博士論文

# CONSTRUCTION AND CLASSIFICATION OF TOPOLOGICAL WEAVES

(トポロジー的織込み構造の構成と分類)

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# CONSTRUCTION AND CLASSIFICATION OF TOPOLOGICAL WEAVES

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# ABSTRACT

*Weaves* are complex entangled objects that mainly differ from general links because they contain no closed components. Although weaves have been investigated for so many years, we still do not have a universal study to describe them. Several interesting attempts have been made to approach them from a mathematical point of view and this thesis contributes to it by introducing a new way to *define*, *construct*, and *classify* topological weaves.

In Chapter 2, based on our paper [21], we state a new definition of weaves as the lift to the thickened Euclidean plane $\mathbb{X}^3$ of a quadrivalent graph $\Gamma$ in $\mathbb{E}^2$ made of colored straight lines, such that each vertex is specified by an over or under information given by a set of *crossing sequences* $\Sigma$.

**Definition 0.1.** [21] *We call untwisted weave the lift to* $\mathbb{X}^3$ *of a pair* $(\Gamma, \Sigma)$ *of* $\mathbb{E}^2$*. Moreover, two threads are said to be in the same set of threads, if they are the lift of straight lines belonging to the same color group.*

More complex weaves, called *twisted weaves*, can be defined from these untwisted ones by introducing *twists* between neighboring threads of the same color, via some local surgeries, called $\pm k$-moves in knot theory.

**Definition 0.2.** [21] *A twisted weave is the lift to* $\mathbb{X}^3$ *of a pair* $(\Gamma, \Sigma)$ *embedded on* $\mathbb{E}^2$ *admitting at least a twisted region. Moreover, if two threads twist their total number of twists is even and they cannot twist with other threads.*

We also define diagrammatic representations to study their properties.

**Definition 0.3.** [21] *The planar projection* $W_0$ *of a weave* W *by* $\pi : \mathbb{X}^3 \to \mathbb{E}^2$, $(x, y, z) \mapsto (x, y, 0)$ *is called a regular projection. When all its vertices are specified by an over or under information, it is called a weaving diagram* $D_{W_0}$*. And if* $D_{W_0}$ *is periodic, then any generating cell is called a weaving motif.*

In Chapter 3, we introduce a systematic way to construct weaving motifs [22], based on the concept of *polyhedral link* defined by W.Y. Qiu et al. [47]. The strategy consists in covering the edges and vertices of a generating cell $U_{\mathcal{T}}$ of a periodic tiling $\mathcal{T}$ of $\mathbb{E}^2$ by crossed strands with respect to the *polygonal link methods*, denoted by $(\Lambda, L)$. However, we have noticed that these methods can generate closed curves and have developed a way to predict the construction of weaving motifs using algebraic and combinatorial arguments.

**Theorem 0.4.** [22] ***(Construction of Weaving Motifs)*** *A weaving motif is created from a pair* $\big(U_{\mathcal{T}}, (\Lambda, L)\big)$ *if and only if all the characteristic loops in* $U_{\mathcal{T}}$ *are nontrivial polygonal chains with at least two non-equivalent ones.*

In Chapter 4, we take a first step towards the classification of *alternating* weaving motifs [48], via the extension of two famous theorems of knot theory. By alternating, we mean that the crossings alternate cyclically between under and over. To achieve this goal, we generalize the concept of *reduced diagram* to periodic weaves and extend the Kauffman *bracket polynomial* [39], defined for weaving motifs on a torus in [25], to higher genus surfaces $\Sigma_g$.

**Theorem 0.5.** [48] ***(Tait's First and Second Conjectures for Weaves)*** *The crossing number and the writhe of a* $\Sigma_g$*-reduced alternating minimal diagram of its periodic alternating weaving diagram are weaving invariants.*

Chapter 5 ends this manuscript with a strong result in terms of classification of the class of doubly periodic untwisted $(p, q)$-weaves, based on our paper [21]. First, we construct a new weaving invariant defined as a set of *crossing matrices*, whose elements are symbols $\pm 1$ characterizing the organization of crossings in a generating cell. Then, we prove our main theorem.

**Theorem 0.6.** [21] ***(Equivalence Classes of Doubly Periodic Untwisted*** $(p, q)$***-Weaves)*** *Let* $\mathrm{W}_1$ *and* $\mathrm{W}_2$ *be two doubly periodic untwisted* $(p, q)$*-weaves with* $N \geq 2$ *sets of threads, such that their corresponding regular projections are equivalent, up to isotopy of* $\mathbb{E}^2$*, and with the same set of crossing sequences. Let* $D_{W_1}$ *and* $D_{W_2}$ *be two weaving motifs of same area of* $\mathrm{W}_1$ *and* $\mathrm{W}_2$*, respectively. Then,* $D_{W_1}$ *and* $D_{W_2}$ *are equivalent if and only if their crossing matrices are pairwise equivalent.*

Then, in a logic of classification in terms of crossings, we state a solution to the open problem of finding the *crossing number* of weaving motifs for doubly periodic untwisted $(p, q)$-weaves. The idea is to use combinatorial arguments on the torus to obtain a formula which depends on $(\Gamma, \Sigma)$.

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# Contents

# Chapter 1

# INTRODUCTION

Objects whose structure can be represented by open or closed intertwined strings are studied and used since ancient times for many purposes in the fields of art, science, labor, and engineering among others. In mathematics, we mainly find some of these entangled objects in the area of *knot theory* which emerged around the $19^{th}$ century as a branch of topology [1], while in *materials science*, they appear at different scales from old traditional textiles to recent nanomolecules. Our modern society faces many important problems, and one of our main challenges is to create innovative multifunctional materials that could contribute to the solution of some of them. Materials with an *entangled* structure are in particular very complex, multi-scale and multi-sciences systems, with hidden mechanisms that characterize functions. When studying such materials, there are many trials and errors based on experience and intuition, which end up being very costly, both in terms of time and money. However, we can now benefit from using big data to explore new materials using computers for simulation and prediction. For this purpose, it is essential to translate this material knowledge into a language understandable by computers, which justifies the interest in creating mathematical models to encode the entangled objects. Indeed many mechanical functions come from the structure of a material that can be described using geometry and topology, which therefore inspire new mathematical developments [36]. On the other hand, materials science is also a great source of inspiration for developing new mathematical theories, such as the early interest in knot theory which came from chemistry. For this thesis, polymer science [63] and textile design [46], [62] have also been driving forces to develop this new topological theory of weaving, and to study weaves as mathematical objects in knot theory, as also suggested by S.A. Grishanov et al. [23], [24].

*Weaves* are structures that can intuitively be described by multiple *threads*, more or less thick or curved, interlaced into each other in our three-dimensional physical space. They differ from general *links* in knot theory mainly because they do not contain any closed components. In this thesis, we will be more particularly interested in weaves that can be described as 'flat', in the sense that they can be embedded in the thickened plane, see Figure 1.1 for examples. Woven objects include traditional and innovative woven fabrics, which were first handmade with a design process that seemed to be already known 27,000 years ago. At the molecular scale, they appear for example in metal organic frameworks (MOFs) and covalent organic frameworks (COFs), which are porous crystalline solids used for gas and energy storage, drug delivery systems, or even semiconductors, among many applications [45], [34]. However, although such structures have been investigated for so many years all over the world and by different groups of people, the scientific community has not yet agreed on a formal description of weaves, so that they can be distinguished from other complex entangled structures exhibiting different functionalities. Moreover, we still do not have a universal study about weaves to identify and classify them. Many inspiring attempts, which will be described below, have been made to study woven structures from a mathematical point of view and many questions still remain open. This thesis contributes to this challenge by proposing a new systematic approach, based on low-dimensional topology, to *define*, *construct*, and *classify* weaves as topological objects.

## 1.1 Mathematical description of weaves

To define entangled structures from a mathematical point of view, different strategies have been considered. One of the first that caught our attention is the research of B. Grünbaum and G.C. Shephard, which introduced an ideal geometric model of existing weaves in the textile industry. They are among the first groups of mathematicians who worked on the mathematics of textiles [28], [29], [30], [32], apart from E. Lucas who targeted a specific subclass of weaves called *satins* [14]. They defined a weave as an unbounded and rigid structure made from *strands*, which are doubly infinite open strips of constant width without thickness, organized into *layers*. By layer is meant

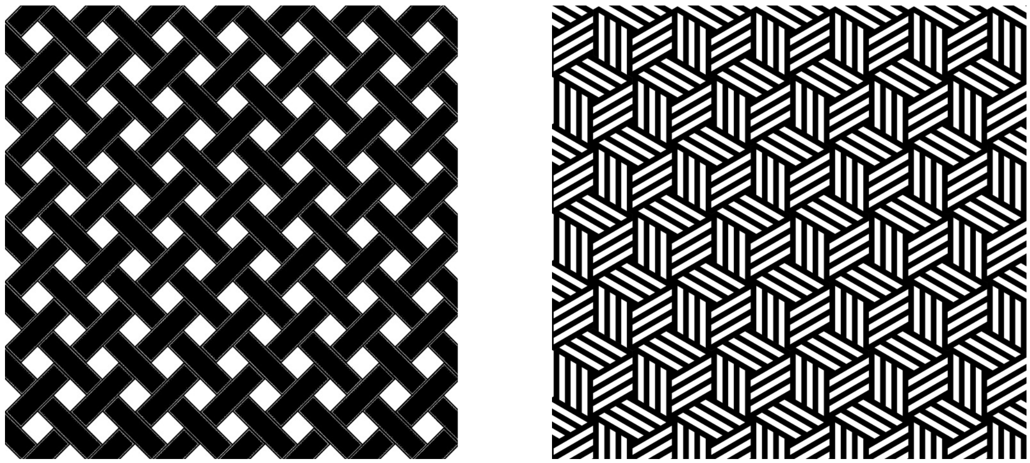


Figure 1.1: Two examples of weaves.

here a set of congruent and disjoint parallel strands such that each point of the plane either belongs to the interior of exactly one of the strands or lies on the boundary of two adjacent strands. In particular, a weave consists of at least two layers of congruent strands on the same plane such that the strands of the different ones are not parallel and intertwine over and under each other respecting a *ranking*, and such that it is impossible to partition the set of all strands into two nonempty subsets, where each strand of the first subset would pass over every strand of the second subset.

Grishanov et al. [23], [24], [25], [26], [27], [54], chose knot theory to study textiles, including weavings as well as other different types of entangled structures. They described them topologically in a single class of *doubly periodic* structures embedded in the thickened Euclidean plane and consisting of interwoven threads. These threads are assumed to be *smooth spatial curves* that can be continuously deformed without breaking or crossing each other or themselves. As pointed out in their study, even if doubly periodic structures seem similar to general knots and links, they cannot contain closed components or knots tied on their threads nor non-interlaced threads, disconnected strips, or layers. This description was formalized and generalized by V. Kurlin et al. with the mathematical definition of a textile structure as a set of continuous *open* or also *closed* curves embedded without intersections in the thickened Euclidean plane, preserved under translations by two linearly independent directions [6]. The periodicity allows a description of the complete structure in a generating cell, which is defined as an embedding of a finite number of closed curves in a thickened *torus*, and thus as a specific type of link which depends on the choice of the periodic cell.

Other scientists with various scientific backgrounds have chosen a different strategy for describing weavings using *graphs* and *tilings*. B. Thompson and S.T. Hyde described a weave as any decomposition of a connected graph, called a *parent net*, into multiple subgraph components, and specified the entanglement of the different components with a *color sequence* characterizing the over or under information at the split vertices [67]. They use tilings of the *Euclidean* and *hyperbolic* planes, as well as tilings of the *two-sphere* to obtain their results. In particular, they constructed entangled networks from two *dual regular nets* of the two-dimensional spaces. The first step is to choose the topology of one of the components and then, enumerate all the ways to color the edges of these nets with two colors, using the *Dress-Delaney symbol* from the combinatorial tiling theory. More generally and for many years, S.T. Hyde, M. Evans, and their coauthors have not limited themselves to the study of weaving of threads-like components but have also considered the entanglements of closed components and nets in different surfaces, including doubly and *triply periodic* structures, as discussed in [8], [16], [17], [18], [34], [35], [67]. Another example is that of E.D. Miro et al. who also considered the theory of hyperbolic tilings to study weaving networks in [52], [53], [72]. The idea is to first build a tiling from the repeating of one triangle, which is then colored using subgroups of the *triangle group*. Furthermore, new colored vertices and edges are constructed to form two different nets, and thus, a woven structure is defined as the union of these two nets together with a *weaving map* describing the position of the two nets at each crossing point.

M. O'Keeffe, O.M. Yaghi et al. gave a different description of the weaving of threads, loops, and two-periodic nets [45]. In their work, such structures are considered *piecewise linear*, which means that they consist of linear segments that meet at *divalent vertices*, which is a characterization that inspired us. In the case of traditional weavings in the textile industry, they noticed that if the threads are pulled straight they intersect on the lattice plane and the points of intersection appear as the vertices of a two-periodic net. In particular, for weaves with threads organized in *two directions*, the net is that of the *square lattice*, and for *three directions*, the net is either that of the *kagome pattern* or the *hexagonal lattice*. They also identified other types of interwoven structures made of threads, such that their projection is not linear, which are defined so that if these threads are stretched and allowed to go accross each other, they become parallel lines. This type of network is called a *chain-link weaving* and the conventional *knitting* is a special case of this class. A different kind of entangled objects consists in the interlacing of $n$ loops, which are called $[n]$-*catenanes*, or *polycatenanes*, while another class described in their work is that of entangled structures consisting of the union of at least two nets, also mentioned in the studies of Hyde et al.

Finally, T. Kechadi et al. found a way to describe textile structures that include *weaves*, *knits*, and *braids*, using the notion of *hypergraph* which do not limit to periodic and infinite patterns [58]. Here a *textile graph* is defined as a hypergraph consisting of a set of vertices belonging to crossings, a set of *hyperedges* of degree four that connect these vertices, a set of terminal nodes that end the threads, a set of regular edges of degree two with information describing which thread is on the top at each crossing, and lastly a set of edges connecting vertices to vertices of other crossings or terminal nodes.

