\ell}(a - c - e + g - 1) \equiv 0 \mod \hat{n}. \tag{A.270}$$

We rewrite the conditions (A.269) and (A.270) in a matrix form as

$$A \begin{bmatrix} \hat{k} \\ \hat{\ell} \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mod \hat{n} \tag{A.271}$$

with

$$A = \begin{bmatrix} a + c - e - g - 1 & -b - d + f + h \\ b - d - f + h & a - c - e + g - 1 \end{bmatrix}. \tag{A.272}$$

This condition is equivalent to the following condition:

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad A \begin{bmatrix} \hat{k} \\ \hat{\ell} \end{bmatrix} = \hat{n} \begin{bmatrix} p \\ q \end{bmatrix}. \tag{A.273}$$

For this condition, we define a set P as

$$P = \{(a, b, \ldots, h) \in \mathbb{Z}_+^8 \mid \text{(A.273) with (A.272)}\}, \tag{A.274}$$

where $\mathbb{Z}_+$ represents the set of nonnegative integers. Note that $(a, b, \ldots, h) \in \mathbb{Z}_+^8$ must belong to P when $A_{ab\cdots h}(\widetilde{\phi}) \neq 0$ in (A.238). Hence, we replace the power series (A.238) with

$$F_1(z_1, z_2, z_3, z_4, \widetilde{\phi}) = \sum_P A_{abcdefgh}(\widetilde{\phi})\, {z_1}^a {z_2}^b {z_3}^c {z_4}^d\, {\bar{z}_1}^e\, {\bar{z}_2}^f\, {\bar{z}_3}^g\, {\bar{z}_4}^h. \tag{A.275}$$

In addition, we have the following proposition:

Proposition A.20. *If $\hat{n} = n/\gcd(n, k, \ell)$ is even, then $(a, b, \ldots, h) \in P$ satisfies $a + b + c + d + e + f + g + h \notin 2\mathbb{Z}$.*

Proof. Since $\hat{n}$ is even, $p\hat{n}$ ($p \in \mathbb{Z}$) in (A.269) and $q\hat{n}$ ($q \in \mathbb{Z}$) in (A.270) are even. Since $\hat{n}$, $\hat{k}$, and $\hat{\ell}$ do not have a common divisor, $(\hat{k}, \hat{\ell}) \neq$ (even, even). To prove the statement by contradiction, assume $a + b + c + d + e + f + g + h \in 2\mathbb{Z}$.

- For the case $a + c + e + g \in 2\mathbb{Z}$ and $b + d - f - h \in 2\mathbb{Z}$, we have the following statements: If $(\hat{k}, \hat{\ell})$ = (odd, even), the left-hand side of (A.269) is odd since it takes the form:
$$(\text{odd}) \times (\text{odd}) + (\text{even}) \times (\text{even}).$$
If $(\hat{k}, \hat{\ell})$ = (even, odd), the left-hand side of (A.270) is odd since it takes the form:
$$(\text{even}) \times (\text{even}) + (\text{odd}) \times (\text{odd}).$$
Thus, the condition (A.269) and (A.270) are cannot be satisfied simultaneously.

- For the case $a + c + e + g \notin 2\mathbb{Z}$ and $b + d + f + h \notin 2\mathbb{Z}$, we have the following statements: If $(\hat{k}, \hat{\ell})$ = (odd, even), the left-hand side of (A.270) is odd since it takes the form: (odd)+(even). If $(\hat{k}, \hat{\ell})$ = (even, odd), the left-hand side of (A.269) is odd since it takes the form: (even) + (odd). Thus, the condition (A.269) and (A.270) are cannot be satisfied simultaneously.

Hence, $a + b + c + d + e + f + g + h \in 2\mathbb{Z}$ is a contradiction. □

Symmetry of Square Patterns

For the irreducible representation $\mu = (8; k, \ell)$, a system of the equations $F_1 = F_2 = F_3 = F_4 = 0$ has the following bifurcating solutions:

$$\text{Type VM square pattern: } (z_1, z_2, z_3, z_4) = (w, w, w, w) \ (w \in \mathbb{R}),$$
$$\text{Type T square pattern: } (z_1, z_2, z_3, z_4) = (w, w, 0, 0) \ (w \in \mathbb{R}).$$

In Section 3.5.6, we showed that the Type VM solution exists for any $(\hat{n}, \hat{k}, \hat{\ell})$, while the Type T solution exists if the values of $(\hat{n}, \hat{k}, \hat{\ell})$ satisfies

$$\overline{\textbf{GCD-div}}\text{: } 2\gcd(\hat{k}, \hat{\ell}) \text{ is not divisible by } \gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}) \tag{A.276}$$

(see Proposition 3.24).

Substituting the Type VM solution $(z_1, z_2, z_3, z_4) = (w, w, w, w)$ into (A.275), we have

$$F_1(w, w, w, w, \widetilde{\phi}) = \sum_P A_{abcdefgh}(\widetilde{\phi}) w^{a+b+d+e+f+g+h}.$$

Proposition A.20 shows that if $\hat{n}$ is even, then $F_1(w, w, w, w, \widetilde{\phi})$ becomes an odd function in w. Thus, the two bifurcating solutions $(w, w, w, w, \widetilde{\phi})$ and $(-w, -w, -w, -w, \widetilde{\phi})$ are conjugate. Substituting the Type T solution $(z_1, z_2, z_3, z_4) = (w, w, 0, 0)$ into (A.275), we have

$$F_1(w, w, 0, 0, \widetilde{\phi}) = \sum_{(a,b,0,0,e,f,0,0) \in P} A_{ab00ef00}(\widetilde{\phi}) w^{a+b+e+f}.$$

Proposition A.20 shows that if $\hat{n}$ is even, then $a + b + e + f \notin 2\mathbb{Z}$ for $(a, b, 0, 0, e, f, 0, 0) \in P$. Thus, $F_1(w, w, 0, 0, \widetilde{\phi})$ becomes an odd function in w, and hence the two bifurcating solutions $(w, w, 0, 0, \widetilde{\phi})$ and $(-w, -w, 0, 0, \widetilde{\phi})$ are conjugate.

To sum up, we have the following proposition on the symmetry of the square patterns.

Proposition A.21. *For a critical point of multiplicity 8 associated with* $\mu = (8; k, \ell)$*, the two bifurcating solutions* $(z, \widetilde{\phi})$ *and* $(-z, \widetilde{\phi})$ *are conjugate for* $z = (w, w, w, w)$*,* $(w, w, 0, 0)$ $(w \in \mathbb{R})$ *if* $\hat{n} = n/\gcd(n, k, \ell)$ *is even and are not conjugate if* $\hat{n}$ *is odd.*

Existence and Symmetry of Stripe Patterns

We would like to show the existence and the symmetry of two types of stripe patterns, which are represented as

$$\text{Type I stripe pattern: } (z_1, z_2, z_3, z_4) = (w, 0, 0, 0) \ (w \in \mathbb{R}),$$
$$\text{Type II stripe pattern: } (z_1, z_2, z_3, z_4) = (\mathrm{i}w, 0, 0, 0) \ (w \in \mathbb{R}).$$

For both cases, we have $(a, b, \dots, h) = (a, 0, 0, 0, e, 0, 0, 0)$, and hence (A.269) and (A.270) leads to

$$\hat{k}(a - e - 1) \equiv 0, \quad \hat{\ell}(a - e - 1) \equiv 0 \mod \hat{n}, \tag{A.277}$$

which imply $a = e + p\hat{n} + 1$ $(p \in \mathbb{Z})$. Then, F_1 in (A.275) is rewritten as

$$\begin{aligned} &F_1(z_1, 0, 0, 0, \widetilde{\phi}) \\ &= \sum_{q=0}^{\infty} A_{q+1,q}(\widetilde{\phi})|z_1|^{2q} z_1 + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} [A_{q+p\hat{n}+1,q}(\widetilde{\phi})|z_1|^{2q} {z_1}^{p\hat{n}+1} + A_{q,q+p\hat{n}-1}(\widetilde{\phi})|z_1|^{2q}\, {\overline{z}_1}^{p\hat{n}-1}] \end{aligned} \tag{A.278}$$

with $A_{ae}(\widetilde{\phi}) = A_{a000e000}(\widetilde{\phi})$.

Substituting the Type I stripe pattern $(z_1, z_2, z_3, z_4) = (w, 0, 0, 0)$ into (A.278), we have

$$\begin{aligned} F_1(w, 0, 0, 0, \widetilde{\phi}) &= w \left\{ \sum_{q=0}^{\infty} A_{q+1,q}(\widetilde{\phi}) w^{2q} + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} [A_{q+p\hat{n}+1,q}(\widetilde{\phi}) w^{2q+p\hat{n}} + A_{q,q+p\hat{n}-1}(\widetilde{\phi}) w^{2q+p\hat{n}-2}] \right\} \\ &\approx w \left\{ A'_{10}(0)\widetilde{\phi} + A_{21}(0) w^2 + A_{0,\hat{n}-1}(0) w^{\hat{n}-2} \right\}. \end{aligned}$$

We see that $F_1(w, 0, 0, 0, \widetilde{\phi}) = 0$ has the trivial solution $(w = 0)$ and a bifurcating solution. Note that $F_1(w, 0, 0, 0, \widetilde{\phi})$ becomes an odd function in w if $\hat{n}$ is even. Then, the two bifurcating solutions $(w, 0, 0, 0, \widetilde{\phi})$ and $(-w, 0, 0, 0, \widetilde{\phi})$ are conjugate.

Substituting $(z_1, z_2, z_3, z_4) = (w, 0, 0, 0)$ into the equivariance conditions (A.255)–(A.258), we have

$$\begin{aligned} F_2(w, 0, 0, 0) &= F_1(0, w, 0, 0), \\ F_3(w, 0, 0, 0) &= F_1(0, 0, w, 0), \\ F_4(w, 0, 0, 0) &= F_1(0, 0, 0, w). \end{aligned} \tag{A.279}$$

With the use of P in (A.274), we have $F_i = 0$ $(i = 2, 3, 4)$ in (A.279) if

$$(0, b, 0, 0, 0, f, 0, 0) \notin P,$$
$$(0, 0, c, 0, 0, 0, g, 0) \notin P,$$
$$(0, 0, 0, d, 0, 0, 0, h) \notin P.$$

The conditions in (A.269) and (A.270) lead to

$$\begin{aligned}
\hat{k}(b-f)-\hat{\ell} &\equiv 0, & \hat{\ell}(b-f)+\hat{k} &\equiv 0 \mod \hat{n} & &\text{for } (a,b,c,d,e,f,g,h)=(0,b,0,0,0,f,0,0),\\
\hat{k}(c-g)-\hat{k} &\equiv 0, & \hat{\ell}(c-g)+\hat{\ell} &\equiv 0 \mod \hat{n} & &\text{for } (a,b,c,d,e,f,g,h)=(0,0,c,0,0,0,g,0),\\
\hat{k}(d-h)+\hat{\ell} &\equiv 0, & \hat{\ell}(d-h)+\hat{k} &\equiv 0 \mod \hat{n} & &\text{for } (a,b,c,d,e,f,g,h)=(0,0,0,d,0,0,0,h).
\end{aligned}$$

These relations can be expressed in a matrix form as

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{with} \quad A = \begin{bmatrix} \hat{k} & -\hat{n} & 0 \\ \hat{\ell} & 0 & -\hat{n} \end{bmatrix}. \tag{A.280}$$

The vectors $\boldsymbol{x}$ and $\boldsymbol{b}$ vary with (a,b,c,d,e,f,g,h) as follows:

$$\begin{aligned}
\boldsymbol{x} &= \begin{bmatrix} b-f \\ p \\ q \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} \hat{\ell} \\ -\hat{k} \end{bmatrix} \quad \text{for } (a,b,c,d,e,f,g,h)=(0,b,0,0,0,f,0,0),\\
\boldsymbol{x} &= \begin{bmatrix} c-g \\ p \\ q \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} \hat{k} \\ -\hat{\ell} \end{bmatrix} \quad \text{for } (a,b,c,d,e,f,g,h)=(0,0,c,0,0,0,g,0),\\
\boldsymbol{x} &= \begin{bmatrix} d-h \\ p \\ q \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} -\hat{\ell} \\ -\hat{k} \end{bmatrix} \quad \text{for } (a,b,c,d,e,f,g,h)=(0,0,0,d,0,0,0,h).
\end{aligned}$$

The existence of an integer solution $\boldsymbol{x}$ of (A.280) is investigated by showing the two conditions (A.281) in Remark A.4 below. The first condition is satisfied since we have

$$\operatorname{rank} A = \operatorname{rank} \begin{bmatrix} \hat{k} & -\hat{n} & 0 \\ \hat{\ell} & 0 & -\hat{n} \end{bmatrix} = 2$$

and

$$\begin{aligned}
\operatorname{rank}\,[A \mid \boldsymbol{b}] &= \operatorname{rank} \begin{bmatrix} \hat{k} & -\hat{n} & 0 & \hat{\ell} \\ \hat{\ell} & 0 & -\hat{n} & -\hat{k} \end{bmatrix} = 2 \quad \text{for } (a,b,c,d,e,f,g,h)=(0,b,0,0,0,f,0,0),\\
\operatorname{rank}\,[A \mid \boldsymbol{b}] &= \operatorname{rank} \begin{bmatrix} \hat{k} & -\hat{n} & 0 & \hat{k} \\ \hat{\ell} & 0 & -\hat{n} & -\hat{\ell} \end{bmatrix} = 2 \quad \text{for } (a,b,c,d,e,f,g,h)=(0,0,c,0,0,0,g,0),\\
\operatorname{rank}\,[A \mid \boldsymbol{b}] &= \operatorname{rank} \begin{bmatrix} \hat{k} & -\hat{n} & 0 & -\hat{\ell} \\ \hat{\ell} & 0 & -\hat{n} & -\hat{k} \end{bmatrix} = 2 \quad \text{for } (a,b,c,d,e,f,g,h)=(0,0,0,d,0,0,0,h).
\end{aligned}$$

For the second condition, we have

$$\begin{aligned}
d_1(A) &= \gcd(\hat{\ell}, \hat{k}, \hat{n}) = 1,\\
d_1([A \mid \boldsymbol{b}]) &= \gcd(\hat{\ell}, \hat{k}, \hat{n}) = 1,\\
d_2(A) &= \gcd(\hat{k}\hat{n}, \hat{\ell}\hat{n}, \hat{n}^2) = \hat{n}.
\end{aligned}$$

The value of $d_2([A \mid \boldsymbol{b}])$ varies with (a, b, c, d, e, f, g, h) as follows:

$$d_2([A \mid \boldsymbol{b}]) = \gcd(\hat{n}, \hat{k}^2 + \hat{\ell}^2) \quad \text{for } (a, b, c, d, e, f, g, h) = (0, b, 0, 0, 0, f, 0, 0),$$
$$d_2([A \mid \boldsymbol{b}]) = \gcd(\hat{n}, 2\hat{k}\hat{\ell}) \quad \text{for } (a, b, c, d, e, f, g, h) = (0, 0, c, 0, 0, 0, g, 0),$$
$$d_2([A \mid \boldsymbol{b}]) = \gcd(\hat{n}, \hat{k}^2 - \hat{\ell}^2) \quad \text{for } (a, b, c, d, e, f, g, h) = (0, 0, 0, d, 0, 0, 0, h).$$

For $(a, b, c, d, e, f, g, h) = (0, b, 0, 0, 0, f, 0, 0)$, we have $d_2(A) = d_2([A \mid \boldsymbol{b}])$ when $\hat{k}^2 + \hat{\ell}^2$ is divisible by $\hat{n}$. Then, the equation (A.280) has an integer solution $\boldsymbol{x}$. Hence, we have $(0, b, 0, 0, 0, f, 0, 0) \in P$ and, in turn, $F_2 \neq 0$. On the contrary, we have $(0, b, 0, 0, 0, f, 0, 0) \notin P$ and, in turn, $F_2 = 0$ when $\hat{k}^2 + \hat{\ell}^2$ is not divisible by $\hat{n}$. In a similar manner, we have $(0, 0, c, 0, 0, 0, g, 0) \notin P$ and, in turn, $F_3 = 0$ when $2\hat{k}\hat{\ell}$ is not divisible by $\hat{n}$. We have $(0, 0, 0, d, 0, 0, 0, h) \notin P$ and, in turn, $F_4 = 0$ when $\hat{k}^2 - \hat{\ell}^2$ is not divisible by $\hat{n}$. Consequently, a system of the equations $F_1 = F_2 = F_3 = F_4 = 0$ holds for $(z_1, z_2, z_3, z_4) = (w, 0, 0, 0)$ when $\hat{k}^2 + \hat{\ell}^2$, $2\hat{k}\hat{\ell}$, and $\hat{k}^2 - \hat{\ell}^2$ are not divisible by $\hat{n}$.

Remark A.4. Let A be an $m \times n$ integer matrix and $\boldsymbol{b}$ an m-dimensional integer vector. A system of equations $A\boldsymbol{x} = \boldsymbol{b}$ admits an integer solution $\boldsymbol{x}$ if and only if two matrices A and $[A \mid \boldsymbol{b}]$ share the same determinantal divisors, i.e.,

$$\operatorname{rank} A = \operatorname{rank} [A \mid \boldsymbol{b}], \quad d_k(A) = d_k([A \mid \boldsymbol{b}]) \tag{A.281}$$

for all k. Here, $d_k(A)$ is the kth determinantal divisor, which is the greatest common divisor of all $k \times k$ minors (subdeterminants) of the integer matrix A.

□

Substituting Type II stripe pattern $(z_1, z_2, z_3, z_4) = (\mathrm{i}w, 0, 0, 0)$ into (A.278), we have

$$\begin{aligned} &F_1(\mathrm{i}w, 0, 0, 0, \widetilde{\phi}) \\ &= \mathrm{i}w \left\{ \sum_{q=0}^{\infty} A_{q+1,q}(\widetilde{\phi}) w^{2q} + \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} [A_{q+p\hat{n}+1,q}(\widetilde{\phi}) \mathrm{i}^{p\hat{n}} w^{2q+p\hat{n}} + A_{q,q+p\hat{n}-1}(\widetilde{\phi})(-\mathrm{i})^{p\hat{n}} w^{2q+p\hat{n}-2}] \right\} \\ &\approx \mathrm{i}w \left\{ A'_{10}(0)\widetilde{\phi} + A_{21}(0) w^2 + A_{0,\hat{n}-1}(0)(-\mathrm{i})^{\hat{n}} w^{\hat{n}-2} \right\}. \end{aligned}$$

Thus, $F_1(\mathrm{i}w, 0, 0, 0, \widetilde{\phi}) = 0$ has a bifurcating solution if $\hat{n}$ is even ($\mathrm{i}^{p\hat{n}}$ and $(-\mathrm{i})^{p\hat{n}}$ are real). Then, a discussion similar to that for the Type I stripe pattern holds.

To sum up, we have the following propositions on the existence and the symmetry of the stripe patterns.

Proposition A.22. *For a critical point of multiplicity 8 associated with $\mu = (8; k, \ell)$, Type I stripe pattern exists if the condition*

$$\hat{k}^2 + \hat{\ell}^2, 2\hat{k}\hat{\ell}, \text{ and } \hat{k}^2 - \hat{\ell}^2 \text{ are not divisible by } \hat{n} \tag{A.282}$$

is satisfied. Therein, $\hat{k} = k/\gcd(n, k, \ell)$, $\hat{\ell} = \ell/\gcd(n, k, \ell)$, and $\hat{n} = n/\gcd(n, k, \ell)$. Type II stripe pattern exists if the condition (A.282) is satisfied and $\hat{n}$ is even.

Proposition A.23. *For a critical point of multiplicity 8 associated with $\mu = (8; k, \ell)$, the two bifurcating solutions $(z, \widetilde{\phi})$ and $(-z, \widetilde{\phi})$ are conjugate for $z = (w, 0, 0, 0)$, $(\mathrm{i}w, 0, 0, 0)$ $(w \in \mathbb{R})$ if $\hat{n} = n/\gcd(n, k, \ell)$ is even and are not conjugate for $z = (w, 0, 0, 0)$ if $\hat{n}$ is odd.*

Existence and Symmetry of Upside-down Patterns

We would like to show the existence and the symmetry of two types of upside-down patterns, which are represented as

$$\text{Type I upside-down pattern: } (z_1, z_2, z_3, z_4) = (w, 0, w, 0)\ (w \in \mathbb{R}),$$
$$\text{Type II upside-down pattern: } (z_1, z_2, z_3, z_4) = (\mathrm{i}w, 0, \mathrm{i}w, 0)\ (w \in \mathbb{R}).$$

For both cases, we have $(a, b, \ldots, h) = (a, 0, c, 0, e, 0, g, 0)$, and hence (A.269) and (A.270) leads to

$$\hat{k}(a - e - 1) + \hat{k}(c - g) \equiv 0 \mod \hat{n},$$
$$\hat{\ell}(a - e - 1) - \hat{\ell}(c - g) \equiv 0 \mod \hat{n},$$

which imply $a = e + p\hat{n} + 1$ and $c = g + q\hat{n}$ $(p, q \in \mathbb{Z})$. Then, F_1 in (A.275) is rewritten as

$$F_1(z_1, 0, z_3, 0, \widetilde{\phi}) = \sum_{e=0}^{\infty}\sum_{g=0}^{\infty}\sum_{p\in\mathbb{Z}, e+p\hat{n}+1\geq 0}\ \sum_{q\in\mathbb{Z}, g+q\hat{n}\geq 0} A_{e+p\hat{n}+1, g+q\hat{n}, e, g}(\widetilde{\phi}) {z_1}^{e+p\hat{n}+1} {z_3}^{g+q\hat{n}}\ {\overline{z}_1}^{e}\ {\overline{z}_3}^{g} \tag{A.283}$$

with $A_{aceg}(\widetilde{\phi}) = A_{a0c0e0g0}(\widetilde{\phi})$.

Substituting Type I upside-down pattern $(z_1, z_2, z_3, z_4) = (w, 0, w, 0)$ into (A.283), we have

$$F_1(w, 0, w, 0, \widetilde{\phi}) = w\left\{\sum_{e=0}^{\infty}\sum_{g=0}^{\infty}\sum_{p\in\mathbb{Z}, e+p\hat{n}+1\geq 0}\ \sum_{q\in\mathbb{Z}, g+q\hat{n}\geq 0} A_{e+p\hat{n}+1, g+q\hat{n}, e, g}(\widetilde{\phi}) w^{2(e+g)+(p+q)\hat{n}}\right\}$$
$$\approx w\left\{A'_{1000}(0)\widetilde{\phi} + (A_{2010}(0) + A_{1101}(0))w^2\right\}.$$

We see that $F_1(w, 0, w, 0, \widetilde{\phi}) = 0$ has the trivial solution $(w = 0)$ and a bifurcating solution. Note that $F_1(w, 0, w, 0, \widetilde{\phi})$ becomes an odd function in w if $\hat{n}$ is even. Then, the two bifurcating solutions $(w, 0, w, 0, \widetilde{\phi})$ and $(-w, 0, -w, 0, \widetilde{\phi})$ are conjugate.

Substituting $(z_1, z_2, z_3, z_4) = (w, 0, w, 0)$ into the equivariance conditions (A.255)–(A.258), we have

$$\begin{aligned} F_2(w, 0, w, 0) &= F_4(w, 0, w, 0) = F_1(0, w, 0, w), \\ F_3(w, 0, w, 0) &= F_1(w, 0, w, 0). \end{aligned} \tag{A.284}$$

With the use of P in (A.274), we have $F_i = 0$ $(i = 2, 4)$ in (A.284) if

$$(0, b, 0, d, 0, f, 0, h) \notin P.$$

The use of $(a, b, \ldots, h) = (0, b, 0, d, 0, f, 0, h)$ in (A.269) and (A.270) leads to

$$-\hat{k} - \hat{\ell}(b + d - f - h) \equiv 0 \mod \hat{n},$$
$$\hat{k}(b - d - f + h) - \hat{\ell} \equiv 0 \mod \hat{n}.$$

This relation can be expressed in a matrix form as

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{with} \quad A = \begin{bmatrix} -\hat{\ell} & -\hat{\ell} & -\hat{n} & 0 \\ \hat{k} & -\hat{k} & 0 & -\hat{n} \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} b-f \\ d-h \\ p \\ q \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} \hat{k} \\ \hat{\ell} \end{bmatrix}. \tag{A.285}$$

The existence of an integer solution $\boldsymbol{x}$ of (A.285) is investigated by showing the two conditions (A.281) in Remark A.4. The first condition is satisfied since

$$\begin{aligned} \text{rank } A &= \text{rank} \begin{bmatrix} -\hat{\ell} & -\hat{\ell} & -\hat{n} & 0 \\ \hat{k} & -\hat{k} & 0 & -\hat{n} \end{bmatrix} = 2, \\ \text{rank } [A \mid \boldsymbol{b}] &= \text{rank} \begin{bmatrix} -\hat{\ell} & -\hat{\ell} & -\hat{n} & 0 & -\hat{k} \\ \hat{k} & -\hat{k} & 0 & -\hat{n} & \hat{\ell} \end{bmatrix} = 2. \end{aligned}$$

For the second condition, we have

$$\begin{aligned} d_1(A) &= \gcd(\hat{\ell}, \hat{k}, \hat{n}) = 1, \\ d_1([A \mid \boldsymbol{b}]) &= \gcd(\hat{\ell}, \hat{k}, \hat{n}) = 1, \\ d_2(A) &= \gcd(2\hat{k}\hat{\ell}, \hat{k}\hat{n}, \hat{\ell}\hat{n}, \hat{n}^2) = \gcd(2\hat{k}\hat{\ell}, \hat{n}), \\ d_2([A \mid \boldsymbol{b}]) &= \gcd(\hat{n}, 2\hat{k}\hat{\ell}, \hat{k}^2 + \hat{\ell}^2, \hat{k}^2 - \hat{\ell}^2). \end{aligned}$$

Hence, $d_2(A) = d_2([A \mid \boldsymbol{b}])$ is satisfied if

$$\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{k}^2 - \hat{\ell}^2) \text{ is divisible by } \gcd(\hat{n}, 2\hat{k}\hat{\ell}),$$

Then, the equation (A.285) has an integer solution $\boldsymbol{x}$, and hence we have $(0, b, 0, d, 0, f, 0, h) \in P$ and, in turn, $F_2 = F_4 \neq 0$. On the contrary, we have $(0, b, 0, d, 0, f, 0, h) \notin P$ and, in turn, $F_2 = F_4 = 0$ if $(\hat{k} + \hat{\ell}) \gcd(\hat{k} + \hat{\ell}, \hat{k} - \hat{\ell})$ is not divisible by $\gcd(\hat{n}, 2\hat{k}\hat{\ell})$.

Substituting Type II upside-down pattern $(z_1, z_2, z_3, z_4) = (\mathrm{i}w, 0, \mathrm{i}w, 0)$ into (A.283), we have

$$\begin{aligned} &F_1(\mathrm{i}w, 0, \mathrm{i}w, 0, \widetilde{\phi}) \\ &= \sum_{e=0}^{\infty} \sum_{g=0}^{\infty} \sum_{p \in \mathbb{Z}, e+p\hat{n}+1 \geq 0} \sum_{q \in \mathbb{Z}, g+q\hat{n} \geq 0} A_{e+p\hat{n}+1, g+q\hat{n}, e, g}(\widetilde{\phi})(\mathrm{i}w)^{e+p\hat{n}+1}(\mathrm{i}w)^{g+q\hat{n}}(-\mathrm{i}w)^{e}(-\mathrm{i}w)^{g} \\ &= \mathrm{i}w \left\{ \sum_{e=0}^{\infty} \sum_{g=0}^{\infty} \sum_{p \in \mathbb{Z}, e+p\hat{n}+1 \geq 0} \sum_{q \in \mathbb{Z}, g+q\hat{n} \geq 0} A_{e+p\hat{n}+1, g+q\hat{n}, e, g}(\widetilde{\phi}) \mathrm{i}^{p\hat{n}} (-\mathrm{i})^{q\hat{n}} w^{2(e+g)+(p+q)\hat{n}} \right\} \\ &\approx \mathrm{i}w \left\{ A'_{1000}(0)\widetilde{\phi} + (A_{2010}(0) + A_{1101}(0))w^2 \right\}. \end{aligned}$$

Thus, $F_1(\mathrm{i}w, 0, \mathrm{i}w, 0, \widetilde{\phi}) = 0$ has a bifurcating solution if $\hat{n}$ is even ($\mathrm{i}^{p\hat{n}}$ and $(-\mathrm{i})^{q\hat{n}}$ are real). Then, a discussion similar to that for Type I stripe pattern holds.

To sum up, we have the following propositions on the existence and the symmetry of the upside-down patterns.

Proposition A.24. *For a critical point of multiplicity 8 associated with $\mu = (8; k, \ell)$, Type I upside-down pattern exists if the condition*

$$\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{k}^2 - \hat{\ell}^2) \text{ is not divisible by } \gcd(\hat{n}, 2\hat{k}\hat{\ell}) \tag{A.286}$$

is satisfied. Therein, $\hat{k} = k/\gcd(n, k, \ell)$, $\hat{\ell} = \ell/\gcd(n, k, \ell)$, and $\hat{n} = n/\gcd(n, k, \ell)$. Type II upside-down pattern exists if the condition (A.286) is satisfied and $\hat{n}$ is even.

Proposition A.25. *For a critical point of multiplicity 8 associated with $\mu = (8; k, \ell)$, the two bifurcating solutions $(z, \widetilde{\phi})$ and $(-z, \widetilde{\phi})$ are conjugate for $z = (w, 0, w, 0)$, $(\mathrm{i}w, 0, \mathrm{i}w, 0)$ $(w \in \mathbb{R})$ if $\hat{n} = n/\gcd(n, k, \ell)$ is even and are not conjugate for $z = (w, 0, w, 0)$ if $\hat{n}$ is odd.*

Stability of Bifurcating Solutions

In Section 3.5.6, we found the square patterns for a critical point of multiplicity 8 by using the equivariant branching lemma. In the previous subsections, we showed the stripe and upside-down patterns by solving the bifurcation equations. These bifurcating solutions are represented for the bifurcation equation in real variables in (A.231) as follows ($w \in \mathbb{R}$):

$$\begin{aligned}
\boldsymbol{w}_{\mathrm{sqVM}} &= (w, 0, w, 0, w, 0, w, 0),\\
\boldsymbol{w}_{\mathrm{sqT}} &= (w, 0, w, 0, 0, 0, 0, 0),\\
\boldsymbol{w}_{\mathrm{stripeI}} &= (w, 0, 0, 0, 0, 0, 0, 0),\\
\boldsymbol{w}_{\mathrm{stripeII}} &= (0, w, 0, 0, 0, 0, 0, 0),\\
\boldsymbol{w}_{\mathrm{upside-downI}} &= (w, 0, 0, 0, w, 0, 0, 0),\\
\boldsymbol{w}_{\mathrm{upside-downII}} &= (0, w, 0, 0, 0, w, 0, 0)
\end{aligned}$$

We would like to evaluate the stability of these bifurcating solutions.

To obtain the the asymptotic form of the bifurcation equation and the Jacobian matrix, we first investigate which $(a, b, \dots, h) \in \mathbb{Z}_+^8$ belongs to P in (A.274). In other words, we investigate which $A_{ab\cdots h}(\widetilde{\phi})$ becomes nonzero in (A.275). We focus on the coefficients of linear terms, quadratic terms, and cubic terms, which play a vital role as leading terms in (A.275). For this purpose, we exhaustively find $(a, b, \dots, h) \in \mathbb{Z}_+^8$ such as

$$(a, b, \dots, h) \in P \text{ with } a + b + \cdots + h \leq 3.$$

Let us take some $(a, b, \dots, h) \in \mathbb{Z}_+^8$ and substitute it into the matrix A in (A.272). Then, A becomes any one of twelve possible forms as shown in Table A.6. The condition (A.273) varies with the form of A.

For the case (i), the elements of A in (A.272) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = 0, \tag{A.287}$$
$$-b - d + f + h = 0, \quad b - d - f + h = 0. \tag{A.288}$$

The condition in (A.273) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = 0, \ q\hat{n} = 0. \tag{A.289}$$

Table A.6: Possible cases for A in (A.272).

Cases		Conditions in (A.273)
(i)	$A = O$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = 0,\ q\hat{n} = 0$
(ii)	$A = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \alpha\hat{k},\ q\hat{n} = 0$
(iii)	$A = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \beta\hat{\ell},\ q\hat{n} = 0$
(iv)	$A = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = 0,\ q\hat{n} = \gamma\hat{k}$
(v)	$A = \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = 0,\ q\hat{n} = \delta\hat{\ell}$
(vi)	$A = \begin{bmatrix} \alpha & 0 \\ \gamma & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \alpha\hat{k},\ q\hat{n} = \gamma\hat{k}$
(vii)	$A = \begin{bmatrix} 0 & \beta \\ 0 & \delta \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \beta\hat{\ell},\ q\hat{n} = \delta\hat{\ell}$
(viii)	$A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \alpha\hat{k} + \beta\hat{\ell},\ q\hat{n} = 0$
(ix)	$A = \begin{bmatrix} 0 & 0 \\ \gamma & \delta \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = 0,\ q\hat{n} = \gamma\hat{k} + \delta\hat{\ell}$
(x)	$A = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \alpha\hat{k},\ q\hat{n} = \delta\hat{\ell}$
(xi)	$A = \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \beta\hat{\ell},\ q\hat{n} = \gamma\hat{k}$
(xii)	$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$	$\exists p, q \in \mathbb{Z}$ s.t. $p\hat{n} = \alpha\hat{k} + \beta\hat{\ell},\ q\hat{n} = \gamma\hat{k} + \delta\hat{\ell}$

Table A.7: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$.