All these studies are very interesting and provide complementary points of view that allow a better comprehension of the hidden structure of complex entangled objects. However, they often do not distinguish weaves from other complex entangled structures such as links, (poly)catenanes, or knits, which seems essential from our point of view. Indeed, as explained by Grishanov et al. [23], entanglement has a direct influence on the physical and mechanical properties of materials. We believe that a weave should be a structure containing components that are neither closed curves, nets, nor threads that become 'parallel' once pulled straight. Moreover, we also note that even if the notion of weaving with more than two directions has been mentioned in the literature, the examples studied in depth mainly consist of threads organized in two or three directions. We think that more complex weaves deserve further investigation, which inspired our work. Indeed, so far the results mentioned in the literature cited above which satisfy our definition of weaves seem relevant for existing materials, but might be limited if one wants to consider more complex patterns.

## 1.2 Mathematical construction of weaves

Different approaches have been considered to create a mathematical algorithm that can generate a weave, as well as other entangled structures. We will only state the methods found in the literature which make it possible to construct a weave within the meaning of our definition.

In [67], Hyde et al. constructed a weave in the thickened Euclidean or hyperbolic plane from a periodic connected graph embedded in an oriented surface, the previously called *parent net*. Then, they color the edges so that distinct components of the weaves are distinguished by *colors*. Therefore, two different types of vertices emerge from a parent net, which are either *monochromatic vertices* if all incident edges have the same color, or *polychromatic vertices* otherwise. Then, these latter are separated into at least two disjoint vertices displaced from each other along the surface normal. We observed from their work that if the distinct colored components satisfy the definition of a thread and that each polychromatic vertex splits into exactly two vertices, then the structure generated by their method is a weave.

Grünbaum and Shephard have chosen to formalize a classic and well-known approach used in the textile industry. This consists in associating each doubly periodic weave built from two sets of threads, namely a *biaxial weave*, with a chessboard of black and white squares describing the entanglement in a periodic cell [28], [29], [30], [32]. Such a chessboard is called the *design* of a weave, and by repeating this design by *translations* in horizontal and vertical directions, the biaxial weave can be reconstructed. They focused on the construction of a particular class of biaxial doubly periodic weaves, called *isonemal fabrics*, which are characterized such that for any pair of strands $(s, s')$, there exist a *symmetry* of the structure that maps $s$ to $s'$. Regarding the design of an isonemal fabric, this implies that any row or column of black and white squares can be mapped into any other row or column by either a symmetry of the design or by such a symmetry combined with the exchange of black and white colors. Different types of biaxial weaves have been constructed from a design. They considered isonemal fabrics associated with designs constructed in am $n \times n$ square block such that each row is deduced from the one above by a shift of $a$ units to the left, for different fixed value of the integer $a$. The case $a = 1$ generates a class of weaves called *twills*, while the case $s^2 = \pm 1 \, mod(n)$ generates different weaves called *satins*. More precisely, it is possible to assign to each square of a design integer coordinates $(x, y)$, where $x$ denotes the column and $y$ denotes the row of the given square. For the case of twills, consider an infinite sequence $A = (a_i)_{-\infty}^{+\infty}$ of 0 and 1, then the design of an $A$-twill is given such that the $(x, y)$-square is colored black if $a_{y-x} = 1$ and white if $a_{y-x} = 0$, or alternatively such that these relations hold after the design has been turned through an angle of $\frac{\pi}{2}$. However, to construct a satin, the design must contain a single black square in each row, and the position of that square is moved from one row to the next over a step of $a$ units to the right. This can be defined so that the $(x, y)$-box of the design is colored black if and only if $sy = x\, mod(n)$. They attempted a generalization to more directions using a tiling defined by sets of parallel and equidistant straight lines with a symbol to describe the ranking of the crossings, as a generalization of the bicolored design.

More recently, Kurlin et al. extended the concept of the *Gauss codes* defined in knot theory, to textile structures including weaves [6]. The idea is to describe the order of over and undercrossings in a doubly periodic structure using a cyclic *word*, called an abstract *textile code*. Then, from this code, a *textile graph* is built starting from the vertices which represent the crossings. The next step is the construction of non-oriented edges of the textile graph from pairs of successive symbols in the abstract code, and to find oriented cycle in the graph following given rules. One must verify that each oriented edge is passed once in each of two opposite directions and if any contradictions appear, then the code is said to be *unrealizable*. Finally, by attaching a topological disk to the resulting oriented cycle along its boundary, the *Euler characteristic* of the structure is computed. If this compact surface is orientable, has an empty boundary, and a Euler characteristic equal to zero, then this algorithm generates a doubly periodic structure, which can possibly be a weave.

Finally, another recent and interesting mathematical approach has been considered by chemists, who have constructed two classes of periodic and symmetric structures, the previously mentioned *biaxial* and *triaxial* weaves [45]. An infinite family of regular and symmetric biaxial weaves can be constructed starting with two parallel square lattices of points on top of each other, defined by perpendicular vectors $a$ and $b$. The next step is to connect two points related by a symmetry without belonging to the same lattice and which are separated by vector $ua + vb$, $u$ and $v$ being integers. Then, an embedding of the structure in a generating cell with an appropriate origin and axis of symmetry will create a weave for which the over and undercrossings will alternate for certain values of $u$ and $v$. Other subclasses of biaxial weaves were constructed with the same method by replacing each thread either by a set of multiple parallel threads or by a *helical pair of entwined threads*. For the case of triaxial weaves, similar strategies have been considered by replacing the square lattices with *trihexagonal lattices*, also called kagome.

## 1.3 Structure of the manuscript

Inspired by the connections between weaving, knot theory, and tiling theory mentioned above, we will introduce in this thesis a new topological model for particular classes of weaves to *define*, *construct* and *classify* them. This thesis is based on our results in [21], [22], and [48].

In Chapter two, which is based on our work in [21], [48], we enter the core of this thesis by introducing our new topological *definition* of weaves to distinguish them from other complex entangled structures. Note that we will not consider geometrical characteristics such as lengths, angles, and curvatures in this thesis. In summary, we will define a *weave* as the lift to the thickened Euclidean plane of a particular type of *planar quadrivalent connected graph* consisting of infinite open curves organized in at least two 'directions', together with a crossing information at each vertex. These curves are straight lines in the case of an *untwisted weave* and we will define a *twisted weave* from an untwisted one to which we apply surgery to include *twists*. The main challenge and contribution of this first step were to describe the crossing information in a *pairwise* fashion. Indeed, for any weave W with $N \geq 2$ disjoint *sets of threads* $T_1, \cdots, T_N$, we will define the notion of *crossing sequence*, which is considered for each pair of sets. The simplest cases that illustrate this consideration are weaves with crossings sequences that can be described by two integers, as defined below.

**Definition 1.1.** [21] *Let $i$, $j$, and $k$ be strictly positive integers. A $(p,q)$-weave* W *is defined such that all its crossing sequences, possibly distinct, can be described by two positive integers $p_k$ and $q_k$. In other words, if $C_{i,j} = (+p_k, -q_k)$ is the crossing sequence of minimal length associated with the disjoint sets of threads $T_i$ and $T_j$ of* W*, then, each thread of $T_i$ is cyclically $p_k$ consecutive times over the threads of $T_j$, followed by $q_k$ consecutive times under.*

A more complete and general definition of crossing sequences will also be given in this chapter. Note that the pairwise condition will be a key element to construct and compute interesting topological parameters to classify weaves in the next chapters. Then, as in knot theory, we will show that there exists a direct correspondence between a weave in the thickened Euclidean plane and its planar representation that covers $\mathbb{E}^2$, called an *infinite diagram*. Moreover, if such an infinite diagram is *periodic*, we will call any generating cell a *weaving motif*. We will discuss some properties of weaving diagrams and will also introduce the notion of *hyperbolic weaving diagrams*, with the aim of formalizing the description of such structures used by the community as mentioned in the sections above.

Chapter three will focus on the *construction* of doubly periodic untwisted and twisted weaving diagrams. Inspired by the construction of links from Platonic and Archimedean polyhedra using the concept of *polyhedral link* defined by W.Y. Qiu et al. in [33], [47], [60], we applied the same strategy to *periodic planar tilings*. In this chapter, which is based on our results in [22], we will indeed start with the mathematical formalization of the three construction methods that they defined using illustrations for polyhedra, and we will call them *polygonal link* methods when applied to the plane. Note that given the interest in weaving using hyperbolic tilings of the plane by other scientists as mentioned above, we can state that these methods can be generalized to any surface tiled by *edge-to-edge* polygons. The main idea is to cover each edge of a polygon with one or two strands, that can possibly twist, and glue them at a neighborhood of each vertex respecting a specific pattern which will be described in detail in this chapter. We observed that this generalization to the plane defines a new systematic way of constructing weaving diagrams, but also diagrams of other types of entangled structures such as the *polycatenanes* mentioned earlier, as well as *mixed* structures made of both open and closed components. These methods can be applied to any *edge-to-edge tiling by polygons* and our main contribution concerns the characterization of *doubly periodic* structures. More specifically, we will show that one can predict the type of entangled structure that can generate a given polygonal link method applied to a chosen doubly periodic tiling of the plane, namely a *weaving motif*, a *polycatenane motif*, or a *mixed motif*. Indeed, each curve created by of the polygonal method $(\Lambda, L)$ will cover a closed path in the original skeleton of any periodic cell of a tiling $\mathcal{T}$, called a *characteristic loop* and denoted by $\delta_{(\Lambda,L)}$.

**Theorem 1.2.** [22] *Let $U_{\mathcal{T}}$ be a periodic cell of $\mathcal{T}$. Then, a pair $\big(U_{\mathcal{T}}, (\Lambda, L)\big)$ will generate,*

- *a polycatenane motif if and only if all the characteristic loops $\delta_{(\Lambda,L)}$ in $U_{\mathcal{T}}$ are trivial polygonal chains.*
- *a mixed motif if and only if the set of all the characteristic loops $\delta_{(\Lambda,L)}$ in $U_{\mathcal{T}}$ contains at least a trivial polygonal chain and a nontrivial one.*

- *a weaving motif if and only if all the characteristic loops $\delta_{(\Lambda,L)}$ in $U_{\mathcal{T}}$ are nontrivial polygonal chains, and such that at least two of them are non-parallel.*

In Chapter four, we will state our first results [48] concerning the *classification* of a particular class of weaves, called *alternating*, that is to say that the crossings of such weaves alternate cyclically between undercrossings and overcrossings, as one travels along each of its components. Classical tools and results of knot theory can be generalized to weaving theory in order to describe and classify them. As for general knots and links, one of the main criteria in which we are interested to study the classification of weaves is the *number of crossings*, since it is a useful parameter to describe the complexity of entangled objects [24]. In particular, we will show that we can extend *Tait's first and second conjectures* on alternating links to our periodic weaves. These conjectures, proven a century later, state that the *crossing number* and the *writhe* of an alternating reduced link diagram are topological *invariants*, meaning properties that hold for all different diagrams of the same link. A *reduced diagram* is a diagram that does not contain any *isthmus*, which is a crossing that does not separate its neighborhood into four distinct regions. By crossing number, we mean the minimal number of crossings that can possibly be found in any projection to the plane, while the writhe is defined as the sum of the signs of all the crossings, where each crossing is given a sign $\pm 1$. Since any generating cell of a periodic weave can be seen as a particular type of link, our proofs are a generalization of existing ones in classic knot theory, from L.H. Kauffman [39] and R. Stong [64] respectively. However, considering that such a link results from the 'closure' of strands embedded in a periodic cell, which are in fact segments of infinite open curves, they will be treated differently. Moreover, to make sense to the notion of weaving invariant applied to motifs, such as the minimal number of crossing, this must apply to a minimal periodic cell, called *minimal motif*, and not to any generating cell made of copies of a smaller one. In this study, our contribution lies in the definition of a *reduced weaving motif*. Besides, depending on the choice of generating cell in the infinite diagram, a weaving motif might contain an *isthmus* that would disappear if one considers a 'bigger' cell made of copies of this first. Thus, by iteration, we will say that a weaving motif is reduced if its corresponding infinite diagram is reduced. Regarding the second conjecture, the main difference with the case of general links is that in a weaving diagram a thread component never crosses itself unless it makes a simple sequence of twists that can be easily undone. Then, we also generalized the *Kauffman bracket* polynomial invariant defined for weaving motifs on the torus by Grishanov et al. [25] to motifs on higher genus surfaces.

**Theorem 1.3.** ***(Tait's First and Second Conjectures for Alternating Periodic Weaves)*** [48]

(1) *A reduced alternating minimal motif is a minimum diagram of its alternating periodic weave.*

(2) *Two reduced alternating minimal motifs of an oriented periodic weave have the same writhe.*

At this point, it must be emphasized that weaving motifs represent periodic structures consisting of open curves, which is why they are treated differently from general link diagrams embedded on compact surfaces, even though weaving motifs are also link diagrams on similar surfaces. These two Tait's conjectures have been proved by T. Fleming et al. [20], or more recently by H.U. Boden and H. Karimi [5] for general link diagrams on compact surfaces with a different approach.

In Chapter five, we will focus on the classification of *doubly periodic untwisted $(p, q)$-weaves of the Euclidean thickened plane*, which is based on our paper [21]. Such a weave can be characterized on $\mathbb{E}^2$ by a planar graph $\Gamma$ composed of straight lines that cross in double points, with respect to a set of crossing sequences $\Sigma$ as defined in Definition 1.1. However, the way to assign the over or under information to each crossing given by $\Sigma$ not unique, as we will see in this chapter. This motivated the characterization of equivalence classes of weaves and the construction of a new topological invariant for this class of weaves. This weaving invariant is a set of *crossing matrices*, denoted $\Pi$ and whose elements are symbols $\pm 1$, which characterize the organization of the crossings on a periodic cell.