(a, b, c, d, e, f, g, h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (A.272)
(0, 0, 0, 0, 0, 0, 0, 0)	-1	0	0	-1	(x)	-
(1, 0, 0, 0, 0, 0, 0, 0)	0	0	0	0	(i)	$p = 0,\ q = 0$ for any $(\hat{n}, \hat{k}, \hat{\ell})$
(0, 1, 0, 0, 0, 0, 0, 0)	-1	-1	1	-1	(xii)	-
(0, 0, 1, 0, 0, 0, 0, 0)	0	0	0	-2	(v)	-
(0, 0, 0, 1, 0, 0, 0, 0)	-1	-1	-1	-1	(xii)	-
(0, 0, 0, 0, 1, 0, 0, 0)	-2	0	0	-2	(x)	-
(0, 0, 0, 0, 0, 1, 0, 0)	-1	1	-1	-1	(xii)	-
(0, 0, 0, 0, 0, 0, 1, 0)	-2	0	0	0	(ii)	-
(0, 0, 0, 0, 0, 0, 0, 1)	-1	1	1	-1	(xii)	-
(2, 0, 0, 0, 0, 0, 0, 0)	1	0	0	1	(x)	-
(0, 2, 0, 0, 0, 0, 0, 0)	-1	-2	2	-1	(xii)	-
(0, 0, 2, 0, 0, 0, 0, 0)	1	0	0	-3	(x)	-
(0, 0, 0, 2, 0, 0, 0, 0)	-1	-2	-2	-1	(xii)	-
(0, 0, 0, 0, 2, 0, 0, 0)	-3	0	0	-3	(x)	-
(0, 0, 0, 0, 0, 2, 0, 0)	-1	2	-2	-1	(xii)	$p = 0,\ q = -1$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$
(0, 0, 0, 0, 0, 0, 2, 0)	-3	0	0	1	(x)	-
(0, 0, 0, 0, 0, 0, 0, 2)	-1	2	2	-1	(xii)	-
(1, 1, 0, 0, 0, 0, 0, 0)	0	-1	1	0	(xi)	-
(1, 0, 1, 0, 0, 0, 0, 0)	1	0	0	-1	(x)	-
(1, 0, 0, 1, 0, 0, 0, 0)	0	-1	-1	0	(xi)	-
(1, 0, 0, 0, 1, 0, 0, 0)	-1	0	0	-1	(x)	-
(1, 0, 0, 0, 0, 1, 0, 0)	0	1	-1	0	(xi)	-
(1, 0, 0, 0, 0, 0, 1, 0)	-1	0	0	1	(x)	-
(1, 0, 0, 0, 0, 0, 0, 1)	0	1	1	0	(xi)	-
(0, 1, 1, 0, 0, 0, 0, 0)	0	-1	1	-2	(xii)	-
(0, 1, 0, 1, 0, 0, 0, 0)	-1	-2	0	-1	(xii)	-
(0, 1, 0, 0, 1, 0, 0, 0)	-2	-1	1	-2	(xii)	$p = -1,\ q = 0$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$
(0, 1, 0, 0, 0, 1, 0, 0)	-1	0	0	-1	(x)	-
(0, 1, 0, 0, 0, 0, 1, 0)	-2	-1	1	0	(xii)	-
(0, 1, 0, 0, 0, 0, 0, 1)	-1	0	2	-1	(xii)	-

$\alpha = a + c - e - g - 1;\ \beta = -b - d + f + h;\ \gamma = b - d - f + h;\ \delta = a - c - e + g - 1$

Table A.8: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$.

(a, b, c, d, e, f, g, h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (A.272)
(0, 0, 1, 1, 0, 0, 0, 0)	0	-1	-1	-2	(xii)	-
(0, 0, 1, 0, 1, 0, 0, 0)	-1	0	0	-3	(x)	-
(0, 0, 1, 0, 0, 1, 0, 0)	0	1	-1	-2	(xii)	-
(0, 0, 1, 0, 0, 0, 1, 0)	-1	0	0	-1	(x)	-
(0, 0, 1, 0, 0, 0, 0, 1)	0	1	1	-2	(xii)	-
(0, 0, 0, 1, 1, 0, 0, 0)	-2	-1	-1	-2	(xii)	-
(0, 0, 0, 1, 0, 1, 0, 0)	-1	0	-2	-1	(xii)	-
(0, 0, 0, 1, 0, 0, 1, 0)	-2	-1	-1	0	(xii)	-
(0, 0, 0, 1, 0, 0, 0, 1)	-1	0	0	-1	(x)	-
(0, 0, 0, 0, 1, 1, 0, 0)	-2	1	-1	-2	(xii)	-
(0, 0, 0, 0, 1, 0, 1, 0)	-3	0	0	-1	(x)	-
(0, 0, 0, 0, 1, 0, 0, 1)	-2	1	1	-2	(xii)	-
(0, 0, 0, 0, 0, 1, 1, 0)	-2	1	-1	0	(xii)	-
(0, 0, 0, 0, 0, 1, 0, 1)	-1	2	0	-1	(xii)	-
(0, 0, 0, 0, 0, 0, 1, 1)	-2	1	1	0	(xii)	-
(3, 0, 0, 0, 0, 0, 0, 0)	2	0	0	2	(x)	-
(0, 3, 0, 0, 0, 0, 0, 0)	-1	-3	3	-1	(xii)	$p = -1,\ q = 1$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$
(0, 0, 3, 0, 0, 0, 0, 0)	2	0	0	-4	(x)	-
(0, 0, 0, 3, 0, 0, 0, 0)	-1	-3	-3	-1	(xii)	-
(0, 0, 0, 0, 3, 0, 0, 0)	-4	0	0	-4	(x)	-
(0, 0, 0, 0, 0, 3, 0, 0)	-1	3	-3	-1	(xii)	$p = 0,\ q = 1$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (10, 3, 1)$
(0, 0, 0, 0, 0, 0, 3, 0)	-4	0	0	2	(x)	-
(0, 0, 0, 0, 0, 0, 0, 3)	-1	3	3	-1	(xii)	$p = 0,\ q = -1$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (8, 3, 1)$
(2, 1, 0, 0, 0, 0, 0, 0)	1	-1	1	1	(xii)	-
(2, 0, 1, 0, 0, 0, 0, 0)	2	0	0	0	(ii)	-
(2, 0, 0, 1, 0, 0, 0, 0)	1	-1	-1	1	(xii)	-
(2, 0, 0, 0, 1, 0, 0, 0)	0	0	0	0	(i)	$p = 0,\ q = 0$ for any $(\hat{n}, \hat{k}, \hat{\ell})$
(2, 0, 0, 0, 0, 1, 0, 0)	1	1	-1	1	(xii)	-
(2, 0, 0, 0, 0, 0, 1, 0)	0	0	0	2	(v)	-
(2, 0, 0, 0, 0, 0, 0, 1)	1	1	1	1	(xii)	-

$\alpha = a + c - e - g - 1;\ \beta = -b - d + f + h;\ \gamma = b - d - f + h;\ \delta = a - c - e + g - 1$

Table A.9: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$.

(a, b, c, d, e, f, g, h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (A.272)
(1, 2, 0, 0, 0, 0, 0, 0)	0	-2	2	0	(xi)	-
(0, 2, 1, 0, 0, 0, 0, 0)	0	-2	2	-2	(xii)	-
(0, 2, 0, 1, 0, 0, 0, 0)	-1	-3	1	-1	(xii)	-
(0, 2, 0, 0, 1, 0, 0, 0)	-2	-2	2	-2	(xii)	-
(0, 2, 0, 0, 0, 1, 0, 0)	-1	-1	1	-1	(xii)	-
(0, 2, 0, 0, 0, 0, 1, 0)	-2	-2	2	0	(xii)	-
(0, 2, 0, 0, 0, 0, 0, 1)	-1	-1	3	-1	(xii)	-
(1, 0, 2, 0, 0, 0, 0, 0)	2	0	0	-2	(x)	-
(0, 1, 2, 0, 0, 0, 0, 0)	1	-1	1	-3	(xii)	-
(0, 0, 2, 1, 0, 0, 0, 0)	1	-1	-1	-3	(xii)	-
(0, 0, 2, 0, 1, 0, 0, 0)	0	0	0	-4	(v)	$p = 0,\ q = -1$ for $\hat{n} = 4\hat{\ell}$
(0, 0, 2, 0, 0, 1, 0, 0)	1	1	-1	-3	(xii)	-
(0, 0, 2, 0, 0, 0, 1, 0)	0	0	0	-2	(v)	-
(0, 0, 2, 0, 0, 0, 0, 1)	1	1	1	-3	(xii)	-
(1, 0, 0, 2, 0, 0, 0, 0)	0	-2	-2	0	(xi)	-
(0, 1, 0, 2, 0, 0, 0, 0)	-1	-3	-1	-1	(xii)	-
(0, 0, 1, 2, 0, 0, 0, 0)	0	-2	-2	-2	(xii)	-
(0, 0, 0, 2, 1, 0, 0, 0)	-2	-2	-2	-2	(xii)	$p = -1,\ q = -1$ for $\hat{n} = 2\hat{k} + 2\hat{\ell}$
(0, 0, 0, 2, 0, 1, 0, 0)	-1	-1	-3	-1	(xii)	-
(0, 0, 0, 2, 0, 0, 1, 0)	-2	-2	-2	0	(xii)	-
(0, 0, 0, 2, 0, 0, 0, 1)	-1	-1	-1	-1	(xii)	-
(1, 0, 0, 0, 2, 0, 0, 0)	-2	0	0	-2	(x)	-
(0, 1, 0, 0, 2, 0, 0, 0)	-3	-1	1	-3	(xii)	$p = -1,\ q = 0$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (10, 3, 1)$
(0, 0, 1, 0, 2, 0, 0, 0)	-2	0	0	-4	(x)	-
(0, 0, 0, 1, 2, 0, 0, 0)	-3	-1	-1	-3	(xii)	-
(0, 0, 0, 0, 2, 1, 0, 0)	-3	1	-1	-3	(xii)	$p = -1,\ q = -1$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$
(0, 0, 0, 0, 2, 0, 1, 0)	-4	0	0	-2	(x)	-
(0, 0, 0, 0, 2, 0, 0, 1)	-3	1	1	-3	(xii)	$p = -1,\ q = 0$ for $(\hat{n}, \hat{k}, \hat{\ell}) = (8, 3, 1)$

$\alpha = a + c - e - g - 1$; $\beta = -b - d + f + h$; $\gamma = b - d - f + h$; $\delta = a - c - e + g - 1$

Table A.10: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$.

(a, b, c, d, e, f, g, h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (A.272)
(1, 0, 0, 0, 0, 2, 0, 0)	0	2	-2	0	(xi)	-
(0, 1, 0, 0, 0, 2, 0, 0)	-1	1	-1	-1	(xii)	-
(0, 0, 1, 0, 0, 2, 0, 0)	0	2	-2	-2	(xii)	-
(0, 0, 0, 1, 0, 2, 0, 0)	-1	1	-3	-1	(xii)	-
(0, 0, 0, 0, 1, 2, 0, 0)	-2	2	-2	-2	(xii)	-
(0, 0, 0, 0, 0, 2, 1, 0)	-2	2	-2	0	(xii)	-
(0, 0, 0, 0, 0, 2, 0, 1)	-1	3	-1	-1	(xii)	-
(1, 0, 0, 0, 0, 0, 2, 0)	-2	0	0	2	(x)	-
(0, 1, 0, 0, 0, 0, 2, 0)	-3	-1	1	1	(xii)	-
(0, 0, 1, 0, 0, 0, 2, 0)	-2	0	0	0	(ii)	-
(0, 0, 0, 1, 0, 0, 2, 0)	-3	-1	-1	1	(xii)	-
(0, 0, 0, 0, 1, 0, 2, 0)	-4	0	0	0	(ii)	$p = -1,\ q = 0$ for $\hat{n} = 4\hat{k}$
(0, 0, 0, 0, 0, 1, 2, 0)	-3	1	-1	1	(xii)	-
(0, 0, 0, 0, 0, 0, 2, 1)	-3	1	1	1	(xii)	-
(1, 0, 0, 0, 0, 0, 0, 2)	0	2	2	0	(xi)	-
(0, 1, 0, 0, 0, 0, 0, 2)	-1	1	3	-1	(xii)	-
(0, 0, 1, 0, 0, 0, 0, 2)	0	2	2	-2	(xii)	-
(0, 0, 0, 1, 0, 0, 0, 2)	-1	1	1	-1	(xii)	-
(0, 0, 0, 0, 1, 0, 0, 2)	-2	2	2	-2	(xii)	-
(0, 0, 0, 0, 0, 1, 0, 2)	-1	3	1	-1	(xii)	-
(0, 0, 0, 0, 0, 0, 1, 2)	-2	2	2	0	(xii)	-
(1, 1, 1, 0, 0, 0, 0, 0)	1	-1	1	-1	(xii)	-
(1, 1, 0, 1, 0, 0, 0, 0)	0	-2	0	0	(iii)	-
(1, 1, 0, 0, 1, 0, 0, 0)	-1	-1	1	-1	(xii)	-
(1, 1, 0, 0, 0, 1, 0, 0)	0	0	0	0	(i)	$p = 0,\ q = 0$ for any $(\hat{n}, \hat{k}, \hat{\ell})$
(1, 1, 0, 0, 0, 0, 1, 0)	-1	-1	1	1	(xii)	-
(1, 1, 0, 0, 0, 0, 0, 1)	0	0	2	0	(iv)	-

$\alpha = a + c - e - g - 1$; $\beta = -b - d + f + h$; $\gamma = b - d - f + h$; $\delta = a - c - e + g - 1$

Table A.11: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$.

(a, b, c, d, e, f, g, h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (A.272)
(1, 0, 1, 1, 0, 0, 0, 0)	1	-1	-1	-1	(xii)	-
(1, 0, 1, 0, 1, 0, 0, 0)	0	0	0	-2	(v)	-
(1, 0, 1, 0, 0, 1, 0, 0)	1	1	-1	-1	(xii)	-
(1, 0, 1, 0, 0, 0, 1, 0)	0	0	0	0	(i)	$p = 0,\ q = 0$ for any $(\hat{n}, \hat{k}, \hat{\ell})$
(1, 0, 1, 0, 0, 0, 0, 1)	1	1	1	-1	(xii)	-
(1, 0, 0, 1, 1, 0, 0, 0)	-1	-1	-1	-1	(xii)	-
(1, 0, 0, 1, 0, 1, 0, 0)	0	0	-2	0	(iv)	-
(1, 0, 0, 1, 0, 0, 1, 0)	-1	-1	-1	1	(xii)	-
(1, 0, 0, 1, 0, 0, 0, 1)	0	0	0	0	(i)	$p = 0,\ q = 0$ for any $(\hat{n}, \hat{k}, \hat{\ell})$
(1, 0, 0, 0, 1, 1, 0, 0)	-1	1	-1	-1	(xii)	-
(1, 0, 0, 0, 1, 0, 1, 0)	-2	0	0	0	(ii)	-
(1, 0, 0, 0, 1, 0, 0, 1)	-1	1	1	-1	(xii)	-
(1, 0, 0, 0, 0, 1, 1, 0)	-1	1	-1	1	(xii)	-
(1, 0, 0, 0, 0, 1, 0, 1)	0	2	0	0	(iii)	-
(1, 0, 0, 0, 0, 0, 1, 1)	-1	1	1	1	(xii)	-
(0, 1, 1, 1, 0, 0, 0, 0)	0	-2	0	-2	(vii)	-
(0, 1, 1, 0, 1, 0, 0, 0)	-1	-1	1	-3	(xii)	-
(0, 1, 1, 0, 0, 1, 0, 0)	0	0	0	-2	(v)	-
(0, 1, 1, 0, 0, 0, 1, 0)	-1	-1	1	-1	(xii)	-
(0, 1, 1, 0, 0, 0, 0, 1)	0	0	2	-2	(ix)	-
(0, 1, 0, 1, 1, 0, 0, 0)	-2	-2	0	-2	(xii)	-
(0, 1, 0, 1, 0, 1, 0, 0)	-1	-1	-1	-1	(xii)	-
(0, 1, 0, 1, 0, 0, 1, 0)	-2	-2	0	0	(viii)	$p = -1,\ q = 0$ for $\hat{n} = 2\hat{k} + 2\hat{\ell}$
(0, 1, 0, 1, 0, 0, 0, 1)	-1	-1	1	-1	(xii)	-
(0, 1, 0, 0, 1, 1, 0, 0)	-2	0	0	-2	(x)	-
(0, 1, 0, 0, 1, 0, 1, 0)	-3	-1	1	-1	(xii)	-
(0, 1, 0, 0, 1, 0, 0, 1)	-2	0	2	-2	(xii)	-
(0, 1, 0, 0, 0, 1, 1, 0)	-2	0	0	0	(ii)	-
(0, 1, 0, 0, 0, 1, 0, 1)	-1	1	1	-1	(xii)	-
(0, 1, 0, 0, 0, 0, 1, 1)	-2	0	2	0	(vi)	-

$\alpha = a + c - e - g - 1;\ \beta = -b - d + f + h;\ \gamma = b - d - f + h;\ \delta = a - c - e + g - 1$

Table A.12: Possible cases for $(a, b, c, d, e, f, g, h) \in \mathbb{Z}_+^8$.

(a, b, c, d, e, f, g, h)	α	β	γ	δ	A	Existence of $p, q \in \mathbb{Z}$ in (A.272)
(0, 0, 1, 1, 1, 0, 0, 0)	-1	-1	-1	-3	(xii)	-
(0, 0, 1, 1, 0, 1, 0, 0)	0	0	-2	-2	(ix)	-
(0, 0, 1, 1, 0, 0, 1, 0)	-1	-1	-1	-1	(xii)	-
(0, 0, 1, 1, 0, 0, 0, 1)	0	0	0	-2	(v)	$p = 0,\ q = -1$ for $\hat{n} = 2\hat{k} + 2\hat{\ell}$
(0, 0, 1, 0, 1, 1, 0, 0)	-1	1	-1	-3	(xii)	-
(0, 0, 1, 0, 1, 0, 1, 0)	-2	0	0	-2	(x)	-
(0, 0, 1, 0, 1, 0, 0, 1)	-1	1	1	-3	(xii)	-
(0, 0, 1, 0, 0, 1, 1, 0)	-1	1	-1	-1	(xii)	-
(0, 0, 1, 0, 0, 1, 0, 1)	0	2	0	-2	(vii)	-
(0, 0, 1, 0, 0, 0, 1, 1)	-1	1	1	-1	(xii)	-
(0, 0, 0, 1, 1, 1, 0, 0)	-2	0	-2	-2	(xii)	-
(0, 0, 0, 1, 1, 0, 1, 0)	-3	-1	-1	-1	(xii)	-
(0, 0, 0, 1, 1, 0, 0, 1)	-2	0	0	-2	(x)	-
(0, 0, 0, 1, 0, 1, 1, 0)	-2	0	-2	0	(vi)	-
(0, 0, 0, 1, 0, 1, 0, 1)	-1	1	-1	-1	(xii)	-
(0, 0, 0, 1, 0, 0, 1, 1)	-2	0	0	0	(ii)	-
(0, 0, 0, 0, 1, 1, 1, 0)	-3	1	-1	-1	(xii)	-
(0, 0, 0, 0, 1, 1, 0, 1)	-2	2	0	-2	(xii)	-
(0, 0, 0, 0, 1, 0, 1, 1)	-3	1	1	-1	(xii)	-
(0, 0, 0, 0, 0, 1, 1, 1)	-2	2	0	0	(viii)	-

$\alpha = a + c - e - g - 1;\ \beta = -b - d + f + h;\ \gamma = b - d - f + h;\ \delta = a - c - e + g - 1$

This condition is satisfied for any values of $(\hat{n}, \hat{k}, \hat{\ell})$ when $p = q = 0$. Recall that $a+b+\ldots+h \leq 3$. Then, $(a, b, \ldots, h)$ which satisfy (A.287) and (A.288) are enumerated as follows:

$$(1,0,0,0,0,0,0,0), (2,0,0,0,1,0,0,0), (1,1,0,0,0,1,0,0),$$
$$(1,0,1,0,0,0,1,0), (1,0,0,1,0,0,0,1) \in P \text{ for any } (\hat{n}, \hat{k}, \hat{\ell}). \tag{A.290}$$

For the case (ii), the elements of A in (A.272) represent

$$a + c - e - g - 1 = \alpha, \quad a - c - e + g - 1 = 0, \tag{A.291}$$
$$-b - d + f + h = 0, \quad b - d - f + h = 0. \tag{A.292}$$

The condition in (A.273) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \alpha\hat{k}, \ q\hat{n} = 0. \tag{A.293}$$

From the conditions for ℓ, k, and n in (A.227), we have necessary conditions $1 \leq \ell < k < n/2$. Dividing each side of these inequalities by $\gcd(k, \ell, n)$, we have $1/\gcd(k, \ell, n) \leq \hat{\ell} < \hat{k} < \hat{n}/2$. Since $\hat{\ell}$, $\hat{k}$, and $\hat{n}$ are integers, we have $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\alpha| \leq 2$, we see that $p\hat{n} = \alpha\hat{k}$ is not satisfied for any p. From this, we have $|\alpha| \geq 3$. The sum of equalities in (A.291) leads to $\alpha = 2(a - e - 1)$. From this, α is even. Recall that $a + b + \ldots + h \leq 3$. From this, $\alpha = a + c - e - g - 1$ takes a value within the range of $-4 \leq \alpha \leq 2$. From this and $|\alpha| \geq 3$, we have $\alpha = -4$. From (A.291) and (A.292), we have $(a, b, c, d, e, f, g, h) = (0, 0, 0, 0, 1, 0, 2, 0)$. From (A.293), we have $-4\hat{k} = p\hat{n}$. This condition is satisfied for $p = -1$. Hence, we have

$$(0,0,0,0,1,0,2,0) \in P \quad \text{for } \hat{n} = 4\hat{k}.$$

For the case (iii), the elements of A in (A.272) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = 0, \tag{A.294}$$
$$-b - d + f + h = \beta, \quad b - d - f + h = 0. \tag{A.295}$$

The condition in (A.273) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \beta\hat{\ell}, \ q\hat{n} = 0. \tag{A.296}$$

Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\beta| \leq 2$, we see that $p\hat{n} = \beta\hat{\ell}$ is not satisfied for any p. From this, we have $|\beta| \geq 3$. The sum of equalities in (A.295) leads to $\beta = -2(d - h)$. From this, β is even. Recall that $a + b + \ldots + h \leq 3$. From this, $\beta = -b - d + f + h$ takes a value within the range of $-3 \leq \beta \leq 3$. Hence, we have $\beta = \pm 2$. This contradicts $|\beta| \geq 3$.

For the case (iv), the elements of A in (A.272) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = 0, \tag{A.297}$$
$$-b - d + f + h = 0, \quad b - d - f + h = \gamma. \tag{A.298}$$

The condition in (A.273) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = 0, \ q\hat{n} = \gamma\hat{k}. \tag{A.299}$$

Recall that $1 \le \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\gamma| \le 2$, we see that $q\hat{n} = \gamma\hat{k}$ is not satisfied for any q. From this, we have $|\gamma| \ge 3$. The sum of equalities in (A.298) leads to $\gamma = -2(d-h)$. From this, γ is even. Recall that $a + b + \ldots + h \le 3$. From this, $\gamma = b - d - f + h$ takes a value within the range of $-3 \le \gamma \le 3$. Hence, we have $\gamma = \pm 2$. This contradicts $|\gamma| \ge 3$.

For the case (v), the elements of A in (A.272) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = \delta, \tag{A.300}$$
$$-b - d + f + h = 0, \quad b - d - f + h = 0. \tag{A.301}$$

The condition in (A.273) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = 0,\ q\hat{n} = \delta\hat{\ell}. \tag{A.302}$$

Recall that $1 \le \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\delta| \le 2$, we see that $q\hat{n} = \delta\hat{\ell}$ is not satisfied for any p. From this, we have $|\delta| \ge 3$. Recall that $a+b+\ldots+h \le 3$. From this, $\delta = a - c - e + g - 1$ takes a value within the range of $-4 \le \delta \le 2$. From this and $|\delta| \ge 3$, we have $\delta = -4$. From (A.300) and (A.301), we have $(a, b, c, d, e, f, g, h) = (0, 0, 2, 0, 1, 0, 0, 0)$. From (A.302), we have $-4\hat{\ell} = p\hat{n}$. This condition is satisfied for $p = -1$. Hence, we have

$$(0, 0, 2, 0, 1, 0, 0, 0) \in P \quad \text{for } \hat{n} = 4\hat{\ell}.$$

For the case (vi), the elements of A in (A.272) represent

$$a + c - e - g - 1 = \alpha, \quad a - c - e + g - 1 = 0, \tag{A.303}$$
$$-b - d + f + h = 0, \quad b - d - f + h = \gamma. \tag{A.304}$$

The condition in (A.273) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \alpha\hat{k},\ q\hat{n} = \gamma\hat{k}. \tag{A.305}$$

Recall that $1 \le \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\gamma| \le 2$, we see that $q\hat{n} = \gamma\hat{k}$ is not satisfied for any q. From this, we have $|\gamma| \ge 3$. The sum of equalities in (A.304) leads to $\gamma = -2(d-h)$. From this, γ is even. Recall that $a + b + \ldots + h \le 3$. From this, $\gamma = b - d - f + h$ takes a value within the range of $-3 \le \gamma \le 3$. Hence, we have $\gamma = \pm 2$. This contradicts $|\gamma| \ge 3$.

For the case (vii), the elements of A in (A.272) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = \delta, \tag{A.306}$$
$$-b - d + f + h = \beta, \quad b - d - f + h = 0. \tag{A.307}$$

The condition in (A.273) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \beta\hat{\ell},\ q\hat{n} = \delta\hat{\ell}. \tag{A.308}$$

Recall that $1 \le \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\beta| \le 2$, we see that $p\hat{n} = \beta\hat{\ell}$ is not satisfied for any p. From this, we have $|\beta| \ge 3$. The sum of equalities in (A.307) leads to $\beta = -2(d-h)$. From this, β is even. Recall that $a+b+\ldots+h \leq 3$. From this, $\beta = -b-d+f+h$ takes a value within the range of $-3 \leq \beta \leq 3$. Hence, we have $\beta = \pm 2$. This contradicts $|\beta| \geq 3$.

For the case (viii), the elements of A in (A.272) represent

$$a+c-e-g-1 = \alpha, \quad a-c-e+g-1 = 0, \tag{A.309}$$

$$-b-d+f+h = \beta, \quad b-d-f+h = 0. \tag{A.310}$$

The condition in (A.273) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \alpha\hat{k} + \beta\hat{\ell}, \ q\hat{n} = 0. \tag{A.311}$$

Recall that $a+b+\ldots+h \leq 3$. From this, $\alpha = a+c-e-g-1$ takes a value within the range of $-4 \leq \alpha \leq 2$. The sum of equalities in (A.309) leads to $\alpha = 2(a-e-1)$. Thus, α is even. Hence, we have $\alpha = \pm 2, -4$. In a similar manner, $\beta = -b-d+f+h$ takes a value within the range of $-3 \leq \beta \leq 3$. The sum of equalities in (A.310) leads to $\beta = -2(d-h)$. Thus, β is even. Hence, we have $\beta = \pm 2$. When we consider $\alpha = -4$, we have $(a,b,c,d,e,f,g,h) = (0,0,0,0,1,0,2,0)$. Hence, we have $\beta = 0$. This contradicts $\beta \neq 0$. When we consider $\alpha = 2$, we have $(a,b,c,d,e,f,g,h) = (2,0,1,0,0,0,0,0)$. Hence, we have $\beta = 0$. This contradicts $\beta \neq 0$. When we consider $\alpha = -2$ with $\beta = 2$, we have $(a,b,c,d,e,f,g,h) = (0,0,0,0,0,1,1,1)$. From (A.311), we have $-2(\hat{k}-\hat{\ell}) = p\hat{n}$. Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. From this, we have $1 \leq \hat{k}-\hat{\ell} < \hat{n}/2$. Thus, the condition $-2(\hat{k}-\hat{\ell}) = p\hat{n}$ is not satisfied for any p. When we consider $\alpha = -2$ with $\beta = -2$, we have $(a,b,c,d,e,f,g,h) = (0,1,0,1,0,0,1,0)$. From (A.311), we have $-2(\hat{k}+\hat{\ell}) = p\hat{n}$. This condition is satisfied for $p = -1$. Hence, we have

$$(0,1,0,1,0,0,1,0) \in P \quad \text{for } \hat{n} = 2\hat{k} + 2\hat{\ell}.$$

For the case (ix), the elements of A in (A.272) represent

$$a+c-e-g-1 = 0, \quad a-c-e+g-1 = \delta, \tag{A.312}$$

$$-b-d+f+h = 0, \quad b-d-f+h = \gamma. \tag{A.313}$$

The condition in (A.273) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = 0, \ q\hat{n} = \gamma\hat{k} + \delta\hat{\ell}. \tag{A.314}$$

Recall that $a+b+\ldots+h \leq 3$. From this, $\alpha = a-c-e+g-1$ takes a value within the range of $-4 \leq \alpha \leq 2$. The sum of equalities in (A.312) leads to $\alpha = 2(a-e-1)$. Thus, α is even. Hence, we have $\alpha = \pm 2, -4$. In a similar manner, $\beta = b-d-f+h$ takes a value within the range of $-3 \leq \beta \leq 3$. The sum of equalities in (A.313) leads to $\beta = -2(d-h)$. Thus, β is even. Hence, we have $\beta = \pm 2$. When we consider $\alpha = -4$, we have $(a,b,c,d,e,f,g,h) = (0,0,2,0,1,0,0,0)$. Hence, we have $\beta = 0$. This contradicts $\beta \neq 0$. When we consider $\alpha = 2$, we have $(a,b,c,d,e,f,g,h) = (2,0,0,0,0,0,1,0)$. Hence, we have $\beta = 0$. This contradicts $\beta \neq 0$. When we consider $\alpha = -2$ with $\beta = 2$, we have $(a,b,c,d,e,f,g,h) = (0,1,1,0,0,0,0,1)$. From (A.314), we have $-2(\hat{k}-\hat{\ell}) = q\hat{n}$. Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. From this, we have $1 \leq \hat{k}-\hat{\ell} < \hat{n}/2$. Thus, the condition $-2(\hat{k}-\hat{\ell}) = q\hat{n}$ is not satisfied for any q. When we consider $\alpha = -2$ with

$\beta = -2$, we have $(a, b, c, d, e, f, g, h) = (0, 0, 1, 1, 0, 1, 0, 0)$. From (A.314), we have $-2(\hat{k} + \hat{\ell}) = q\hat{n}$. This condition is satisfied for $q = -1$. Hence, we have

$$(0, 0, 1, 1, 0, 1, 0, 0) \in P \quad \text{for } \hat{n} = 2\hat{k} + 2\hat{\ell}.$$

For the case (x), the elements of A in (A.272) represent

$$a + c - e - g - 1 = \alpha, \quad a - c - e + g - 1 = \delta, \tag{A.315}$$

$$-b - d + f + h = 0, \quad b - d - f + h = 0. \tag{A.316}$$

The condition in (A.273) is equivalent to

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \alpha\hat{k}, \ q\hat{n} = \delta\hat{\ell}. \tag{A.317}$$

Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\alpha| \leq 2$, we see that $p\hat{n} = \alpha\hat{k}$ is not satisfied for any p. From this, we have $|\alpha| \geq 3$. Similarly, for the case $|\delta| \leq 2$, we see that $q\hat{n} = \delta\hat{\ell}$ is not satisfied for any q. From this, we have $|\delta| \geq 3$. According to the results in Table A.12–A.12, only $(a, b, c, d, e, f, g, h) = (0, 0, 0, 0, 2, 0, 0, 0)$ corresponds to this case. From (A.317), we have $p\hat{n} = -3\hat{k}$ and $q\hat{n} = -3\hat{\ell}$. From $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$, we have $p = -1$ and $q = -1$. Thus, we have $\hat{n} = 3\hat{k}$ and $\hat{n} = 3\hat{\ell}$. This contradicts $\hat{k} \neq \hat{\ell}$.

For the case (xi), the elements of A in (A.272) represent

$$a + c - e - g - 1 = 0, \quad a - c - e + g - 1 = 0, \tag{A.318}$$

$$-b - d + f + h = \beta, \quad b - d - f + h = \gamma. \tag{A.319}$$

The condition in (A.273) is equivalent to

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \beta\hat{\ell}, \ q\hat{n} = \gamma\hat{k}. \tag{A.320}$$

Recall that $1 \leq \hat{\ell} < \hat{k} < \hat{n}/2$. Thus, for the case $|\beta| \leq 2$, we see that $p\hat{n} = \beta\hat{\ell}$ is not satisfied for any p. From this, we have $|\beta| \geq 3$. Similarly, for the case $|\gamma| \leq 2$, we see that $q\hat{n} = \gamma\hat{k}$ is not satisfied for any q. From this, we have $|\gamma| \geq 3$. According to the results in Table A.7–A.12, no (a, b, c, d, e, f, g, h) corresponds to this case.

For the case (xii), the elements of A in (A.272) represent

$$a + c - e - g - 1 = \alpha, \quad a - c - e + g - 1 = \delta, \tag{A.321}$$

$$-b - d + f + h = \beta, \quad b - d - f + h = \gamma. \tag{A.322}$$

The condition in (A.273) is rewritten as

$$\exists p, q \in \mathbb{Z} \quad \text{s.t.} \quad p\hat{n} = \alpha\hat{k} + \beta\hat{\ell}, \ q\hat{n} = \gamma\hat{k} + \delta\hat{\ell}. \tag{A.323}$$

All (a, b, c, d, e, f, g, h) that correspond to this case are shown in Table A.7–A.12.

Based on the above discussion, F_i $(i = 1, \ldots, 4)$ is restricted to the form of

$$F_i = a_1\widetilde{\phi}z_i + F_i^C + (\text{other terms}), \quad i = 1, \ldots, 4, \tag{A.324}$$

Table A.13: Nonzero coefficients of leading terms which belong to "other terms" in (A.324).

$(\hat{n}, \hat{k}, \hat{\ell})$	Nonzero coefficients
General $(\hat{n}, \hat{k}, \hat{\ell})$	None
$(5, 2, 1)$	$A_{01001000}(0)$, $A_{00000200}(0)$, $A_{03000000}(0)$, $A_{00002100}(0)$
$(8, 3, 1)$	$A_{01010010}(0)$, $A_{00110100}(0)$, $A_{00021000}(0)$, $A_{00002001}(0)$, $A_{00000003}(0)$
$(10, 3, 1)$	$A_{01002000}(0)$, $A_{00000300}(0)$
$(4\hat{k}, \hat{k}, \hat{\ell})$	$A_{00001020}(0)$
$(4\hat{\ell}, \hat{k}, \hat{\ell})$	$A_{00201000}(0)$
$(2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$ with $(\hat{k}, \hat{\ell}) \neq (3, 1)$	$A_{01010010}(0)$, $A_{00110100}(0)$, $A_{00021000}(0)$

where

$$F_1^C = z_1(a_2|z_1|^2 + a_3|z_2|^2 + a_4|z_3|^2 + a_5|z_4|^2), \tag{A.325}$$
$$F_2^C = z_2(a_2|z_2|^2 + a_3|z_1|^2 + a_4|z_4|^2 + a_5|z_3|^2), \tag{A.326}$$
$$F_3^C = z_3(a_2|z_3|^2 + a_3|z_4|^2 + a_4|z_1|^2 + a_5|z_2|^2), \tag{A.327}$$
$$F_4^C = z_4(a_2|z_4|^2 + a_3|z_3|^2 + a_4|z_2|^2 + a_5|z_1|^2) \tag{A.328}$$

with the following notations:[21]

$$\begin{aligned} &a_1 = A'_{10000000}(0), \quad a_2 = A_{20001000}(0), \quad a_3 = A_{11000100}(0), \\ &a_4 = A_{10100010}(0), \quad a_5 = A_{10010001}(0). \end{aligned} \tag{A.329}$$

F_2, F_3, and F_4 are obtained by (A.255), (A.256), and (A.258), respectively.

In (A.324), F_i^C corresponds to cubic terms, and the form of "(other terms)" varies with the values of $(\hat{n}, \hat{k}, \hat{\ell})$. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$, we have quadratic terms as leading terms. For any other cases, we have cubic terms as leading terms that vary with the values of $(\hat{n}, \hat{k}, \hat{\ell})$. From this point of view, we classify the form of the bifurcation equation as shown in Table A.13 by the values of $(\hat{n}, \hat{k}, \hat{\ell})$.

The form of "(other terms)" in (A.324) depends on the values of $(\hat{n}, \hat{k}, \hat{\ell})$ in (A.228). All the possible cases and stability conditions for the bifurcating solutions are summarized in Tables A.14–A.16. The main finding of this section is as follows:

Proposition A.26. *For a critical point of multiplicity 8 associated with $\mu = (8; k, \ell)$, we have the following statements:*

- *For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$, the bifurcating solution $\boldsymbol{w}_{\mathrm{sqT}}$ is always unstable in the neighborhood of the critical point, and the bifurcating curve takes the form $\widetilde{\phi} \approx cw$ for some constant c.*

[21] These notations are local and should not be confused with (A.180) used in Appendix A.4.4.

Table A.14: Stability conditions of bifurcating solutions for group-theoretic critical points with multiplicity 8.

$(\hat{n}, \hat{k}, \hat{\ell})$	Solutions	Stability conditions (necessary conditions)
General $(\hat{n}, \hat{k}, \hat{\ell})$	$\boldsymbol{w}_{\mathrm{stripeI}}, \boldsymbol{w}_{\mathrm{stripeII}}$	$\max(a_3, a_4, a_5) < a_2 < 0$
	$\boldsymbol{w}_{\mathrm{upside-downI}}, \boldsymbol{w}_{\mathrm{upside-downII}}$	$a_3 - a_4 + a_5 < a_2 < -\lvert a_4 \rvert$
	$\boldsymbol{w}_{\mathrm{sqT}}$	$-a_3 + a_4 + a_5 < a_2 < -\lvert a_3 \rvert$
	$\boldsymbol{w}_{\mathrm{sqVM}}$	$a_2 + a_3 < -\lvert a_4 + a_5 \rvert, a_2 - a_3 < -\lvert a_4 - a_5 \rvert$

Table A.15: Stability conditions of bifurcating solutions for group-theoretic critical points with multiplicity 8.

$(\hat{n}, \hat{k}, \hat{\ell})$	Solutions	Stability conditions	
$(5, 2, 1)$	$\boldsymbol{w}_{\mathrm{stripeI}}, \boldsymbol{w}_{\mathrm{stripeII}}$	Does not exist	
	$\boldsymbol{w}_{\mathrm{upside-downI}}, \boldsymbol{w}_{\mathrm{upside-downII}}$	Does not exist	
	$\boldsymbol{w}_{\mathrm{sqT}}$	Always unstable	
	$\boldsymbol{w}_{\mathrm{sqVM}}$	$a_6 + a_7 < 0, 3a_6 + a_7 > 0, 2a_6 + a_7 > 0$	if $w > 0$
		$a_6 + a_7 > 0, 3a_6 + a_7 > 0, 2a_6 + a_7 > 0$	if $w < 0$
$(8, 3, 1)$	$\boldsymbol{w}_{\mathrm{stripeI}}, \boldsymbol{w}_{\mathrm{stripeII}}$	Does not exist	
	$\boldsymbol{w}_{\mathrm{upside-downI}}, \boldsymbol{w}_{\mathrm{upside-downII}}$	Does not exist	
	$\boldsymbol{w}_{\mathrm{sqT}}$	Does not exist	
	$\boldsymbol{w}_{\mathrm{sqVM}}$	$a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} < 0$	
		$a_2 + a_3 - a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14} < 0$	
		$a_2 - a_3 + a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14} < 0$	
		$a_2 - a_3 - a_4 + a_5 - a_{10} - a_{11} + a_{12} + a_{13} + a_{14} < 0$	
		$a_{10} + a_{11} + 2a_{12} + a_{13} - a_{14} > 0$	
		$a_{13} + a_{14} > 0$	
$(10, 3, 1)$	$\boldsymbol{w}_{\mathrm{stripeI}}, \boldsymbol{w}_{\mathrm{stripeII}}$	Does not exist	
	$\boldsymbol{w}_{\mathrm{upside-downI}}, \boldsymbol{w}_{\mathrm{upside-downII}}$	Does not exist	
	$\boldsymbol{w}_{\mathrm{sqT}}$	$a_2 + a_3 + a_{15} + a_{16} < 0$	
		$a_2 - a_3 - 2a_{16} < 0$	
		$3a_{15} + a_{16} > 0$	
		$a_2 + a_3 - a_4 - a_5 + a_{15} + a_{16} < 0$	
	$\boldsymbol{w}_{\mathrm{sqVM}}$	$a_2 + a_3 + a_4 + a_5 + a_{15} + a_{16} < 0$	
		$a_2 + a_3 - a_4 - a_5 - a_{15} - a_{16} < 0$	
		$a_2 - a_3 + a_4 - a_5 - 2a_{16} < 0$	
		$a_2 - a_3 - a_4 + a_5 - 2a_{16} < 0$	
		$3a_{15} + a_{16} > 0$	

Table A.16: Stability conditions of bifurcating solutions for group-theoretic critical points with multiplicity 8.

$(\hat{n}, \hat{k}, \hat{\ell})$	Solutions	Stability conditions (necessary conditions)
$(4\hat{k}, \hat{k}, \hat{\ell})$	$\boldsymbol{w}_{\text{stripeI}}, \boldsymbol{w}_{\text{stripeII}}$	$\max(a_3, a_4 + \lvert a_{17}\rvert, a_5) < a_2 < 0$
	$\boldsymbol{w}_{\text{upside-downI}}, \boldsymbol{w}_{\text{upside-downII}}$	$a_3 - a_4 + a_5 - a_{17} < a_2 < -\lvert a_4 + a_{17}\rvert$
		$a_4 > 0$
	$\boldsymbol{w}_{\text{SqT}}$	Does not exist
	$\boldsymbol{w}_{\text{sqVM}}$	$a_2 + a_3 + a_4 + a_5 + a_{17} < 0$
		$a_2 + a_3 - a_4 - a_5 - a_{17} < 0$
		$a_2 - a_3 + a_4 - a_5 + a_{17} < 0$
		$a_2 - a_3 - a_4 + a_5 - a_{17} < 0$
		$a_{17} > 0$
$(4\hat{\ell}, \hat{k}, \hat{\ell})$	$\boldsymbol{w}_{\text{stripeI}}, \boldsymbol{w}_{\text{stripeII}}$	$\max(a_3, a_4 + \lvert a_{18}\rvert, a_5) < a_2 < 0$
	$\boldsymbol{w}_{\text{upside-downI}}, \boldsymbol{w}_{\text{upside-downII}}$	$a_3 - a_4 + a_5 + a_{17} < a_2 < -\lvert a_4 + a_{18}\rvert$
		$a_{18} > 0$
	$\boldsymbol{w}_{\text{SqT}}$	Does not exist
	$\boldsymbol{w}_{\text{SqVM}}$	$a_2 + a_3 + a_4 + a_5 + a_{18} < 0$
		$a_2 + a_3 - a_4 - a_5 - a_{18} < 0$
		$a_2 - a_3 + a_4 - a_5 + a_{18} < 0$
		$a_2 - a_3 - a_4 + a_5 - a_{18} < 0$
		$a_{18} > 0$
$(2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$	$\boldsymbol{w}_{\text{stripeI}}, \boldsymbol{w}_{\text{stripeII}}$	$\max(a_3, a_4, a_5 - \lvert a_{12}\rvert) < a_2 < 0$
with $(\hat{k}, \hat{\ell}) \neq (3, 1)$	$\boldsymbol{w}_{\text{upside-downI}}, \boldsymbol{w}_{\text{upside-downII}}$	$a_2 < -\lvert a_4\rvert$
		$a_2 + a_3 - a_4 - a_5 - a_{12} > -\lvert a_{10} + a_{11}\rvert$
		$a_2 + a_3 - a_4 - a_5 + a_{12} > -\lvert a_{10} - a_{11}\rvert$
	$\boldsymbol{w}_{\text{sqT}}$	Does not exist
	$\boldsymbol{w}_{\text{sqVM}}$	$a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12} < 0$
		$a_2 + a_3 - a_4 - a_5 - a_{10} - a_{11} - a_{12} < 0$
		$a_2 - a_3 + a_4 - a_5 - a_{10} - a_{11} - a_{12} < 0$
		$a_2 - a_3 - a_4 + a_5 - a_{10} - a_{11} + a_{12} < 0$
		$a_{10} + a_{11} + 2a_{12} > 0$

- *For any other cases, the stability of the bifurcating solutions $\boldsymbol{w}_{\mathrm{stripeI}}$, $\boldsymbol{w}_{\mathrm{stripeII}}$, $\boldsymbol{w}_{\mathrm{upside-downI}}$, $\boldsymbol{w}_{\mathrm{upside-downII}}$, $\boldsymbol{w}_{\mathrm{sqT}}$, and $\boldsymbol{w}_{\mathrm{sqVM}}$ depends on the values of the coefficients of the power series expansion of the bifurcation equation in (A.275), and the bifurcating curve takes the form $\widetilde{\phi} \approx cw^2$ for some constant c.*

To show these results, we focus on each case and study stability conditions for the bifurcating solutions in the remainder of this section.

Case 1: General $(\hat{n}, \hat{k}, \hat{\ell})$

For general cases, other than special cases to be treated in the sequel, the asymptotic form of F_i $(i = 1, \dots, 4)$ in (A.324) becomes

$$F_i \approx a_1\widetilde{\phi}z_i + F_i^C, \tag{A.330}$$

where F_i^C $(i = 1, \dots, 4)$ are given in (A.325)–(A.328). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \dots, 8)$ in (A.234)–(A.237) becomes

$$\widetilde{F}_i \approx a_1\widetilde{\phi}w_i + \widetilde{F}_i^C \tag{A.331}$$

with

$$\widetilde{F}_1^C = w_1\{a_2({w_1}^2 + {w_2}^2) + a_3({w_3}^2 + {w_4}^2) + a_4({w_5}^2 + {w_6}^2) + a_5({w_7}^2 + {w_8}^2)\}, \tag{A.332}$$
$$\widetilde{F}_2^C = w_2\{a_2({w_1}^2 + {w_2}^2) + a_3({w_3}^2 + {w_4}^2) + a_4({w_5}^2 + {w_6}^2) + a_5({w_7}^2 + {w_8}^2)\}, \tag{A.333}$$
$$\widetilde{F}_3^C = w_3\{a_2({w_3}^2 + {w_4}^2) + a_3({w_1}^2 + {w_2}^2) + a_4({w_7}^2 + {w_8}^2) + a_5({w_5}^2 + {w_6}^2)\}, \tag{A.334}$$
$$\widetilde{F}_4^C = w_4\{a_2({w_3}^2 + {w_4}^2) + a_3({w_1}^2 + {w_2}^2) + a_4({w_7}^2 + {w_8}^2) + a_5({w_5}^2 + {w_6}^2)\}, \tag{A.335}$$
$$\widetilde{F}_5^C = w_5\{a_2({w_5}^2 + {w_6}^2) + a_3({w_7}^2 + {w_8}^2) + a_4({w_1}^2 + {w_2}^2) + a_5({w_3}^2 + {w_4}^2)\}, \tag{A.336}$$
$$\widetilde{F}_6^C = w_6\{a_2({w_5}^2 + {w_6}^2) + a_3({w_7}^2 + {w_8}^2) + a_4({w_1}^2 + {w_2}^2) + a_5({w_3}^2 + {w_4}^2)\}, \tag{A.337}$$
$$\widetilde{F}_7^C = w_7\{a_2({w_7}^2 + {w_8}^2) + a_3({w_5}^2 + {w_6}^2) + a_4({w_3}^2 + {w_4}^2) + a_5({w_1}^2 + {w_2}^2)\}, \tag{A.338}$$
$$\widetilde{F}_8^C = w_8\{a_2({w_7}^2 + {w_8}^2) + a_3({w_5}^2 + {w_6}^2) + a_4({w_3}^2 + {w_4}^2) + a_5({w_1}^2 + {w_2}^2)\}, \tag{A.339}$$

Hence, the asymptotic form of the Jacobian matrix in (A.232) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\widetilde{\phi}I_8 + B_C \tag{A.340}$$

with the following notations:[22]

$$B_C = a_2B_2 + a_3B_3 + a_4B_4 + a_5B_5, \tag{A.341}$$

$$B_2 = \begin{bmatrix} B_1^2 & O \\ O & B_2^2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} B_1^3 & O \\ O & B_2^3 \end{bmatrix}, \quad B_4 = \begin{bmatrix} B_1^4 & B_3^4 \\ (B_3^4)^\top & B_2^4 \end{bmatrix}, \quad B_5 = \begin{bmatrix} B_1^5 & B_3^5 \\ (B_3^5)^\top & B_2^5 \end{bmatrix},$$

[22] The notations here are local and should not be confused with (A.193) used in Appendix A.4.4.

$$B_1^2 = \begin{bmatrix} 3{w_1}^2 + {w_2}^2 & 2w_1w_2 & 0 & 0 \\ 2w_1w_2 & {w_1}^2 + 3{w_2}^2 & 0 & 0 \\ 0 & 0 & 3{w_3}^2 + {w_4}^2 & 2w_3w_4 \\ 0 & 0 & 2w_3w_4 & {w_3}^2 + 3{w_4}^2 \end{bmatrix},$$

$$B_2^2 = \begin{bmatrix} 3{w_5}^2 + {w_6}^2 & 2w_5w_6 & 0 & 0 \\ 2w_5w_6 & {w_5}^2 + 3{w_6}^2 & 0 & 0 \\ 0 & 0 & 3{w_7}^2 + {w_8}^2 & 2w_7w_8 \\ 0 & 0 & 2w_7w_8 & {w_7}^2 + 3{w_8}^2 \end{bmatrix},$$

$$B_1^3 = \begin{bmatrix} {w_3}^2 + {w_4}^2 & 0 & 2w_1w_3 & 2w_1w_4 \\ 0 & {w_3}^2 + {w_4}^2 & 2w_2w_3 & 2w_2w_4 \\ 2w_1w_3 & 2w_2w_3 & {w_1}^2 + {w_2}^2 & 0 \\ 2w_1w_4 & 2w_2w_4 & 0 & {w_1}^2 + {w_2}^2 \end{bmatrix},$$

$$B_2^3 = \begin{bmatrix} {w_7}^2 + {w_8}^2 & 0 & 2w_5w_7 & 2w_5w_8 \\ 0 & {w_7}^2 + {w_8}^2 & 2w_6w_7 & 2w_6w_8 \\ 2w_5w_7 & 2w_6w_7 & {w_5}^2 + {w_6}^2 & 0 \\ 2w_5w_8 & 2w_6w_8 & 0 & {w_5}^2 + {w_6}^2 \end{bmatrix},$$

$$B_1^4 = \begin{bmatrix} ({w_5}^2 + {w_6}^2)I_2 & O \\ O & ({w_7}^2 + {w_8}^2)I_2 \end{bmatrix}, \quad B_2^4 = \begin{bmatrix} ({w_1}^2 + {w_2}^2)I_2 & O \\ O & ({w_3}^2 + {w_4}^2)I_2 \end{bmatrix},$$

$$B_3^4 = 2\begin{bmatrix} w_1w_5 & w_1w_6 & 0 & 0 \\ w_2w_5 & w_2w_6 & 0 & 0 \\ 0 & 0 & w_3w_7 & w_3w_8 \\ 0 & 0 & w_4w_7 & w_4w_8 \end{bmatrix}, \quad B_1^5 = \begin{bmatrix} ({w_7}^2 + {w_8}^2)I_2 & O \\ O & ({w_5}^2 + {w_6}^2)I_2 \end{bmatrix},$$

$$B_2^5 = \begin{bmatrix} ({w_3}^2 + {w_4}^2)I_2 & O \\ O & ({w_1}^2 + {w_2}^2)I_2 \end{bmatrix}, \quad B_3^5 = 2\begin{bmatrix} 0 & 0 & w_1w_7 & w_1w_8 \\ 0 & 0 & w_2w_7 & w_2w_8 \\ w_3w_5 & w_3w_6 & 0 & 0 \\ w_4w_5 & w_4w_6 & 0 & 0 \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\text{stripeI}} = (w, 0, 0, 0, 0, 0, 0, 0)$ into (A.331) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeI}} \approx -\frac{a_2}{a_1}w^2.$$

Evaluating the Jacobian matrix (A.340) at $(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}}) = \widetilde{J_C}^{\text{stripeI}} \approx w^2 \begin{bmatrix} C_1 & O \\ O & C_2 \end{bmatrix} \tag{A.342}$$

with

$$C_1 = \begin{bmatrix} 2a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 + a_3 & 0 \\ 0 & 0 & 0 & -a_2 + a_3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} (-a_2 + a_4)I_2 & O \\ O & (-a_2 + a_5)I_2 \end{bmatrix}. \tag{A.343}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeI}}, \widetilde{\phi}_{\mathrm{stripeI}})$ are given by

$$\begin{aligned} \lambda_1 &\approx 2a_2 w^2, \\ \lambda_2 &\approx O(w^3), \\ \lambda_3 &\approx -(a_2 - a_3)w^2 \quad \text{(repeated twice)}, \\ \lambda_4 &\approx -(a_2 - a_4)w^2 \quad \text{(repeated twice)}, \\ \lambda_5 &\approx -(a_2 - a_5)w^2 \quad \text{(repeated twice)}. \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 < 0, \quad a_2 - a_3 > 0, \quad a_2 - a_4 > 0, \quad a_2 - a_5 > 0.$$

These conditions are equivalent to

$$\max(a_3, a_4, a_5) < a_2 < 0. \tag{A.344}$$

Thus, the stability of $\boldsymbol{w}_{\mathrm{stripeI}}$ is conditional and depends on the values of $a_2, \ldots, a_5$.

Substituting $\boldsymbol{w}_{\mathrm{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$ into (A.331) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{stripeII}} \approx -\frac{a_2}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.340) at $(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}}) = \widetilde{J_C}^{\mathrm{stripeII}} \approx w^2 \begin{bmatrix} C_3 & O \\ O & C_2 \end{bmatrix} \tag{A.345}$$

with

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2a_2 & 0 & 0 \\ 0 & 0 & -a_2 + a_3 & 0 \\ 0 & 0 & 0 & -a_2 + a_3 \end{bmatrix}, \tag{A.346}$$

where C_2 is given in (A.343). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{stripeII}}, \widetilde{\phi}_{\mathrm{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\mathrm{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\mathrm{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\mathrm{stripeI}}$.

Substituting $\boldsymbol{w}_{\mathrm{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$ into (A.331) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{upside-downI}} \approx -\frac{a_2 + a_4}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.340) at $(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}}) = \widetilde{J_C}^{\text{upside-downI}} \approx w^2 \begin{bmatrix} C_4 & C_5 \\ C_5 & C_4 \end{bmatrix} \tag{A.347}$$

with

$$C_4 = \begin{bmatrix} 2a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 + a_3 - a_4 + a_5 & 0 \\ 0 & 0 & 0 & -a_2 + a_3 - a_4 + a_5 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 2a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{A.348}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$ are given by

$$\begin{aligned} &\lambda_1, \lambda_2 \approx 2(a_2 \pm a_4)w^2, \\ &\lambda_3 \approx O(w^3) \quad \text{(repeated twice)}, \\ &\lambda_4 \approx -(a_2 - a_3 + a_4 - a_5)w^2 \quad \text{(repeated 4 times)}. \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 < -|a_4|, \quad a_2 - a_3 + a_4 - a_5 > 0.$$

These conditions are equivalent to

$$a_3 - a_4 + a_5 < a_2 < -|a_4|. \tag{A.349}$$

Thus, the stability of $\boldsymbol{w}_{\text{upside-downI}}$ is conditional and depends on the values of $a_2, \dots, a_5$.

Substituting $\boldsymbol{w}_{\text{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$ into (A.331) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downII}} \approx -\frac{a_2 + a_4}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.340) at $(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}}) = \widetilde{J_C}^{\text{upside-downII}} \approx w^2 \begin{bmatrix} C_6 & C_7 \\ C_7 & C_6 \end{bmatrix} \tag{A.350}$$

with

$$C_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2a_2 & 0 & 0 \\ 0 & 0 & -a_2 + a_3 - a_4 + a_5 & 0 \\ 0 & 0 & 0 & -a_2 + a_3 - a_4 + a_5 \end{bmatrix}, \quad C_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{A.351}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{upside-downII}}$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$.

Substituting $\boldsymbol{w}_{\text{sqT}} = (w, 0, w, 0, 0, 0, 0, 0)$ into (A.331) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqT}} \approx -\frac{a_2 + a_3}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.340) at $(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}}) = \widetilde{J_C}^{\text{sqT}} \approx w^2 \begin{bmatrix} C_8 & O \\ O & C_9 \end{bmatrix} \tag{A.352}$$

$$C_8 = 2 \begin{bmatrix} a_2 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \\ a_3 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_9 = -(a_2 + a_3 - a_4 - a_5) I_4. \tag{A.353}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}})$ are given by

$$\begin{aligned} &\lambda_1, \lambda_2 \approx 2(a_2 \pm a_3) w^2, \\ &\lambda_3 \approx O(w^3) \quad \text{(repeated twice)}, \\ &\lambda_4 \approx -(a_2 + a_3 - a_4 - a_5) w^2 \quad \text{(repeated 4 times)}. \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 < -|a_3|, \quad a_2 + a_3 - a_4 - a_5 > 0.$$

These conditions are equivalent to

$$-a_3 + a_4 + a_5 < a_2 < -|a_3|. \tag{A.354}$$

Thus, the stability of $\boldsymbol{w}_{\text{sqT}}$ is conditional and depends on the values of $a_2, \ldots, a_5$.

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (A.331) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.340) at $(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}}) = \widetilde{J_C}^{\text{sqVM}} \approx w^2 \begin{bmatrix} C_8 & C_{10} \\ C_{10} & C_8 \end{bmatrix} \tag{A.355}$$

with

$$C_{10} = 2 \begin{bmatrix} a_4 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 \\ a_5 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{A.356}$$

where C_8 is given in (A.353). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$ are given by

$$\begin{aligned}\lambda_1, \lambda_2 &\approx 2\{a_2 + a_3 \pm (a_4 + a_5)\}w^2,\\ \lambda_3, \lambda_4 &\approx 2\{a_2 - a_3 \pm (a_4 - a_5)\}w^2,\\ \lambda_5 &\approx O(w^3) \quad \text{(repeated 4 times)}.\end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 + a_3 < -|a_4 + a_5|, \tag{A.357}$$
$$a_2 - a_3 < -|a_4 - a_5|. \tag{A.358}$$

Thus, the stability of $\boldsymbol{w}_{\text{sqVM}}$ is conditional and depends on the values of $a_2, \ldots, a_5$.

Case 2: $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$

For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$, we have

$$(0,1,0,0,1,0,0,0), (0,0,0,0,0,2,0,0), (0,3,0,0,0,0,0,0), (0,0,0,0,2,1,0,0) \in P$$

as well as

$$\begin{aligned}&(1,0,0,0,0,0,0,0), (2,0,0,0,1,0,0,0), (1,1,0,0,0,1,0,0),\\ &(1,0,1,0,0,0,1,0), (1,0,0,1,0,0,0,1) \in P\end{aligned}$$

in (A.290). Then, the asymptotic form of F_i $(i = 1, \ldots, 4)$ in (A.324) becomes

$$F_1 \approx a_1\widetilde{\phi}z_1 + a_6 z_2\overline{z}_1 + a_7\overline{z}_2{}^2 + a_8 z_2{}^3 + a_9\overline{z}_1{}^2\overline{z}_2 + F_1^C, \tag{A.359}$$
$$F_2 \approx a_1\widetilde{\phi}z_2 + a_6\overline{z}_1\overline{z}_2 + a_7 z_1{}^2 + a_8\overline{z}_1{}^3 + a_9\overline{z}_2{}^2 z_1 + F_2^C, \tag{A.360}$$
$$F_3 \approx a_1\widetilde{\phi}z_3 + a_6 z_4\overline{z}_3 + a_7\overline{z}_4{}^2 + a_8 z_4{}^3 + a_9\overline{z}_3{}^2\overline{z}_4 + F_3^C, \tag{A.361}$$
$$F_4 \approx a_1\widetilde{\phi}z_4 + a_6\overline{z}_3\overline{z}_4 + a_7 z_3{}^2 + a_8\overline{z}_3{}^3 + a_9\overline{z}_4{}^2 z_3 + F_4^C \tag{A.362}$$

with

$$a_6 = A_{01001000}(0), \quad a_7 = A_{00000200}(0), \quad a_8 = A_{03000000}(0), \quad a_9 = A_{00002100}(0),$$

where F_i^C $(i = 1, \ldots, 4)$ is given in (A.325)–(A.328). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \ldots, 8)$ in (A.234)–(A.237) becomes

$$\begin{aligned}\widetilde{F}_1 \approx{}& a_1\widetilde{\phi}w_1 + a_6(w_1w_3 + w_2w_4) + a_7(w_3{}^2 - w_4{}^2)\\ &+ a_8w_3(w_3{}^2 - 3w_4{}^2) + a_9\{w_3(w_1{}^2 - w_2{}^2) - 2w_1w_2w_4\} + \widetilde{F}_1^C,\end{aligned} \tag{A.363}$$
$$\begin{aligned}\widetilde{F}_2 \approx{}& a_1\widetilde{\phi}w_2 + a_6(w_1w_4 - w_2w_3) - 2a_7w_3w_4\\ &+ a_8w_4(3w_3{}^2 - w_4{}^2) + a_9\{-w_4(w_1{}^2 - w_2{}^2) - 2w_1w_2w_3\} + \widetilde{F}_2^C,\end{aligned} \tag{A.364}$$

$$\widetilde{F}_3 \approx a_1\widetilde{\phi}w_3 + a_6(w_1w_3 - w_2w_4) + a_7({w_1}^2 - {w_2}^2)$$
$$+ a_8w_1({w_1}^2 - 3{w_2}^2) + a_9\{w_1({w_3}^2 - {w_4}^2) + 2w_3w_4w_2\} + \widetilde{F}_3^C, \quad \text{(A.365)}$$
$$\widetilde{F}_4 \approx a_1\widetilde{\phi}w_4 + a_6(-w_1w_4 - w_2w_3) + 2a_7w_1w_2$$
$$+ a_8w_2(-3{w_1}^2 + {w_2}^2) + a_9\{w_2({w_3}^2 - {w_4}^2) - 2w_3w_4w_1\} + \widetilde{F}_4^C, \quad \text{(A.366)}$$
$$\widetilde{F}_5 \approx a_1\widetilde{\phi}w_5 + a_6(w_5w_7 + w_6w_8) + a_7({w_7}^2 - {w_8}^2)$$
$$+ a_8w_7({w_7}^2 - 3{w_8}^2) + a_9\{w_7({w_5}^2 - {w_6}^2) - 2w_5w_6w_8\} + \widetilde{F}_5^C, \quad \text{(A.367)}$$
$$\widetilde{F}_6 \approx a_1\widetilde{\phi}w_6 + a_6(w_5w_8 - w_6w_7) - 2a_7w_7w_8$$
$$+ a_8w_8(3{w_7}^2 - {w_8}^2) + a_9\{-w_8({w_5}^2 - {w_6}^2) - 2w_5w_6w_7\} + \widetilde{F}_6^C, \quad \text{(A.368)}$$
$$\widetilde{F}_7 \approx a_1\widetilde{\phi}w_7 + a_6(w_5w_7 - w_6w_8) + a_7({w_5}^2 - {w_6}^2)$$
$$+ a_8w_5({w_5}^2 - 3{w_6}^2) + a_9\{w_5({w_7}^2 - {w_8}^2) + 2w_7w_8w_6\} + \widetilde{F}_7^C, \quad \text{(A.369)}$$
$$\widetilde{F}_8 \approx a_1\widetilde{\phi}w_8 + a_6(-w_5w_8 - w_6w_7) + 2a_7w_5w_6$$
$$+ a_8w_6(-3{w_5}^2 + {w_6}^2) + a_9\{w_6({w_7}^2 - {w_8}^2) - 2w_7w_8w_5\} + \widetilde{F}_8^C, \quad \text{(A.370)}$$

where $\widetilde{F}_i^C$ $(i = 1, \ldots, 8)$ is given in (A.332)–(A.339). Hence, the asymptotic form of the Jacobian matrix in (A.232) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\tilde{\phi}I_8 + a_6B_6 + a_7B_7 + a_8B_8 + a_9B_9 + B_C, \quad \text{(A.371)}$$

where B_C is given in (A.341) and

$$B_6 = \begin{bmatrix} B_1^6 & O \\ O & B_2^6 \end{bmatrix}, \quad B_7 = \begin{bmatrix} B_1^7 & O \\ O & B_2^7 \end{bmatrix}, \quad B_8 = \begin{bmatrix} B_1^8 & O \\ O & B_2^8 \end{bmatrix}, \quad B_9 = \begin{bmatrix} B_1^9 & O \\ O & B_2^9 \end{bmatrix},$$

$$B_1^6 = \begin{bmatrix} w_3 & w_4 & w_1 & w_2 \\ w_4 & -w_3 & -w_2 & w_1 \\ w_3 & -w_4 & w_1 & -w_2 \\ -w_4 & -w_3 & -w_2 & -w_1 \end{bmatrix}, \quad B_2^6 = \begin{bmatrix} w_7 & w_8 & w_5 & w_6 \\ w_8 & -w_7 & -w_6 & w_5 \\ w_7 & -w_8 & w_5 & -w_6 \\ -w_8 & -w_7 & -w_6 & -w_5 \end{bmatrix},$$

$$B_1^7 = 2\begin{bmatrix} 0 & 0 & w_3 & -w_4 \\ 0 & 0 & -w_4 & -w_3 \\ w_1 & -w_2 & 0 & 0 \\ w_2 & w_1 & 0 & 0 \end{bmatrix}, \quad B_2^7 = 2\begin{bmatrix} 0 & 0 & w_7 & -w_8 \\ 0 & 0 & -w_8 & -w_7 \\ w_5 & -w_6 & 0 & 0 \\ w_6 & w_5 & 0 & 0 \end{bmatrix},$$

$$B_1^8 = 3\begin{bmatrix} 0 & 0 & {w_3}^2 - {w_4}^2 & -2w_3w_4 \\ 0 & 0 & 2w_3w_4 & {w_3}^2 - {w_4}^2 \\ {w_1}^2 - {w_2}^2 & -2w_1w_2 & 0 & 0 \\ -2w_1w_2 & -{w_1}^2 + {w_2}^2 & 0 & 0 \end{bmatrix},$$

$$B_2^8 = 3\begin{bmatrix} 0 & 0 & {w_7}^2 - {w_8}^2 & -2w_7w_8 \\ 0 & 0 & 2w_7w_8 & {w_7}^2 - {w_8}^2 \\ {w_5}^2 - {w_6}^2 & -2w_5w_6 & 0 & 0 \\ -2w_5w_6 & -{w_5}^2 + {w_6}^2 & 0 & 0 \end{bmatrix},$$

$$B_1^9 = \begin{bmatrix} 2(w_1w_3 - w_2w_4) & 2(-w_1w_4 - w_2w_3) & {w_1}^2 - {w_2}^2 & -2w_1w_2 \\ 2(-w_1w_4 - w_2w_3) & 2(-w_1w_3 + w_2w_4) & -2w_1w_2 & -{w_1}^2 + {w_2}^2 \\ {w_3}^2 - {w_4}^2 & 2w_3w_4 & 2(w_1w_3 + w_2w_4) & 2(-w_1w_4 + w_2w_3) \\ -2w_3w_4 & {w_3}^2 - {w_4}^2 & 2(-w_1w_4 + w_2w_3) & 2(-w_1w_3 - w_2w_4) \end{bmatrix},$$

$$B_2^9 = \begin{bmatrix} 2(w_5w_7 - w_6w_8) & 2(-w_5w_8 - w_6w_7) & {w_5}^2 - {w_6}^2 & -2w_5w_6 \\ 2(-w_5w_8 - w_6w_7) & 2(-w_5w_7 + w_6w_8) & -2w_5w_6 & -{w_5}^2 + {w_6}^2 \\ {w_7}^2 - {w_8}^2 & 2w_7w_8 & 2(w_5w_7 + w_6w_8) & 2(-w_5w_8 + w_6w_7) \\ -2w_7w_8 & {w_7}^2 - {w_8}^2 & 2(-w_5w_8 + w_6w_7) & 2(-w_5w_7 - w_6w_8) \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\text{sqT}} = (w, 0, w, 0, 0, 0, 0, 0)$ into (A.363) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqT}} \approx -\frac{a_6 + a_7}{a_1} w.$$

Evaluating the Jacobian matrix (A.371) at $(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}}) \approx w \begin{bmatrix} C_{11} & O \\ O & C_{12} \end{bmatrix} \tag{A.372}$$

with

$$C_{11} = \begin{bmatrix} -a_7 & 0 & a_6 + 2a_7 & 0 \\ 0 & -2a_6 - a_7 & 0 & a_6 - 2a_7 \\ a_6 + 2a_7 & 0 & -a_7 & 0 \\ 0 & -a_6 + 2a_7 & 0 & -2a_6 - a_7 \end{bmatrix}, \quad C_{12} = -(a_6 + a_7)I_4. \tag{A.373}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqT}}, \widetilde{\phi}_{\text{sqT}})$ are given by

$$\lambda_1 \approx (a_6 + a_7)w, \tag{A.374}$$

$$\lambda_2 \approx -(a_6 + 3a_7)w, \tag{A.375}$$

$$\lambda_3, \lambda_4 \approx -(2a_6 + a_7)w \pm \mathrm{i}(a_6 - 2a_7)w, \tag{A.376}$$

$$\lambda_5 \approx -(a_6 + a_7)w \quad \text{(repeated 4 times)}. \tag{A.377}$$

Since the eigenvalues λ_1 and λ_5 have opposite signs, there is at least one positive eigenvalue. Thus, the bifurcating solution $\boldsymbol{w}_{\text{sqT}}$ is always unstable.

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (A.363) with (A.332) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_6 + a_7}{a_1} w.$$

Evaluating the Jacobian matrix (A.371) at $(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}}) \approx w \begin{bmatrix} C_{11} & O \\ O & C_{11} \end{bmatrix}, \tag{A.378}$$

where C_{11} is given in (A.373). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$ are given by

$$\begin{aligned} \lambda_1 &\approx (a_6 + a_7)w, \\ \lambda_2 &\approx -(3a_6 + a_7)w, \\ \lambda_3, \lambda_4 &\approx -\{2a_6 + a_7 \pm \mathrm{i}(a_6 - 2a_7)\}w \end{aligned}$$

and are all repeated twice. Assume that all eigenvalues have negative real parts. If $w < 0$, we have the following stability conditions:

$$a_6 + a_7 < 0, \tag{A.379}$$
$$3a_6 + a_7 > 0, \tag{A.380}$$
$$2a_6 + a_7 > 0. \tag{A.381}$$

If $w < 0$, we have the following stability conditions:

$$a_6 + a_7 > 0, \tag{A.382}$$
$$3a_6 + a_7 < 0, \tag{A.383}$$
$$2a_6 + a_7 < 0. \tag{A.384}$$

Thus, the stability of $\boldsymbol{w}_{\text{sqVM}}$ depends on the direction w of the bifurcating solution and the values of a_6 and a_7.