**Theorem 1.4.** ***(Equivalence Classes of doubly periodic untwisted $(p, q)$-weaves)*** [21] *Let* $\mathrm{W}_1$ *and* $\mathrm{W}_2$ *be two doubly periodic untwisted* $(p, q)$-*weaves with* $N \geq 2$ *sets of threads, such that their corresponding regular projections* $\Gamma$ *are equivalent, up to isotopy of* $\mathbb{E}^2$*, and with the same set of crossing sequences* $\Sigma$*. Let* $D_{W_1}$ *and* $D_{W_2}$ *be two weaving motifs with the same number of crossings of* $\mathrm{W}_1$ *and* $\mathrm{W}_2$*, respectively. Then,* $D_{W_1}$ *and* $D_{W_2}$ *are equivalent if and only if their crossing matrices are pairwise equivalent.*

Then, in the continuity of the study of the number of crossings of weaves, we will address the question of finding the *crossing number* of this class of weaves, denoted by $\mathcal{C}$. Proving that the crossing number is a weaving invariant in Chapter four was an important first step. However, computing this number has been a difficult open problem, since there exists an infinite number of ways to choose a periodic cell for such weaves, possibly with a different number of crossings, as stated in [25]. Nevertheless, an approach using combinatorial arguments allowed us to find a *total crossing number* formula [21], which depends on a *pairwise crossing number* formula that relates itself to the pair $(\Gamma, \Sigma)$. We also give a characterization of *minimal diagrams* in terms of the slopes of the strands. Therefore, with the results stated in this thesis, we are now able to classify doubly periodic untwisted $(p, q)$-weaves using $\Gamma, \Sigma, \Pi, \mathcal{C}$, as well as the *bracket polynomial* and other classical knot invariants that easily generalize to doubly periodic structures, which are listed in [24].

Finally, we will conclude this thesis with some future perspectives for the mathematical study of weaves.

# Chapter 2

# WEAVES, WEAVING DIAGRAMS AND WEAVING MOTIFS

The aim of this chapter, which is based on our work in [21], [48], is to define a weave as a topological object in such a way that it can be easily distinguished from another type of entangled structure consisting of one-dimensional open curves, like a *knit* [68], [69], or a *braid* [42] for example, which are constructed from a single set of threads.

First, define a set of *straight lines* embedded on the Euclidean plane $\mathbb{E}^2$ and belonging to at least two disjoint color groups. We will say that two such lines belong to different color groups if they intersect. Moreover, we only admit intersections corresponding to double points, which means vertices of degree four. Note here that knits and braids would be characterized by only one color group. Then, we specify each vertex with an over or under information using arcs, called *crossing*, with respect to a set of *crossing sequences*. By a crossing sequence, we mean a sequence of *minimal length* consisting of integers that characterize in a *pairwise* fashion the number of consecutive over or under information that appear while walking on a given colored line. In other words, this means that each such crossing sequence is associated with only two colors. Then, we will define an *untwisted weave* as the lift to the Euclidean thickened plane of these curves, called *threads*, such that they do not intersect in $\mathbb{X}^3$. This simple class of weaves will serve as a basis to describe more complex structures. By introducing *twists* between nearest neighboring threads of the same color via some local surgeries, we will indeed define the class of *twisted weaves*. We will also define the notion of trivial weave, called *unweave*, by specifying the crossings such that the structure 'fall apart' or is not *entangled*. In the second part of this chapter, we will discuss diagrammatic representations of weaving. Weaves being objects embedded in the thickened Euclidean plane, a simpler way to approach them is by studying their regular projections onto $\mathbb{E}^2$, which encode the crossing information, as for general knots and links. Finally, in the last section, we will study a particular class of weaves having the property of being periodic and with simple cyclic crossing sequences, called $(p, q)$*-weaves*. The periodicity allows a full description of these objects in a periodic cell, which can be seen as particular links embedded in a thickened surface of genus $g \geq 1$.

## 2.1 Preliminaries on general knots and links

In this section, we recall basic definitions and properties of general knots and links that were essential to introduce weaves as topological objects. We refer to the classic books on knot theory by C.Adams [1], K. Murasugi [57], and R. Lickorish [44] for further details.

### 2.1.1 Knots and links

Intuitively, a *knot* is an entangled circle with no thickness nor self-intersection in a three-dimensional ambient space, without any starting or endpoint. In other words, it is a smooth embedding of $\mathbb{S}^1$ in $\mathbb{E}^3$ or $\mathbb{S}^3$. A *link* is defined as a finite union of knots. From an aesthetic point of view, these objects are drawn as smooth curves, as illustrated in Figure 2.1, however, each knot must be considered as a *polygonal closed curve*. This piecewise linear condition implies that each curve is made up of a finite number of straight line segments placed end to end, which prevents a link from having any pathology such as a sharp twist or a part of a knot converging to a point. Such a link isotopic to a polygonal link is called *tame*. Otherwise, it is said to be *wild*.

**Definition 2.1.** *A link* $\mathrm{L}$ *is a finite union of disjoint piecewise linear simple closed curves embedded in* $\mathbb{S}^3$ *or* $\mathbb{R}^3$*. Moreover, a link consisting of a single component is called a knot.*

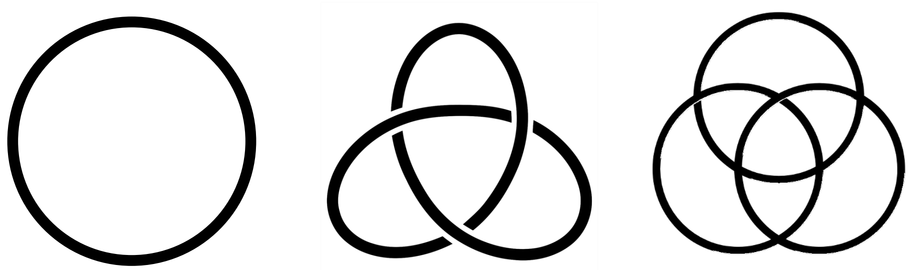


Figure 2.1: Left: unknot. Center: trefoil knot. Right: boromean ring.

**Definition 2.2.** *A knot is said to be trivial if it bounds an embedded piecewise linear disk in* $\mathbb{S}^3$ *or* $\mathbb{R}^3$*. We also call such a trivial knot the unknot. Moreover, a* $k$*-component link is said to be trivial if it consists of* $k$ *disjoint trivial knots.*

Then, we can easily assume that the shape of a knot or link, defined as polygonal curves, can easily be changed by the following basic transformations described in [57], that do not alter its nature, see Figure 2.2.

**Definition 2.3.** [57] *On a given knot* K*,*

- (1) *we may divide an edge,* AB*, in space of* K *into two edges,* AC*,* CB*, by placing a point* C *on the edge* AB*.*
- (1′) *if* AC *and* CB *are two adjacent edges of* K *such that if* C *is erased* AB *becomes a straight line, then we may remove the point* C*.*
- (2) *if* C *is a point in space that does not lie on* K *such that the triangle* ABC *does not intersect* K*, with the exception of the edge* AB*, then we may remove* AB *and add the two edges* AC *and* CB*.*
- (2′) *if there exists in space a triangle* ABC *that contains two adjacent edges* AC *and* CB *of* K*, and this triangle does not intersect* K*, except at the edges* AC *and* CB*, then we may delete the two edges* AC*,* CB *and add the edge* AB*.*

*These four operations are called the elementary knot moves.*

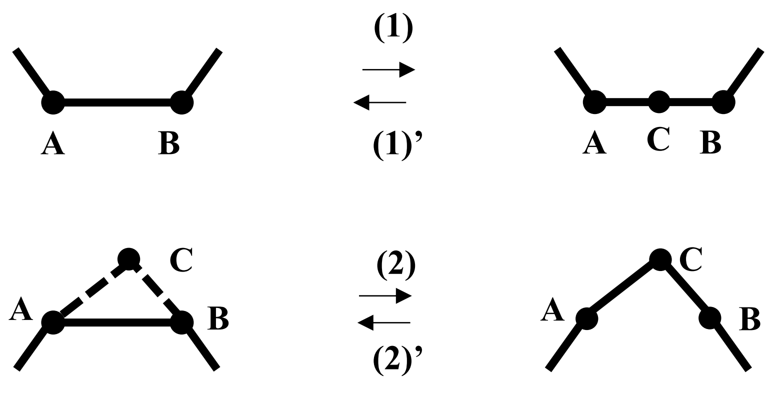



Figure 2.2: Elementary knot moves [57].

Therefore, knots or links that are related by these elementary knot moves are said to be *equivalent*.

**Definition 2.4.** *Two knots* K *and* K′ *are called equivalent if we can obtain* K′ *from* K *by applying a finite sequence of elementary knot moves. Moreover, two links* $\mathrm{L} = \mathrm{K}_1, \cdots, \mathrm{K}_m$ *and* $\mathrm{L}' = \mathrm{K'}_1, \cdots, \mathrm{K'}_n$ *are equivalent if* $m = n$ *and that we can transform* L *into* L′ *by applying a finite sequence of elementary knot moves pairwise between* $K_i$ *and* $K_i'$*, for all* $i \in \{1, \cdots, m = n\}$.

However, these elementary moves can be seen as 'local' transformations only applied to a small region of the knot or link, and one would prefer a definition of equivalence more 'global'. The idea is therefore to consider transformations of the whole three-dimensional space in which the knot or link exists. Let $\phi$ be a continuous bijective map from a topological space $X$ to a topological space $Y$, where X and Y are considered to be three-dimensional Euclidean spaces or subspaces thereof here, and $\phi^{-1}$ be the continuous inverse map. Then, we say that $\phi$ is a *homeomorphism*, and X and Y are said to be *homeomorphic*, meaning that they are considered to be the same space from an algebraic topology point of view. In the case where $X = Y$, we call $\phi$ an *auto-homeomorphism*. Moreover, if an orientation is assigned to the two topological spaces, then $\phi$ is said to be an *orientation-preserving homeomorphism* if it maps the corresponding orientations to each other.

Therefore, Definition 2.4 can be reformulated globally as done by Lickorish in [44]. Two knots or links are indeed said to be equivalent if we can deform one continuously into the other without generating any self-intersections, or including operations consisting of cutting and gluing the curves. Such a continuous deformation that does not change the nature of the transformed object is called an *ambient isotopy*.

**Definition 2.5.** [44] *Two links are said to be equivalent if there exists an orientation-preserving piecewise linear homeomorphism of the three-dimensional ambient space that maps one onto the other.*

### 2.1.2 Diagrammatic representation of knots and links

Considering the notion of equivalence stated in the previous subsection, any link L can be transformed to be in *general position* in the three-dimensional Euclidean space. Let $\pi : \mathbb{E}^3 \to \mathbb{E}^2$ be the standard projection map. By general position, we mean that when we project L onto the Euclidean plane, its image must satisfy the three following conditions,

- each line segment of L projects to a line segment onto $\mathbb{E}^2$,
- the projection of two such segments intersect in at most one point, different from an endpoint for disjoint segments,
- no point belongs to the projection of three or more segments.

The planar projection to $\mathbb{E}^2$ of a link L in general position in $\mathbb{E}^3$ (Figure 2.3, on the top) is called a *regular projection* and is a quadrivalent planar connected graph (Figure 2.3, on the bottom left). Then, to encode the different relative heights in $\mathbb{E}^3$ of the inverse images of two intersecting segments of $\pi(\mathrm{L})$, we assign to each such point of intersection an 'over' or 'under' information by breaking each under-passing segment into two arcs, and call such a region a *crossing*. When the crossing information has been assigned to every vertex of $\pi(\mathrm{L})$, we call this planar representation a *link diagram* of L (Figure 2.3, on the bottom right). It is important to notice at this point that if a knot projection has only a single crossing, then it can be untwisted and leads to an unknot. Therefore, a nontrivial knot must have more than one crossing in a projection.

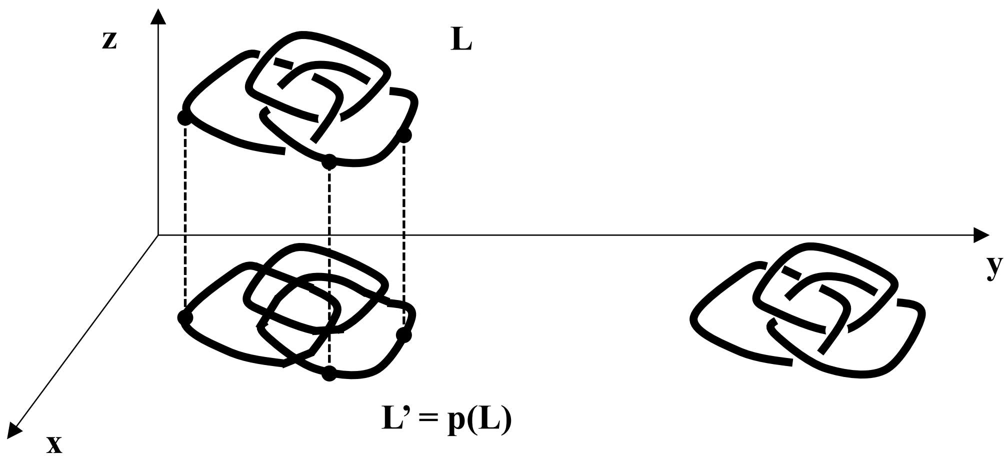



Figure 2.3: On the top, a link in general position in $\mathbb{E}^3$. On the bottom left, its regular projection to $\mathbb{E}^2$. On the bottom right, the corresponding diagram.

It is also possible to assign an *orientation* to a knot or a link, by choosing a direction to travel around the curves. The classic convention is to denote such an orientation by placing directed arrows on the curves of a diagram in a chosen direction among the two possibilities for each component. Finally, we can define a sign for each crossing in a diagram of an oriented link according to the standard convention illustrated in Figure 4.2, which uses the orientations of the two strands implied in the crossing as well as the orientation of the space. The crossing is said to be either positive or negative.