Remark A.5. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (5, 2, 1)$, we have the following statements:

- The solutions $\boldsymbol{w}_{\text{stripeI}}$ and $\boldsymbol{w}_{\text{stripeII}}$ do not exist. See Proposition A.22. In fact, $\hat{k}^2 + \hat{\ell} = 5$. This is divisible by $\hat{n} = 5$. Hence, the condition (A.282) is not satisfied.
- The solutions $\boldsymbol{w}_{\text{upside-downI}}$ and $\boldsymbol{w}_{\text{upside-downII}}$ do not exist. See Proposition A.24. In fact, $\gcd(\hat{k}^2 + \hat{\ell}, \hat{k}^2 - \hat{\ell}) = \gcd(5, 3) = 1$. This is divisible by $\gcd(\hat{n}, 2\hat{k}\hat{\ell}) = \gcd(5, 4) = 1$. Hence, the condition (A.286) is not satisfied.

□

Case 3: $(\hat{n}, \hat{k}, \hat{\ell}) = (8, 3, 1)$

For the case of $(\hat{n}, \hat{k}, \hat{\ell}) = (8, 3, 1)$, we have

$$\begin{aligned} &(0, 1, 0, 1, 0, 0, 1, 0), (0, 0, 1, 1, 0, 1, 0, 0), (0, 0, 0, 2, 1, 0, 0, 0), \\ &(0, 0, 0, 0, 2, 0, 0, 1), (0, 0, 0, 0, 0, 0, 0, 3) \in P \end{aligned}$$

as well as

$$(1,0,0,0,0,0,0,0), (2,0,0,0,1,0,0,0), (1,1,0,0,0,1,0,0),$$
$$(1,0,1,0,0,0,1,0), (1,0,0,1,0,0,0,1) \in P$$

in (A.290). Then, the asymptotic form of F_i $(i = 1,\dots,4)$ in (A.324) becomes

$$F_1 \approx a_1\widetilde{\phi}z_1 + a_{10}z_2z_4\overline{z}_3 + a_{11}z_3z_4\overline{z}_2 + a_{12}{z_4}^2\overline{z}_1 + a_{13}{\overline{z}_1}^2\overline{z}_4 + a_{14}{\overline{z}_4}^3 + F_1^C, \tag{A.385}$$
$$F_2 \approx a_1\widetilde{\phi}z_2 + a_{10}\overline{z}_1z_3z_4 + a_{11}\overline{z}_4z_3z_1 + a_{12}{z_3}^2\overline{z}_2 + a_{13}{\overline{z}_2}^2\overline{z}_3 + a_{14}{\overline{z}_3}^3 + F_2^C, \tag{A.386}$$
$$F_3 \approx a_1\widetilde{\phi}z_3 + a_{10}z_4z_2\overline{z}_1 + a_{11}z_1z_2\overline{z}_4 + a_{12}{z_2}^2\overline{z}_3 + a_{13}{\overline{z}_3}^2\overline{z}_2 + a_{14}{\overline{z}_2}^3 + F_3^C, \tag{A.387}$$
$$F_4 \approx a_1\widetilde{\phi}z_4 + a_{10}\overline{z}_3z_1z_2 + a_{11}\overline{z}_2z_1z_3 + a_{12}{z_1}^2\overline{z}_4 + a_{13}{\overline{z}_4}^2\overline{z}_1 + a_{14}{\overline{z}_1}^3 + F_4^C \tag{A.388}$$

with

$$a_{10} = A_{01010010}(0), \quad a_{11} = A_{00110100}(0), \quad a_{12} = A_{00021000}(0),$$
$$a_{13} = A_{00002001}(0), \quad a_{14} = A_{00000003}(0), \tag{A.389}$$

where F_i^C $(i = 1,\dots,4)$ is given in (A.325)–(A.328). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1,\dots,8)$ in (A.234)–(A.237) becomes

$$\begin{aligned}\widetilde{F}_1 \approx\ & a_1\widetilde{\phi}w_1 + a_{10}\{w_5(w_3w_7 - w_4w_8) + w_6(w_3w_8 + w_4w_7)\}\\ & + a_{11}\{w_3(w_5w_7 - w_6w_8) + w_4(w_5w_8 + w_6w_7)\}\\ & + a_{12}\{w_1({w_7}^2 - {w_8}^2) + 2w_2w_7w_8\} + a_{13}\{w_7({w_1}^2 - {w_2}^2) - 2w_8w_1w_2\}\\ & + a_{14}w_7({w_7}^2 - 3{w_8}^2) + \widetilde{F}_1^C,\end{aligned} \tag{A.390}$$

$$\begin{aligned}\widetilde{F}_2 \approx\ & a_1\widetilde{\phi}w_2 + a_{10}\{w_5(w_3w_8 + w_4w_7) - w_6(w_3w_7 - w_4w_8)\}\\ & + a_{11}\{w_3(w_5w_8 + w_6w_7) - w_4(w_5w_7 - w_6w_8)\}\\ & + a_{12}\{-w_2({w_7}^2 - {w_8}^2) + 2w_1w_7w_8\} + a_{13}\{-w_8({w_1}^2 - {w_2}^2) - 2w_7w_1w_2\}\\ & + a_{14}w_8(-3{w_7}^2 + {w_8}^2) + \widetilde{F}_2^C,\end{aligned} \tag{A.391}$$

$$\begin{aligned}\widetilde{F}_3 \approx\ & a_1\widetilde{\phi}w_3 + a_{10}\{w_1(w_5w_7 - w_6w_8) + w_2(w_5w_8 + w_6w_7)\}\\ & + a_{11}\{w_7(w_1w_5 - w_2w_6) + w_8(w_1w_6 + w_2w_5)\}\\ & + a_{12}\{w_3({w_5}^2 - {w_6}^2) + 2w_4w_5w_6\} + a_{13}\{w_5({w_3}^2 - {w_4}^2) - 2w_6w_3w_4\}\\ & + a_{14}w_5({w_5}^2 - 3{w_6}^2) + \widetilde{F}_3^C,\end{aligned} \tag{A.392}$$

$$\begin{aligned}\widetilde{F}_4 \approx\ & a_1\widetilde{\phi}w_4 + a_{10}\{w_1(w_5w_8 + w_6w_7) - w_2(w_5w_7 - w_6w_8)\}\\ & + a_{11}\{w_7(w_1w_6 + w_2w_5) - w_8(w_1w_5 - w_2w_6)\}\\ & + a_{12}\{-w_4({w_5}^2 - {w_6}^2) + 2w_3w_5w_6\} + a_{13}\{-w_6({w_3}^2 - {w_4}^2) - 2w_5w_3w_4\}\\ & + a_{14}w_6(-3{w_5}^2 + {w_6}^2) + \widetilde{F}_4^C,\end{aligned} \tag{A.393}$$

$$\begin{aligned}\widetilde{F}_5 \approx\ & a_1\widetilde{\phi}w_5 + a_{10}\{w_1(w_3w_7 - w_4w_8) + w_2(w_3w_8 + w_4w_7)\}\\ & + a_{11}\{w_7(w_1w_3 - w_2w_4) + w_8(w_1w_4 + w_2w_3)\}\end{aligned}$$

$$+ a_{12}\{w_5({w_3}^2 - {w_4}^2) + 2w_3w_4w_6\} + a_{13}\{w_3({w_5}^2 - {w_6}^2) - 2w_4w_5w_6\}$$
$$+ a_{14}w_3({w_3}^2 - 3{w_4}^2) + \widetilde{F}_5^C, \tag{A.394}$$
$$\widetilde{F}_6 \approx a_1\widetilde{\phi}w_6 + a_{10}\{w_1(w_3w_8 + w_4w_7) - w_2(w_3w_7 - w_4w_8)\}$$
$$+ a_{11}\{w_7(w_1w_4 + w_2w_3) - w_8(w_1w_3 - w_2w_4)\}$$
$$+ a_{12}\{-w_6({w_3}^2 - {w_4}^2) + 2w_3w_4w_5\} + a_{13}\{-w_4({w_5}^2 - {w_6}^2) - 2w_3w_5w_6\}$$
$$+ a_{14}w_4(-3{w_3}^2 + {w_4}^2) + \widetilde{F}_6^C, \tag{A.395}$$
$$\widetilde{F}_7 \approx a_1\widetilde{\phi}w_7 + a_{10}\{w_5(w_1w_3 - w_2w_4) + w_6(w_1w_4 + w_2w_3)\}$$
$$+ a_{11}\{w_3(w_1w_5 - w_2w_6) + w_4(w_1w_6 + w_2w_5)\}$$
$$+ a_{12}\{w_7({w_1}^2 - {w_2}^2) + 2w_8w_1w_2\} + a_{13}\{w_1({w_7}^2 - {w_8}^2) - 2w_2w_7w_8\}$$
$$+ a_{14}w_1({w_1}^2 - 3{w_2}^2) + \widetilde{F}_7^C, \tag{A.396}$$
$$\widetilde{F}_8 \approx a_1\widetilde{\phi}w_8 + a_{10}\{w_5(w_1w_4 + w_2w_3) - w_6(w_1w_3 - w_2w_4)\}$$
$$+ a_{11}\{w_3(w_1w_6 + w_2w_5) - w_4(w_1w_5 - w_2w_6)\}$$
$$+ a_{12}\{-w_8({w_1}^2 - {w_2}^2) + 2w_7w_1w_2\} + a_{13}\{-w_2({w_7}^2 - {w_8}^2) - 2w_1w_7w_8\}$$
$$+ a_{14}w_2(-3{w_1}^2 + {w_2}^2) + \widetilde{F}_8^C, \tag{A.397}$$

where $\widetilde{F}_i^C$ $(i = 1, \dots, 8)$ is given in (A.332)–(A.339). Hence, the asymptotic form of the Jacobian matrix in (A.232) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\tilde{\phi}I_8 + a_{10}B_{10} + a_{11}B_{11} + a_{12}B_{12} + a_{13}B_{13} + a_{14}B_{14} + B_C, \tag{A.398}$$

where B_C is given in (A.341) and

$$B_{10} = \begin{bmatrix} B_1^{10} & B_3^{10} \\ B_4^{10} & B_2^{10} \end{bmatrix}, \quad B_{11} = \begin{bmatrix} B_1^{11} & B_3^{11} \\ B_4^{11} & B_2^{11} \end{bmatrix}, \quad B_{12} = \begin{bmatrix} B_1^{12} & B_3^{12} \\ (B_3^{12})^\top & B_2^{12} \end{bmatrix}, \tag{A.399}$$

$$B_{13} = \begin{bmatrix} B_1^{13} & B_3^{13} \\ B_4^{13} & B_2^{13} \end{bmatrix}, \quad B_{14} = \begin{bmatrix} O & B_1^{14} \\ B_2^{14} & O \end{bmatrix}, \tag{A.400}$$

$$B_1^{10} = \begin{bmatrix} 0 & 0 & w_5w_7 + w_6w_8 & -w_5w_8 + w_6w_7 \\ 0 & 0 & w_5w_8 - w_6w_7 & w_5w_7 + w_6w_8 \\ w_5w_7 - w_6w_8 & w_5w_8 + w_6w_7 & 0 & 0 \\ w_5w_8 + w_6w_7 & -w_5w_7 + w_6w_8 & 0 & 0 \end{bmatrix},$$

$$B_2^{10} = \begin{bmatrix} 0 & 0 & w_1w_3 + w_2w_4 & -w_1w_4 + w_2w_3 \\ 0 & 0 & w_1w_4 - w_2w_3 & w_1w_3 + w_2w_4 \\ w_1w_3 - w_2w_4 & w_1w_4 + w_2w_3 & 0 & 0 \\ w_1w_4 + w_2w_3 & -w_1w_3 + w_2w_4 & 0 & 0 \end{bmatrix},$$

$$B_3^{10} = \begin{bmatrix} w_3w_7 - w_4w_8 & w_3w_8 + w_4w_7 & w_3w_5 + w_4w_6 & w_3w_6 - w_4w_5 \\ w_3w_8 + w_4w_7 & -w_3w_7 + w_4w_8 & -w_3w_6 + w_4w_5 & w_3w_5 + w_4w_6 \\ w_1w_7 + w_2w_8 & -w_1w_8 + w_2w_7 & w_1w_5 + w_2w_6 & -w_1w_6 + w_2w_5 \\ w_1w_8 - w_2w_7 & w_1w_7 + w_2w_8 & w_1w_6 - w_2w_5 & w_1w_5 + w_2w_6 \end{bmatrix},$$

$$B_4^{10} = \begin{bmatrix} w_3w_7 - w_4w_8 & w_3w_8 + w_4w_7 & w_1w_7 + w_2w_8 & -w_1w_8 + w_2w_7 \\ w_3w_8 + w_4w_7 & -w_3w_7 + w_4w_8 & w_1w_8 - w_2w_7 & w_1w_7 + w_2w_8 \\ w_3w_5 + w_4w_6 & w_3w_6 - w_4w_5 & w_1w_5 + w_2w_6 & w_1w_6 - w_2w_5 \\ -w_3w_6 + w_4w_5 & w_3w_5 + w_4w_6 & -w_1w_6 + w_2w_5 & w_1w_5 + w_2w_6 \end{bmatrix},$$

$$B_1^{11} = \begin{bmatrix} 0 & 0 & w_5w_7 - w_6w_8 & w_5w_8 + w_6w_7 \\ 0 & 0 & w_5w_8 + w_6w_7 & -w_5w_7 + w_6w_8 \\ w_5w_7 + w_6w_8 & w_5w_8 - w_6w_7 & 0 & 0 \\ -w_5w_8 + w_6w_7 & w_5w_7 + w_6w_8 & 0 & 0 \end{bmatrix},$$

$$B_2^{11} = \begin{bmatrix} 0 & 0 & w_1w_3 - w_2w_4 & w_1w_4 + w_2w_3 \\ 0 & 0 & w_1w_4 + w_2w_3 & -w_1w_3 + w_2w_4 \\ w_1w_3 + w_2w_4 & w_1w_4 - w_2w_3 & 0 & 0 \\ -w_1w_4 + w_2w_3 & w_1w_3 + w_2w_4 & 0 & 0 \end{bmatrix},$$

$$B_3^{11} = \begin{bmatrix} w_3w_7 + w_4w_8 & -w_3w_8 + w_4w_7 & w_3w_5 + w_4w_6 & -w_3w_6 + w_4w_5 \\ w_3w_8 - w_4w_7 & w_3w_7 + w_4w_8 & w_3w_6 - w_4w_5 & w_3w_5 + w_4w_6 \\ w_1w_7 + w_2w_8 & w_1w_8 - w_2w_7 & w_1w_5 - w_2w_6 & w_1w_6 + w_2w_5 \\ -w_1w_8 + w_2w_7 & w_1w_7 + w_2w_8 & w_1w_6 + w_2w_5 & -w_1w_5 + w_2w_6 \end{bmatrix},$$

$$B_4^{11} = \begin{bmatrix} w_3w_7 + w_4w_8 & w_3w_8 - w_4w_7 & w_1w_7 + w_2w_8 & w_1w_8 - w_2w_7 \\ -w_3w_8 + w_4w_7 & w_3w_7 + w_4w_8 & -w_1w_8 + w_2w_7 & w_1w_7 + w_2w_8 \\ w_3w_5 + w_4w_6 & -w_3w_6 + w_4w_5 & w_1w_5 - w_2w_6 & w_1w_6 + w_2w_5 \\ w_3w_6 - w_4w_5 & w_3w_5 + w_4w_6 & w_1w_6 + w_2w_5 & -w_1w_5 + w_2w_6 \end{bmatrix},$$

$$B_1^{12} = \begin{bmatrix} {w_7}^2 - {w_8}^2 & 2w_7w_8 & 0 & 0 \\ 2w_7w_8 & -{w_7}^2 + {w_8}^2 & 0 & 0 \\ 0 & 0 & {w_5}^2 - {w_6}^2 & 2w_5w_6 \\ 0 & 0 & 2w_5w_6 & -{w_5}^2 + {w_6}^2 \end{bmatrix},$$

$$B_2^{12} = \begin{bmatrix} {w_3}^2 - {w_4}^2 & 2w_3w_4 & 0 & 0 \\ 2w_3w_4 & -{w_3}^2 + {w_4}^2 & 0 & 0 \\ 0 & 0 & {w_1}^2 - {w_2}^2 & 2w_1w_2 \\ 0 & 0 & 2w_1w_2 & -{w_1}^2 + {w_2}^2 \end{bmatrix},$$

$$B_3^{12} = 2\begin{bmatrix} 0 & 0 & w_1w_7 + w_2w_8 & -w_1w_8 + w_2w_7 \\ 0 & 0 & w_1w_8 - w_2w_7 & w_1w_7 + w_2w_8 \\ w_3w_5 + w_4w_6 & -w_3w_6 + w_4w_5 & 0 & 0 \\ w_3w_6 - w_4w_5 & w_3w_5 + w_4w_6 & 0 & 0 \end{bmatrix},$$

$$B_1^{13} = 2\begin{bmatrix} w_1w_7 - w_2w_8 & -w_1w_8 - w_2w_7 & 0 & 0 \\ -w_1w_8 - w_2w_7 & -w_1w_7 + w_2w_8 & 0 & 0 \\ 0 & 0 & w_3w_5 - w_4w_6 & -w_3w_6 - w_4w_5 \\ 0 & 0 & -w_3w_6 - w_4w_5 & -w_3w_5 + w_4w_6 \end{bmatrix},$$

$$B_2^{13} = 2\begin{bmatrix} w_3w_5 - w_4w_6 & -w_3w_6 - w_4w_5 & 0 & 0 \\ -w_3w_6 - w_4w_5 & -w_3w_5 + w_4w_6 & 0 & 0 \\ 0 & 0 & w_1w_7 - w_2w_8 & -w_1w_8 - w_2w_7 \\ 0 & 0 & -w_1w_8 - w_2w_7 & -w_1w_7 + w_2w_8 \end{bmatrix},$$

$$B_3^{13} = \begin{bmatrix} 0 & 0 & {w_1}^2 - {w_2}^2 & -2w_1w_2 \\ 0 & 0 & -2w_1w_2 & -{w_1}^2 + {w_2}^2 \\ {w_3}^2 - {w_4}^2 & -2w_3w_4 & 0 & 0 \\ -2w_3w_4 & -{w_3}^2 + {w_4}^2 & 0 & 0 \end{bmatrix},$$

$$B_4^{13} = \begin{bmatrix} 0 & 0 & {w_5}^2 - {w_6}^2 & -2w_5w_6 \\ 0 & 0 & -2w_5w_6 & -{w_5}^2 + {w_6}^2 \\ {w_7}^2 - {w_8}^2 & -2w_7w_8 & 0 & 0 \\ -2w_7w_8 & -{w_7}^2 + {w_8}^2 & 0 & 0 \end{bmatrix},$$

$$B_1^{14} = 3\begin{bmatrix} 0 & 0 & {w_7}^2 - {w_8}^2 & -2w_7w_8 \\ 0 & 0 & -2w_7w_8 & -{w_7}^2 + {w_8}^2 \\ {w_5}^2 - {w_6}^2 & -2w_5w_6 & 0 & 0 \\ -2w_5w_6 & -{w_5}^2 + {w_6}^2 & 0 & 0 \end{bmatrix},$$

$$B_2^{14} = 3\begin{bmatrix} 0 & 0 & {w_3}^2 - {w_4}^2 & -2w_3w_4 \\ 0 & 0 & -2w_3w_4 & -{w_3}^2 + {w_4}^2 \\ {w_1}^2 - {w_2}^2 & -2w_1w_2 & 0 & 0 \\ -2w_1w_2 & -{w_1}^2 + {w_2}^2 & 0 & 0 \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (A.390) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14}}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.398) at $(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}}) \approx w^2 \begin{bmatrix} C_{12} & C_{13} \\ C_{13} & C_{12} \end{bmatrix} + \widetilde{J_C}^{\text{sqVM}}, \tag{A.401}$$

where $\widetilde{J_C}^{\text{sqVM}}$ is given in (A.355) and

$$C_{12} = \begin{bmatrix} c_1 & 0 & c_3 & 0 \\ 0 & c_2 & 0 & c_4 \\ c_3 & 0 & c_1 & 0 \\ 0 & -c_4 & 0 & c_2 \end{bmatrix}, \quad C_{13} = \begin{bmatrix} c_3 & 0 & c_5 & 0 \\ 0 & -c_4 & 0 & c_6 \\ c_5 & 0 & c_3 & 0 \\ 0 & c_6 & 0 & c_4 \end{bmatrix},$$

$$c_1 = -a_{10} - a_{11} + a_{13} - a_{14}, \quad c_2 = -a_{10} - a_{11} - 2a_{12} - 3a_{13} - a_{14}, \quad c_3 = a_{10} + a_{11},$$
$$c_4 = a_{10} - a_{11}, \quad c_5 = a_{10} + a_{11} + 2a_{12} + a_{13} + 3a_{14}, \quad c_6 = a_{10} + a_{11} + 2a_{12} - a_{13} - 3a_{14}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$ are given by

$$\begin{aligned} \lambda_1, \lambda_2 &\approx \{(c_1 + c_3) \pm (c_5 + c_6)\}w^2, \\ \lambda_3, \lambda_4 &\approx \{(c_1 - c_3) \pm (c_5 - c_6)\}w^2, \\ \lambda_5, \lambda_6 &\approx (c_2 \pm c_7)w^2 \quad \text{(repeated twice)}, \end{aligned}$$

which are rewritten as

$$\begin{aligned} \lambda_1 &\approx 2(a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14})w^2, \\ \lambda_2 &\approx 2(a_2 + a_3 - a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14})w^2, \\ \lambda_3 &\approx 2(a_2 - a_3 + a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14})w^2, \\ \lambda_4 &\approx 2(a_2 - a_3 - a_4 + a_5 - a_{10} - a_{11} + a_{12} + a_{13} + a_{14})w^2, \\ \lambda_5 &\approx -2(a_{10} + a_{11} + 2a_{12} + a_{13} - a_{14})w^2 \quad \text{(repeated twice)}, \\ \lambda_6 &\approx -4(a_{13} + a_{14})w^2 \quad \text{(repeated twice)}. \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions:

$$a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} < 0, \tag{A.402}$$
$$a_2 + a_3 - a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14} < 0, \tag{A.403}$$
$$a_2 - a_3 + a_4 - a_5 - a_{10} - a_{11} - a_{12} - 2a_{14} < 0, \tag{A.404}$$
$$a_2 - a_3 - a_4 + a_5 - a_{10} - a_{11} + a_{12} + a_{13} + a_{14} < 0, \tag{A.405}$$
$$a_{10} + a_{11} + 2a_{12} + a_{13} - a_{14} > 0, \tag{A.406}$$
$$a_{13} + a_{14} > 0. \tag{A.407}$$

Thus, the stability of $\boldsymbol{w}_{\text{sqVM}}$ depends on the values of $a_2, \ldots, a_5$ and $a_{10}, \ldots, a_{14}$.

Remark A.6. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (8, 3, 1)$, we have the following statements:

- The solutions $\boldsymbol{w}_{\text{stripeI}}$ and $\boldsymbol{w}_{\text{stripeII}}$ do not exist. See Proposition A.22. In fact, $\hat{k}^2 - \hat{\ell} = 8$. This is divisible by $\hat{n} = 8$. Hence, the condition (A.282) is not satisfied.
- The solutions $\boldsymbol{w}_{\text{upside-downI}}$ and $\boldsymbol{w}_{\text{upside-downII}}$ do not exist. See Proposition A.24. In fact, $\gcd(\hat{k}^2 + \hat{\ell}, \hat{k}^2 - \hat{\ell}) = 2\gcd(10, 8) = 2$. This is divisible by $\gcd(\hat{n}, 2\hat{k}\hat{\ell}) = \gcd(8, 6) = 2$. Hence, the condition (A.286) is not satisfied.
- The solution $\boldsymbol{w}_{\text{sqT}}$ does not exist. See Proposition 3.32. This case corresponds to the case $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$. In fact, $2\gcd(\hat{k}, \hat{\ell}) = 2\gcd(3, 1) = 2$. This is divisible by $\gcd(\hat{k}^2 + \hat{\ell}^2, \hat{n}) = \gcd(10, 8) = 2$. Hence, $\overline{\textbf{GCD-div}}$ in (3.194) is not satisfied.

□

Case 4: $(\hat{n}, \hat{k}, \hat{\ell}) = (10, 3, 1)$

For the case of $(\hat{n}, \hat{k}, \hat{\ell}) = (10, 3, 1)$, we have

$$(0, 1, 0, 0, 2, 0, 0, 0), (0, 0, 0, 0, 0, 3, 0, 0) \in P$$

as well as

$$(1, 0, 0, 0, 0, 0, 0, 0), (2, 0, 0, 0, 1, 0, 0, 0), (1, 1, 0, 0, 0, 1, 0, 0),$$
$$(1, 0, 1, 0, 0, 0, 1, 0), (1, 0, 0, 1, 0, 0, 0, 1) \in P$$

in (A.290). Then, the asymptotic form of F_i $(i = 1, \dots, 4)$ in (A.324) becomes

$$F_1 \approx a_1\widetilde{\phi}z_1 + a_{15}z_2\overline{z}_1{}^2 + a_{16}\overline{z}_2{}^3 + F_1^C, \tag{A.408}$$
$$F_2 \approx a_1\widetilde{\phi}z_2 + a_{15}\overline{z}_1\overline{z}_2{}^2 + a_{16}z_1{}^3 + F_2^C, \tag{A.409}$$
$$F_3 \approx a_1\widetilde{\phi}z_3 + a_{15}z_4\overline{z}_3{}^2 + a_{16}\overline{z}_4{}^3 + F_3^C, \tag{A.410}$$
$$F_4 \approx a_1\widetilde{\phi}z_4 + a_{15}\overline{z}_3\overline{z}_4{}^2 + a_{16}z_3{}^3 + F_4^C \tag{A.411}$$

with

$$a_{15} = A_{01002000}(0), \quad a_{16} = A_{00000300}(0),$$

where F_i^C $(i = 1, \dots, 4)$ is given in (A.325)–(A.328). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \dots, 8)$ in (A.234)–(A.237) becomes

$$\widetilde{F}_1 \approx a_1\widetilde{\phi}w_1 + a_{15}\{w_3(w_1{}^2 - w_2{}^2) + 2w_4w_1w_2\} + a_{16}w_3(w_3{}^2 - 3w_4{}^2) + \widetilde{F}_1^C, \tag{A.412}$$
$$\widetilde{F}_2 \approx a_1\widetilde{\phi}w_2 + a_{15}\{w_4(w_1{}^2 - w_2{}^2) - 2w_3w_1w_2\} + a_{16}w_4(-3w_3{}^2 + w_4{}^2) + \widetilde{F}_2^C, \tag{A.413}$$
$$\widetilde{F}_3 \approx a_1\widetilde{\phi}w_3 + a_{15}\{w_1(w_3{}^2 - w_4{}^2) - 2w_2w_3w_4\} + a_{16}w_1(w_1{}^2 - 3w_2{}^2) + \widetilde{F}_3^C, \tag{A.414}$$
$$\widetilde{F}_4 \approx a_1\widetilde{\phi}w_4 + a_{15}\{-w_2(w_3{}^2 - w_4{}^2) - 2w_1w_3w_4\} + a_{16}w_2(3w_1{}^2 - w_2{}^2) + \widetilde{F}_4^C, \tag{A.415}$$
$$\widetilde{F}_5 \approx a_1\widetilde{\phi}w_5 + a_{15}\{w_7(w_5{}^2 - w_6{}^2) + 2w_8w_5w_6\} + a_{16}w_7(w_7{}^2 - 3w_8{}^2) + \widetilde{F}_5^C, \tag{A.416}$$

$$\widetilde{F}_6 \approx a_1\widetilde{\phi}w_6 + a_{15}\{w_8({w_5}^2 - {w_6}^2) - 2w_7w_5w_6\} + a_{16}w_8(-3{w_7}^2 + {w_8}^2) + \widetilde{F}_6^C, \tag{A.417}$$

$$\widetilde{F}_7 \approx a_1\widetilde{\phi}w_7 + a_{15}\{w_5({w_7}^2 - {w_8}^2) - 2w_6w_7w_8\} + a_{16}w_5({w_5}^2 - 3{w_6}^2) + \widetilde{F}_7^C, \tag{A.418}$$

$$\widetilde{F}_8 \approx a_1\widetilde{\phi}w_8 + a_{15}\{-w_6({w_7}^2 - {w_8}^2) - 2w_5w_7w_8\} + a_{16}w_6(3{w_5}^2 - {w_6}^2) + \widetilde{F}_8^C, \tag{A.419}$$

where $\widetilde{F}_i^C$ $(i = 1, \dots, 8)$ is given in (A.332)–(A.339). Hence, the asymptotic form of the Jacobian matrix in (A.232) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\tilde{\phi}I_8 + a_{15}B_{15} + a_{16}B_{16} + B_C, \tag{A.420}$$

where B_C is given in (A.341) and

$$B_{15} = \begin{bmatrix} B_1^{15} & O \\ O & B_2^{15} \end{bmatrix}, \quad B_{16} = \begin{bmatrix} B_1^{16} & O \\ O & B_2^{16} \end{bmatrix},$$

$$B_1^{15} = \begin{bmatrix} 2(w_1w_3 + w_2w_4) & 2(w_1w_4 - w_2w_3) & {w_1}^2 - {w_2}^2 & 2w_1w_2 \\ 2(w_1w_4 - w_2w_3) & 2(-w_1w_3 - w_2w_4) & -2w_1w_2 & {w_1}^2 - {w_2}^2 \\ {w_3}^2 - {w_4}^2 & -2w_3w_4 & 2(w_1w_3 - w_2w_4) & 2(-w_1w_4 - w_2w_3) \\ -2w_3w_4 & -{w_3}^2 + {w_4}^2 & 2(-w_1w_4 - w_2w_3) & 2(-w_1w_3 + w_2w_4) \end{bmatrix},$$

$$B_2^{15} = \begin{bmatrix} 2(w_5w_7 + w_6w_8) & 2(w_5w_8 - w_6w_7) & {w_5}^2 - {w_6}^2 & 2w_5w_6 \\ 2(w_5w_8 - w_6w_7) & 2(-w_5w_7 - w_6w_8) & -2w_5w_6 & {w_5}^2 - {w_6}^2 \\ {w_7}^2 - {w_8}^2 & -2w_7w_8 & 2(w_5w_7 - w_6w_8) & 2(-w_5w_8 - w_6w_7) \\ -2w_7w_8 & -{w_7}^2 + {w_8}^2 & 2(-w_5w_8 - w_6w_7) & 2(-w_5w_7 + w_6w_8) \end{bmatrix},$$

$$B_1^{16} = 3\begin{bmatrix} 0 & 0 & {w_3}^2 - {w_4}^2 & -2w_3w_4 \\ 0 & 0 & -2w_3w_4 & -{w_3}^2 + {w_4}^2 \\ {w_1}^2 - {w_2}^2 & -2w_1w_2 & 0 & 0 \\ 2w_1w_2 & {w_1}^2 - {w_2}^2 & 0 & 0 \end{bmatrix},$$

$$B_2^{16} = 3\begin{bmatrix} 0 & 0 & {w_7}^2 - {w_8}^2 & -2w_7w_8 \\ 0 & 0 & -2w_7w_8 & -{w_7}^2 + {w_8}^2 \\ {w_5}^2 - {w_6}^2 & -2w_5w_6 & 0 & 0 \\ 2w_5w_6 & {w_5}^2 - {w_6}^2 & 0 & 0 \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\mathrm{sqT}} = (w, 0, w, 0, 0, 0, 0, 0)$ into (A.412) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{sqT}} \approx -\frac{a_2 + a_3 + a_{15} + a_{16}}{a_1}w^2.$$

Evaluating the Jacobian matrix (A.420) at $(\boldsymbol{w}_{\mathrm{sqT}}, \widetilde{\phi}_{\mathrm{sqT}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{sqT}}, \widetilde{\phi}_{\mathrm{sqT}}) \approx w^2 \begin{bmatrix} C_{14} & O \\ O & C_{15} \end{bmatrix} + \widetilde{J_C}^{\mathrm{sqT}}, \tag{A.421}$$

where $\widetilde{J_C}^{\mathrm{sqT}}$ is given in (A.342) and

$$C_{14} = \begin{bmatrix} a_{15} - a_{16} & 0 & a_{15} + 3a_{16} & 0 \\ 0 & -3a_{15} - a_{16} & 0 & a_{15} - 3a_{16} \\ a_{15} + 3a_{16} & 0 & a_{15} - a_{16} & 0 \\ 0 & -a_{15} + 3a_{16} & 0 & -3a_{15} - a_{16} \end{bmatrix}, \tag{A.422}$$

$$C_{15} = -(a_{15} + a_{16})I_4. \tag{A.423}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{sqT}}, \widetilde{\phi}_{\mathrm{sqT}})$ are given by

$$\begin{aligned} &\lambda_1 \approx 2(a_2 + a_3 + a_{15} + a_{16})w^2, \\ &\lambda_2 \approx 2(a_2 - a_3 - 2a_{16})w^2, \\ &\lambda_3, \lambda_4 \approx -\{3a_{15} + a_{16} \pm \mathrm{i}(a_{15} - 3a_{16})\}w^2, \\ &\lambda_5 \approx -(a_2 + a_3 - a_4 - a_5 + a_{15} + a_{16})w^2 \quad \text{(repeated 4 times)}. \end{aligned}$$

Assuming that all eigenvalues have negative real parts, we have the following stability conditions:

$$a_2 + a_3 + a_{15} + a_{16} < 0, \tag{A.424}$$
$$a_2 - a_3 - 2a_{16} < 0, \tag{A.425}$$
$$3a_{15} + a_{16} > 0, \tag{A.426}$$
$$a_2 + a_3 - a_4 - a_5 + a_{15} + a_{16} < 0. \tag{A.427}$$

Thus, the stability of $\boldsymbol{w}_{\mathrm{sqT}}$ depends on the values of $a_2, \ldots, a_5, a_{15}$ and a_{16}.

Substituting $\boldsymbol{w}_{\mathrm{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (A.412) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\mathrm{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5 + a_{15} + a_{16}}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.420) at $(\boldsymbol{w}_{\mathrm{sqVM}}, \widetilde{\phi}_{\mathrm{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{sqVM}}, \widetilde{\phi}_{\mathrm{sqVM}}) \approx w^2 \begin{bmatrix} C_{14} & O \\ O & C_{14} \end{bmatrix} + \widetilde{J_C}^{\mathrm{sqVM}}, \tag{A.428}$$

where C_{14} is given in (A.422), and $\widetilde{J_C}^{\mathrm{sqVM}}$ is given in (A.355). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{sqVM}}, \widetilde{\phi}_{\mathrm{sqVM}})$ are given by

$$\begin{aligned} &\lambda_1 \approx 2(a_2 + a_3 + a_4 + a_5 + a_{15} + a_{16})w^2, \\ &\lambda_2 \approx 2(a_2 + a_3 - a_4 - a_5 - a_{15} - a_{16})w^2, \\ &\lambda_3 \approx 2(a_2 - a_3 + a_4 - a_5 - 2a_{16})w^2, \\ &\lambda_4 \approx 2(a_2 - a_3 - a_4 + a_5 - 2a_{16})w^2, \\ &\lambda_5, \lambda_6 \approx -\{3a_{15} + a_{16} \pm \mathrm{i}(a_{15} - 3a_{16})\}w^2 \quad \text{(repeated twice)}. \end{aligned}$$

Assuming that all eigenvalues have negative real parts, we have the following stability conditions:

$$a_2 + a_3 + a_4 + a_5 + a_{15} + a_{16} < 0, \tag{A.429}$$
$$a_2 + a_3 - a_4 - a_5 - a_{15} - a_{16} < 0, \tag{A.430}$$
$$a_2 - a_3 + a_4 - a_5 - 2a_{16} < 0, \tag{A.431}$$
$$a_2 - a_3 - a_4 + a_5 - 2a_{16} < 0, \tag{A.432}$$
$$3a_{15} + a_{16} > 0. \tag{A.433}$$

Thus, the stability of $\boldsymbol{w}_{\mathrm{sqVM}}$ depends on the values of $a_2, \dots, a_5, a_{15}$ and a_{16}.

Remark A.7. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (10, 3, 1)$, we have the following statements:

- The solutions $\boldsymbol{w}_{\mathrm{stripeI}}$ and $\boldsymbol{w}_{\mathrm{stripeII}}$ do not exist. See Proposition A.22. In fact, $\hat{k}^2 + \hat{\ell} = 10$. This is divisible by $\hat{n} = 10$. Hence, the condition (A.282) is not satisfied.
- The solutions $\boldsymbol{w}_{\mathrm{upside-downI}}$ and $\boldsymbol{w}_{\mathrm{upside-downII}}$ do not exist. See Proposition A.24. In fact, $\gcd(\hat{k}^2 + \hat{\ell}, \hat{k}^2 - \hat{\ell}) = 2\gcd(10, 8) = 2$. This is divisible by $\gcd(\hat{n}, 2\hat{k}\hat{\ell}) = \gcd(10, 6) = 2$. Hence, the condition (A.286) is not satisfied.

□

Case 5: $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell})$

For the case of $(\hat{n}, \hat{k}, \hat{\ell})$ with $\hat{n} = 4\hat{k}$, we have

$$(0, 0, 0, 0, 1, 0, 2, 0) \in P$$

as well as

$$(1, 0, 0, 0, 0, 0, 0, 0), (2, 0, 0, 0, 1, 0, 0, 0), (1, 1, 0, 0, 0, 1, 0, 0),$$
$$(1, 0, 1, 0, 0, 0, 1, 0), (1, 0, 0, 1, 0, 0, 0, 1) \in P$$

in (A.290). Then, the asymptotic form of F_i $(i = 1, \dots, 4)$ in (A.324) becomes

$$F_1 \approx a_1\widetilde{\phi}z_1 + a_{17}\overline{z}_1\overline{z}_3{}^2 + F_1^C, \tag{A.434}$$
$$F_2 \approx a_1\widetilde{\phi}z_2 + a_{17}\overline{z}_2 z_4{}^2 + F_2^C, \tag{A.435}$$
$$F_3 \approx a_1\widetilde{\phi}z_3 + a_{17}\overline{z}_3\overline{z}_1{}^2 + F_3^C, \tag{A.436}$$
$$F_4 \approx a_1\widetilde{\phi}z_4 + a_{17}\overline{z}_4 z_2{}^2 + F_4^C \tag{A.437}$$

with

$$a_{17} = A_{00001020}(0),$$

where F_i^C $(i = 1, \dots, 4)$ is given in (A.325)–(A.328). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \dots, 8)$ in (A.234)–(A.237) becomes

$$\widetilde{F}_1 \approx a_1\widetilde{\phi}w_1 + a_{17}\{\ \ w_1(w_5{}^2 - w_6{}^2) - 2w_2w_5w_6)\} + \widetilde{F}_1^C, \tag{A.438}$$

$$\widetilde{F}_2 \approx a_1\widetilde{\phi}w_2 + a_{17}\{-w_2({w_5}^2 - {w_6}^2) - 2w_1w_5w_6\} + \widetilde{F}_2^C, \tag{A.439}$$
$$\widetilde{F}_3 \approx a_1\widetilde{\phi}w_3 + a_{17}\{\ \ w_3({w_7}^2 - {w_8}^2) + 2w_4w_7w_8\} + \widetilde{F}_3^C, \tag{A.440}$$
$$\widetilde{F}_4 \approx a_1\widetilde{\phi}w_4 + a_{17}\{-w_4({w_7}^2 - {w_8}^2) + 2w_3w_7w_8\} + \widetilde{F}_4^C, \tag{A.441}$$
$$\widetilde{F}_5 \approx a_1\widetilde{\phi}w_5 + a_{17}\{\ \ w_5({w_1}^2 - {w_2}^2) - 2w_6w_1w_2\} + \widetilde{F}_5^C, \tag{A.442}$$
$$\widetilde{F}_6 \approx a_1\widetilde{\phi}w_6 + a_{17}\{-w_6({w_1}^2 - {w_2}^2) - 2w_5w_1w_2\} + \widetilde{F}_6^C, \tag{A.443}$$
$$\widetilde{F}_7 \approx a_1\widetilde{\phi}w_7 + a_{17}\{\ \ w_7({w_3}^2 - {w_4}^2) + 2w_8w_3w_4\} + \widetilde{F}_7^C, \tag{A.444}$$
$$\widetilde{F}_8 \approx a_1\widetilde{\phi}w_8 + a_{17}\{-w_8({w_3}^2 - {w_4}^2) + 2w_7w_3w_4\} + \widetilde{F}_8^C, \tag{A.445}$$

where $\widetilde{F}_i^C$ $(i = 1, \dots, 8)$ is given in (A.332)–(A.339). Hence, the asymptotic form of the Jacobian matrix in (A.232) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\tilde{\phi}I_8 + a_{17}B_{17} + B_C, \tag{A.446}$$

where B_C is given in (A.341) and

$$B_{17} = \begin{bmatrix} B_1^{17} & B_3^{17} \\ (B_3^{17})^\top & B_2^{17} \end{bmatrix},$$

$$B_1^{17} = \begin{bmatrix} {w_5}^2 - {w_6}^2 & -2w_5w_6 & 0 & 0 \\ -2w_5w_6 & -{w_5}^2 + {w_6}^2 & 0 & 0 \\ 0 & 0 & {w_7}^2 - {w_8}^2 & 2w_7w_8 \\ 0 & 0 & 2w_7w_8 & -{w_7}^2 + {w_8}^2 \end{bmatrix},$$

$$B_2^{17} = \begin{bmatrix} {w_1}^2 - {w_2}^2 & -2w_1w_2 & 0 & 0 \\ -2w_1w_2 & -{w_1}^2 + {w_2}^2 & 0 & 0 \\ 0 & 0 & {w_3}^2 - {w_4}^2 & 2w_3w_4 \\ 0 & 0 & 2w_3w_4 & -{w_3}^2 + {w_4}^2 \end{bmatrix},$$

$$B_3^{17} = 2\begin{bmatrix} w_1w_5 - w_2w_6 & -w_1w_6 - w_2w_5 & 0 & 0 \\ -w_1w_6 - w_2w_5 & -w_1w_5 + w_2w_6 & 0 & 0 \\ 0 & 0 & w_3w_7 + w_4w_8 & -w_3w_8 + w_4w_7 \\ 0 & 0 & w_3w_8 - w_4w_7 & w_3w_7 + w_4w_8 \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\text{stripeI}} = (w, 0, 0, 0, 0, 0, 0, 0)$ into (A.438) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeI}} \approx -\frac{a_2}{a_1}w^2.$$

Evaluating the Jacobian matrix (A.446) at $(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}}) \approx w^2 \begin{bmatrix} O & O \\ O & C_{16} \end{bmatrix} + \widetilde{J_C}^{\text{stripeI}}, \tag{A.447}$$

where $\widetilde{J_C}^{\text{stripeI}}$ is given in (A.342) and

$$C_{16} = \begin{bmatrix} a_{17} & 0 & 0 & 0 \\ 0 & -a_{17} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$ are given by

$$\begin{aligned}
&\lambda_1 \approx 2a_2 w^2, \\
&\lambda_2 \approx O(w^3), \\
&\lambda_3, \lambda_4 \approx -(a_2 - a_4 \pm a_{17})w^2, \\
&\lambda_5 \approx -(a_2 - a_3)w^2 \quad \text{(repeated twice)}, \\
&\lambda_6 \approx -(a_2 - a_5)w^2 \quad \text{(repeated twice)}.
\end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 < 0, \quad a_2 - a_4 \pm a_{17} > 0, \quad a_2 - a_3 > 0, \quad a_2 - a_5 > 0.$$

These conditions are equivalent to

$$\max(a_3, a_4 + |a_{17}|, a_5) < a_2 < 0. \tag{A.448}$$

Thus, the stability of $\boldsymbol{w}_{\text{stripeI}}$ depends on the values of $a_2, \ldots, a_5$ and a_{17}.

Substituting $\boldsymbol{w}_{\text{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$ into (A.438) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeII}} \approx -\frac{a_2}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.446) at $(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}}) \approx w^2 \begin{bmatrix} O & O \\ O & -C_{16} \end{bmatrix} + \widetilde{J_C}^{\text{stripeII}}, \tag{A.449}$$

where C_{16} is given in (A.4.5), and $\widetilde{J_C}^{\text{stripeII}}$ is given in (A.345). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$.

Substituting $\boldsymbol{w}_{\text{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$ into (A.331) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downI}} \approx -\frac{a_2 + a_4 + a_{17}}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.340) at $(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}}) \approx w^2 \begin{bmatrix} C_{17} & C_{18} \\ C_{18} & C_{17} \end{bmatrix} + \widetilde{J_C}^{\text{upside-downI}} \tag{A.450}$$

with

$$C_{17} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2a_{17} & 0 & 0 \\ 0 & 0 & -a_{17} & 0 \\ 0 & 0 & 0 & -a_{17} \end{bmatrix}, \quad C_{18} = \begin{bmatrix} 2a_{17} & 0 & 0 & 0 \\ 0 & -2a_{17} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{A.451}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$ are given by

$$\begin{aligned} &\lambda_1, \lambda_2 \approx 2a_2 \pm (a_4 + a_{17})w^2, \\ &\lambda_3 \approx -4a_{17}w^2, \\ &\lambda_4 \approx O(w^3), \\ &\lambda_5 \approx -(a_2 - a_3 + a_4 - a_5 + a_{17})w^2 \quad \text{(repeated 4 times).} \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 < -|a_4 + a_{17}|, \quad a_{17} > 0, \quad a_2 - a_3 + a_4 - a_5 + a_{17} > 0.$$

These conditions are equivalent to

$$a_3 - a_4 + a_5 - a_{17} < a_2 < -|a_4 + a_{17}| \tag{A.452}$$

$$a_4 > 0. \tag{A.453}$$

Thus, the stability of $\boldsymbol{w}_{\text{upside-downI}}$ is conditional and depends on the values of $a_2, \ldots, a_5$ and a_{17}.

Substituting $\boldsymbol{w}_{\text{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$ into (A.331) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downII}} \approx -\frac{a_2 + a_4 + a_{17}}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.340) at $(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}}) = \widetilde{J_C}^{\text{upside-downII}} \approx w^2 \begin{bmatrix} C_{19} & -C_{18} \\ -C_{18} & C_{19} \end{bmatrix} \tag{A.454}$$

with

$$C_{19} = \begin{bmatrix} -2a_{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{17} & 0 \\ 0 & 0 & 0 & -a_{17} \end{bmatrix}, \tag{A.455}$$

where C_{18} is given in (A.451). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{upside-downII}}$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$.

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (A.438) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5 + a_{17}}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.446) at $(\boldsymbol{w}_{\mathrm{sqVM}}, \widetilde{\phi}_{\mathrm{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{sqVM}}, \widetilde{\phi}_{\mathrm{sqVM}}) \approx w^2 \begin{bmatrix} C_{20} & C_{21} \\ C_{21} & C_{20} \end{bmatrix} + \widetilde{J_C}^{\mathrm{sqVM}}, \tag{A.456}$$

where $\widetilde{J_C}^{\mathrm{sqVM}}$ is given in (A.355) and

$$C_{20} = 2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -a_{17} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{17} \end{bmatrix}, \quad C_{21} = 2\begin{bmatrix} a_{17} & 0 & 0 & 0 \\ 0 & -a_{17} & 0 & 0 \\ 0 & 0 & a_{17} & 0 \\ 0 & 0 & 0 & a_{17} \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{sqVM}}, \widetilde{\phi}_{\mathrm{sqVM}})$ are given by

$$\begin{aligned}
\lambda_1 &\approx 2(a_2 + a_3 + a_4 + a_5 + a_{17})w^2, \\
\lambda_2 &\approx 2(a_2 + a_3 - a_4 - a_5 - a_{17})w^2, \\
\lambda_3 &\approx 2(a_2 - a_3 + a_4 - a_5 + a_{17})w^2, \\
\lambda_4 &\approx 2(a_2 - a_3 - a_4 + a_5 - a_{17})w^2, \\
\lambda_5 &\approx -4a_{17}w^2, \quad \text{(repeated twice)} \\
\lambda_6 &\approx O(w^3) \quad \text{(repeated twice)}.
\end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 + a_3 + a_4 + a_5 + a_{17} < 0, \tag{A.457}$$
$$a_2 + a_3 - a_4 - a_5 - a_{17} < 0, \tag{A.458}$$
$$a_2 - a_3 + a_4 - a_5 + a_{17} < 0, \tag{A.459}$$
$$a_2 - a_3 - a_4 + a_5 - a_{17} < 0, \tag{A.460}$$
$$a_{17} > 0. \tag{A.461}$$

Thus, the stability of $\boldsymbol{w}_{\mathrm{sqVM}}$ depends on the values of $a_2, \ldots, a_5$ and a_{17}.

Remark A.8. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{k}, \hat{k}, \hat{\ell})$, $\boldsymbol{w}_{\mathrm{sqT}}$ does not exist. See Proposition 3.32.

□

Case 6: $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{\ell}, \hat{k}, \hat{\ell})$

For the case of $(\hat{n}, \hat{k}, \hat{\ell})$ with $\hat{n} = 4\hat{\ell}$, we have

$$(0, 0, 2, 0, 1, 0, 0, 0) \in P$$

as well as

$$(1, 0, 0, 0, 0, 0, 0, 0), (2, 0, 0, 0, 1, 0, 0, 0), (1, 1, 0, 0, 0, 1, 0, 0),$$

$$(1,0,1,0,0,0,1,0), (1,0,0,1,0,0,0,1) \in P$$

in (A.290). Then, the asymptotic form of F_i $(i = 1, \dots, 4)$ in (A.324) becomes

$$F_1 \approx a_1\widetilde{\phi}z_1 + a_{18}{z_3}^2\overline{z}_1 + F_1^C, \tag{A.462}$$
$$F_2 \approx a_1\widetilde{\phi}z_2 + a_{18}{\overline{z}_4}^2\overline{z}_2 + F_2^C, \tag{A.463}$$
$$F_3 \approx a_1\widetilde{\phi}z_3 + a_{18}{z_1}^2\overline{z}_3 + F_3^C, \tag{A.464}$$
$$F_4 \approx a_1\widetilde{\phi}z_4 + a_{18}{\overline{z}_2}^2\overline{z}_4 + F_4^C \tag{A.465}$$

with

$$a_{18} = A_{00201000}(0).$$

where F_i^C $(i = 1, \dots, 4)$ is given in (A.325)–(A.328). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \dots, 8)$ in (A.234)–(A.237) becomes

$$\widetilde{F}_1 \approx a_1\widetilde{\phi}w_1 + a_{18}\{\ \ w_1({w_5}^2 - {w_6}^2) + 2w_2w_5w_6)\} + \widetilde{F}_1^C, \tag{A.466}$$
$$\widetilde{F}_2 \approx a_1\widetilde{\phi}w_2 + a_{18}\{-w_2({w_5}^2 - {w_6}^2) + 2w_1w_5w_6)\} + \widetilde{F}_2^C, \tag{A.467}$$
$$\widetilde{F}_3 \approx a_1\widetilde{\phi}w_3 + a_{18}\{\ \ w_3({w_7}^2 - {w_8}^2) - 2w_4w_7w_8)\} + \widetilde{F}_3^C, \tag{A.468}$$
$$\widetilde{F}_4 \approx a_1\widetilde{\phi}w_4 + a_{18}\{-w_4({w_7}^2 - {w_8}^2) - 2w_3w_7w_8)\} + \widetilde{F}_4^C, \tag{A.469}$$
$$\widetilde{F}_5 \approx a_1\widetilde{\phi}w_5 + a_{18}\{\ \ w_5({w_1}^2 - {w_2}^2) + 2w_6w_1w_2)\} + \widetilde{F}_5^C, \tag{A.470}$$
$$\widetilde{F}_6 \approx a_1\widetilde{\phi}w_6 + a_{18}\{-w_6({w_1}^2 - {w_2}^2) + 2w_5w_1w_2)\} + \widetilde{F}_6^C, \tag{A.471}$$
$$\widetilde{F}_7 \approx a_1\widetilde{\phi}w_7 + a_{18}\{\ \ w_7({w_3}^2 - {w_4}^2) - 2w_8w_3w_4)\} + \widetilde{F}_7^C, \tag{A.472}$$
$$\widetilde{F}_8 \approx a_1\widetilde{\phi}w_8 + a_{18}\{-w_8({w_3}^2 - {w_4}^2) - 2w_7w_3w_4)\} + \widetilde{F}_8^C, \tag{A.473}$$

where $\widetilde{F}_i^C$ $(i = 1, \dots, 8)$ is given in (A.332)–(A.339). Hence, the asymptotic form of the Jacobian matrix in (A.232) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\tilde{\phi}I_8 + a_{18}B_{18} + B_C, \tag{A.474}$$

where B_C is given in (A.341) and

$$B_{18} = \begin{bmatrix} B_1^{18} & B_3^{18} \\ (B_3^{18})^\top & B_2^{18} \end{bmatrix},$$

$$B_1^{18} = \begin{bmatrix} {w_5}^2 - {w_6}^2 & 2w_5w_6 & 0 & 0 \\ 2w_5w_6 & -{w_5}^2 + {w_6}^2 & 0 & 0 \\ 0 & 0 & {w_7}^2 - {w_8}^2 & -2w_7w_8 \\ 0 & 0 & -2w_7w_8 & -{w_7}^2 + {w_8}^2 \end{bmatrix},$$

$$B_2^{18} = \begin{bmatrix} {w_1}^2 - {w_2}^2 & 2w_1w_2 & 0 & 0 \\ 2w_1w_2 & -{w_1}^2 + {w_2}^2 & 0 & 0 \\ 0 & 0 & {w_3}^2 - {w_4}^2 & -2w_3w_4 \\ 0 & 0 & -2w_3w_4 & -{w_3}^2 + {w_4}^2 \end{bmatrix},$$

$$B_3^{18} = 2\begin{bmatrix} w_1w_5 + w_2w_6 & -w_1w_6 + w_2w_5 & 0 & 0 \\ w_1w_6 - w_2w_5 & w_1w_5 + w_2w_6 & 0 & 0 \\ 0 & 0 & w_3w_7 - w_4w_8 & -w_3w_8 - w_4w_7 \\ 0 & 0 & -w_3w_8 - w_4w_7 & -w_3w_7 + w_4w_8 \end{bmatrix}.$$

Substituting $\boldsymbol{w}_{\text{stripeI}}$ into (A.466) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeI}} \approx -\frac{a_2}{a_1}w^2.$$

Evaluating the Jacobian matrix (A.474) at $(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}}) \approx w^2 \begin{bmatrix} O & O \\ O & C_{22} \end{bmatrix} + \widetilde{J_C}^{\text{stripeI}}, \tag{A.475}$$

where $\widetilde{J_C}^{\text{stripeI}}$ is given in (A.342) and

$$C_{22} = \begin{bmatrix} a_{18} & 0 & 0 & 0 \\ 0 & -a_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$ are given by

$$\begin{aligned} \lambda_1 &\approx 2a_2w^2, \\ \lambda_2 &\approx O(w^3), \\ \lambda_3, \lambda_4 &\approx -(a_2 - a_4 \pm a_{18})w^2, \\ \lambda_5 &\approx -(a_2 - a_3)w^2 \quad \text{(repeated twice)}, \\ \lambda_6 &\approx -(a_2 - a_5)w^2 \quad \text{(repeated twice)}. \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 < 0, \quad a_2 - a_4 \pm a_{18} > 0, \quad a_2 - a_3 > 0, \quad a_2 - a_5 > 0.$$

These conditions are equivalent to

$$\max(a_3, a_4 + |a_{18}|, a_5) < a_2 < 0. \tag{A.476}$$

Thus, the stability of $\boldsymbol{w}_{\text{stripeI}}$ depends on the values of $a_2, \ldots, a_5$ and a_{18}.

Substituting $\boldsymbol{w}_{\text{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$ into (A.438) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeII}} \approx -\frac{a_2}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.474) at $(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}}) \approx w^2 \begin{bmatrix} O & O \\ O & -C_{22} \end{bmatrix} + \widetilde{J_C}^{\text{stripeII}}, \tag{A.477}$$

where C_{22} is given in (A.4.5), and $\widetilde{J_C}^{\text{stripeII}}$ is given in (A.345). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$.

Substituting $\boldsymbol{w}_{\text{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$ into (A.331) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downI}} \approx -\frac{a_2 + a_4 + a_{18}}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.340) at $(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}}) \approx w^2 \begin{bmatrix} C_{23} & C_{24} \\ C_{24} & C_{23} \end{bmatrix} + \widetilde{J_C}^{\text{upside-downI}} \tag{A.478}$$

with

$$C_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2a_{18} & 0 & 0 \\ 0 & 0 & -a_{18} & 0 \\ 0 & 0 & 0 & -a_{18} \end{bmatrix}, \quad C_{24} = \begin{bmatrix} 2a_{18} & 0 & 0 & 0 \\ 0 & 2a_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{A.479}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$ are given by

$$\begin{aligned} &\lambda_1, \lambda_2 \approx 2a_2 \pm (a_4 + a_{18}) w^2, \\ &\lambda_3 \approx -a_{18} w^2, \\ &\lambda_4 \approx O(w^3), \\ &\lambda_5 \approx -(a_2 - a_3 + a_4 - a_5 + a_{18}) w^2 \quad \text{(repeated 4 times)}. \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 < -|a_4 + a_{18}|, \quad a_{18} > 0, \quad a_2 - a_3 + a_4 - a_5 + a_{18} > 0.$$

These conditions are equivalent to

$$a_3 - a_4 + a_5 + a_{17} < a_2 < -|a_4 + a_{18}| \tag{A.480}$$

$$a_{18} > 0. \tag{A.481}$$

Thus, the stability of $\boldsymbol{w}_{\text{upside-downI}}$ is conditional and depends on the values of $a_2, \ldots, a_5$ and a_{18}.

Substituting $\boldsymbol{w}_{\text{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$ into (A.331) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downII}} \approx -\frac{a_2 + a_4 + a_{18}}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.340) at $(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}}) = \widetilde{J_C}^{\text{upside-downII}} \approx w^2 \begin{bmatrix} C_{25} & C_{24} \\ C_{24} & C_{25} \end{bmatrix} \tag{A.482}$$

with

$$C_{25} = \begin{bmatrix} -2a_{18} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{18} & 0 \\ 0 & 0 & 0 & -a_{18} \end{bmatrix}, \tag{A.483}$$

where C_{18} is given in (A.451). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{upside-downII}}$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$.

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (A.466) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5 + a_{18}}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.474) at $(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}}) \approx w^2 \begin{bmatrix} C_{21} & C_{22} \\ C_{22} & C_{21} \end{bmatrix} + \widetilde{J_C}^{\text{sqVM}}, \tag{A.484}$$

where $\widetilde{J_C}^{\text{sqVM}}$ is given in (A.355) and

$$C_{21} = 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -a_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{18} \end{bmatrix}, \quad C_{22} = 2 \begin{bmatrix} a_{18} & 0 & 0 & 0 \\ 0 & a_{18} & 0 & 0 \\ 0 & 0 & a_{18} & 0 \\ 0 & 0 & 0 & -a_{18} \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{sqVM}}, \widetilde{\phi}_{\text{sqVM}})$ are given by

$$\lambda_1 \approx 2(a_2 + a_3 + a_4 + a_5 + a_{18}) w^2,$$
$$\lambda_2 \approx 2(a_2 + a_3 - a_4 - a_5 - a_{18}) w^2,$$
$$\lambda_3 \approx 2(a_2 - a_3 + a_4 - a_5 + a_{18}) w^2,$$
$$\lambda_4 \approx 2(a_2 - a_3 - a_4 + a_5 - a_{18}) w^2,$$

$$\lambda_5 \approx -4a_{18}w^2 \quad \text{(repeated twice)},$$
$$\lambda_6 \approx O(w^3) \quad \text{(repeated twice)}.$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 + a_3 + a_4 + a_5 + a_{18} < 0, \tag{A.485}$$
$$a_2 + a_3 - a_4 - a_5 - a_{18} < 0, \tag{A.486}$$
$$a_2 - a_3 + a_4 - a_5 + a_{18} < 0, \tag{A.487}$$
$$a_2 - a_3 - a_4 + a_5 - a_{18} < 0, \tag{A.488}$$
$$a_{18} > 0. \tag{A.489}$$

Thus, the stability of $\boldsymbol{w}_{\mathrm{sqVM}}$ depends on the values of $a_2, \ldots, a_5$ and a_{18}.

Remark A.9. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (4\hat{\ell}, \hat{k}, \hat{\ell})$, $\boldsymbol{w}_{\mathrm{sqT}}$ does not exist. See Proposition 3.32.

□

Case 7: $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k} + 2\hat{\ell}, \hat{k}, \hat{\ell})$, $(\hat{k}, \hat{\ell}) \neq (3, 1)$

For the case of $(\hat{n}, \hat{k}, \hat{\ell})$ with $\hat{n} = 2(\hat{k} + \hat{\ell})$ and $(\hat{k}, \hat{\ell}) \neq (3, 1)$, we have

$$(0, 1, 0, 1, 0, 0, 1, 0), (0, 0, 1, 1, 0, 1, 0, 0), (0, 0, 0, 2, 1, 0, 0, 0) \in P.$$

as well as

$$(1, 0, 0, 0, 0, 0, 0, 0), (2, 0, 0, 0, 1, 0, 0, 0), (1, 1, 0, 0, 0, 1, 0, 0),$$
$$(1, 0, 1, 0, 0, 0, 1, 0), (1, 0, 0, 1, 0, 0, 0, 1) \in P$$

in (A.290). Then, the asymptotic form of F_i $(i = 1, \ldots, 4)$ in (A.324) becomes

$$F_1 \approx a_1\widetilde{\phi} z_1 + a_{10} z_2 z_4 \overline{z}_3 + a_{11} z_3 z_4 \overline{z}_2 + a_{12} {z_4}^2 \overline{z}_1 + F_1^C, \tag{A.490}$$
$$F_2 \approx a_1\widetilde{\phi} z_2 + a_{10} \overline{z}_1 z_3 z_4 + a_{11} \overline{z}_4 z_3 z_1 + a_{12} {z_3}^2 \overline{z}_2 + F_2^C, \tag{A.491}$$
$$F_3 \approx a_1\widetilde{\phi} z_3 + a_{10} z_4 z_2 \overline{z}_1 + a_{11} z_1 z_2 \overline{z}_4 + a_{12} {z_2}^2 \overline{z}_3 + F_3^C, \tag{A.492}$$
$$F_4 \approx a_1\widetilde{\phi} z_4 + a_{10} \overline{z}_3 z_1 z_2 + a_{11} \overline{z}_2 z_1 z_3 + a_{12} {z_1}^2 \overline{z}_4 + F_4^C \tag{A.493}$$

with a_{10}, a_{11}, a_{12} given in (A.389), and F_i^C $(i = 1, \ldots, 4)$ given in (A.325)–(A.328). Then, the asymptotic form of $\widetilde{F}_i$ $(i = 1, \ldots, 8)$ in (A.234)–(A.237) becomes

$$\begin{aligned}
\widetilde{F}_1 \approx{}& a_1\widetilde{\phi} w_1 + a_{10}\{w_5(w_3w_7 - w_4w_8) + w_6(w_3w_8 + w_4w_7)\} \\
&+ a_{11}\{w_3(w_5w_7 - w_6w_8) + w_4(w_5w_8 + w_6w_7)\} \\
&+ a_{12}\{w_1({w_7}^2 - {w_8}^2) + 2w_2w_7w_8\} + \widetilde{F}_1^C,
\end{aligned} \tag{A.494}$$

$$\begin{aligned}
\widetilde{F}_2 \approx{}& a_1\widetilde{\phi} w_2 + a_{10}\{w_5(w_3w_8 + w_4w_7) - w_6(w_3w_7 - w_4w_8)\} \\
&+ a_{11}\{w_3(w_5w_8 + w_6w_7) - w_4(w_5w_7 - w_6w_8)\}
\end{aligned}$$

$$+ a_{12}\{-w_2({w_7}^2 - {w_8}^2) + 2w_1w_7w_8\} + \widetilde{F}_2^C, \tag{A.495}$$

$$\widetilde{F}_3 \approx a_1\widetilde{\phi}w_3 + a_{10}\{w_1(w_5w_7 - w_6w_8) + w_2(w_5w_8 + w_6w_7)\}$$
$$+ a_{11}\{w_7(w_1w_5 - w_2w_6) + w_8(w_1w_6 + w_2w_5)\}$$
$$+ a_{12}\{w_3({w_5}^2 - {w_6}^2) + 2w_4w_5w_6\} + \widetilde{F}_3^C, \tag{A.496}$$

$$\widetilde{F}_4 \approx a_1\widetilde{\phi}w_4 + a_{10}\{w_1(w_5w_8 + w_6w_7) - w_2(w_5w_7 - w_6w_8)\}$$
$$+ a_{11}\{w_7(w_1w_6 + w_2w_5) - w_8(w_1w_5 - w_2w_6)\}$$
$$+ a_{12}\{-w_4({w_5}^2 - {w_6}^2) + 2w_3w_5w_6\} + \widetilde{F}_4^C, \tag{A.497}$$

$$\widetilde{F}_5 \approx a_1\widetilde{\phi}w_5 + a_{10}\{w_1(w_3w_7 - w_4w_8) + w_2(w_3w_8 + w_4w_7)\}$$
$$+ a_{11}\{w_7(w_1w_3 - w_2w_4) + w_8(w_1w_4 + w_2w_3)\}$$
$$+ a_{12}\{w_5({w_3}^2 - {w_4}^2) + 2w_3w_4w_6\} + \widetilde{F}_5^C, \tag{A.498}$$

$$\widetilde{F}_6 \approx a_1\widetilde{\phi}w_6 + a_{10}\{w_1(w_3w_8 + w_4w_7) - w_2(w_3w_7 - w_4w_8)\}$$
$$+ a_{11}\{w_7(w_1w_4 + w_2w_3) - w_8(w_1w_3 - w_2w_4)\}$$
$$+ a_{12}\{-w_6({w_3}^2 - {w_4}^2) + 2w_3w_4w_5\} + \widetilde{F}_6^C, \tag{A.499}$$

$$\widetilde{F}_7 \approx a_1\widetilde{\phi}w_7 + a_{10}\{w_5(w_1w_3 - w_2w_4) + w_6(w_1w_4 + w_2w_3)\}$$
$$+ a_{11}\{w_3(w_1w_5 - w_2w_6) + w_4(w_1w_6 + w_2w_5)\}$$
$$+ a_{12}\{w_7({w_1}^2 - {w_2}^2) + 2w_8w_1w_2\} + \widetilde{F}_7^C, \tag{A.500}$$

$$\widetilde{F}_8 \approx a_1\widetilde{\phi}w_8 + a_{10}\{w_5(w_1w_4 + w_2w_3) - w_6(w_1w_3 - w_2w_4)\}$$
$$+ a_{11}\{w_3(w_1w_6 + w_2w_5) - w_4(w_1w_5 - w_2w_6)\}$$
$$+ a_{12}\{-w_8({w_1}^2 - {w_2}^2) + 2w_7w_1w_2\} + \widetilde{F}_8^C, \tag{A.501}$$

where $\widetilde{F}_i^C$ $(i = 1, \ldots, 8)$ is given in (A.332)–(A.339). Hence, the asymptotic form of the Jacobian matrix in (A.232) becomes

$$\widetilde{J}(\boldsymbol{w}, \widetilde{\phi}) \approx a_1\tilde{\phi}I_8 + a_{10}B_{10} + a_{11}B_{11} + a_{12}B_{12} + B_C, \tag{A.502}$$

with B_C given in (A.341), B_{10}, B_{11} and B_{12} given in (A.399).

Substituting $\boldsymbol{w}_{\text{stripeI}} = (w, 0, 0, 0, 0, 0, 0, 0)$ into (A.494) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeI}} \approx -\frac{a_2}{a_1}w^2.$$

Evaluating the Jacobian matrix (A.502) at $(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}}) \approx w^2 \begin{bmatrix} O & O \\ O & C_{23} \end{bmatrix} + \widetilde{J_C}^{\text{stripeI}}, \tag{A.503}$$

where $\widetilde{J_C}^{\text{stripeI}}$ is given in (A.342) and

$$C_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12} & 0 \\ 0 & 0 & 0 & -a_{12} \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeI}}, \widetilde{\phi}_{\text{stripeI}})$ are given by

$$\begin{aligned} \lambda_1 &\approx 2a_2 w^2, \\ \lambda_2, \lambda_3 &\approx -(a_2 - a_5 \pm a_{12}) w^2, \\ \lambda_4 &\approx O(w^3), \\ \lambda_5 &\approx -(a_2 - a_3) w^2, \quad \text{(repeated twice)} \\ \lambda_6 &\approx -(a_2 - a_4) w^2, \quad \text{(repeated twice)}. \end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 < 0, \quad a_2 - a_5 > -|a_{12}|, \quad a_2 - a_3 > 0, \quad a_2 - a_4 > 0.$$

These conditions are equivalent to

$$\max(a_3, a_4, a_5 - |a_{12}|) < a_2 < 0. \tag{A.504}$$

Thus, the stability of $\boldsymbol{w}_{\text{stripeI}}$ depends on the values of $a_2, \ldots, a_5$ and a_{12}.