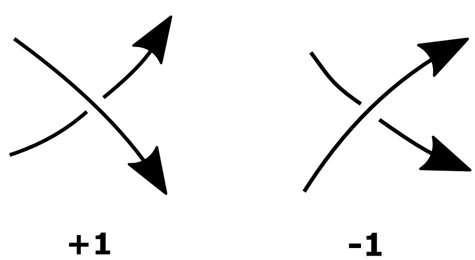



Figure 2.4: Sign convention.

Finally, we end this subsection by recalling the work of K. Reidemeister, who discovered three simple moves, illustrated in Figure 2.5, that can be applied at the diagrammatic scale and that encode the notion of ambient isotopy of links in the three-dimensional space. This approach, which provides a direct correspondence between links and their diagrams, is stated in his famous theorem, whose proof can be found in [50]. The interest of this theorem lies in the fact that it allows a complete combinatorial description of the topology of links.

**Theorem 2.6.** ***(Reidemeister Theorem)****, Two links in $\mathbb{S}^3$ or $\mathbb{E}^3$ are ambient isotopic if their exists a sequence of Reidemeister moves $\Omega_1$, $\Omega_2$, and $\Omega_3$ taking a diagram of one link to a diagram of the other.*

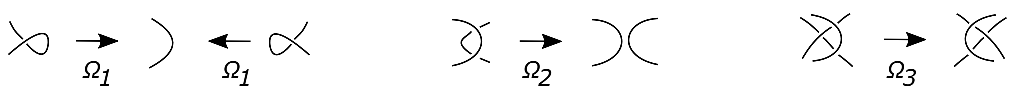



Figure 2.5: Reidemeister moves.

### 2.1.3 Sum of knots and links

The set of all knots can be approached from a viewpoint of group theory by defining an operation of sum of knots, which allows one to obtain a single knot from two originally disjoint knots. Considering two knot diagrams $D_{\mathrm{K}_1}$ and $D_{\mathrm{K}_2}$ that do not overlap, it is possible to construct a new knot by deleting a small arc from each diagram, then joining the four endpoints with two new arcs as in Figure 2.6 to obtain a new diagram $D_{\mathrm{K}_1} + D_{\mathrm{K}_2}$ and finally, apply the inverse projection map $\pi^{-1}(D_{\mathrm{K}_1} + D_{\mathrm{K}_2})$. The generated knot $\mathrm{K}_1 + \mathrm{K}_2$ is called the *sum* of the two knots. Note that the two arcs which are removed are assumed to be 'outside' of each diagram in order to avoid any crossings and that the two new arcs are chosen so they do not cross either the original knot diagrams or each other, as illustrated in Figure 2.6. Conversely, one may want to decompose a knot into two disjoint knots. Then, a knot is called a *composite knot* if it can be expressed as the sum of two knots, neither of which being isotopic to the unknot. The knot components used to create the composite knot are called *factor knots*. In particular, if a non-trivial decomposition cannot be found for a knot K, then K is said to be *prime*.

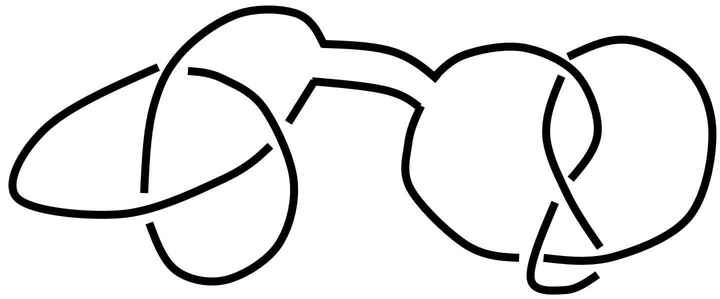


Figure 2.6: Sum of two knots.

**Definition 2.7.** [44] *A knot* K *is said to be prime if is not equivalent to the unknot, and if* K *can be decomposed in a sum of two knots* $K = K_1 + K_2$*, then it implies that* $K_1$ *or* $K_2$ *is the unknot.*

Notice that the way to sum two knots is not unique. First, there is a possibility to choose the regions on the diagrams where the small arcs are deleted. Then, it is also possible to define the sum operation for oriented knots and to construct therefore two different composite knots from the same pair of knots $K_1$ and $K_2$. Indeed, either the orientation of $K_1$ corresponds to the orientation of $K_2$ in $K_1 + K_2$, or does not match.

In a more general way, a knot K is either prime or is a composition of at least two nontrivial knots, which are themselves either prime or composite, and so forth finitely many times. This implies that a knot can be uniquely decomposed into prime knots.

**Theorem 2.8.** ***(The uniqueness and existence of a decomposition of knots)*** [57] *Any knot can be decomposed into a finite number of prime knots and this decomposition, excluding the order, is unique.*

Finally, it is possible to conclude about the group structure of the set of all knots for the sum operation.

**Proposition 2.9.** [57] *The sum of knots defines an associative and commutative law.*

However, this sum operation on knots cannot make the set of all oriented knots a group, since even if the unknot can be considered as the identity element, the existence of inverse elements is however compromised. It has therefore only the property of a semi-group, called the *semi-group formed under the operation of the sum of knots*.

## 2.2 Untwisted and Twisted Weaves

The introduction of weaves as mathematical objects has been motivated by the study of weaving that can be found in materials science, like the early interest in knot theory which came from chemistry. Since the theory aims to correspond to the physical reality, initial definitions need to be both formal and precise mathematically, while excluding unwanted pathology contradicting existing structures.

### 2.2.1 Untwisted Weaves

In this subsection, we will define an *untwisted weave* W (Figure 2.7(C)) as the lift to the topological ambient space $\mathbb{X}^3 = \mathbb{E}^2 \times \mathrm{I}$, with $\mathrm{I} = [-1, 1]$, namely the *Euclidean thickened plane*, of a particular quadrivalent connected graph embedded on $\mathbb{E}^2$ with a crossing replacing each vertex. Such a graph arises from a set of straight lines of the Euclidean plane $\mathbb{E}^2$ (Figure 2.7(A)), and intersecting only in vertices of degree four in which we specify an over or under information (Figure 2.7(B)). Let $\pi$ be the natural projection map from the Euclidean thickened plane to the Euclidean plane $\pi : \mathbb{X}^3 \to \mathbb{E}^2$, $(x, y, z) \mapsto (x, y, 0)$. The first step to define an untwisted weave is to organize the graph components in different sets.

**Definition 2.10.** [21] *Let $\Gamma_i$ be a set of straight lines embedded on $\mathbb{E}^2$ with color $i \in \{1, \cdots, N\}$, where $N$ is a positive integer. Then, $\Gamma = (\Gamma_1, \cdots, \Gamma_N)$ is a set of colored straight lines that belong to $N \geq 2$ disjoint color groups such that two lines either intersect in a vertex of degree four and are in different color groups, or belong to the same color group.*

Next, recall that to define a weave W, we will lift these colored lines to $\mathbb{X}^3$ such that the infinite curve components do not intersect each other, as in

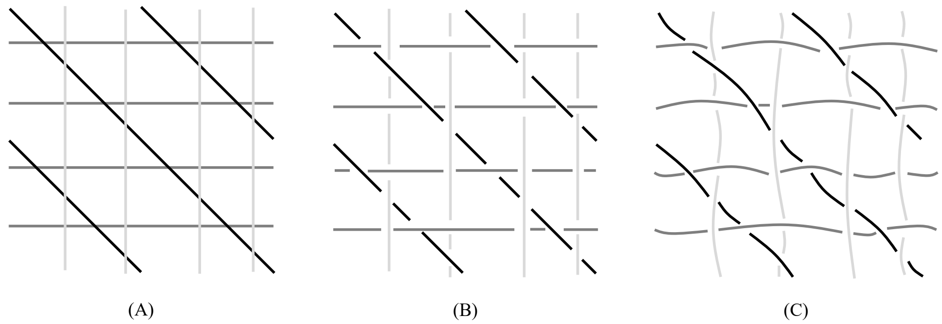



Figure 2.7: (A) Sets of straight lines belonging to different color groups on the Euclidean plane. (B) Crossing information to each intersection on the Euclidean plane. (C) Untwisted weave in the thickened Euclidean plane [21].

general knot theory. Therefore, to represent this entanglement at the planar scale, we use two intersecting arcs on $\mathbb{E}^2$, as illustrated in Figure 2.7(B). So, if $P$ is such an intersection on $\mathbb{E}^2$, then the inverse image $\pi^{-1}(P) \cap \mathrm{W}$ of this point in $\mathbb{X}^3$ has exactly two distinct points, and $P$ is called a *double point*. Then, after specifying the over or under information using arcs, it is called a *crossing*. We use a set of sequences consisting of positive integers to describe the number of consecutive over and undercrossings for each curve on $\mathbb{E}^2$.

**Definition 2.11.** [21] *Let $i$, $j$, $k$, $l$ be strictly positive integers, and let $\Gamma_i$ and $\Gamma_j$ be two disjoint sets of colored straight lines of $\Gamma$ on $\mathbb{E}^2$. Then by walking on an oriented line $\gamma_i^k \in \Gamma_i$, the crossing sequence $C_{i,j}^k$ of $\gamma_i^k$ with $\Gamma_j$ is defined either by,*

(1) *a sequence $(+1, 0)$ (resp. $(0, -1)$) if $\gamma_i^k$ is always over (resp. under) all the components of $\Gamma_j$.*

(2) *a sequence $(\cdots, +p_l, -p_{l+1}, +p_{l+2}, \cdots)$ of minimal length, where $p_l$ are strictly positive integers, such that there exists a crossing $c$ between $\gamma_i^k$ and $\gamma_j^{k_i} \in \Gamma_j$ whose closest neighboring crossing in the opposite direction is an undercrossing, and from which $\gamma_i^k$ will have $p_l$ consecutive overcrossings with the colored lines of $\Gamma_j$, followed by $p_{l+1}$ consecutive undercrossings, followed by $p_{l+2}$ consecutive overercrossings and so forth.*

*Moreover, we denote by* $\Sigma_{i,j} = (C_{i,j}^k)_{k>0}$*, with* $i, j \in (1, ..., N), i < j$*, the set of crossing sequences associated to the pair* $(\Gamma_i, \Gamma_j)$*,* $k$ *being the index of the straight lines of* $\Gamma_i$*.*

**Remark 2.12.** [21] The set of crossing sequences $\Sigma_{j,i} = (C_{j,i}^{k'})_{k'>0}$, with $i, j \in (1, ..., N), i < j$ is deduced from $\Sigma_{i,j}$ for any pair $(\Gamma_i, \Gamma_j)$, and conversely.

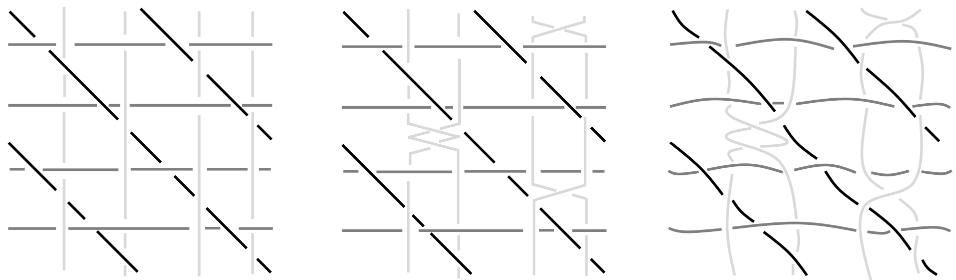
We are now ready to lift our graph with crossings to the thickened Euclidean plane to define one of the simplest classes of general weaves.

**Definition 2.13.** [21] *Let* $\Gamma = (\Gamma_1, \cdots, \Gamma_N)$ *be a set of colored straight lines on* $\mathbb{E}^2$*, belonging to* $N \geq 2$ *color groups, with a crossing information at each vertex according to a set of crossing sequences* $\Sigma = (\Sigma_{i,j})_{i>j}$ *with* $i, j \in (1, ..., N)$*. Then, we call untwisted weave the lift to* $\mathbb{X}^3$ *by* $\pi^{-1}$ *of the pair* $(\Gamma, \Sigma)$*, which is an embedding of non-intersecting infinite curves in the thickened Euclidean plane. Each lifted straight line is called a thread and two threads are said to be in the same set of threads* $T_i$ *if they are the lift of lines belonging to the same color group* $\Gamma_i$*. Moreover, we call strand any compact non-degenerate subset* $s \subseteq t$ *of a thread* $t$*.*

**Remark 2.14.** [21] The crossing sequence $C_{i,j}^k$ of a straight line $\gamma_i^k$ with the set of colored lines $\Gamma_j$ is also the crossing sequence of its lift $t_i^k = \pi^{-1}(\gamma_i^k)$ with the set of threads $T_j$. We therefore use the same notation for both spaces.

### 2.2.2 Twisted Weaves

The definition of an untwisted weave naturally implies that we can define a class of twisted weaves. To do so, we will use a pair $(\Gamma, \Sigma)$ on $\mathbb{E}^2$ that can originally be lifted in the three-dimensional space to become an untwisted weave (Figure 2.8(A)), as described in the previous subsection. However, before lifting to $\mathbb{X}^3$, we will transform such a planar representation of an untwisted weave to create new crossings between two nearest neighboring straight lines of the same color by a surgery operation (Figure 2.8(B)). This transformation, called *twist*, happened locally and the lift of a pair $(\Gamma, \Sigma)$ on $\mathbb{E}^2$ admitting at least a *twisted region* such that the axis of direction of the twisted lines are preserved will be called a *twisted weave* (Figure 2.8(C). These local twisted regions can each be characterized by a $\pm k$*-move*, meaning that two untwisted strands are replaced by two strands that twist around each other with $k$ crossings, in a right ($+k$) or left-handed ($-k$) manner, as defined in [1] for general knots.