Substituting $\boldsymbol{w}_{\text{stripeII}} = (0, w, 0, 0, 0, 0, 0, 0)$ into (A.494) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{stripeII}} \approx -\frac{a_2}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.502) at $(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}}) \approx w^2 \begin{bmatrix} O & O \\ O & -C_{23} \end{bmatrix} + \widetilde{J_C}^{\text{stripeII}}, \tag{A.505}$$

where C_{23} is given in (A.4.5), and $\widetilde{J_C}^{\text{stripeII}}$ is given in (A.345). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{stripeII}}, \widetilde{\phi}_{\text{stripeII}})$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{stripeII}}$ are equivalent to that for $\boldsymbol{w}_{\text{stripeI}}$.

Substituting $\boldsymbol{w}_{\text{upside-downI}} = (w, 0, 0, 0, w, 0, 0, 0)$ into (A.494) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downI}} \approx -\frac{a_2 + a_4}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.502) at $(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}}) \approx w^2 \begin{bmatrix} C_{24} & C_{25} \\ C_{25} & C_{24} \end{bmatrix} + \widetilde{J_C}^{\text{upside-downI}} \tag{A.506}$$

with

$$C_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12} & 0 \\ 0 & 0 & 0 & -a_{12} \end{bmatrix}, \quad C_{25} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{10}+a_{11} & 0 \\ 0 & 0 & 0 & a_{10}-a_{11} \end{bmatrix}. \tag{A.507}$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downI}}, \widetilde{\phi}_{\text{upside-downI}})$ are given by

$$\begin{aligned}
&\lambda_1, \lambda_2 \approx 2(a_2 \pm a_4)w^2,\\
&\lambda_3, \lambda_4 \approx \{-(a_2 + a_3 - a_4 - a_5 - a_{12}) \pm (a_{10} + a_{11})\}w^2,\\
&\lambda_5, \lambda_6 \approx \{-(a_2 + a_3 - a_4 - a_5 + a_{12}) \pm (a_{10} - a_{11})\}w^2,\\
&\lambda_7 \approx O(w^3) \quad \text{(repeated twice)}.
\end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2 < -|a_4|, \tag{A.508}$$

$$a_2 + a_3 - a_4 - a_5 - a_{12} > -|a_{10} + a_{11}|, \tag{A.509}$$

$$a_2 + a_3 - a_4 - a_5 + a_{12} > -|a_{10} - a_{11}|. \tag{A.510}$$

Thus, the stability of $\boldsymbol{w}_{\text{upside-downI}}$ depends on the values of $a_2, \ldots, a_5$ and $a_{10}, \ldots, a_{12}$.

Substituting $\boldsymbol{w}_{\text{upside-downII}} = (0, w, 0, 0, 0, w, 0, 0)$ into (A.331) and solving $F_2 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{upside-downII}} \approx -\frac{a_2 + a_4}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.340) at $(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}}) \approx w^2 \begin{bmatrix} -C_{24} & C_{26} \\ C_{26} & -C_{24} \end{bmatrix} + \widetilde{J_C}^{\text{upside-downII}} \tag{A.511}$$

with

$$C_{26} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{10}-a_{11} & 0 \\ 0 & 0 & 0 & a_{10}+a_{11} \end{bmatrix}, \tag{A.512}$$

where C_{24} is given in (A.507). The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\text{upside-downII}}, \widetilde{\phi}_{\text{upside-downII}})$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$. Hence, stability conditions for $\boldsymbol{w}_{\text{upside-downII}}$ are equivalent to that for $\boldsymbol{w}_{\text{upside-downI}}$.

Substituting $\boldsymbol{w}_{\text{sqVM}} = (w, 0, w, 0, w, 0, w, 0)$ into (A.494) and solving $F_1 = 0$ for $\widetilde{\phi}$, we have

$$\widetilde{\phi} = \widetilde{\phi}_{\text{sqVM}} \approx -\frac{a_2 + a_3 + a_4 + a_5 + a_{10} + a_{11} + a_{12}}{a_1} w^2.$$

Evaluating the Jacobian matrix (A.502) at $(\boldsymbol{w}_{\mathrm{sqVM}}, \widetilde{\phi}_{\mathrm{sqVM}})$, we have

$$\widetilde{J}(\boldsymbol{w}_{\mathrm{sqVM}}, \widetilde{\phi}_{\mathrm{sqVM}}) \approx w^2 \begin{bmatrix} C_{27} & C_{28} \\ C_{28} & C_{27} \end{bmatrix} + \widetilde{J_C}^{\mathrm{sqVM}}, \tag{A.513}$$

where $\widetilde{J_C}^{\mathrm{sqVM}}$ is given in (A.355) and

$$C_{27} = \begin{bmatrix} -a_{10}-a_{11} & 0 & a_{10}+a_{11} & 0 \\ 0 & -a_{10}-a_{11}-2a_{12} & 0 & a_{10}-a_{11} \\ a_{10}+a_{11} & 0 & -a_{10}-a_{11} & 0 \\ 0 & -a_{10}+a_{11} & 0 & -a_{10}-a_{11}-2a_{12} \end{bmatrix},$$

$$C_{28} = \begin{bmatrix} a_{10}+a_{11} & 0 & a_{10}+a_{11}+2a_{12} & 0 \\ 0 & -a_{10}+a_{11} & 0 & a_{10}+a_{11}+2a_{12} \\ a_{10}+a_{11}+2a_{12} & 0 & a_{10}+a_{11} & 0 \\ 0 & a_{10}+a_{11}+2a_{12} & 0 & a_{10}-a_{11} \end{bmatrix}.$$

The eigenvalues of the matrix $\widetilde{J}(\boldsymbol{w}_{\mathrm{sqVM}}, \widetilde{\phi}_{\mathrm{sqVM}})$ are given by

$$\begin{aligned}
\lambda_1 &\approx 2(a_2+a_3+a_4+a_5+a_{10}+a_{11}+a_{12})w^2, \\
\lambda_2 &\approx 2(a_2+a_3-a_4-a_5-a_{10}-a_{11}-a_{12})w^2, \\
\lambda_3 &\approx 2(a_2-a_3+a_4-a_5-a_{10}-a_{11}-a_{12})w^2, \\
\lambda_4 &\approx 2(a_2-a_3-a_4+a_5-a_{10}-a_{11}+a_{12})w^2, \\
\lambda_5 &\approx -2(a_{10}+a_{11}+2a_{12})w^2 \quad \text{(repeated twice)}, \\
\lambda_6 &\approx O(w^3) \quad \text{(repeated twice)}.
\end{aligned}$$

Assuming that all eigenvalues are negative, we have the following stability conditions (necessary conditions):

$$a_2+a_3+a_4+a_5+a_{10}+a_{11}+a_{12} < 0, \tag{A.514}$$

$$a_2+a_3-a_4-a_5-a_{10}-a_{11}-a_{12} < 0, \tag{A.515}$$

$$a_2-a_3+a_4-a_5-a_{10}-a_{11}-a_{12} < 0, \tag{A.516}$$

$$a_2-a_3-a_4+a_5-a_{10}-a_{11}+a_{12} < 0, \tag{A.517}$$

$$a_{10}+a_{11}+2a_{12} > 0. \tag{A.518}$$

Thus, the stability of $\boldsymbol{w}_{\mathrm{sqVM}}$ depends on the values of $a_2, \ldots, a_5$ and $a_{10}, \ldots, a_{12}$.

Remark A.10. For the case $(\hat{n}, \hat{k}, \hat{\ell}) = (2\hat{k}+2\hat{\ell}, \hat{k}, \hat{\ell})$, $\boldsymbol{w}_{\mathrm{sqT}}$ does not exist. See Proposition 3.32. □

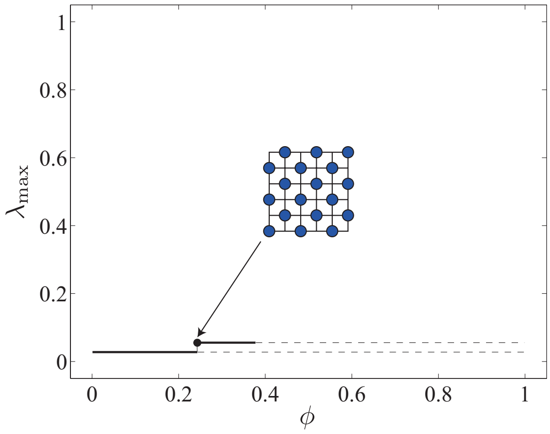
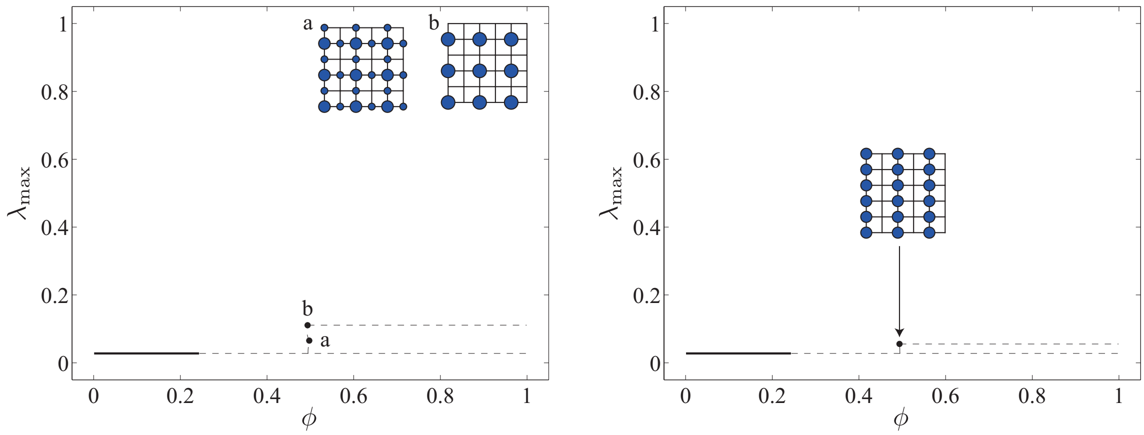


Figure A.2: Equilibrium curves for $\mu = (1; +, +, -)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

A.5. Bifurcation Behaviour of the FO Model

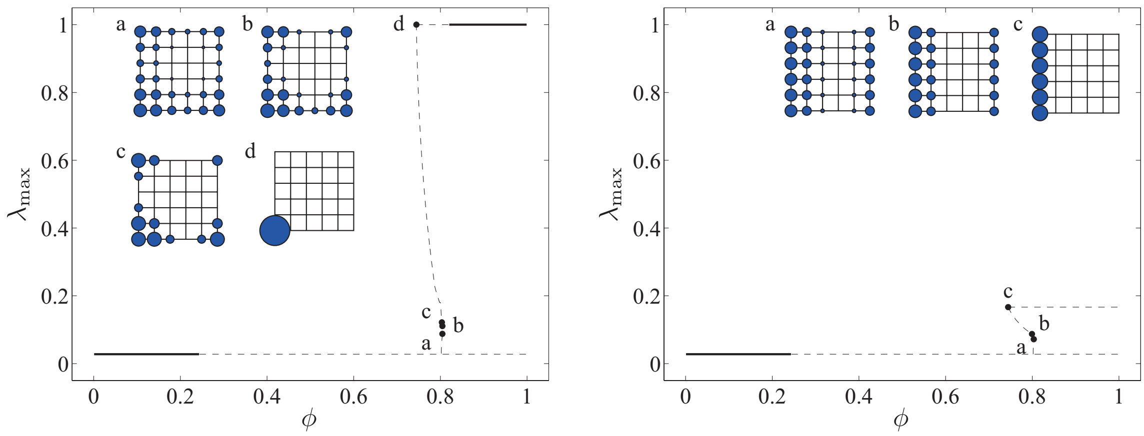
We identified bifurcating solutions from the uniform state on the 6×6 square lattice and demonstrated the emergence of some typical solutions for three types of economic geography models in Section 3.7. In this section, we compute equilibrium curves of all the bifurcating solutions for the FO model (Forslid and Ottaviano, 2003).

Figures A.2–A.10 show bifurcating solution curves for each μ. We see that all the bifurcating solutions are unstable just after bifurcation although stable ones are theoretically possible. For almost all the bifurcating solution curves, population tend to be agglomerated completely to places with the largest positive or negative components of the bifurcating solution after the bifurcation. Note that $\boldsymbol{w}_{\mathrm{sq}}$ with $\mu = (4; 3, 2, +)$ in Fig. A.9 and $\boldsymbol{w}_{\mathrm{sqVM}}$ with $\mu = (8; 2, 1)$ in Fig. A.10 are exceptions to this tendency. These solutions have a common property that some places have a zero component. For solutions with such a property, computing the bifurcating solution curves is troublesome since we cannot predict increase and decrease in population in places with a zero component.



(a) $\boldsymbol{w}_{\mathrm{sq}}$ (b) $\boldsymbol{w}_{\mathrm{stripe}}$

Figure A.3: Equilibrium curves for $\mu = (2; +, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.



(a) $\boldsymbol{w}_{\mathrm{sq}}$ (b) $\boldsymbol{w}_{\mathrm{stripeI}}$

Figure A.4: Equilibrium curves for $\mu = (4; 1, 0, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

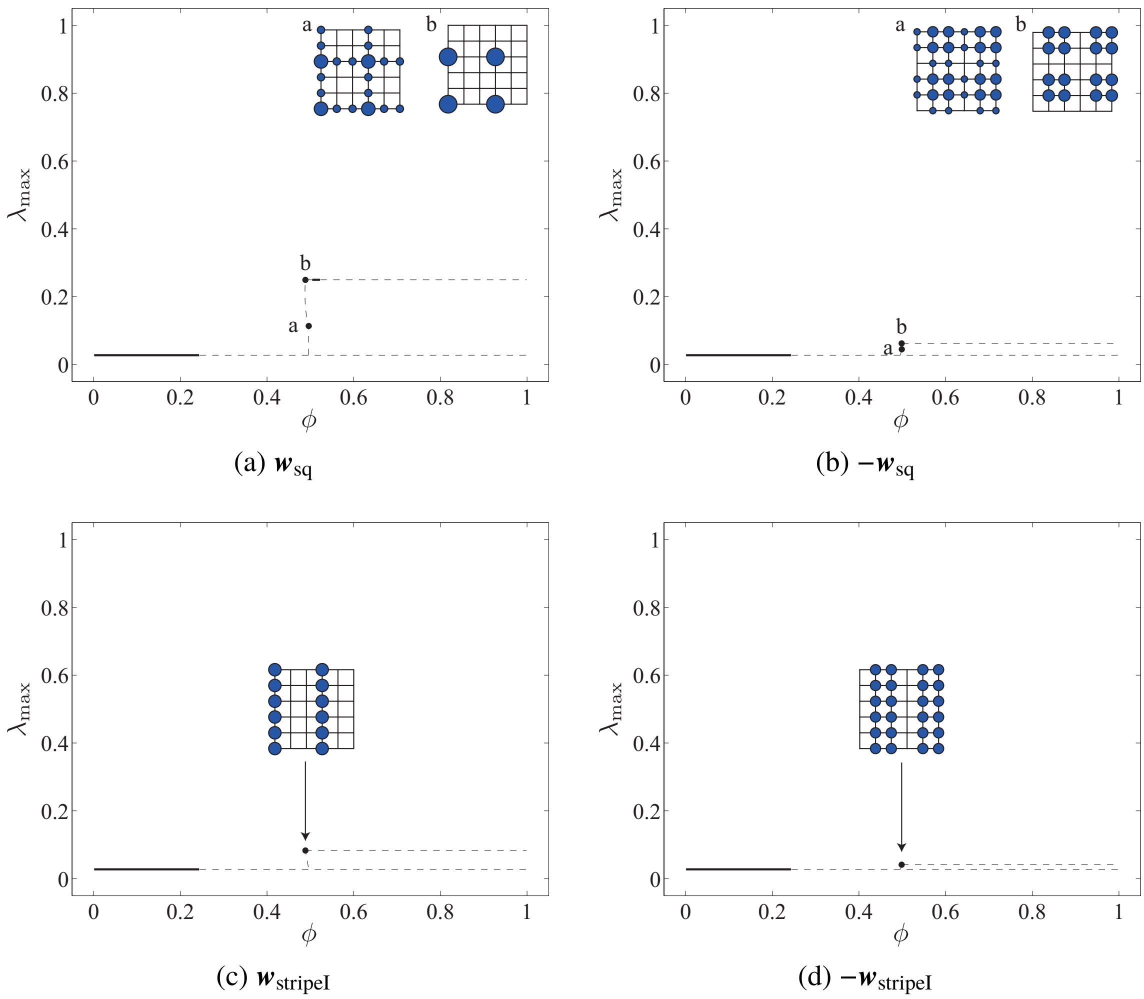
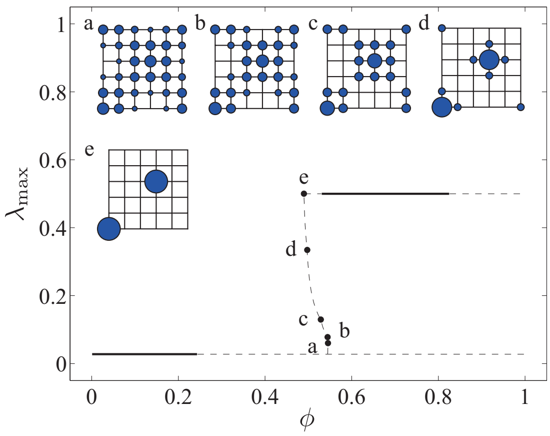
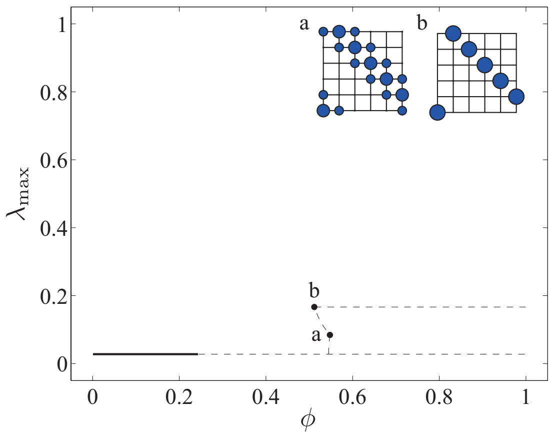


Figure A.5: Equilibrium curves for $\mu = (4; 2, 0, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

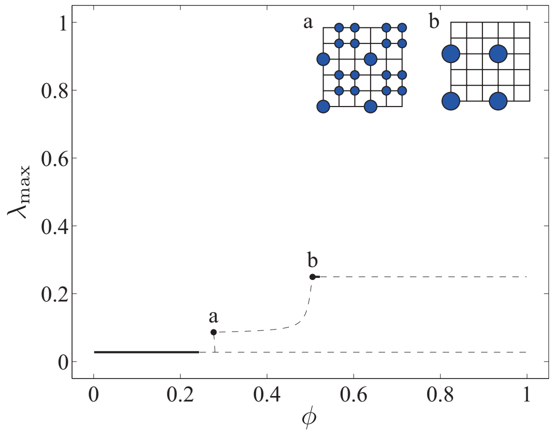


(a) $\boldsymbol{w}_{\mathrm{sq}}$

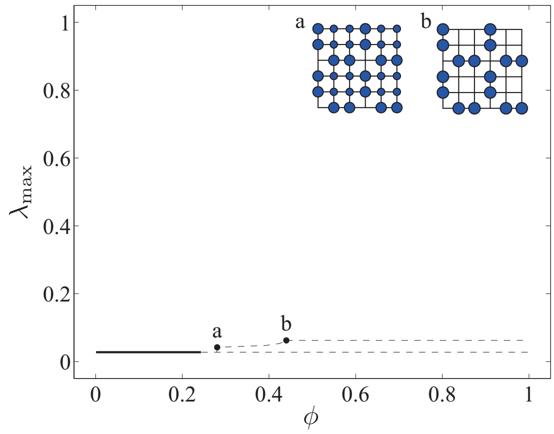


(b) $\boldsymbol{w}_{\mathrm{stripeI}}$

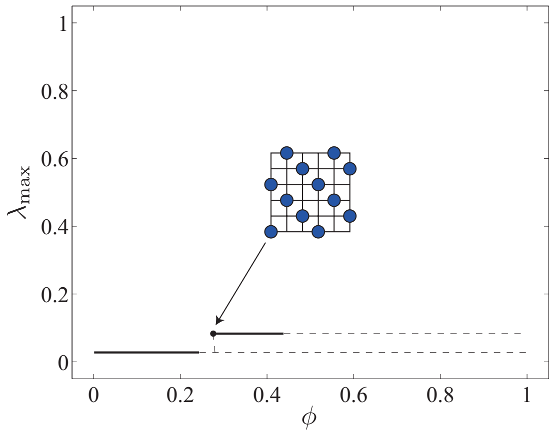
Figure A.6: Equilibrium curves for $\mu = (4; 1, 1, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.



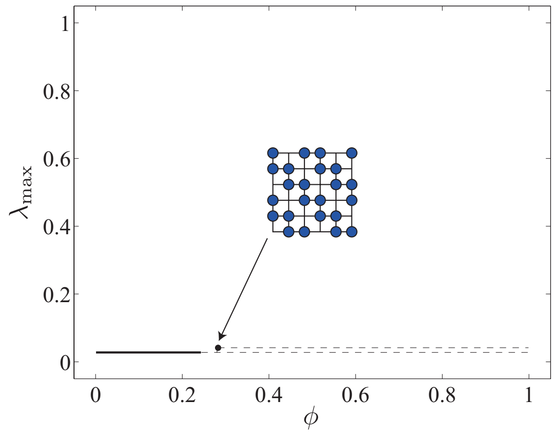
(a) $\boldsymbol{w}_{\mathrm{sq}}$



(b) $-\boldsymbol{w}_{\mathrm{sq}}$



(c) $\boldsymbol{w}_{\mathrm{stripeI}}$



(d) $-\boldsymbol{w}_{\mathrm{stripeI}}$

Figure A.7: Equilibrium curves for $\mu = (4; 2, 2, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

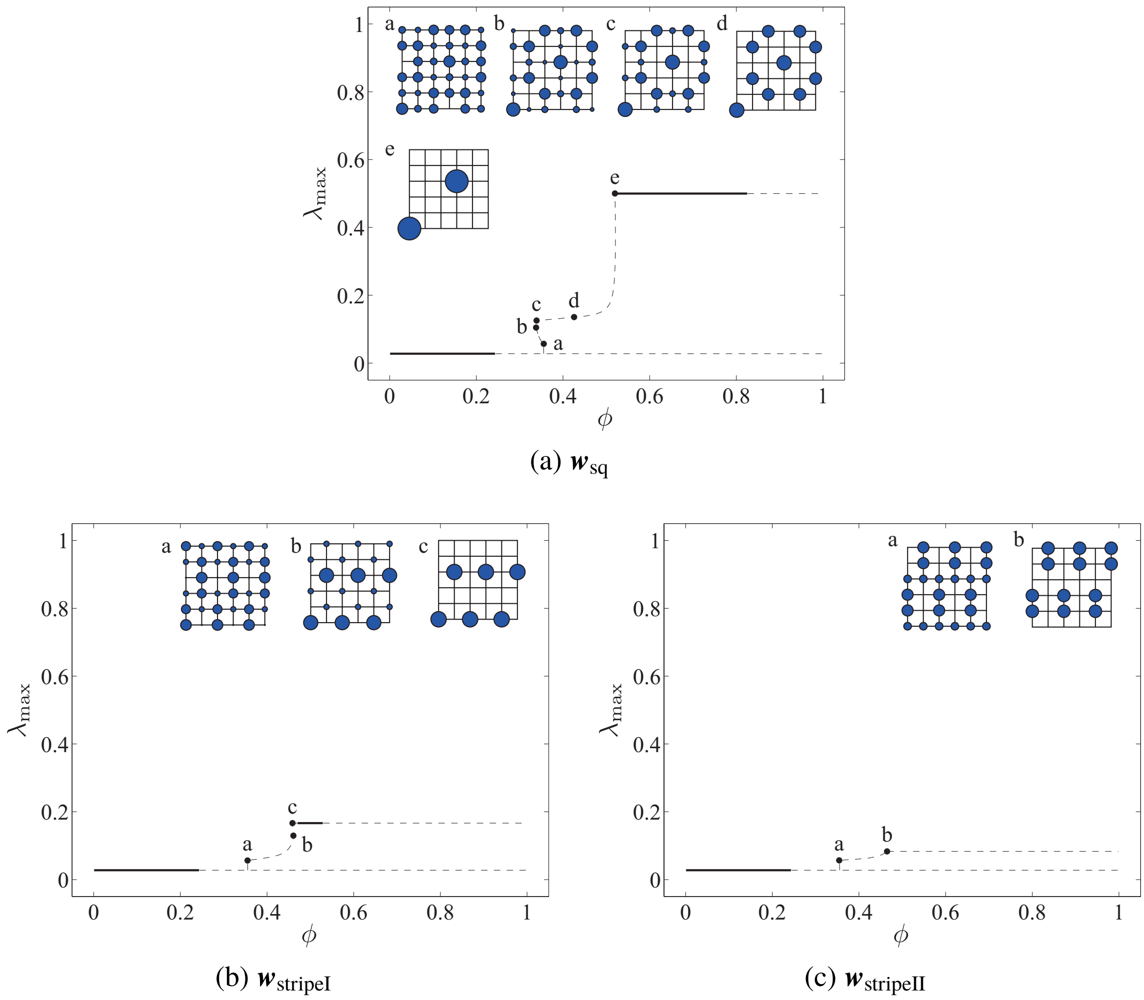
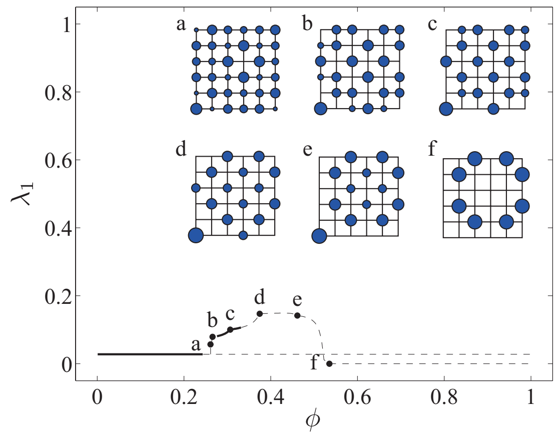
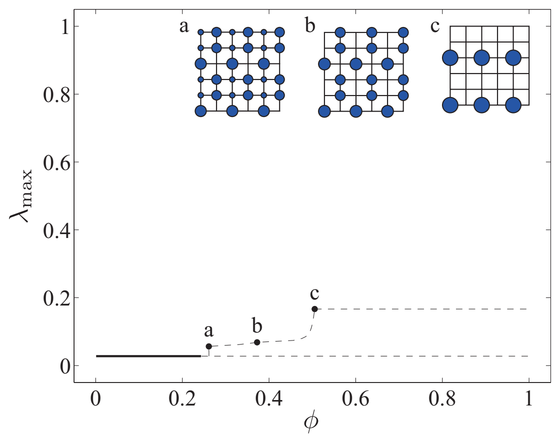


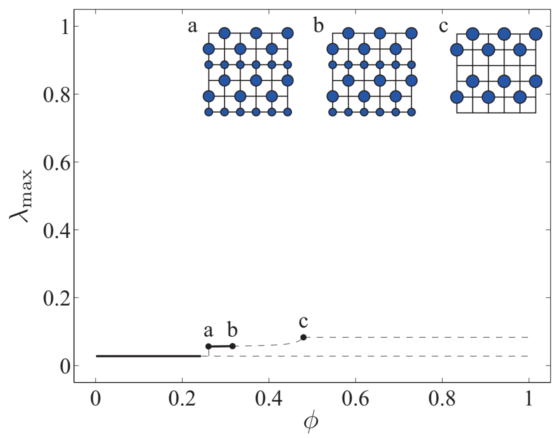
Figure A.8: Equilibrium curves for $\mu = (4; 3, 1, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.



(a) $\boldsymbol{w}_{\mathrm{sq}}$

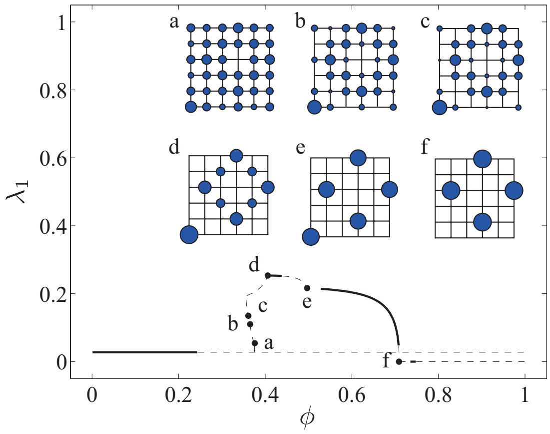


(b) $\boldsymbol{w}_{\mathrm{stripeI}}$

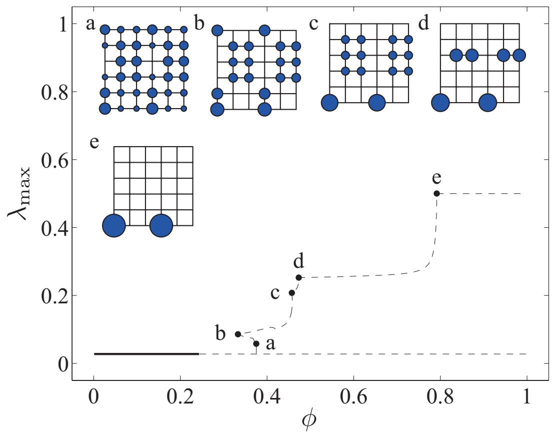


(c) $\boldsymbol{w}_{\mathrm{stripeII}}$

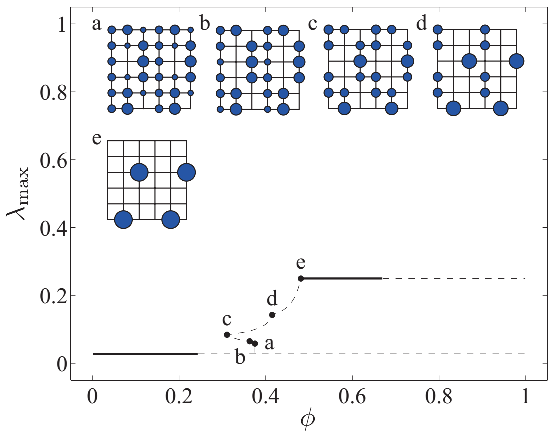
Figure A.9: Equilibrium curves for $\mu = (4; 3, 2, +)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.



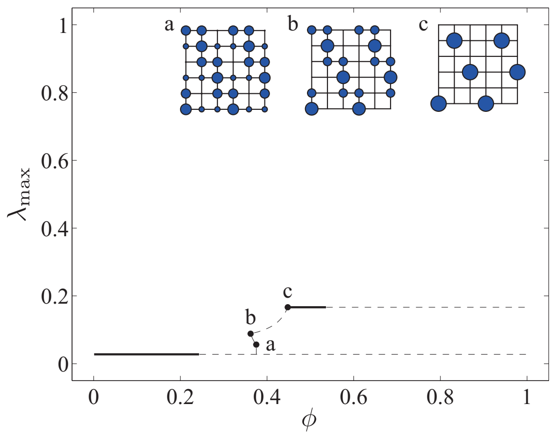
(a) $\boldsymbol{w}_{\text{sqVM}}$



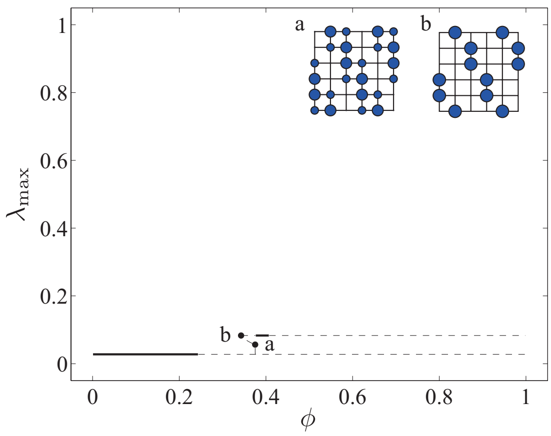
(b) $\boldsymbol{w}_{\text{upside-downI}}$



(c) $\boldsymbol{w}_{\text{upside-downII}}$



(d) $\boldsymbol{w}_{\text{stripeI}}$



(e) $\boldsymbol{w}_{\text{stripeII}}$

Figure A.10: Equilibrium curves for $\mu = (8; 2, 1)$. Solid curves represent stable stationary points, and dashed curves represent unstable ones.

B. Appendices for Chapter 5

We describe details of theoretical analysis in Chapter 5. Using the governing equation

$$F_i(\boldsymbol{\lambda}, \phi) = \lambda_i(v_i(\boldsymbol{\lambda}, \phi) - \bar{v}(\boldsymbol{\lambda}, \phi)) = 0, \quad i \in P \tag{B.1}$$

in (5.5), we derive bifurcation equations. Solving the bifurcation equations, we show the existence of bifurcating solutions from the the mono-centric distribution $\boldsymbol{\lambda}^{\mathrm{FA}} = (1, 0, \ldots, 0)$.

B.1. Bifurcation Point with Type α_i Orbit

We investigate critical points associated with Type α_i orbit.

B.1.1. Derivation of Bifurcation Equations

We focus on a critical point associated with Type α_1 orbit. We can investigate critical points associated with Type α_i ($i = 2, \ldots, n_1$) orbits in a similar manner. Note that n_1 is dependent on the number of places K (cf., Fig. 5.3 for $K = 25$).

Let $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_{\mathrm{c}}^{\alpha_1})$ be a critical point associated with Type α_1 orbit:

$$\alpha_1 = \{2, 3, 4, 5\}. \tag{B.2}$$

By the definition of $\phi_{\mathrm{c}}^{\alpha_1}$, we assume that $v_{\alpha_1} - v_1 = 0$. Hence, the Jacobian matrix $J_{\mathrm{c}} \equiv J(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_{\mathrm{c}}^{\alpha_1})$ takes the following form:

$$J_{\mathrm{c}} = \begin{pmatrix} -v_1 & -v_1 \mathbf{1}_4 & -v_{\alpha_2}\mathbf{1}_4 & \cdots & -v_{\alpha_{n_1}}\mathbf{1}_4 & -v_{\beta_1}\mathbf{1}_8 & \cdots & -v_{\beta_{n_2}}\mathbf{1}_8 \\ & 0 \times I_4 & & & & & & \\ & & (v_{\alpha_2} - v_1)I_4 & & & & & \\ & & & \ddots & & & & \\ & & & & (v_{\alpha_{n_1}} - v_1)I_4 & & & \\ & & & & & (v_{\beta_1} - v_1)I_8 & & \\ & & & & & & \ddots & \\ & & & & & & & (v_{\beta_{n_2}} - v_1)I_8 \end{pmatrix}, \tag{B.3}$$

where I_j is the $j \times j$ identity matrix, and $\mathbf{1}_j$ is the j-dimensional all-one row vector.