(A) (B) (C)

Figure 2.8: (A) Set of straight lines with crossing information on the Euclidean plane. (B) Introduction of twists on the Euclidean plane. (C) Twisted weave in the thickened Euclidean plane [21].

**Definition 2.15.** [21] *Let $(\Gamma, \Sigma)$ be a graph with crossing information satisfying Definition 2.13, and let $\Gamma_i \in \Gamma$ be a set of lines with the same color $(\cdots, \gamma_i^{j-1}, \gamma_i^j, \gamma_i^{j+1}, \cdots)$, indexed by a positive integer $j$ in terms of closest neighboring components, for any $i = 1, \cdots, N$. Let $D$ be a disk whose boundary circle intersects $(\Gamma, \Sigma)$ exactly four times, and containing only two closest neighboring parallel segments of $\gamma_i^j$ and $\gamma_i^{j+1}$ of the color group $\Gamma_i$. Then, we say that $\gamma_i^j$ and $\gamma_i^{j+1}$ twist $k$ times if there exists a $\pm k$-move in $D$, meaning that they twist around each other with $k$ crossings in a right-handed way for a $+k$-move or in a left-handed manner for a $-k$-move. Such a disk $D$ containing twists is then called a twisted region.*

The total number of twists between two curves that twist in $n$ disjoint disks $D_1, \cdots D_n$ is the sum of the local number of twists for each of these disks. Thus, we can define a twisted weave from any pair $(\Gamma, \Sigma)$ that admits at least a twisted region, and such that the color groups associated with each set of threads is preserved.

**Definition 2.16.** [21] *A twisted weave is the lift to $\mathbb{X}^3$ of a pair $(\Gamma, \Sigma)$ satisfying Definition 2.13 admitting at least a twisted region. Moreover, if two threads twist their total number of twists is even and they cannot twist with any other threads.*

### 2.2.3 Unweave and entanglement

The next step to consider regarding the definition of untwisted or twisted weaves, which we will from now call weaves if there is no ambiguity, is to study the continuous deformation of these objects in $\mathbb{X}^3$. Note that we consider the existence of other classes of weaves, which are neither untwisted nor twisted. Examples could be weaves admitting crossings between threads of the same set that are more complex than twists between two threads, such as braided threads of the same color, as in braid theory. However, these objects go beyond the scope of this manuscript and will not be studied here. The same notion of ambient isotopy defined above for knot theory is applied here for the case of weaves. Considering that one of the essential properties of a weave is that it 'hangs together', which in other words we call *entangled*. This implies one cannot partition the set of all strands into nonempty subsets so that each strand of the first subset passes over every strands of a second subset, and so forth, which defines the *trivial weave*, also called the *unweave* illustrated in Figure 2.9, as described in [32] for the case of two sets of threads. We will say that a weave is *entangled* if it is not equivalent to the unweave.

**Definition 2.17.** [21] *A weave with $N$ sets of threads is said to be the unweave if there exists an isotopy of $\mathbb{X}^3$ that separates the thickened Euclidean plane into $N$ disjoint layer planes, each of them containing a single set of threads at a height coordinate $z \in [-1, 1]$.*

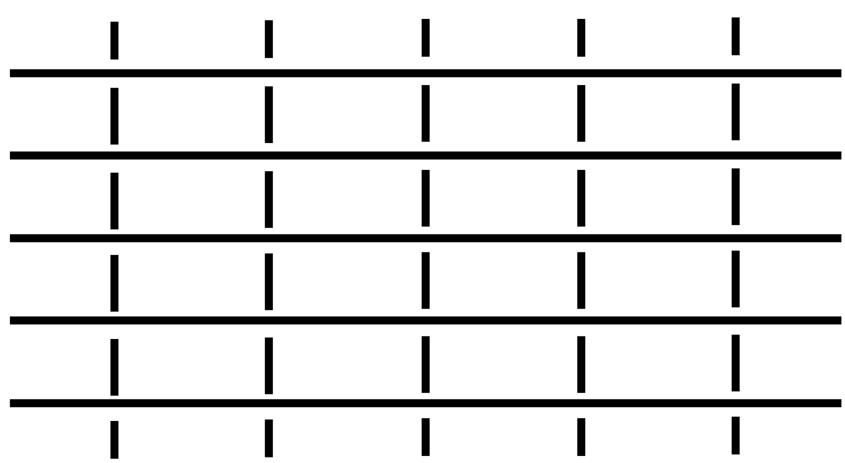


Figure 2.9: An example of unweave with two sets of threads.

The definition of entangled weave follows directly from the definition of the unweave, as mentioned above.

**Definition 2.18.** [21] *An untwisted weave* W *is said to be entangled, if it is not isotopic to the unweave.*

## 2.3 Diagrammatic Representation of Weaves

As in knot theory, we are interested in studying the properties of weaves on the plane instead of $\mathbb{X}^3$. The notion of a diagrammatic representation of weaves follows immediately from that of general links, and we will see that if a weave is periodic, then any diagrammatic translational generating cell can be seen as a particular type of link diagram embedded in a surface.

### 2.3.1 Regular projection and weaving diagrams

Ambient isotopy allows deforming a weave in $\mathbb{X}^3$ such that it becomes in a general position, that we can then project onto the Euclidean plane by the map $\pi : \mathbb{X}^3 \to \mathbb{E}^2$, $(x, y, z) \mapsto (x, y, 0)$. A weave is said to be in general position if the projection of two threads by $\pi$ to $\mathbb{E}^2$ are distinct and such that all the crossings are double points. The projection of a weave to the plane is therefore a quadrivalent connected graph isotopic to the original building graph $\Gamma$, meaning that an edge does not have to be a straight line as long as it does not intersect itself. In the particular case of a doubly periodic weave, that will be discussed below, any planar periodic cell can be seen as a link diagram in the torus as described in Figure 2.10.

**Definition 2.19.** [21] *The projection $W_0$ of a weave* W *in general position onto $\mathbb{E}^2$ by the map $\pi : \mathbb{X}^3 \to \mathbb{E}^2$, $(x, y, z) \mapsto (x, y, 0)$ is called a regular projection, and once an over or under information is given at each vertex of $W_0$, we say that this structure is an infinite weaving diagram $D_{W_0}$. Moreover, if $D_{W_0}$ is periodic, then a generating cell in only contains essential simple closed curve components on a torus $\mathbb{T}^2$ and is called a weaving motif.*

As in knot theory, it is also possible to describe the notion of ambient isotopy at the diagrammatic scale using the Reidemeister moves (Figure 2.5).

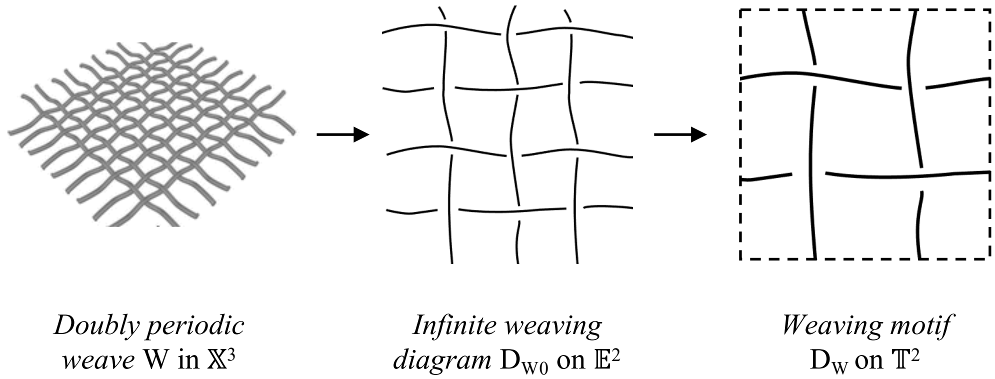



Figure 2.10: Weave, weaving diagram and weaving motif.

Recal that since our weaves lie on the thickened plane $\mathbb{X}^3$, then the ambient isotopy for weaves only allows continuous deformations in $\mathbb{X}^3$.

**Definition 2.20.** [21] *Two weaves in* $\mathbb{X}^3$ *are said to be equivalent if their weaving diagrams can be obtained from each other by a sequence of Reidemeister moves* $\Omega_1$*,* $\Omega_2$*,* $\Omega_3$ *and planar isotopies. Moreover, an equivalence class of weaves is also called a weave.*

### 2.3.2 Diagrammatic representation of entanglement

It is now also possible to characterize the entangled property of a weave at the diagrammatic scale. To do so, we must define the notion of *blocking crossings* (see Figure 2.11), which can be described in terms of *Reidemeister moves* of type $\Omega_3$ illustrated in Figure 2.5. Such a move is applied to a region of a given weave containing three strands crossing each other in a pairwise way. Thus, it contains three crossings such that one of the strands involved is either over or under the two other strands.

**Definition 2.21.** [21] *Let* $t_i \in T_i$*,* $t_j \in T_j$*, and* $t_k \in T_k$ *be three threads of a weave* W*, with* $N \geq 2$ *sets of threads* $(T_1, \cdots, T_N)$*, for all* $j, k \in (1, \cdots, N)$ *distinct. Then, if there exists a crossing* $c = \pi(t_j) \cap \pi(t_k)$ *on a weaving diagram of* W*, such that a Reidemeister move* $\Omega_3$ *is not admissible for* $\pi(t_i)$ *at the neighborhood of* $c$*, we say that* $c$ *is a blocking crossing.*

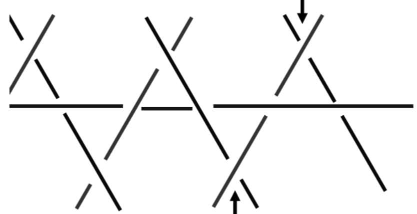



Figure 2.11: Blocking crossing [21]

Next, observe that any projection of a thread divides $\mathbb{E}^2$ in two disjoint regions, namely its *right* and its *left*, which can be arbitrarily labeled depending on the orientation of the plane and the threads. Therefore, it is possible to characterize the entangled property of a weave W in terms of the existence of blocking crossings at the diagrammatic scale by the following statement, whose proof follows directly from the definition of the unweave and of the blocking crossing, since each thread must at least admit an overcrossing and an undercrossing at the diagrammatic scale to be entangled.

**Proposition 2.22.** [21] *A weave with $N \geq 2$ sets of threads $T_1, \cdots, T_N$ is entangled if and only if for all $i \in (1, ..., N)$, each thread $t_i \in T_i$ projected on $\mathbb{E}^2$ admits a blocking crossing $c = \pi(t_j) \cap \pi(t_k)$ on its left, and a blocking crossing $c' = \pi(t'_j) \cap \pi(t'_k)$ on its right, where $t_j, t'_j \in T_j$ and $t_k, t'_k \in T_k$ are disjoint threads, for all $j, k \in (1, \cdots, N)$ distinct.*

### 2.3.3 Weaving motifs

Many weaves that physically exist as material structures show the property of being doubly periodic. Thus, as mentioned earlier, by taking a diagrammatic periodic cell, namely a weaving motif, which can be described as a set of crossed strands embedded on a parallelogram, and by gluing the endpoints of the strands lying on its opposite sides, we end up with a link diagram embedded in a torus. It is therefore natural to approach these structures using knot theory, as done by S. Grishanov, H. Morton et al. in [23], [24],

[25], [26], [27], [54], as well as A. Kawauchi in [40]. Then, being inspired by the use of graphs of the hyperbolic plane in the study of other types of entangled structures, as mentioned in the introduction, we also consider a generalization of weaving motifs to *higher genus surfaces*, which encodes the properties of a periodic weaving diagram on the *hyperbolic plane*. In this hyperbolic case, since the notion of parallel lines that we used to define the sets of threads in the Euclidean case cannot be extended, we will restrict our definition to the case of a hyperbolic diagram whose associated regular projection is isotopic to a *kaleidoscopic tiling* of the hyperbolic plane by convex regular polygons. This means that such a periodic tiling can be reconstructed starting from a single hyperbolic convex polygon $\mathcal{P}$ with $n$ sides, that is mirrored along its sides recursively to tile the entire plane, as illustrated in Figure 2.12.

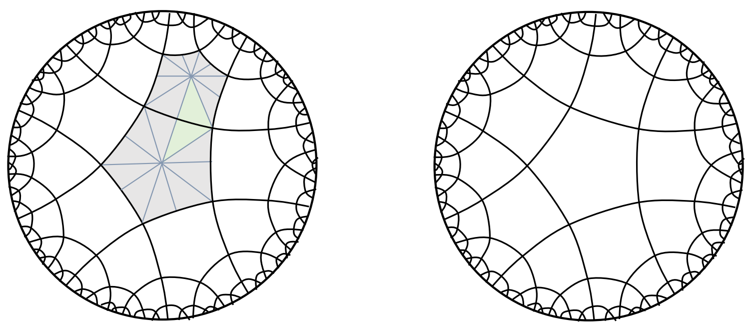


Figure 2.12: Example of kaleidoscopic tiling of the hyperbolic plane.

If the number of sides $n$ of $\mathcal{P}$ is odd, then each side will be assigned a different color, which leads to a set of $n$ color groups. However, if $n$ is even, then each pair of opposite sides, meaning parallel sides in the abstract Euclidean sense, will be assigned the same color, which leads to a set of $\frac{n}{2}$ color groups. The mirror symmetry preserves the color, which defines the color groups in a coherent fashion that generalizes the definition of sets of threads in the Euclidean case, see Figure 2.13 for an illustration.

This defines the class of *hyperbolic untwisted weaving diagrams* (Figure 2.13, left) and since the notion of twists also applies here, the definition of hyperbolic twisted weaving diagrams follows directly (Figure 2.13, right).

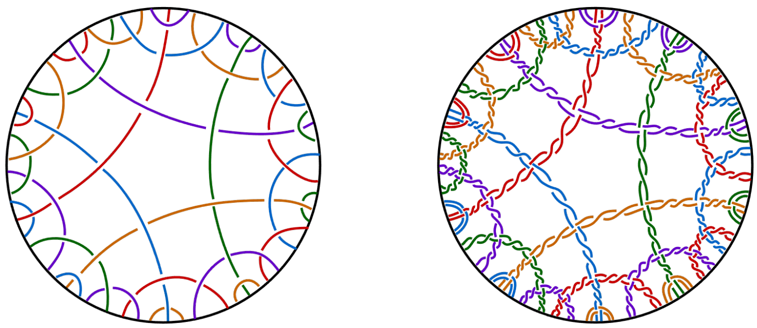


Figure 2.13: Example of color groups for a weaving diagram of the hyperbolic plane [48]

In this section, we will therefore consider the topological ambient space $\mathbb{X}^3 = \mathbb{X}^2 \times \mathrm{I}$, with $\mathrm{I} = [-1, 1]$, where $\mathbb{X}^2 = \mathbb{E}^2$ or $\mathbb{X}^2 = \mathbb{H}^2$, and we will use the Poincare disk model as a representation of the hyperbolic plane.

**Weaving motifs on a torus and extension to higher genus surfaces**

In the case of periodic weaves, instead of analyzing a planar diagram containing an infinite number of crossings, we prefer to study a weaving motif $D_{\mathrm{W}}$ containing a finite number of crossings, and that can also be seen as a link diagram embedded on a surface $\Sigma_g$ of genus $g \geq 1$. Such a link results from the pairwise identification of the sides of a periodic cell of the planar diagram considered. More precisely, a weaving motif is a set of crossed strands embedded on a polygon, and there is an infinite way to choose the shape of such a polygon, the easiest being a regular $4g$-gons of $\mathbb{X}^2$. The particular case for $g = 1$ corresponds to doubly periodic weaving diagrams embedded on the Euclidean plane and is detailed in the work of Grishanov and his coauthors [23], [24], [25]. Here we generalize the results to a closed orientable surface of any genus. The diagram $D_{\mathrm{W}}$ obtained on the surface $\Sigma_g$ of genus $g \geq 1$ is still called a weaving motif. Such a diagram consists of several simple closed polygonal curves drawn on $\Sigma_g$, as for classic link diagrams on a similar surface, and each curve is a *component* of the diagram. Moreover, we will also assume that $\Sigma_g - D_{\mathrm{W}}$ consists of open discs.

As mentioned previously, regular polygons of the plane are not the unique possibility, when selecting a generating cell of an infinite periodic weaving diagram $D_{W_0}$ on $\mathbb{X}^2$. For example, if $\mathbb{X}^2 = \mathbb{E}^2$, then any parallelogram of unit area which has for sides integer vectors can be chosen as a unit cell, as explained by Grishanov et al [25]. Moreover, if $\mathbb{X}^2 = \mathbb{H}^2$, it is not necessary to consider the area since two hyperbolic surfaces with the same topology always have the same area. To extend the results of Euclidean weaving motifs on a torus to hyperbolic weaving motifs on higher genus surfaces, we need to take into account the *Teichmüller space* of a surface $\Sigma_g$ of genus $g \geq 2$, and its *Mapping Class Group*. For more details, we refer to [19]. First, we need to consider a geodesic hyperbolic $4g$-gon on $\mathbb{H}^2$, meaning a polygon such that the sum of its interior angles is equal to $2\pi$. Then, we label its edges such that they can be identified pairwise, which results in a closed marked hyperbolic surface of genus $g$, as illustrated in Figure 2.14.

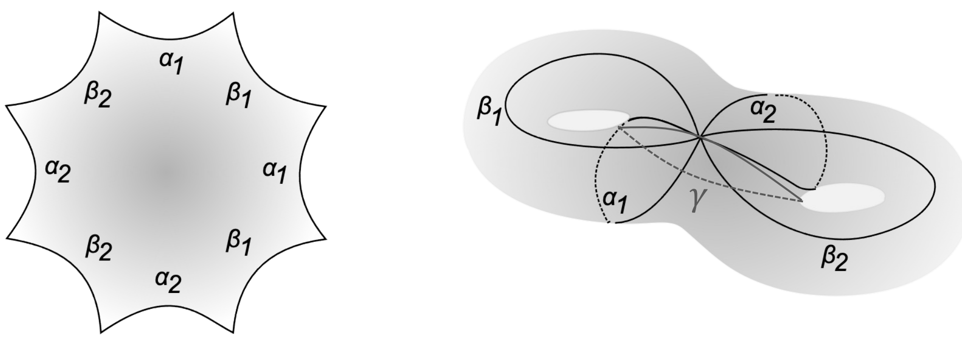



Figure 2.14: A regular $\Sigma_2$-tile and its corresponding marked hyperbolic surface [48].

We call such a polygon a $\Sigma_g$-*tile* and the Teichmüller space of the corresponding surface $\Sigma_g$ can be seen as the space of marked surfaces homeomorphic to it. Moreover, it is well-known that this space is in bijection with the set of equivalence classes of hyperbolic $\Sigma_g$-tiles. Note that two $\Sigma_g$-tiles are said to be *equivalent* if they differ by a marked, orientation-preserving isometry and by 'pushing the basepoint', which is the point on the surface where all the vertices of a $\Sigma_g$-tile meet after gluing. The details of this bijection are given in [19].

We will now prove the existence of an infinite number of $\Sigma_g$-tiles, which implies the existence an infinite number of possibilities to choose a generating cell for any periodic weaving diagrams. We have seen that an element of the Teichmüller space of $\Sigma_g$, denoted by Teich($\Sigma_g$), represents the equivalence class of a marked surface $\Sigma_g$ of genus $g$. In other words, such an element corresponds to equivalent $\Sigma_g$-tiles that can generate isometric tilings as periodic cells. However, recall that any chosen $\Sigma_g$-tile, which corresponds to a unique element in Teich($\Sigma_g$) up to equivalence, can be taken as a generating cell of a hyperbolic weaving diagram, namely a hyperbolic weaving motif. This means that different weaving motifs correspond to different marked surfaces, and are thus not isometric to the chosen $\Sigma_g$-tile. Moreover, let $\Sigma_g$ be a marked surface of genus $g$ in Teich($\Sigma_g$) and let $MCG(\Sigma_g)$ be its mapping class group. The simplest infinite-order elements of $MCG(\Sigma_g)$ are known to be the *Dehn twists* of $\Sigma_g$, and their action on $\Sigma_g$ will change the marking of the surface. Therefore, by applying an infinite number of times to a given marked surface $\Sigma_g$ of Teich($\Sigma_g$) the same Dehn twist, we never obtain equivalent marked surfaces, nor isometric $\Sigma_g$-tiles. This justifies the existence of an infinite number of weaving motifs in $\mathbb{X}^2$ corresponding to the same weave. Moreover, any pair of weaving motifs of the same fixed periodic weaving diagram of $\mathbb{X}^2$, which only differ by their marking, can be obtained from each other by a sequence of Dehn twists of $\Sigma_g$ along their generating curves [19], [43], denoted by $\alpha_i$, $\beta_i$, and $\gamma_i$ in Figure 2.14.

As mentioned in [25] for the Euclidean case, an *ambient isotopy* of a periodic weave is any continuous deformation that must also preserve its periodic property. Moreover, for the diagrammatic representations, the equivalence concerns all the different weaving motifs with the same topology corresponding to a single weave. Since we have seen that there exists an infinite number of them, the continuous deformation and the Dehn twists of the surface $\Sigma_g$ must also be considered here. Therefore, the Reidemeister theorem for weaving motifs on a torus stated in [25] generalizes immediately to higher genus surfaces, with a proof similar to the original one using the arguments on the Teichmüller space and Mapping Class Groups.

**Theorem 2.23.** ***(Reidemeister Theorem for Weaves)*** [25], [48] *Two periodic weaves in $\mathbb{X}^3$ are ambient isotopic if and only if their weaving motifs be obtained from each other by a sequence of Reidemeister moves $\Omega_1$, $\Omega_2$, $\Omega_3$, isotopies on the surface $\Sigma_g$ of genus $g$, and $\Sigma_g$-twists.*

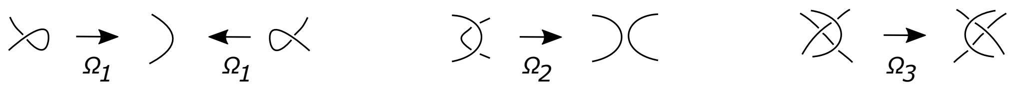



Figure 2.15: Reidemeister moves.

### Particular properties of weaving diagrams

Interesting properties of classical links can often be generalized to weaves.

**Definition 2.24.** [48] *A weave, a weaving diagram, or a weaving motif is said to be alternating, if all its crossing sequences are equal to* $(+1, -1)$.

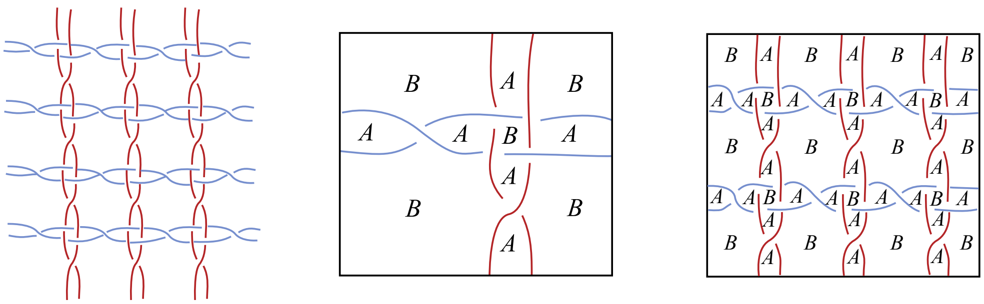
The notion of alternating weaving diagrams is a fundamental point that will be used in Chapter 4. It indeed characterizes an interesting class of weaves that can be classified by generalizing famous theorems for alternating knots and links. Next, we will say that a weaving motif $D_{\mathrm{W}}$ is *prime* if any simple closed curve $\gamma$ in $\Sigma_g$ bound a disk that intersects $D_{\mathrm{W}}$ exactly twice transversely away from crossings and without crossings in its interior. This intuitively means that a prime weaving motif cannot be decomposed into a 'connected sum' of weaving motifs. However, this notion of sum operation on weaving motifs is not trivial since it depends on specific boundary conditions. This is currently a work in progress.

Finally, for the purpose of studying the classification of periodic weaves, we need to define the notion of reduced motifs. Here this notion differs slightly from the definition of a reduced link in classic knot theory, since one need to take into consideration that a weaving motif is a link diagram on a surface that encodes an infinite and periodic planar diagram whose component are simple open curves.

**Definition 2.25.** [48] *A reduced weaving motif* $D_{\mathrm{W}}$ *in* $\Sigma_g \subset \mathbb{X}^2$ *is one that does not contain an isthmus in its corresponding infinite planar diagram. An isthmus is a crossing in the diagram that does not separate its neighborhood into four distinct regions in the associated infinite diagram. We say that* $D_{\mathrm{W}}$ *is* $\Sigma_g$*-reduced.*

Moreover, we will call a crossing $c$ *proper* if the four regions around $c$ in $D_{\mathrm{W}}$, delimited by the projection of the strands, are all distinct. When every crossing of $D_{\mathrm{W}}$ is proper, $D_{\mathrm{W}}$ is said to be *proper*.

We illustrate these notions of reduced and proper in Figure 2.16.




(a) (b) (c)

Figure 2.16: Distinction between proper and unproper $\Sigma_g$-reduced weaving motifs. On the left, an infinite reduced (without isthmus) planar diagram (a). On the center, a $\Sigma_g$-reduced unproper weaving motif (b), On the right, a $\Sigma_g$-reduced proper weaving motif (c) [48].

# Chapter 3

# CONSTRUCTION OF WEAVING MOTIFS FROM PERIODIC TILINGS OF THE PLANE

Being inspired by the construction of polyhedral links from Platonic and Archimedean polyhedron, using the concept of *polyhedral link* defined by W.Y. Qiu et al. in [33], [47], [60], we tried to apply the same strategy to periodic planar tilings, as presented in our paper [22]. The polyhedral link methods consist in transforming a polyhedron into a link by replacing each edge with a *single* or *double line*, possibly with *twists*, and each vertex by a set of *branched curves* or *crossed curves*. We will call these three methods applied in $\mathbb{E}^2$ *polygonal link* methods. By applying these methods to different types of tessellations of the plane, we observed that not only weaving diagrams can be constructed, but also diagrams that contain closed curve components.

The first main purpose of this chapter is to describe a methodology to construct untwisted and twisted weaving motifs using the polygonal link methods. Note that we will construct alternating weaving motifs by convention, meaning that each crossing sequence is equal to $(+1, -1)$, and consider that an overcrossing can be turned into an undercrossing afterward, and conversely. Then, our second objective is to define a systematic way to predict whether the motif constructed from a given periodic tiling and a chosen polygonal link method will be a weaving motif or not.

## 3.1 Background: construction of polyhedral links

Motivated by the study of new molecular structures in chemistry through knot theory, and in particular by linked structures called *catenanes*, W.Y. Qiu et al. developed a new methodology to understand the construction of such links on the basis of the Platonic and Archimedean polyhedra, used as scaffold for molecular design. The resulting object defined an interlocked cage called a *polyhedral link* [33], [47], [60]. Note that by a polyhedron, we mean an embedded connected graph in the three-dimensional Euclidean space, consisting of a finite number of faces, vertices, and edges. The first methodology created to build such polyhedral links is called *$n$-cross-curve and $m$-twisted double-lines covering* and is illustrated in Figure 3.1. In this case an *$n$-crossed curve* building block (Figure 3.1(a)) covers a vertex, while either a $2n$ or $2n+1$ *$m$-twisted double-lines* building block covers an edge, with $n$ and $m$ positive integers (Figure 3.1 (b), (c), and (d)).