We decompose the increment $\boldsymbol{\lambda} - \boldsymbol{\lambda}^{\mathrm{FA}}$ into two components as

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{FA}} + \boldsymbol{w} + \bar{\boldsymbol{w}}, \tag{B.4}$$

where $\boldsymbol{w} \in \ker(J_{\mathrm{c}})$ and $\bar{\boldsymbol{w}} \in \ker(J_{\mathrm{c}})^{\perp}$. Note that $\ker(J_{\mathrm{c}})$ represents the kernel space of J_{c}, which is generated by a basis satisfying $J_{\mathrm{c}}\boldsymbol{\eta} = \mathbf{0}$:

$$\ker(J_{\mathrm{c}}) = \{\boldsymbol{\eta} \in \mathbb{R}^K \mid \eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5 = 0,\ \eta_j = 0,\ j = 6, \ldots, K\}, \tag{B.5}$$

where η_j denotes the jth component of $\boldsymbol{\eta}$. We take a basis $\{\boldsymbol{\eta}_j \mid j = 1, \ldots, 4\}$ of $\ker(J_{\mathrm{c}})$ as

$$\boldsymbol{\eta}_1 = (-1, 1, 0, 0, 0, 0, \ldots, 0), \tag{B.6}$$

$$\boldsymbol{\eta}_2 = (-1, 0, 1, 0, 0, 0, \dots, 0), \tag{B.7}$$
$$\boldsymbol{\eta}_3 = (-1, 0, 0, 1, 0, 0, \dots, 0), \tag{B.8}$$
$$\boldsymbol{\eta}_4 = (-1, 0, 0, 0, 1, 0, \dots, 0). \tag{B.9}$$

Then, we can represent $\boldsymbol{w}$ as

$$\boldsymbol{w} = x_1\boldsymbol{\eta}_1 + x_2\boldsymbol{\eta}_2 + x_3\boldsymbol{\eta}_3 + x_4\boldsymbol{\eta}_4. \tag{B.10}$$

We take a basis $\{\bar{\boldsymbol{\eta}}_j \mid j = 1, \dots, K-4\}$ of $\ker(J_c)^\perp$ as

$$\bar{\boldsymbol{\eta}}_1 = (1, 1, 1, 1, 1, \dots, 1), \tag{B.11}$$
$$\bar{\boldsymbol{\eta}}_j = (0, 0, 0, 0, 0, \underbrace{0, \dots, 0}_{j-2 \text{ times}}, 1, \underbrace{0, \dots, 0}_{K-(j+4) \text{ times}}), \quad j = 2, \dots, K-4. \tag{B.12}$$

Then, we can represent $\bar{\boldsymbol{w}}$ as

$$\bar{\boldsymbol{w}} = \bar{x}_1\bar{\boldsymbol{\eta}}_1 + \sum_{k=2}^{K-4} \bar{x}_k\bar{\boldsymbol{\eta}}_k. \tag{B.13}$$

Combining (B.4), (B.10), and (B.13), we can represent $\boldsymbol{\lambda}$ as

$$\boldsymbol{\lambda} = (1 + \bar{x}_1 - x_1 - x_2 - x_3 - x_4, \bar{x}_1 + x_1, \dots, \bar{x}_1 + x_4, \bar{x}_1 + \bar{x}_2, \dots, \bar{x}_1 + \bar{x}_{K-4}). \tag{B.14}$$

Substituting (B.14) into the governing equation (5.5) with (5.4), we have

$$(\bar{x}_1 + \bar{x}_j)(v_{j+4} - \bar{v}) = 0, \quad j = 2, \dots, K-4. \tag{B.15}$$

Note that by the definition of the critical point $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_{\mathrm{c}}^{\alpha_1})$, we have $v_j - \bar{v} \neq 0$ $(j \notin \alpha_1)$ at $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_{\mathrm{c}}^{\alpha_1})$, which means $v_{j+4} - \bar{v} \neq 0$ $(j = 2, \dots, K-4)$. Then, the continuity of the payoff function ensures $v_{j+4} - \bar{v} \neq 0$ in a neighborhood of $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_{\mathrm{c}}^{\alpha_1})$. Hence, we have

$$\bar{x}_1 + \bar{x}_j = 0, \quad j = 2, \dots, K-4. \tag{B.16}$$

Substituting (B.14) and (B.16) into the condition (5.2), we have

$$1 + 5\bar{x}_1 = 1. \tag{B.17}$$

Hence, we have

$$\bar{x}_1 = 0. \tag{B.18}$$

By the conditions (B.16) and (B.18), we can represent v_j as a function of x_1, x_2, x_3, x_4, and ψ:

$$v_j = v_j(1 - x_1 - x_2 - x_3 - x_4, x_1, x_2, x_3, x_4, \boldsymbol{0}_{K-5}, \psi). \tag{B.19}$$

We take a set of vectors $\{\boldsymbol{\xi}_j \mid j = 1, \dots, 4\}$ that satisfies $\boldsymbol{\xi}_j^\top J_c = \boldsymbol{0}^\top$:

$$\boldsymbol{\xi}_1 = (0, 1, 0, 0, 0, 0, \dots, 0)^\top, \tag{B.20}$$

$$\boldsymbol{\xi}_2 = (0, 0, 1, 0, 0, 0, \ldots, 0)^\top, \tag{B.21}$$
$$\boldsymbol{\xi}_3 = (0, 0, 0, 1, 0, 0, \ldots, 0)^\top, \tag{B.22}$$
$$\boldsymbol{\xi}_4 = (0, 0, 0, 0, 1, 0, \ldots, 0)^\top. \tag{B.23}$$

We can obtain the bifurcation equation for Type α_1 orbit as the inner product between $\boldsymbol{F}$ and $\boldsymbol{\xi}_j$:

$$\tilde{F}_1(x_1, x_2, x_3, x_4, \psi) = x_1(v_2 - \bar{v}) = x_1(v_2 - v_1), \tag{B.24}$$
$$\tilde{F}_2(x_1, x_2, x_3, x_4, \psi) = x_2(v_3 - \bar{v}) = x_2(v_3 - v_1), \tag{B.25}$$
$$\tilde{F}_3(x_1, x_2, x_3, x_4, \psi) = x_3(v_4 - \bar{v}) = x_3(v_4 - v_1), \tag{B.26}$$
$$\tilde{F}_4(x_1, x_2, x_3, x_4, \psi) = x_4(v_5 - \bar{v}) = x_4(v_5 - v_1), \tag{B.27}$$

where $\psi = \phi - \phi_{\mathrm{c}}^{\alpha_1}$ represents the increment of ϕ. Therein, we used $\bar{v} = v_1$ since

$$F_1 = (1 - x_1 - x_2 - x_3 - x_4)(v_1 - \bar{v}) = 0. \tag{B.28}$$

The bifurcation equation inherits the equivariance in (5.8) as

$$\tilde{T}(g)\tilde{\boldsymbol{F}}(\boldsymbol{x}, \psi) = \tilde{\boldsymbol{F}}(\tilde{T}(g)\boldsymbol{x}, \psi), \quad g \in G, \tag{B.29}$$

where $\tilde{T}$ is a subrepresentation of T on $\ker(J_{\mathrm{c}})$. The equivariance condition for $\tilde{T}(r)$ imposes

$$\tilde{F}_2(x_1, x_2, x_3, x_4) = \tilde{F}_1(x_2, x_3, x_4, x_1), \tag{B.30}$$
$$\tilde{F}_3(x_1, x_2, x_3, x_4) = \tilde{F}_2(x_2, x_3, x_4, x_1), \tag{B.31}$$
$$\tilde{F}_4(x_1, x_2, x_3, x_4) = \tilde{F}_3(x_2, x_3, x_4, x_1), \tag{B.32}$$
$$\tilde{F}_1(x_1, x_2, x_3, x_4) = \tilde{F}_4(x_2, x_3, x_4, x_1). \tag{B.33}$$

Combining (B.30) and (B.31), we have

$$\tilde{F}_3(x_1, x_2, x_3, x_4) = \tilde{F}_1(x_3, x_4, x_1, x_2). \tag{B.34}$$

Combining (B.34) and (B.32), we have

$$\tilde{F}_4(x_1, x_2, x_3, x_4) = \tilde{F}_1(x_4, x_1, x_2, x_3). \tag{B.35}$$

The remaining condition (B.33) is equivalent to (B.35). To sum up, we have the conditions (5.23) and (5.24) in Lemma 4.

Let R be a function as

$$R(\boldsymbol{x}, \psi) \equiv v_2 - v_1. \tag{B.36}$$

We expand R into a power series as

$$R(\boldsymbol{x}, \psi) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} A_{abcd}(\psi) {x_1}^a {x_2}^b {x_3}^c {x_4}^d \tag{B.37}$$

with coefficients $A_{abcd}(\psi) \in \mathbb{R}$. Then, we can represent $\tilde{F}_1$ as

$$\tilde{F}_1(\boldsymbol{x}, \psi) = x_1 R(\boldsymbol{x}, \psi) = x_1 \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} A_{abcd}(\psi) {x_1}^a {x_2}^b {x_3}^c {x_4}^d. \tag{B.38}$$

We conclude

$$\tilde{F}_2(\boldsymbol{x}, \psi) = x_2 \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} A_{abcd}(\psi) {x_2}^a {x_3}^b {x_4}^c {x_1}^d, \tag{B.39}$$

$$\tilde{F}_3(\boldsymbol{x}, \psi) = x_3 \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} A_{abcd}(\psi) {x_3}^a {x_4}^b {x_1}^c {x_2}^d, \tag{B.40}$$

$$\tilde{F}_4(\boldsymbol{x}, \psi) = x_4 \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} A_{abcd}(\psi) {x_4}^a {x_1}^b {x_2}^c {x_3}^d. \tag{B.41}$$

On the other hand, the equivariance condition for $\tilde{T}(s)$ imposes

$$\tilde{F}_1(x_1, x_2, x_3, x_4) = \tilde{F}_1(x_1, x_4, x_3, x_2), \tag{B.42}$$

$$\tilde{F}_4(x_1, x_2, x_3, x_4) = \tilde{F}_2(x_1, x_4, x_3, x_2), \tag{B.43}$$

$$\tilde{F}_3(x_1, x_2, x_3, x_4) = \tilde{F}_3(x_1, x_4, x_3, x_2), \tag{B.44}$$

$$\tilde{F}_2(x_1, x_2, x_3, x_4) = \tilde{F}_4(x_1, x_4, x_3, x_2). \tag{B.45}$$

Combining (B.38) and (B.42), we have

$$\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} A_{abcd}(\psi) {x_1}^a {x_2}^b {x_3}^c {x_4}^d = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} A_{abcd}(\psi) {x_1}^a {x_4}^b {x_2}^c {x_3}^d, \tag{B.46}$$

which means $A_{abcd} = A_{adcb}$. The remaining conditions (B.43)–(B.45) lead to the same result as (B.46).

Since $(\boldsymbol{x}, \psi) = (0, 0, 0, 0, 0)$ corresponds to the critical point, we have

$$A_{0000}(0) = \left. \frac{\partial \tilde{F}_1}{\partial x_1} \right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0)} = 0. \tag{B.47}$$

Since $A'_{0000}(0)$ is generically nonzero, we have $A_{0000}(\psi) \approx a_0 \psi$ with

$$a_0 = A'_{0000}(0) = \left. \frac{\partial R}{\partial \psi} \right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0)}. \tag{B.48}$$

Then, the asymptotic form of the bifurcation equation becomes

$$\tilde{F}_1(\boldsymbol{x}, \psi) \approx x_1 \{a_0 \psi + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4\}, \tag{B.49}$$

$$\tilde{F}_2(\boldsymbol{x}, \psi) \approx x_2 \{a_0 \psi + a_1 x_2 + a_2 x_3 + a_3 x_4 + a_4 x_1\}, \tag{B.50}$$

$$\tilde{F}_3(\boldsymbol{x}, \psi) \approx x_3\{a_0\psi + a_1x_3 + a_2x_4 + a_3x_1 + a_4x_2\}, \tag{B.51}$$

$$\tilde{F}_4(\boldsymbol{x}, \psi) \approx x_4\{a_0\psi + a_1x_4 + a_2x_1 + a_3x_2 + a_4x_3\}, \tag{B.52}$$

where

$$a_1 = A_{1000}(0) = \left.\frac{\partial R}{\partial x_1}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0)}, \tag{B.53}$$

$$a_2 = A_{0100}(0) = \left.\frac{\partial R}{\partial x_2}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0)}, \tag{B.54}$$

$$a_3 = A_{0010}(0) = \left.\frac{\partial R}{\partial x_3}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0)}, \tag{B.55}$$

$$a_4 = A_{0001}(0) = \left.\frac{\partial R}{\partial x_4}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0)}. \tag{B.56}$$

B.1.2. Existence of Bifurcating Solutions

We can predict the following bifurcating solutions (cf., Fig. 5.5):

$$\begin{cases} \boldsymbol{x}_{\text{Square-I}} = w(1,1,1,1), \\ \boldsymbol{x}_{\text{Duo-I}} = w(1,1,0,0), \\ \boldsymbol{x}_{\text{Duo-II}} = w(1,0,1,0), \\ \boldsymbol{x}_{\text{Mono-I}} = w(1,0,0,0) \end{cases} \tag{B.57}$$

for some $w > 0$.

We first show the existence of Square-I solution. Substituting $\boldsymbol{x}_{\text{Square-I}} = w(1,1,1,1)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Square-I}}, \psi) = w \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty} A_{abcd}(\psi) w^{a+b+c+d} \approx w\{a_0\psi + (a_1 + a_2 + a_3 + a_4)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Square-I}}, \psi) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution:

$$\psi = \psi_{\text{Square-I}} \approx -\frac{a_1 + a_2 + a_3 + a_4}{a_0} w. \tag{B.58}$$

Substituting $\boldsymbol{x}_{\text{Square-I}}$ into $\tilde{F}_2$, $\tilde{F}_3$, and $\tilde{F}_4$ in (B.39), (B.40), and (B.41), we see that $\tilde{F}_2 = \tilde{F}_3 = \tilde{F}_4 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Square-I}}$. The other solutions can be treated similarly as explained below.

Substituting $\boldsymbol{x}_{\text{Duo-I}} = w(1,1,0,0)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Duo-I}}, \psi) = w \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} A_{ab00}(\psi) w^{a+b} \approx w\{a_0\psi + (a_1 + a_2)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Duo-I}}, \psi) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution:

$$\psi = \psi_{\text{Duo-I}} \approx -\frac{a_1 + a_2}{a_0} w. \tag{B.59}$$

Substituting $\boldsymbol{x}_{\text{Duo-I}}$ into $\tilde{F}_2$ in (B.39), we see that $\tilde{F}_2 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Duo-I}}$.

Substituting $\boldsymbol{x}_{\text{Duo-II}} = w(1, 0, 1, 0)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Duo-II}}, \psi) = w \sum_{a=0}^{\infty} \sum_{c=0}^{\infty} A_{a0c0}(\psi) w^{a+c} \approx w\{a_0\psi + (a_1 + a_3)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Duo-II}}, \psi) = 0$ has the trivial solution $(w = 0)$ and a bifurcating solution:

$$\psi = \psi_{\text{Duo-II}} \approx -\frac{a_1 + a_3}{a_0} w. \tag{B.60}$$

Substituting $\boldsymbol{x}_{\text{Duo-II}}$ into $\tilde{F}_3$ in (B.40), we see that $\tilde{F}_3 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Duo-II}}$.

Substituting $\boldsymbol{x}_{\text{Mono-I}} = w(1, 0, 0, 0)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Mono-I}}, \psi) = w \sum_{a=0}^{\infty} A_{a000}(\psi) w^{a} \approx w(a_0\psi + a_1 w).$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Mono-I}}, \psi) = 0$ has the trivial solution $(w = 0)$ and a bifurcating solution:

$$\psi = \psi_{\text{Mono-I}} \approx -\frac{a_1}{a_0} w. \tag{B.61}$$

Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Mono-I}}$.

B.1.3. Stability of Bifurcating Solutions

The asymptotic form of the Jacobian matrix $\tilde{J} = \partial\tilde{\boldsymbol{F}}/\partial\boldsymbol{x}$ becomes

$$\tilde{J}(\boldsymbol{x}, \psi) \approx \psi a_0 I_4 + x_1\tilde{J}_1 + x_2\tilde{J}_2 + x_3\tilde{J}_3 + x_4\tilde{J}_4, \tag{B.62}$$

where

$$\tilde{J}_1 = \begin{bmatrix} 2a_1 & a_2 & a_3 & a_4 \\ 0 & a_4 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_2 \end{bmatrix}, \quad \tilde{J}_2 = \begin{bmatrix} a_2 & 0 & 0 & 0 \\ a_4 & 2a_1 & a_2 & a_3 \\ 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & a_3 \end{bmatrix},$$

$$\tilde{J}_3 = \begin{bmatrix} a_3 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ a_3 & a_4 & 2a_1 & a_2 \\ 0 & 0 & 0 & a_4 \end{bmatrix}, \quad \tilde{J}_4 = \begin{bmatrix} a_4 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ a_2 & a_3 & a_4 & 2a_1 \end{bmatrix}.$$

To begin with, we investigate the stability of Square-I solution. Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Square-I}}, \psi_{\text{Square-I}}) = (w, w, w, w, -\frac{a_1 + a_2 + a_3 + a_4}{a_0} w),$$

Table B.1: Stability conditions of bifurcating solutions for critical points associated with Type α_i orbit.

Solution	Case	Stability conditions
$\boldsymbol{x}_{\text{Square-I}}$	$w > 0$	$a_1 + a_3 < -\lvert a_2 + a_4 \rvert$
		$a_1 - a_3 < 0$
	$w < 0$	$a_1 + a_3 > \lvert a_2 + a_4 \rvert$
		$a_1 - a_3 > 0$
$\boldsymbol{x}_{\text{Duo-I}}$	$w > 0$	$a_1 - a_2 < 0$
		$a_1 + a_4 < 0$
		$\max(-a_2 + a_4, 0) < a_1 - a_3$
	$w < 0$	$a_1 - a_2 > 0$
		$a_1 + a_4 > 0$
		$\min(-a_2 + a_4, 0) > a_1 - a_3$
$\boldsymbol{x}_{\text{Duo-II}}$	$w > 0$	$a_1 - a_3 < 0$
		$a_2 + a_4 < a_1 + a_3 < 0$
	$w < 0$	$a_1 - a_3 > 0$
		$a_2 + a_4 > a_1 + a_3 > 0$
$\boldsymbol{x}_{\text{Mono-I}}$	$w > 0$	$\max(a_2, a_3, a_4) < a_1 < 0$
	$w < 0$	$\min(a_2, a_3, a_4) > a_1 > 0$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Square-I}}, \psi_{\text{Square-I}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_4 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_1 \end{bmatrix}. \tag{B.63}$$

The eigenvalues of this matrix are given as follows:

$$\lambda_1, \lambda_2 \approx w(a_1 + a_3) \pm w(a_2 + a_4),$$
$$\lambda_3, \lambda_4 \approx w(a_1 - a_3) \pm \mathrm{i}w(a_2 - a_4).$$

Thus, the stability of $\boldsymbol{x}_{\text{Square-I}}$ depends on the values of a_1, a_2, a_3, and a_4. The other solutions can be treated similarly. Table B.1 summarizes the stability conditions of bifurcating solutions.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Duo-I}}, \psi_{\text{Duo-I}}) = (w, w, 0, 0, -\frac{a_1 + a_2}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Duo-I}}, \psi_{\text{Duo-I}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_4 & a_1 - a_2 + a_4 & a_2 & a_3 \\ 0 & 0 & -a_1 - a_2 + a_3 + a_4 & 0 \\ 0 & 0 & 0 & -a_1 + a_3 \end{bmatrix}. \tag{B.64}$$

The eigenvalues of this matrix are given as follows:

$$\lambda_1 \approx w(a_1 - a_2),$$

$$\lambda_2 \approx w(a_1 + a_4),$$
$$\lambda_3 \approx w(-a_1 + a_3),$$
$$\lambda_4 \approx w(-a_1 + a_3 - a_2 + a_4).$$

Thus, the stability of $\boldsymbol{x}_{\text{Duo-I}}$ depends on the values of a_1, a_2, a_3, and a_4.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Duo-II}}, \psi_{\text{Duo-II}}) = (w, 0, w, 0, -\frac{a_1 + a_3}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Duo-II}}, \psi_{\text{Duo-II}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & -a_1 + a_2 - a_3 + a_4 & 0 & 0 \\ a_3 & a_4 & a_1 & a_2 \\ 0 & 0 & 0 & -a_1 + a_2 - a_3 + a_4 \end{bmatrix}. \tag{B.65}$$

The eigenvalues of this matrix are given as follows:

$$\lambda_1, \lambda_2 \approx w(a_1 \pm a_3),$$
$$\lambda_3 \approx w(-a_1 - a_3 + a_2 + a_4) \quad \text{(repeated twice)}.$$

Thus, the stability of $\boldsymbol{x}_{\text{Duo-II}}$ depends on the values of a_1, a_2, a_3, and a_4.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Mono-I}}, \psi_{\text{Mono-I}}) = (w, 0, 0, 0, -\frac{a_1}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Mono-I}}, \psi_{\text{Mono-I}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & -a_1 + a_4 & 0 & 0 \\ 0 & 0 & -a_1 + a_3 & 0 \\ 0 & 0 & 0 & -a_1 + a_2 \end{bmatrix}. \tag{B.66}$$

The eigenvalues of this matrix are given as follows:

$$\lambda_1 \approx w a_1,$$
$$\lambda_2 \approx w(-a_1 + a_2),$$
$$\lambda_3 \approx w(-a_1 + a_3),$$
$$\lambda_4 \approx w(-a_1 + a_4).$$

Thus, the stability of $\boldsymbol{x}_{\text{Mono-I}}$ depends on the values of a_1, a_2, a_3, and a_4.

B.2. Bifurcation Point with Type β_i Orbit

We investigate critical points associated with Type β_i orbit.

B.2.1. Derivation of Bifurcation Equations

We focus on a critical point associated with Type β_1 orbit. We can investigate critical points associated with Type β_i ($i = 2, \dots, n_2$) orbits in a similar manner. Note that n_1 and n_2 are dependent on the number of places K (cf., Fig. 5.3 for $K = 25$).

Let $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_{\mathrm{c}}^{\beta_1})$ be a critical point associated with Type β_1 orbit:

$$\beta_1 = \{(4n_1 + 1) + 1, \dots, (4n_1 + 1) + 8\}. \tag{B.67}$$

Note that we can investigate critical points associated with Type β_i ($i = 2, \dots, n_2$) orbits in a similar manner. By the definition of $\phi_{\mathrm{c}}^{\beta_1}$, we assume that $v_{\beta_1} - v_1 = 0$. Hence, the Jacobian matrix $J_{\mathrm{c}} \equiv J(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_{\mathrm{c}}^{\beta_1})$ takes the following form:

$$J_{\mathrm{c}} = \begin{pmatrix} -v_1 & -v_{\alpha_1}\mathbf{1}_4 & \cdots & -v_{\alpha_{n_1}}\mathbf{1}_4 & -v_1\mathbf{1}_8 & -v_{\beta_2}\mathbf{1}_8 & \cdots & -v_{\beta_{n_2}}\mathbf{1}_8 \\ & (v_{\alpha_1} - v_1)I_4 & & & & & & \\ & & \ddots & & & & & \\ & & & (v_{\alpha_{n_1}} - v_1)I_4 & & & & \\ & & & & 0 \times I_8 & & & \\ & & & & & (v_{\beta_2} - v_1)I_8 & & \\ & & & & & & \ddots & \\ & & & & & & & (v_{\beta_{n_2}} - v_1)I_8 \end{pmatrix}, \tag{B.68}$$

where I_j is the $j \times j$ identity matrix, and $\mathbf{1}_j$ is the j-dimensional all-one row vector.

We decompose the increment $\boldsymbol{\lambda} - \boldsymbol{\lambda}^{\mathrm{FA}}$ into two components as

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{FA}} + \boldsymbol{w} + \bar{\boldsymbol{w}}, \tag{B.69}$$

where $\boldsymbol{w} \in \ker(J_{\mathrm{c}})$ and $\bar{\boldsymbol{w}} \in \ker(J_{\mathrm{c}})^{\perp}$. Note that $\ker(J_{\mathrm{c}})$ represents the kernel space of J_{c}, which is generated by a basis satisfying $J_{\mathrm{c}}\boldsymbol{\eta} = \mathbf{0}$:

$$\ker(J_{\mathrm{c}}) = \{\boldsymbol{\eta} \in \mathbb{R}^K \mid \eta_1 + \sum_{k \in \beta_1} \eta_k = 0,\ \eta_j = 0,\ j \notin \{1\} \cup \beta_1\}, \tag{B.70}$$

where η_j denotes the jth component of $\boldsymbol{\eta}$. We take a basis $\{\boldsymbol{\eta}_j \mid j = 1, \dots, 8\}$ of $\ker(J_{\mathrm{c}})$ as

$$\boldsymbol{\eta}_j = (-1, \underbrace{0, \dots, 0}_{j+4n_1-1 \text{ times}}, 1, \underbrace{0, \dots, 0}_{K-(j+4n_1+1) \text{ times}}). \tag{B.71}$$

Note that $\boldsymbol{\eta}_j$ is a vector whose component corresponding to jth place of the orbit β_1 is 1. Then, we can represent $\boldsymbol{w}$ as

$$\boldsymbol{w} = \sum_{k=1}^{8} x_k \boldsymbol{\eta}_k. \tag{B.72}$$

We take a basis $\{\bar{\boldsymbol{\eta}}_j \mid j = 1, \ldots, K-8\}$ of $\ker(J_c)^\perp$ as

$$\bar{\boldsymbol{\eta}}_1 = (1, 1, \ldots, 1), \tag{B.73}$$

$$\bar{\boldsymbol{\eta}}_j = (0, \underbrace{0, \ldots, 0}_{j-2 \text{ times}}, 1, \underbrace{0, \ldots, 0}_{K-j \text{ times}}), \quad j = 2, \ldots, 4n_1 + 1, \tag{B.74}$$

$$\bar{\boldsymbol{\eta}}_j = (0, \underbrace{0, \ldots, 0}_{j+6 \text{ times}}, 1, \underbrace{0, \ldots, 0}_{K-(j+8) \text{ times}}), \quad j = 4n_1 + 2, \ldots, K-8. \tag{B.75}$$

Then, we can represent $\bar{\boldsymbol{w}}$ as

$$\bar{\boldsymbol{w}} = \bar{x}_1 \bar{\boldsymbol{\eta}}_1 + \sum_{k=2}^{K-8} \bar{x}_k \bar{\boldsymbol{\eta}}_k. \tag{B.76}$$

Combining (B.69), (B.72), and (B.76), we can represent $\boldsymbol{\lambda}$ as

$$\boldsymbol{\lambda} = (1 + \bar{x}_1 - \sum_{k=1}^{8} x_k, \bar{x}_1 + \bar{x}_2, \ldots, \bar{x}_1 + \bar{x}_{4n_1+1}, \bar{x}_1 + x_1, \ldots, \bar{x}_1 + x_8, \bar{x}_1 + \bar{x}_{4n_1+2}, \ldots, \bar{x}_1 + \bar{x}_{K-8}). \tag{B.77}$$

Substituting (B.77) into the governing equation (5.5), we have

$$(\bar{x}_1 + \bar{x}_j)(v_j - \bar{v}) = 0, \quad j = 2, \ldots, 4n_1 + 1. \tag{B.78}$$

$$(\bar{x}_1 + \bar{x}_j)(v_{j+8} - \bar{v}) = 0, \quad j = 4n_1 + 2, \ldots, K-8. \tag{B.79}$$

Note that by the definition of the critical point $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_c^{\beta_1})$, we have $v_j - \bar{v} \neq 0$ $(j \notin \beta_1)$ at $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_c^{\beta_1})$. Then, the continuity of the payoff function ensures $v_j - \bar{v} \neq 0$ $(j \notin \beta_1)$ in a neighborhood of $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_c^{\beta_1})$, which means $v_j - \bar{v} \neq 0$ $(j = 2, \ldots, 4n_1 + 1)$ and $v_{j+8} - \bar{v} \neq 0$ $(j = 4n_1 + 2, \ldots, K-8)$. Hence, we have

$$\bar{x}_1 + \bar{x}_j = 0, \quad j = 2, \ldots, K-8. \tag{B.80}$$

Substituting (B.77) and (B.80) into the condition (5.2), we have

$$1 + 9\bar{x}_1 = 1. \tag{B.81}$$

Hence, we have

$$\bar{x}_1 = 0. \tag{B.82}$$

By the conditions (B.80) and (B.82), we can represent v_j as a function of x_1, x_2, x_3, x_4, and ψ:

$$v_j = v_j(1 - \sum_{k=1}^{8} x_k, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \mathbf{0}_{K-9}, \psi). \tag{B.83}$$

We take a set of vectors $\{\boldsymbol{\xi}_j \mid j = 1, \ldots, 8\}$ that satisfies $\boldsymbol{\xi}_j^\top J_c = \mathbf{0}^\top$:

$$\boldsymbol{\xi}_j = (0, \underbrace{0, \ldots, 0}_{j+4n_1-1 \text{ times}}, 1, \underbrace{0, \ldots, 0}_{K-(j+4n_1+1) \text{ times}}). \tag{B.84}$$

We can obtain the bifurcation equation for Type β_1 orbit as the inner product between $\boldsymbol{F}$ and $\boldsymbol{\xi}_j$:

$$\tilde{F}_j(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \psi) = x_j(v_{j+4n_1+1} - \bar{v}) = x_j(v_{j+4n_1+1} - v_1), \tag{B.85}$$

where $\psi = \phi - \phi_{\mathrm{c}}^{\beta_1}$ represents the increment of ϕ. Therein, we used $\bar{v} = v_1$ since

$$F_1 = (1 - \sum_{k=1}^{8} x_k)(v_1 - \bar{v}) = 0. \tag{B.86}$$

The bifurcation equation inherits the equivariance in (5.8) as

$$\tilde{T}(g)\tilde{\boldsymbol{F}}(\boldsymbol{x}, \psi) = \tilde{\boldsymbol{F}}(\tilde{T}(g)\boldsymbol{x}, \psi), \quad g \in G, \tag{B.87}$$

where $\tilde{T}$ is a subrepresentation of T on $\ker(J_{\mathrm{c}})$. The equivariance condition for $\tilde{T}(r)$ imposes

$$\tilde{F}_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_1(x_3, x_4, x_5, x_6, x_7, x_8, x_1, x_2), \tag{B.88}$$
$$\tilde{F}_4(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_2(x_3, x_4, x_5, x_6, x_7, x_8, x_1, x_2), \tag{B.89}$$
$$\tilde{F}_5(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_3(x_3, x_4, x_5, x_6, x_7, x_8, x_1, x_2), \tag{B.90}$$
$$\tilde{F}_6(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_4(x_3, x_4, x_5, x_6, x_7, x_8, x_1, x_2), \tag{B.91}$$
$$\tilde{F}_7(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_5(x_3, x_4, x_5, x_6, x_7, x_8, x_1, x_2), \tag{B.92}$$
$$\tilde{F}_8(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_6(x_3, x_4, x_5, x_6, x_7, x_8, x_1, x_2), \tag{B.93}$$
$$\tilde{F}_1(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_7(x_3, x_4, x_5, x_6, x_7, x_8, x_1, x_2), \tag{B.94}$$
$$\tilde{F}_2(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_8(x_3, x_4, x_5, x_6, x_7, x_8, x_1, x_2). \tag{B.95}$$

Using (B.88), we obtain $\tilde{F}_3$ from $\tilde{F}_1$. Combining (B.88) and (B.90), we have

$$\tilde{F}_5(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_1(x_5, x_6, x_7, x_8, x_1, x_2, x_3, x_4). \tag{B.96}$$

Combining (B.92) and (B.96), we have

$$\tilde{F}_7(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_1(x_7, x_8, x_1, x_2, x_3, x_4, x_5, x_6). \tag{B.97}$$

Using (B.89), we obtain $\tilde{F}_4$ from $\tilde{F}_2$. Combining (B.89) and (B.91), we have

$$\tilde{F}_6(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_2(x_5, x_6, x_7, x_8, x_1, x_2, x_3, x_4). \tag{B.98}$$

Combining (B.93) and (B.98), we have

$$\tilde{F}_8(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_2(x_7, x_8, x_1, x_2, x_3, x_4, x_5, x_6). \tag{B.99}$$

The conditions (B.94) and (B.95) are equivalent to (B.97) and (B.99). On the other hand, the equivariance condition for $\tilde{T}(s)$ imposes

$$\tilde{F}_8(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_1(x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1), \tag{B.100}$$

$$\tilde{F}_7(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_2(x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1), \tag{B.101}$$
$$\tilde{F}_6(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_3(x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1), \tag{B.102}$$
$$\tilde{F}_5(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_4(x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1), \tag{B.103}$$
$$\tilde{F}_4(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_5(x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1), \tag{B.104}$$
$$\tilde{F}_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_6(x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1), \tag{B.105}$$
$$\tilde{F}_2(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_7(x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1), \tag{B.106}$$
$$\tilde{F}_1(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_8(x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1). \tag{B.107}$$

Using (B.100), we obtain $\tilde{F}_8$ from $\tilde{F}_1$. Combining (B.102) and (B.88), we have

$$\tilde{F}_6(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_1(x_6, x_5, x_4, x_3, x_2, x_1, x_8, x_7). \tag{B.108}$$

Combining (B.104) and (B.96), we have

$$\tilde{F}_4(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_1(x_4, x_3, x_2, x_1, x_8, x_7, x_6, x_5). \tag{B.109}$$

Combining (B.106) and (B.97), we have

$$\tilde{F}_2(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \tilde{F}_1(x_2, x_1, x_8, x_7, x_6, x_5, x_4, x_3). \tag{B.110}$$

The conditions (B.101), (B.103), (B.105), and (B.107) are equivalent to (B.106), (B.104), (B.102), and (B.100). The remaining conditions (B.89), (B.98), and (B.99) for $\tilde{T}(r)$ are equivalent to (B.109), (B.108), and (B.100) for $\tilde{T}(s)$. To sum up, we have the condition (5.26) in Lemma 5.