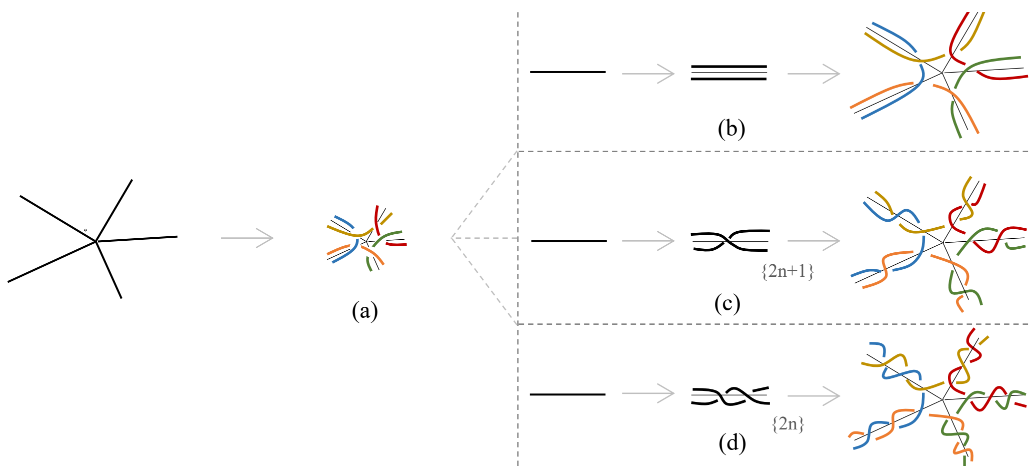



Figure 3.1: 5-cross-curve and $m$-twisted double-lines covering.

Then, these two types of blocks are glued together, and the operation is repeated until all the vertices and edges of the polyhedron are covered, as in Figure 3.2.

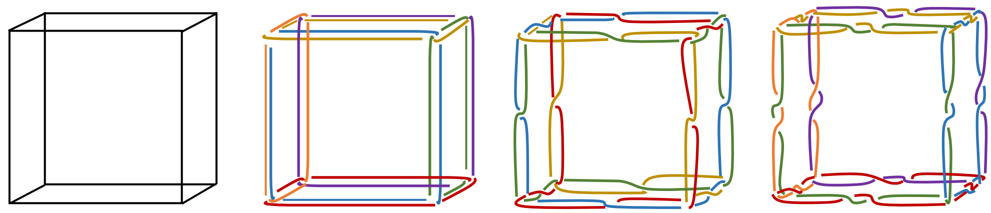


Figure 3.2: Example of polyhedral links constructed from a $n$-cross-curve and $m$-twisted double-lines covering.

In a second paper, they introduced another methodology called *$n$-branched curves and $m$-twisted double-lines covering*. Here, the *$n$-cross-curve* block is replaced by a *$n$-branched curves* building block that covers the vertices of a given polyhedron. See Figure 3.3 for an illustration.

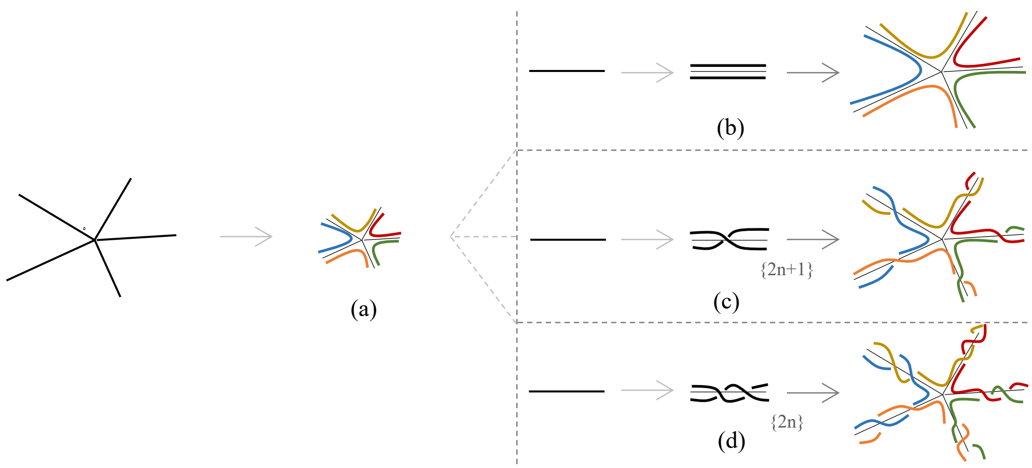



Figure 3.3: 5-branched curves and $m$-twisted double-lines covering.

Finally, in a later study, they presented their last methodology called *cross-curve and single-line covering* for polyhedra with vertices of degree four. In this case, each vertex and edge are covered by a *cross-curve* or *single-line* building block, respectively, as shown in Figure 3.4.

Figure 3.4: Cross-curve and single-line covering.

## 3.2 Polygonal link methods

In the previous chapter, we have seen that a weaving diagram can be described from a particular type of planar connected quadrivalent graph with crossing information at each vertex. Using a graph-theoretic approach to study the properties of entangled structures has indeed shown to be very useful in many cases, such as in [9], [10], [15], [35], [41]. From now, we will extend and formalize the construction methods described above to planar tilings by polygons, which can be illustrated in a similar way as in Figure 3.5, and called them the *polygonal link* methods [22].

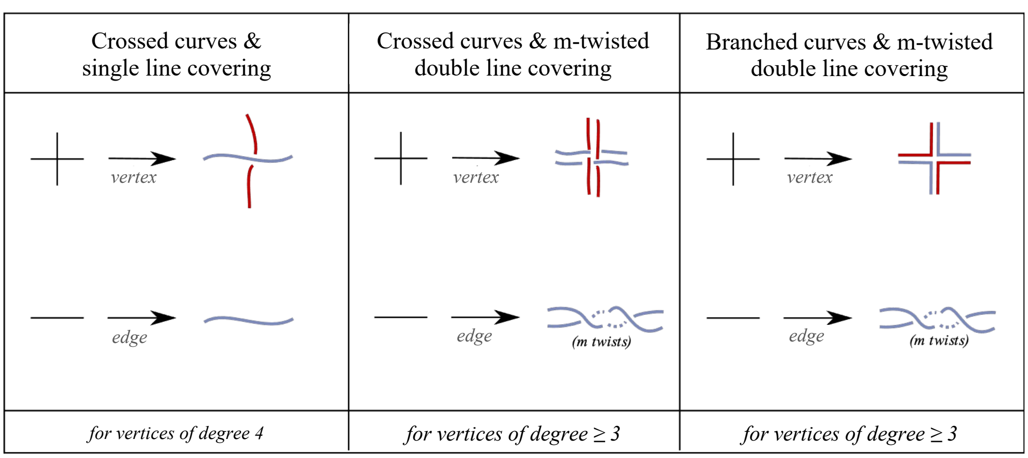



Figure 3.5: Polygonal link methods: transformation of the vertex and the edges of a building block [22].

We indeed noticed that the transformations applied to polyhedra to construct polyhedral links were an efficient way to construct weaving motifs from doubly periodic planar tilings. However, we observed cases where closed curve components were created in the universal cover, which do not satisfy our definition of weaves. This motivates the introduction and characterization of two other types of doubly periodic entangled structures, the *polycatenanes* and the *mixed* motifs.

In chemistry, polycatenanes are described as multiple linked rings [45], [70], where each such ring can be seen as an unknot. We can thus formally define a polycatenane as a particular type of link whose components are all trivial knots. Let $\Omega = \{R_1, R_2, \cdots\}$ be a set of infinitely many intersecting simple closed curves that covers all $\mathbb{E}^2$. Here the rings do not need to be organized in different sets, so the notion of color groups used for weaves is not applied here. However, we will assign to each pair of such loops $R_i$ and $R_j$ a crossing sequence $C_{i,j}$, where $i$ and $j$ are distinct positive integers. The definition of a crossing sequence follows from the one from weaves.

**Definition 3.1.** [22] *Let $i$, $j$, $k$, $l$ be strictly positive integers, and let $R_i$ and $R_j$ be two distinct oriented simple closed curves of $\Omega = \{R_1, R_2, \cdots\}$ on $\mathbb{E}^2$, intersecting exactly $k$ times. Then by walking on $R_i$, the crossing sequence $C_{i,j}$ of $R_i$ with $R_j$ is defined either by,*

(1) *a sequence $(+1, 0)$ (resp. $(0, -1)$) if all the $k$ intersections are assigned an over (resp. under) information for $R_i$.*

(2) *a finite sequence $(+p_1, -p_2, \cdots, +p_{l-1}, -p_l)$ of minimal length, where $p_l$ are strictly positive integers, such that there exists a crossing $c$ between $R_i$ and $R_j$ whose closest neighboring crossing in the opposite direction is an undercrossing, and from which $R_i$ will have $p_1$ consecutive overcrossings with $R_j$, followed by $p_2$ consecutive undercrossings, followed by $p_3$ consecutive overercrossings and so forth.*

*Moreover, we denote by $\Sigma = (C_{i,j})_{i<j}$, with $i, j$ distinct, the set of crossing sequences associated to the pair $(R_i, R_j)$.*

**Remark 3.2.** [22] As for weaves, the crossing sequence $C_{j,i}$ is naturally deduced from $C_{i,j}$ for any pair $(R_i, R_j)$, and conversely.

Then a *polycatenane* can also be defined as the lift to the thickened Euclidean plane of these planar linked rings, respecting a set of crossing sequences, as illustrated in Figure 3.6.

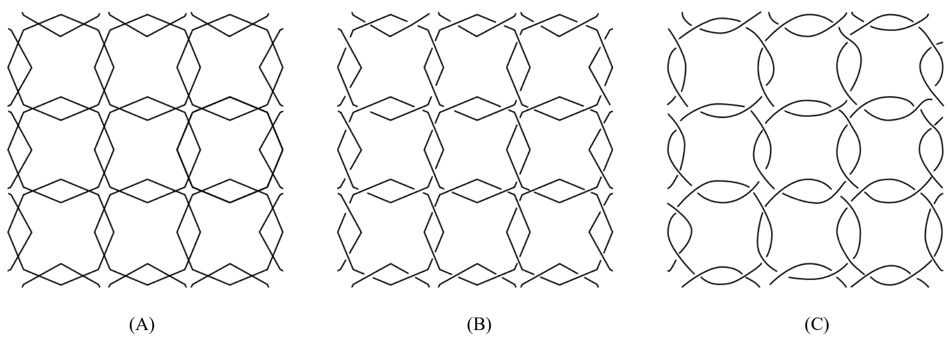


Figure 3.6: (A) Sets of intersecting simple closed curves on the Euclidean plane. (B) Crossing information to each intersection on the Euclidean plane. (C) Polycatenane in the thickened Euclidean plane view from the top [22].

**Definition 3.3.** [22] *Let* $\Omega = \{R_1, R_2, \cdots\}$ *be a set of infinitely many simple closed curves embedded on* $\mathbb{E}^2$*, with a crossing information at each vertex given by a set of crossing sequences* $\Sigma = (C_{i,j})_{i<j}$*, with* $i, j$ *distinct positive integers. Then, we call polycatenane the lift to* $\mathbb{X}^3$ *by* $\pi^{-1}$ *of the pair* $(\Omega, \Sigma)$*, which is an embedding of non-intersecting simple closed curves in* $\mathbb{X}^3$*, called rings.*

As for weaves, a polycatenane W in a general position in $\mathbb{X}^3$ can be projected onto the Euclidean plane by the map $\pi$, and for doubly periodic structure, a generating cell is called a *polycatenane motif.* Moreover, since a motif is a periodic cell of a doubly periodic diagram of $\mathbb{E}^2$, it is thus a diagram embedded on a torus $\mathbb{T}^2$. We can therefore characterize the different types of structures in terms of trivial and nontrivial simple closed curves on $\mathbb{T}^2$.

**Proposition 3.4.** ***(Link diagram of a doubly periodic entangled structure)*** [22]

- *a weaving motif is a link diagram whose components are essential closed curves which lift to simple open curves in the universal cover.*
- *a polycatenane motif is a link diagram whose components are null-homotopic curves which lift to simple closed curves in the universal cover.*

*Moreover, we call mixed motif a link diagram containing both types of components.*

Now that we have defined our new objects, we are ready to describe the methodology to construct weaving, polycatenane, and mixed motifs, by transforming simultaneously all the vertices and edges of an arbitrary generating cell of a given doubly periodic topological *polygonal tessellation* of the Euclidean plane with the same polygonal link method. Here a polygonal tessellation is defined as a covering of $\mathbb{E}^2$ by polygons such that their interior points are pairwise disjoint. We restrict to tilings which are also *edge-to-edge*, meaning that the vertices are corners of all the incident polygons. However, we do not consider geometric properties such as lengths and angles, but only the degree of the vertices and the number of sides of its adjacent polygons, meaning that tilings by squares or parallelograms are considered topologically equivalent here, as described in [31]. We will call a doubly periodic tessellation satisfying all these conditions a $\mathcal{T}$-tiling, and any periodic cell is called a $\mathcal{T}$-cell. Our first step is to translate mathematically the polygonal methods illustrated in Figure 3.5, starting with the transformation of edges.

**Definition 3.5.** [22] *Let $e_{v,v'}$ be the edge of a given $\mathcal{T}$-cell connecting the vertices $v$ and $v'$. Then $e_{v,v'}$ is said to be transformed by,*

- *a single line covering, if each is replaced by a unique strand $s_{v,v'}$;*
- *an $m$-twisted double line covering, if each is replaced by a pair of strands $s_{v,v'}$ and $s'_{v,v'}$ crossing $m$ times in an alternating fashion, with $m$ a positive integer.*

**Remark 3.6.** [22] In the case of an $m$-twisted double line covering, we will decompose the strands into three parts, as illustrated in Figure 3.7,

$$s_{v,v'} = s_{v_\mu,v'} \cup S_{v,v'} \cup s_{v,v'_\mu},$$
$$s'_{v,v'} = s'_{v_\mu,v'} \cup S'_{v,v'} \cup s'_{v,v'_\mu},$$

with $\mu \in \{l, r\}$, where $l$ stand for *left* and $r$ for *right*. Consider the Euclidean plane with a right-handed orientation. To assign the value $\mu$, we consider that at a neighborhood $D$ of a vertex $v$, we give an orientation to $e_{v,v'}$ toward the vertex $v$, which implies that at a neighborhood $D'$ of $v'$, the orientation of $e_{v,v'}$ is opposite, since it is toward the vertex $v'$. Such an edge segment separates these neighborhoods into a left and right region, with respect to its orientation. Therefore, once an edge is transformed by an $m$-twisted double line covering, for each neighborhood, one of the strand is said to be on the left and the other on the right.