Let R be a function as

$$R(\boldsymbol{x}, \psi) \equiv v_{4n_1+2} - v_1. \tag{B.111}$$

We expand R into a power series as

$$R(\boldsymbol{x}, \psi) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{e=0}^{\infty} \sum_{f=0}^{\infty} \sum_{g=0}^{\infty} \sum_{h=0}^{\infty} A_{abcdefgh}(\psi) {x_1}^a {x_2}^b {x_3}^c {x_4}^d {x_5}^e {x_6}^f {x_7}^g {x_8}^h \tag{B.112}$$

with coefficients $A_{abcdefgh}(\psi) \in \mathbb{R}$. Then, we can represent $\tilde{F}_1$ as

$$\tilde{F}_1(\boldsymbol{x}, \psi) = x_1 R(\boldsymbol{x}, \psi) = x_1 \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{e=0}^{\infty} \sum_{f=0}^{\infty} \sum_{g=0}^{\infty} \sum_{h=0}^{\infty} A_{abcdefgh}(\psi) {x_1}^a {x_2}^b {x_3}^c {x_4}^d {x_5}^e {x_6}^f {x_7}^g {x_8}^h. \tag{B.113}$$

We conclude

$$\tilde{F}_2(\boldsymbol{x}, \psi) = x_2 \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{e=0}^{\infty} \sum_{f=0}^{\infty} \sum_{g=0}^{\infty} \sum_{h=0}^{\infty} A_{abcdefgh}(\psi) {x_2}^a {x_1}^b {x_8}^c {x_7}^d {x_6}^e {x_5}^f {x_4}^g {x_3}^h, \tag{B.114}$$

$$\tilde{F}_3(\boldsymbol{x}, \psi) = x_3 \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \sum_{e=0}^{\infty} \sum_{f=0}^{\infty} \sum_{g=0}^{\infty} \sum_{h=0}^{\infty} A_{abcdefgh}(\psi) {x_3}^a {x_4}^b {x_5}^c {x_6}^d {x_7}^e {x_8}^f {x_1}^g {x_2}^h, \tag{B.115}$$

$$\tilde{F}_4(\boldsymbol{x},\psi) = x_4 \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty}\sum_{g=0}^{\infty}\sum_{h=0}^{\infty} A_{abcdefgh}(\psi) {x_4}^a {x_3}^b {x_2}^c {x_1}^d {x_8}^e {x_7}^f {x_6}^g {x_5}^h, \quad \text{(B.116)}$$

$$\tilde{F}_5(\boldsymbol{x},\psi) = x_5 \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty}\sum_{g=0}^{\infty}\sum_{h=0}^{\infty} A_{abcdefgh}(\psi) {x_5}^a {x_6}^b {x_7}^c {x_8}^d {x_1}^e {x_2}^f {x_3}^g {x_4}^h, \quad \text{(B.117)}$$

$$\tilde{F}_6(\boldsymbol{x},\psi) = x_6 \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty}\sum_{g=0}^{\infty}\sum_{h=0}^{\infty} A_{abcdefgh}(\psi) {x_6}^a {x_5}^b {x_4}^c {x_3}^d {x_2}^e {x_1}^f {x_8}^g {x_7}^h, \quad \text{(B.118)}$$

$$\tilde{F}_7(\boldsymbol{x},\psi) = x_7 \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty}\sum_{g=0}^{\infty}\sum_{h=0}^{\infty} A_{abcdefgh}(\psi) {x_7}^a {x_8}^b {x_1}^c {x_2}^d {x_3}^e {x_4}^f {x_5}^g {x_6}^h, \quad \text{(B.119)}$$

$$\tilde{F}_8(\boldsymbol{x},\psi) = x_8 \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty}\sum_{g=0}^{\infty}\sum_{h=0}^{\infty} A_{abcdefgh}(\psi) {x_8}^a {x_7}^b {x_6}^c {x_5}^d {x_4}^e {x_3}^f {x_2}^g {x_1}^h. \quad \text{(B.120)}$$

Since $(\boldsymbol{x},\psi) = (0,0,0,0,0,0,0,0,0)$ corresponds to the critical point, we have

$$A_{00000000}(0) = \left.\frac{\partial \tilde{F}_1}{\partial x_1}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0,0,0,0,0)} = 0. \quad \text{(B.121)}$$

Since $A'_{00000000}(0)$ is generically nonzero, we have $A_{00000000}(\psi) \approx a_0\psi$ with

$$a_0 = A'_{00000000}(0) = \left.\frac{\partial R}{\partial \psi}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0,0,0,0,0)}. \quad \text{(B.122)}$$

Then, the asymptotic form of the bifurcation equation becomes

$$\tilde{F}_1(\boldsymbol{x},\psi) \approx x_1\{a_0\psi + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 + a_6x_6 + a_7x_7 + a_8x_8\}, \quad \text{(B.123)}$$

$$\tilde{F}_2(\boldsymbol{x},\psi) \approx x_2\{a_0\psi + a_1x_2 + a_2x_1 + a_3x_8 + a_4x_7 + a_5x_6 + a_6x_5 + a_7x_4 + a_8x_3\}, \quad \text{(B.124)}$$

$$\tilde{F}_3(\boldsymbol{x},\psi) \approx x_3\{a_0\psi + a_1x_3 + a_2x_4 + a_3x_5 + a_4x_6 + a_5x_7 + a_6x_8 + a_7x_1 + a_8x_2\}, \quad \text{(B.125)}$$

$$\tilde{F}_4(\boldsymbol{x},\psi) \approx x_4\{a_0\psi + a_1x_4 + a_2x_3 + a_3x_2 + a_4x_1 + a_5x_8 + a_6x_7 + a_7x_6 + a_8x_5\}, \quad \text{(B.126)}$$

$$\tilde{F}_5(\boldsymbol{x},\psi) \approx x_5\{a_0\psi + a_1x_5 + a_2x_6 + a_3x_7 + a_4x_8 + a_5x_1 + a_6x_2 + a_7x_3 + a_8x_4\}, \quad \text{(B.127)}$$

$$\tilde{F}_6(\boldsymbol{x},\psi) \approx x_6\{a_0\psi + a_1x_6 + a_2x_5 + a_3x_4 + a_4x_3 + a_5x_2 + a_6x_1 + a_7x_8 + a_8x_7\}, \quad \text{(B.128)}$$

$$\tilde{F}_7(\boldsymbol{x},\psi) \approx x_7\{a_0\psi + a_1x_7 + a_2x_8 + a_3x_1 + a_4x_2 + a_5x_3 + a_6x_4 + a_7x_5 + a_8x_6\}, \quad \text{(B.129)}$$

$$\tilde{F}_8(\boldsymbol{x},\psi) \approx x_8\{a_0\psi + a_1x_8 + a_2x_7 + a_3x_6 + a_4x_5 + a_5x_4 + a_6x_3 + a_7x_2 + a_8x_1\}, \quad \text{(B.130)}$$

where

$$a_1 = A_{10000000}(0) = \left.\frac{\partial R}{\partial x_1}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0,0,0,0,0)}, \quad \text{(B.131)}$$

$$a_2 = A_{01000000}(0) = \left.\frac{\partial R}{\partial x_2}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0,0,0,0,0)}, \quad \text{(B.132)}$$

$$a_3 = A_{00100000}(0) = \left.\frac{\partial R}{\partial x_3}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0,0,0,0,0)}, \tag{B.133}$$

$$a_4 = A_{00010000}(0) = \left.\frac{\partial R}{\partial x_4}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0,0,0,0,0)}, \tag{B.134}$$

$$a_5 = A_{00001000}(0) = \left.\frac{\partial R}{\partial x_5}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0,0,0,0,0)}, \tag{B.135}$$

$$a_6 = A_{00000100}(0) = \left.\frac{\partial R}{\partial x_6}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0,0,0,0,0)}, \tag{B.136}$$

$$a_7 = A_{00000010}(0) = \left.\frac{\partial R}{\partial x_7}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0,0,0,0,0)}, \tag{B.137}$$

$$a_8 = A_{00000001}(0) = \left.\frac{\partial R}{\partial x_8}\right|_{(\boldsymbol{x},\psi)=(0,0,0,0,0,0,0,0,0)}. \tag{B.138}$$

B.2.2. Existence of Bifurcating Solutions

We can predict the following bifurcating solutions (cf., Fig. 5.6):

$$\begin{cases} \boldsymbol{x}_{\text{Square-II}} = w(1,1,1,1,1,1,1,1), \\ \boldsymbol{x}_{\text{Square-III}} = w(1,0,1,0,1,0,1,0), \\ \boldsymbol{x}_{\text{Quad-I}} = w(1,1,0,0,1,1,0,0), \\ \boldsymbol{x}_{\text{Quad-II}} = w(1,0,0,1,1,0,0,1), \\ \boldsymbol{x}_{\text{Duo-III}} = w(1,1,0,0,0,0,0,0), \\ \boldsymbol{x}_{\text{Duo-IV}} = w(1,0,0,1,0,0,0,0), \\ \boldsymbol{x}_{\text{Duo-V}} = w(1,0,0,0,1,0,0,0), \\ \boldsymbol{x}_{\text{Duo-VI}} = w(1,0,0,0,0,1,0,0), \\ \boldsymbol{x}_{\text{Duo-VII}} = w(1,0,0,0,0,0,0,1), \\ \boldsymbol{x}_{\text{Mono-II}} = w(1,0,0,0,0,0,0,0), \end{cases} \tag{B.139}$$

for some $w > 0$.

We first show the existence of Square-II solution. Substituting $\boldsymbol{x}_{\text{Square-II}} = w(1,1,1,1,1,1,1,1)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Square-I}},\psi) = w \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty}\sum_{g=0}^{\infty}\sum_{h=0}^{\infty} A_{abcdefgh}(\psi) w^{a+b+c+d+e+f+g+h}$$
$$\approx w\{a_0\psi + (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Square-II}},\psi) = 0$ has the trivial solution $(w = 0)$ and a bifurcating solution:

$$\psi = \psi_{\text{Square-II}} \approx -\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8}{a_0} w. \tag{B.140}$$

Substituting $\boldsymbol{x}_{\text{Square-II}}$ into (B.114)–(B.120), we see that $\tilde{F}_2 = \tilde{F}_3 = \tilde{F}_4 = \tilde{F}_5 = \tilde{F}_6 = \tilde{F}_7 = \tilde{F}_8 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Square-II}}$.

Substituting $\boldsymbol{x}_{\text{Square-III}} = w(1,0,1,0,1,0,1,0)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Square-III}}, \psi) = w \sum_{a=0}^{\infty}\sum_{c=0}^{\infty}\sum_{e=0}^{\infty}\sum_{g=0}^{\infty} A_{a0c0e0g0}(\psi) w^{a+c+e+g} \approx w\{a_0\psi + (a_1 + a_3 + a_5 + a_7)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Square-III}}, \psi) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution:

$$\psi = \psi_{\text{Square-III}} \approx -\frac{a_1 + a_3 + a_5 + a_7}{a_0} w. \tag{B.141}$$

Substituting $\boldsymbol{x}_{\text{Square-III}}$ into $\tilde{F}_3$, $\tilde{F}_5$, and $\tilde{F}_7$ in (B.115), (B.117), and (B.119), we see that $\tilde{F}_3 = \tilde{F}_5 = \tilde{F}_7 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Square-III}}$.

Substituting $\boldsymbol{x}_{\text{Quad-I}} = w(1,1,0,0,1,1,0,0)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Quad-I}}, \psi) = w \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty} A_{ab00ef00}(\psi) w^{a+b+e+f} \approx w\{a_0\psi + (a_1 + a_2 + a_5 + a_6)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Quad-I}}, \psi) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution:

$$\psi = \psi_{\text{Quad-I}} \approx -\frac{a_1 + a_2 + a_5 + a_6}{a_0} w. \tag{B.142}$$

Substituting $\boldsymbol{x}_{\text{Quad-I}}$ into $\tilde{F}_2$, $\tilde{F}_5$, and $\tilde{F}_6$ in (B.114), (B.117), and (B.118), we see that $\tilde{F}_2 = \tilde{F}_5 = \tilde{F}_6 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Quad-I}}$.

Substituting $\boldsymbol{x}_{\text{Quad-II}} = w(1,0,0,1,1,0,0,1)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Quad-II}}, \psi) = w \sum_{a=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{h=0}^{\infty} A_{a00de00h}(\psi) w^{a+d+e+h} \approx w\{a_0\psi + (a_1 + a_4 + a_5 + a_8)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Quad-II}}, \psi) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution:

$$\psi = \psi_{\text{Quad-II}} \approx -\frac{a_1 + a_4 + a_5 + a_8}{a_0} w. \tag{B.143}$$

Substituting $\boldsymbol{x}_{\text{Quad-II}}$ into $\tilde{F}_4$, $\tilde{F}_5$, and $\tilde{F}_8$ in (B.116), (B.117), and (B.120), we see that $\tilde{F}_4 = \tilde{F}_5 = \tilde{F}_8 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Quad-II}}$.

Substituting $\boldsymbol{x}_{\text{Duo-III}} = w(1,1,0,0,0,0,0,0)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Duo-III}}, \psi) = w \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} A_{ab000000}(\psi) w^{a+b} \approx w\{a_0\psi + (a_1 + a_2)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Duo-III}}, \psi) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution:

$$\psi = \psi_{\text{Duo-III}} \approx -\frac{a_1 + a_2}{a_0} w. \tag{B.144}$$

Substituting $\boldsymbol{x}_{\text{Duo-III}}$ into $\tilde{F}_2$ in (B.114), we see that $\tilde{F}_2 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Duo-III}}$.

Substituting $\boldsymbol{x}_{\text{Duo-IV}} = w(1,0,0,1,0,0,0,0)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Duo-IV}}, \psi) = w \sum_{a=0}^{\infty} \sum_{d=0}^{\infty} A_{a00d0000}(\psi) w^{a+d} \approx w\{a_0\psi + (a_1 + a_4)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Duo-IV}}, \psi) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution:

$$\psi = \psi_{\text{Duo-IV}} \approx -\frac{a_1 + a_4}{a_0} w. \tag{B.145}$$

Substituting $\boldsymbol{x}_{\text{Duo-IV}}$ into $\tilde{F}_4$ in (B.116), we see that $\tilde{F}_4 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Duo-IV}}$.

Substituting $\boldsymbol{x}_{\text{Duo-V}} = w(1,0,0,0,1,0,0,0)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Duo-V}}, \psi) = w \sum_{a=0}^{\infty} \sum_{e=0}^{\infty} A_{a000e000}(\psi) w^{a+e} \approx w\{a_0\psi + (a_1 + a_5)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Duo-V}}, \psi) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution:

$$\psi = \psi_{\text{Duo-V}} \approx -\frac{a_1 + a_5}{a_0} w. \tag{B.146}$$

Substituting $\boldsymbol{x}_{\text{Duo-V}}$ into $\tilde{F}_5$ in (B.117), we see that $\tilde{F}_5 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Duo-V}}$.

Substituting $\boldsymbol{x}_{\text{Duo-VI}} = w(1,0,0,0,0,1,0,0)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Duo-VI}}, \psi) = w \sum_{a=0}^{\infty} \sum_{f=0}^{\infty} A_{a0000f00}(\psi) w^{a+f} \approx w\{a_0\psi + (a_1 + a_6)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Duo-VI}}, \psi) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution:

$$\psi = \psi_{\text{Duo-VI}} \approx -\frac{a_1 + a_6}{a_0} w. \tag{B.147}$$

Substituting $\boldsymbol{x}_{\text{Duo-VI}}$ into $\tilde{F}_6$ in (B.118), we see that $\tilde{F}_6 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Duo-VI}}$.

Substituting $\boldsymbol{x}_{\text{Duo-VII}} = w(1,0,0,0,0,0,0,1)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Duo-VII}}, \psi) = w \sum_{a=0}^{\infty} \sum_{h=0}^{\infty} A_{a000000h}(\psi) w^{a+h} \approx w\{a_0\psi + (a_1 + a_8)w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Duo-VII}}, \psi) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution:

$$\psi = \psi_{\text{Duo-VII}} \approx -\frac{a_1 + a_8}{a_0} w. \tag{B.148}$$

Substituting $\boldsymbol{x}_{\text{Duo-VII}}$ into $\tilde{F}_8$ in (B.120), we see that $\tilde{F}_8 = \tilde{F}_1 = 0$. Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Duo-VII}}$.

Substituting $\boldsymbol{x}_{\text{Mono-II}} = w(1, 0, 0, 0, 0, 0, 0, 0)$ into $\tilde{F}_1$, we have

$$\tilde{F}_1(\boldsymbol{x}_{\text{Mono-II}}, \psi) = w \sum_{a=0}^{\infty} A_{a0000000}(\psi) w^a \approx w\{a_0\psi + a_1 w\}.$$

We see that $\tilde{F}_1(\boldsymbol{x}_{\text{Mono-II}}, \psi) = 0$ has the trivial solution ($w = 0$) and a bifurcating solution:

$$\psi = \psi_{\text{Mono-II}} \approx -\frac{a_1}{a_0} w. \tag{B.149}$$

Hence, the bifurcation equation is satisfied for $\boldsymbol{x}_{\text{Mono-II}}$.

B.2.3. Stability of Bifurcating Solutions

The asymptotic form of the Jacobian matrix $\tilde{J} = \partial \tilde{\boldsymbol{F}} / \partial \boldsymbol{x}$ becomes

$$\tilde{J}(\boldsymbol{x}, \psi) \approx \psi a_0 I_8 + x_1 \tilde{J}_1 + x_2 \tilde{J}_2 + x_3 \tilde{J}_3 + x_4 \tilde{J}_4 + x_5 \tilde{J}_5 + x_6 \tilde{J}_6 + x_7 \tilde{J}_7 + x_8 \tilde{J}_8, \tag{B.150}$$

where

$$\tilde{J}_1 = \begin{bmatrix} 2a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 \end{bmatrix}, \quad \tilde{J}_2 = \begin{bmatrix} a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 2a_1 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 \\ 0 & 0 & a_8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_7 \end{bmatrix},$$

$$\tilde{J}_3 = \begin{bmatrix} a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_7 & a_8 & 2a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix}, \quad \tilde{J}_4 = \begin{bmatrix} a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ a_4 & a_3 & a_2 & 2a_1 & a_8 & a_7 & a_6 & a_5 \\ 0 & 0 & 0 & 0 & a_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 \end{bmatrix},$$

$$\tilde{J}_5 = \begin{bmatrix} a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 & 2a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 \end{bmatrix}, \quad \tilde{J}_6 = \begin{bmatrix} a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ a_6 & a_5 & a_4 & a_3 & a_2 & 2a_1 & a_8 & a_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 \end{bmatrix},$$

$$\tilde{J}_7 = \begin{bmatrix} a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_8 & 0 & 0 \\ a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & 2a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 \end{bmatrix}, \quad \tilde{J}_8 = \begin{bmatrix} a_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & 2a_1 \end{bmatrix}.$$

To begin with, we investigate the stability of Square-II solution. Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Square-II}}, \psi_{\text{Square-II}}) = (w, w, w, w, w, w, w, w, -\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Square-II}}, \psi_{\text{Square-II}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_2 & a_1 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 \\ a_7 & a_8 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_4 & a_3 & a_2 & a_1 & a_8 & a_7 & a_6 & a_5 \\ a_5 & a_6 & a_7 & a_8 & a_1 & a_2 & a_3 & a_4 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_8 & a_7 \\ a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_1 & a_2 \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \end{bmatrix}. \tag{B.151}$$

The eigenvalues of this matrix are given as follows:

$$\begin{aligned} \lambda_1, \lambda_2 &\approx w(a_1 + a_2 + a_5 + a_6) \pm w(a_3 + a_4 + a_7 + a_8), \\ \lambda_3, \lambda_4 &\approx w(a_1 - a_2 + a_5 - a_6) \pm w(a_3 - a_4 + a_7 - a_8), \\ \lambda_5, \lambda_6 &\approx w(a_1 - a_5) \pm w\sqrt{(a_2 - a_6)^2 - (a_3 - a_7)^2 + (a_4 - a_8)^2} \quad \text{(repeated twice)}. \end{aligned}$$

Thus, the stability of $\boldsymbol{x}_{\text{Square-II}}$ depends on the values of $a_1, \dots, a_8$. The other solutions can be treated similarly. Tables B.2 and B.3 summarize the stability conditions of bifurcating solutions.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Square-III}}, \psi_{\text{Square-III}}) = (w, 0, w, 0, w, 0, w, 0, -\frac{a_1 + a_3 + a_5 + a_7}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Square-III}}, \psi_{\text{Square-III}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ a_7 & a_8 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{bmatrix}, \tag{B.152}$$

Table B.2: Stability conditions of bifurcating solutions for critical points associated with Type β_i orbit.

Solution	Case	Stability conditions
$\boldsymbol{x}_{\text{Square-II}}$	$w > 0$, $(a_2 - a_6)^2 - (a_3 - a_7)^2 + (a_4 - a_8)^2 \geq 0$	$a_1 + a_2 + a_5 + a_6 < -\lvert a_3 + a_4 + a_7 + a_8 \rvert$ $a_1 - a_2 + a_5 - a_6 < -\lvert a_3 - a_4 + a_7 - a_8 \rvert$ $a_1 - a_5 < -\sqrt{(a_2 - a_6)^2 - (a_3 - a_7)^2 + (a_4 - a_8)^2}$
	$w > 0$, $(a_2 - a_6)^2 - (a_3 - a_7)^2 + (a_4 - a_8)^2 < 0$	$a_1 + a_2 + a_5 + a_6 < -\lvert a_3 + a_4 + a_7 + a_8 \rvert$ $a_1 - a_2 + a_5 - a_6 < -\lvert a_3 - a_4 + a_7 - a_8 \rvert$ $a_1 - a_5 < 0$
	$w < 0$, $(a_2 - a_6)^2 - (a_3 - a_7)^2 + (a_4 - a_8)^2 \geq 0$	$a_1 + a_2 + a_5 + a_6 > \lvert a_3 + a_4 + a_7 + a_8 \rvert$ $a_1 - a_2 + a_5 - a_6 > \lvert a_3 - a_4 + a_7 - a_8 \rvert$ $a_1 - a_5 > \sqrt{(a_2 - a_6)^2 - (a_3 - a_7)^2 + (a_4 - a_8)^2}$
	$w < 0$, $(a_2 - a_6)^2 - (a_3 - a_7)^2 + (a_4 - a_8)^2 < 0$	$a_1 + a_2 + a_5 + a_6 > \lvert a_3 + a_4 + a_7 + a_8 \rvert$ $a_1 - a_2 + a_5 - a_6 > \lvert a_3 - a_4 + a_7 - a_8 \rvert$ $a_1 - a_5 > 0$
$\boldsymbol{x}_{\text{Square-III}}$	$w > 0$	$-a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 < 0$ $a_1 + a_5 < -\lvert a_1 - a_3 \rvert$ $a_1 - a_5 < 0$
	$w < 0$	$-a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 > 0$ $a_1 + a_5 > \lvert a_1 - a_3 \rvert$ $a_1 - a_5 > 0$
$\boldsymbol{x}_{\text{Quad-I}}$	$w > 0$	$-a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_7 + a_8 < 0$ $a_1 + a_6 < -\lvert a_2 + a_5 \rvert$ $a_1 - a_6 < -\lvert a_2 - a_5 \rvert$
	$w < 0$	$-a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_7 + a_8 > 0$ $a_1 + a_6 > \lvert a_2 + a_5 \rvert$ $a_1 - a_6 > \lvert a_2 - a_5 \rvert$
$\boldsymbol{x}_{\text{Quad-II}}$	$w > 0$	$-a_1 + a_2 + a_3 - a_4 - a_5 + a_6 + a_7 - a_8 < 0$ $a_1 + a_5 < -\lvert a_4 + a_8 \rvert$ $a_1 - a_5 < -\lvert a_4 - a_8 \rvert$
	$w < 0$	$-a_1 + a_2 + a_3 - a_4 - a_5 + a_6 + a_7 - a_8 > 0$ $a_1 + a_5 > \lvert a_4 + a_8 \rvert$ $a_1 - a_5 > \lvert a_4 - a_8 \rvert$

Table B.3: Stability conditions of bifurcating solutions for critical points associated with Type β_i orbit.

Solution	Case	Stability conditions
$\boldsymbol{x}_{\text{Duo-III}}$	$w > 0$	$\max(a_3 + a_4, a_5 + a_6, a_7 + a_8) < a_1 + a_2$ $a_1 < -\lvert a_2 \rvert$
	$w < 0$	$\min(a_3 + a_4, a_5 + a_6, a_7 + a_8) > a_1 + a_2$ $a_1 > \lvert a_2 \rvert$
$\boldsymbol{x}_{\text{Duo-IV}}$	$w > 0$	$\max(a_2 + a_7, a_3 + a_6, a_5 + a_8) < a_1 + a_4$ $a_1 < -\lvert a_4 \rvert$
	$w < 0$	$\min(a_2 + a_7, a_3 + a_6, a_5 + a_8) > a_1 + a_4$ $a_1 > \lvert a_4 \rvert$
$\boldsymbol{x}_{\text{Duo-V}}$	$w > 0$	$\max(a_2 + a_6, a_3 + a_7, a_4 + a_8) < a_1 + a_5$ $a_1 < -\lvert a_5 \rvert$
	$w < 0$	$\min(a_2 + a_6, a_3 + a_7, a_4 + a_8) > a_1 + a_5$ $a_1 > \lvert a_5 \rvert$
$\boldsymbol{x}_{\text{Duo-VI}}$	$w > 0$	$\max(a_2 + a_5, a_3 + a_8, a_4 + a_7) < a_1 + a_6$ $a_1 < -\lvert a_6 \rvert$
	$w < 0$	$\min(a_2 + a_5, a_3 + a_8, a_4 + a_7) > a_1 + a_6$ $a_1 > \lvert a_6 \rvert$
$\boldsymbol{x}_{\text{Duo-VII}}$	$w > 0$	$\max(a_2 + a_3, a_4 + a_5, a_6 + a_7) < a_1 + a_8$ $a_1 < -\lvert a_8 \rvert$
	$w < 0$	$\min(a_2 + a_3, a_4 + a_5, a_6 + a_7) > a_1 + a_8$ $a_1 > \lvert a_8 \rvert$
$\boldsymbol{x}_{\text{Mono-II}}$	$w > 0$	$\max(a_2, a_3, a_4, a_5, a_6, a_7, a_8) < a_1 < 0$
	$w < 0$	$\min(a_2, a_3, a_4, a_5, a_6, a_7, a_8) > a_1 > 0$

where $\alpha = -a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8$. The eigenvalues of this matrix are given as follows:

$$\begin{aligned}
&\lambda_1, \lambda_2 \approx w(a_1 + a_5) \pm w(a_3 + a_7),\\
&\lambda_3, \lambda_4 \approx w(a_1 - a_5) \pm \mathrm{i}w(a_3 - a_7),\\
&\lambda_5 \approx w(-a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8) \quad \text{(repeated 4 times)}.
\end{aligned}$$

Thus, the stability of $\boldsymbol{x}_{\text{Square-III}}$ depends on the values of $a_1, \ldots, a_8$.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Quad-I}}, \psi_{\text{Quad-I}}) = (w, w, 0, 0, w, w, 0, 0, -\frac{a_1 + a_2 + a_5 + a_6}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Quad-I}}, \psi_{\text{Quad-I}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_2 & a_1 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 & a_1 & a_2 & a_3 & a_4 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_8 & a_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{bmatrix}, \tag{B.153}$$

where $\alpha = -a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_7 + a_8$. The eigenvalues of this matrix are given as follows:

$$\begin{aligned}
&\lambda_1, \lambda_2 \approx w(a_1 + a_6) \pm w(a_2 + a_5),\\
&\lambda_3, \lambda_4 \approx w(a_1 - a_6) \pm w(a_2 - a_5),\\
&\lambda_5 \approx w(-a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_7 + a_8) \quad \text{(repeated 4 times)}.
\end{aligned}$$

Thus, the stability of $\boldsymbol{x}_{\text{Quad-I}}$ depends on the values of $a_1, \ldots, a_8$.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Quad-II}}, \psi_{\text{Quad-II}}) = (w, 0, 0, w, w, 0, 0, w, -\frac{a_1 + a_4 + a_5 + a_8}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Quad-II}}, \psi_{\text{Quad-II}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ a_4 & a_3 & a_2 & a_1 & a_8 & a_7 & a_6 & a_5 \\ a_5 & a_6 & a_7 & a_8 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \end{bmatrix}, \tag{B.154}$$

where $\alpha = -a_1 + a_2 + a_3 - a_4 - a_5 + a_6 + a_7 - a_8$. The eigenvalues of this matrix are given as follows:

$$\begin{aligned}
&\lambda_1, \lambda_2 \approx w(a_1 + a_5) \pm w(a_4 + a_8),\\
&\lambda_3, \lambda_4 \approx w(a_1 - a_5) \pm w(a_4 - a_8),\\
&\lambda_5 \approx w(-a_1 + a_2 + a_3 - a_4 - a_5 + a_6 + a_7 - a_8) \quad \text{(repeated 4 times)}.
\end{aligned}$$

Thus, the stability of $\boldsymbol{x}_{\text{Quad-II}}$ depends on the values of $a_1, \ldots, a_8$.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Duo-III}}, \psi_{\text{Duo-III}}) = (w, w, 0, 0, 0, 0, 0, 0, -\frac{a_1 + a_2}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Duo-III}}, \psi_{\text{Duo-III}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_2 & a_1 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix}, \tag{B.155}$$

where $\alpha = -a_1 - a_2 + a_7 + a_8$, $\beta = -a_1 - a_2 + a_3 + a_4$, and $\gamma = -a_1 - a_2 + a_5 + a_6$. The eigenvalues of this matrix are given as follows:

$$\begin{aligned}
&\lambda_1, \lambda_2 \approx w(a_1 \pm a_2) \quad \text{(repeated twice)},\\
&\lambda_3 \approx w(-a_1 - a_2 + a_3 + a_4),\\
&\lambda_4 \approx w(-a_1 - a_2 + a_5 + a_6),\\
&\lambda_5 \approx w(-a_1 - a_2 + a_7 + a_8).
\end{aligned}$$

Thus, the stability of $\boldsymbol{x}_{\text{Duo-III}}$ depends on the values of $a_1, \ldots, a_8$.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Duo-IV}}, \psi_{\text{Duo-IV}}) = (w, 0, 0, w, 0, 0, 0, 0, -\frac{a_1 + a_4}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Duo-IV}}, \psi_{\text{Duo-IV}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ a_4 & a_3 & a_2 & a_1 & a_8 & a_7 & a_6 & a_5 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta \end{bmatrix}, \tag{B.156}$$

where $\alpha = -a_1 - a_4 + a_2 + a_7$, $\beta = -a_1 - a_4 + a_5 + a_8$, and $\gamma = -a_1 - a_4 + a_3 + a_6$. The eigenvalues of this matrix are given as follows:

$$\begin{aligned}
&\lambda_1, \lambda_2 \approx w(a_1 \pm a_4) \quad \text{(repeated twice)},\\
&\lambda_3 \approx w(-a_1 - a_4 + a_2 + a_7),\\
&\lambda_4 \approx w(-a_1 - a_4 + a_3 + a_6),\\
&\lambda_5 \approx w(-a_1 - a_4 + a_5 + a_8).
\end{aligned}$$

Thus, the stability of $\boldsymbol{x}_{\text{Duo-IV}}$ depends on the values of $a_1, \ldots, a_8$.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Duo-V}}, \psi_{\text{Duo-V}}) = (w, 0, 0, 0, w, 0, 0, 0, -\frac{a_1 + a_5}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Duo-V}}, \psi_{\text{Duo-V}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix}, \tag{B.157}$$

where $\alpha = -a_1 - a_5 + a_2 + a_6$, $\beta = -a_1 - a_5 + a_3 + a_7$, and $\gamma = -a_1 - a_5 + a_4 + a_8$. The eigenvalues of this matrix are given as follows:

$$\begin{aligned}
&\lambda_1, \lambda_2 \approx w(a_1 \pm a_5) \quad \text{(repeated twice)},\\
&\lambda_3 \approx w(-a_1 - a_5 + a_2 + a_6),\\
&\lambda_4 \approx w(-a_1 - a_5 + a_3 + a_7),\\
&\lambda_5 \approx w(-a_1 - a_5 + a_4 + a_8).
\end{aligned}$$

Thus, the stability of $\boldsymbol{x}_{\text{Duo-V}}$ depends on the values of $a_1, \ldots, a_8$.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Duo-V}}, \psi_{\text{Duo-VI}}) = (w, 0, 0, 0, 0, w, 0, 0, -\frac{a_1 + a_6}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Duo-VI}}, \psi_{\text{Duo-VI}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_8 & a_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix}, \tag{B.158}$$

where $\alpha = -a_1 - a_6 + a_2 + a_5$, $\beta = -a_1 - a_6 + a_4 + a_7$, and $\gamma = -a_1 - a_6 + a_3 + a_8$. The eigenvalues of this matrix are given as follows:

$$\begin{aligned} &\lambda_1, \lambda_2 \approx w(a_1 \pm a_6) \quad \text{(repeated twice)},\\ &\lambda_3 \approx w(-a_1 - a_6 + a_2 + a_5),\\ &\lambda_4 \approx w(-a_1 - a_6 + a_3 + a_8),\\ &\lambda_5 \approx w(-a_1 - a_6 + a_4 + a_7). \end{aligned}$$

Thus, the stability of $\boldsymbol{x}_{\text{Duo-VI}}$ depends on the values of $a_1, \ldots, a_8$.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Duo-V}}, \psi_{\text{Duo-VII}}) = (w, 0, 0, 0, 0, 0, 0, w, -\frac{a_1 + a_8}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Duo-VII}}, \psi_{\text{Duo-VII}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \end{bmatrix}, \tag{B.159}$$

where $\alpha = -a_1 - a_8 + a_2 + a_3$, $\beta = -a_1 - a_8 + a_6 + a_7$, and $\gamma = -a_1 - a_6 + a_4 + a_5$. The eigenvalues of this matrix are given as follows:

$$\begin{aligned} &\lambda_1, \lambda_2 \approx w(a_1 \pm a_8) \quad \text{(repeated twice)},\\ &\lambda_3 \approx w(-a_1 - a_8 + a_2 + a_3),\\ &\lambda_4 \approx w(-a_1 - a_8 + a_4 + a_5),\\ &\lambda_5 \approx w(-a_1 - a_8 + a_6 + a_7). \end{aligned}$$

Thus, the stability of $\boldsymbol{x}_{\text{Duo-VII}}$ depends on the values of $a_1, \ldots, a_8$.

Evaluating the Jacobian matrix at the point

$$(\boldsymbol{x}_{\text{Mono-II}}, \psi_{\text{Mono-II}}) = (w, 0, 0, 0, 0, 0, 0, 0, -\frac{a_1}{a_0} w),$$

we have

$$\widetilde{J}(\boldsymbol{x}_{\text{Mono-II}}, \psi_{\text{Mono-II}}) \approx w \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \end{bmatrix}. \tag{B.160}$$

The eigenvalues of this matrix are given as follows:

$$\begin{aligned}
\lambda_1 &\approx wa_1, \\
\lambda_2 &\approx w(-a_1 + a_2), \\
\lambda_3 &\approx w(-a_1 + a_3), \\
\lambda_4 &\approx w(-a_1 + a_4), \\
\lambda_5 &\approx w(-a_1 + a_5), \\
\lambda_6 &\approx w(-a_1 + a_6), \\
\lambda_7 &\approx w(-a_1 + a_7), \\
\lambda_8 &\approx w(-a_1 + a_8).
\end{aligned}$$

Thus, the stability of $\boldsymbol{x}_{\text{Mono-II}}$ depends on the values of $a_1, \ldots, a_8$.

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