We will use the following convention.

- If $m = 0$ or $m$ is even,

$$s_{v,v'} = s_{v_r,v'} \cup S_{v,v'} \cup s_{v,v'_l},$$

$$s'_{v,v'} = s_{v_l,v'} \cup S_{v,v'} \cup s_{v,v'_r}.$$

- If $m$ is odd,

$$s_{v,v'} = s_{v_r,v'} \cup S_{v,v'} \cup s_{v,v'_r},$$

$$s'_{v,v'} = s_{v_l,v'} \cup S_{v,v'} \cup s_{v,v'_l}.$$

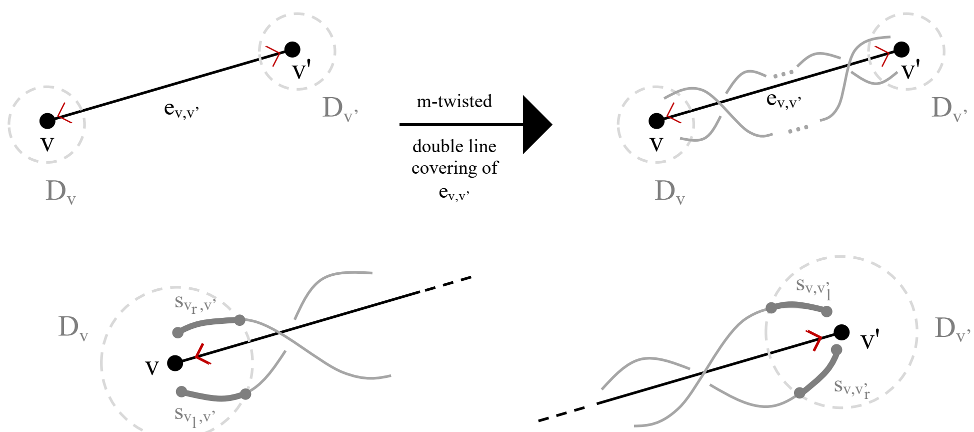



Figure 3.7: Strand labelling for an $m$-twisted double line covering [22].

Now we formalize the covering of the vertices. In Figure 3.5, we observe that each vertex v of degree $n \geq 3$ belonging to the given $\mathcal{T}$-cell can be transformed into an entangled or trivial region in three ways. At the neighborhood $D_v$ of $v$, we label its adjacent edge segments starting from an arbitrary edge $(e_v)_0$. Then, its first counterclockwise adjacent edge is $(e_v)_1$, the second adjacent edge is $(e_v)_2$, and so forth such that $(e_v)_0 = (e_v)_n$. Conversely, by reading clockwise, its first adjacent edge is $(e_v)_{-1}$, the second adjacent edge is $(e_v)_{-2}$, and so forth such that $(e_v)_0 = (e_v)_{-n}$, as illustrated in Figure 3.8.

For the edges, in the case of a single line covering, the strands can only connect at a vertex in a *crossed curves* pattern. This method can only be applied if all the vertices of the tiling have degree four. In that case, each pair

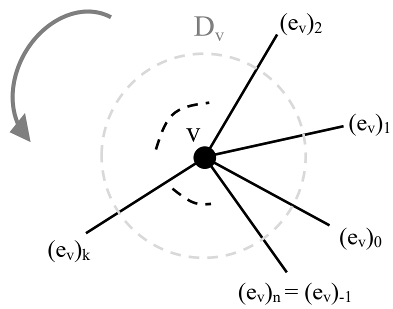



Figure 3.8: Edge labelling at a neighborhood of a vertex $v$ of degree $n$ [22].

of opposite strands are connected into a single strand, and we will use the convention that the strand being the union of $(s_v)_1$ and $(s_v)_3$ will be over the second strand, where a strand $(s_v)_i$ replaced the edge segment $(e_v)_i$ in $D_v$, for $i = 0, 1, 2, 3$. Such crossing information can be reverse to possibly obtain non-equivalent structures. However, if an edge is covered by an $m$-twisted double line then we have two possibilities. We can either connect each right strand $(s_{v_r,v'})_i$ in $D_v$ to its closest neighboring left strand $(s_{v_l,v''})_{i+1}$, constructed from the edge adjacent to the vertices $v$ and $v'' \neq v'$, by a *branched curves* transformation. Otherwise, we can use a *crossed curves* transformation by connected each right strand $(s_{v_r,v'})_i$ to its second counterclockwise adjacent left strand $(s_{v_l,v''})_{i+2}$. For these two transformations, we will use the convention that at the resulting crossings, the left strand is always over, and the right strand under, with once again the possibility to reverse the crossing information afterward.

**Definition 3.7.** [22] *Each vertex* v *of degree* $\mathrm{n}_v \geq 3$ *of a given* $\mathcal{T}$*-cell is said to be transformed by,*

- *crossed curves, if* $\mathrm{n}_v = 4$ *and if each strand* $(s_{v,v'})_i$ *is glued to the strand* $(s_{v,v''})_{i+2}$ *to become a single strand, where each edge adjacent to* v *has been replaced by a single line covering, and* $v, v', v''$ *are distinct adjacent vertices;*
- *crossed curves, if each strand* $(s_{v_r,v'})_i$ *connects with the strand* $(s_{v_l,v''})_{i+1}$

*to become a single strand, where each edge adjacent to* v *has been replaced by an $m$-twisted double line covering, $m \geq 0$, and $v, v', v''$ are distinct adjacent vertices;*

- *branched curves, if each strand $(s_{v_r,v'})_i$ connects with the strand $(s_{v_l,v''})_{i+2}$ to become a single strand, where each edge adjacent to* v *has been replaced by an $m$-twisted double line covering, $m \geq 0$, and $v, v', v''$ are distinct adjacent vertices.*

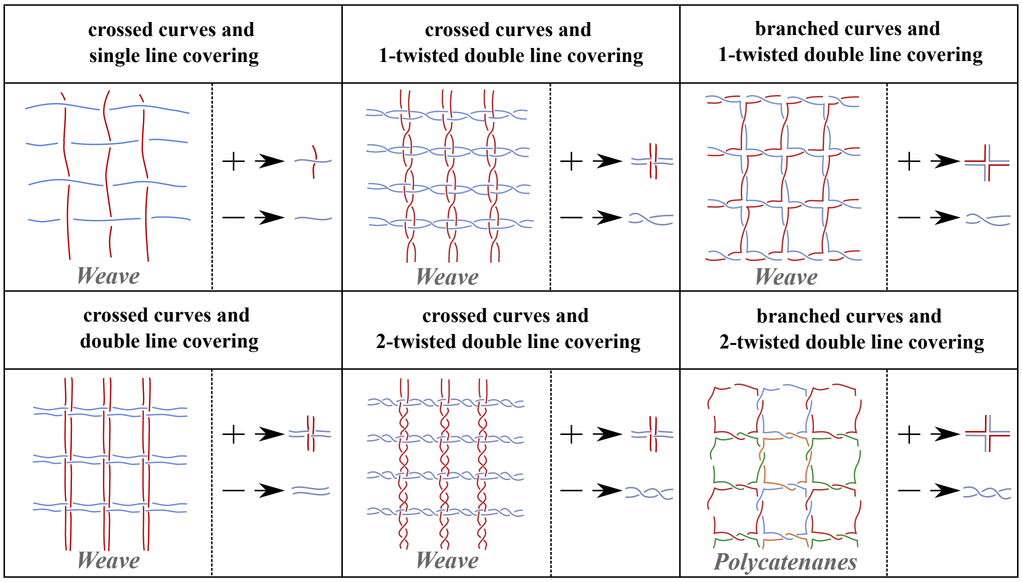
We can summarize the transformation of the vertices and edges of our given $\mathcal{T}$-cell by one of the three methods, and use the following notation.

**Definition 3.8.** [22] *A $\mathcal{T}$-cell is said to be transformed by the polygonal link method $(\Lambda, L)$, $L \in \{s, m\}$, $m$ being a positive integer, and $\Lambda \in \{C_r, B_r\}$, if all its vertices and all its edges are transformed by the same method with,*

$$(\Lambda, L) = \begin{cases} (C_r, s) & : \textit{crossed curves and single line covering}, \\ (C_r, m) & : \textit{crossed curves and } m\textit{-twisted double line covering}, \\ (B_r, m) & : \textit{branched curves and } m\textit{-twisted double line covering.} \end{cases}$$

We show examples in Figure 3.9 (top) of weaving and polycatenane diagrams constructed by applying the polygonal methods to a square tiling.

**Remark 3.9.** In this chapter, we discussed the transformation of doubly periodic tilings of the Euclidean plane. However, we can also apply the same polygonal link methods to hyperbolic periodic tilings to construct hyperbolic motifs and diagrams, as mentioned in the introduction and illustrated in in Figure 3.9 (bottom). Moreover, such transformations can also be applied to non-periodic planar tilings or graphs.



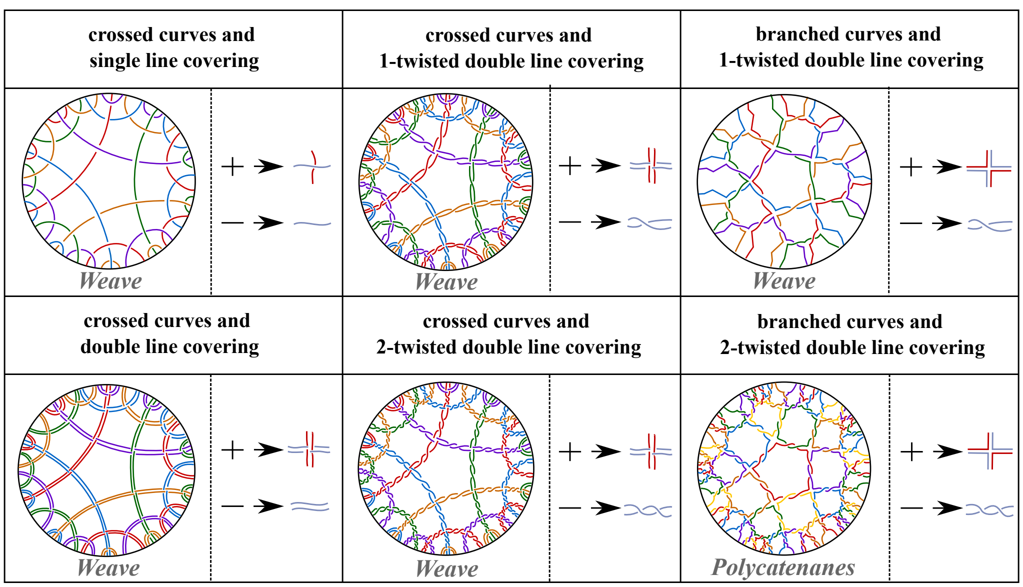




Figure 3.9: Examples of weaves and polycatenanes constructed from a Euclidean tiling and a hyperbolic tiling [22], [48].

## 3.3 Characteristic loops of polygonal links

In the second part of this chapter, we want to define a systematic way to construct, in particular, weaving motifs from tilings using the polygonal link methods. This means that we need a way to predict whether a $(\Lambda, L)$ method applied to a chosen $\mathcal{T}$-cell will generate a weaving motif, a polycatenane motif or a mixed motif.

To do so, we notice that the curve components of the motifs created by the polygonal link methods cover paths in the tiling periodic cell, called *polygonal chains* [19]. After identifying the opposite sides of the boundary of a $\mathcal{T}$-cell, each polygonal chain becomes homotopic to a closed curve on a torus by definition, and is therefore either null-homotopic or essential. We call such a closed polygonal chain a *characteristic loop of* $(\Lambda, L)$ and we introduce a combinatorial description of this object.

**Definition 3.10.** [22] *Let $U_{\mathcal{T}}$ be an arbitrary $\mathcal{T}$-cell and $(\Lambda, L)$ be a chosen polygonal link method. A characteristic loop of $(\Lambda, L)$ is an oriented curve isotopic to a polygonal chain in $U_{\mathcal{T}}$ defined by an ordered and reduced sequence of edges arbitrary oriented,*

$$\Delta_{(\Lambda,L)} = (\delta_0^{\pm}, \delta_1^{\pm}, \cdots, \delta_k^{\pm}),$$

*where $\delta_k^+$ and $\delta_k^-$ denote the same edge with opposite orientation, and $\delta_0$ and $\delta_k$ being edges of $U_{\mathcal{T}}$ sharing a common vertex, with $k$ a positive integer.*

We must now identify the characteristic loops for each of the three distinct $(\Lambda, L)$ methods independently, see Figure 3.10 and Figure 3.11 for two examples of characteristic loops on a triangular tiling.

### 3.3.1 Crossed curves and single line covering: $(\Lambda, L) = (C_r, s)$

Let $U_{\mathcal{T}}$ be an arbitrary $\mathcal{T}$-cell satisfying the above conditions. First, we choose an arbitrary vertex $v$ of $U_{\mathcal{T}}$ that will be the start and endpoint of a characteristic polygonal chain $\Delta_{(C_r,s)}$. From the definition of the crossed curves and single line covering method, and using the above notations, we know that the strand covering the edge adjacent to the arbitrary vertices $v$ and $v'$ must be glued to the strand covering its second counterclockwise adjacent edge in $v'$, also adjacent to another vertex $v''$.

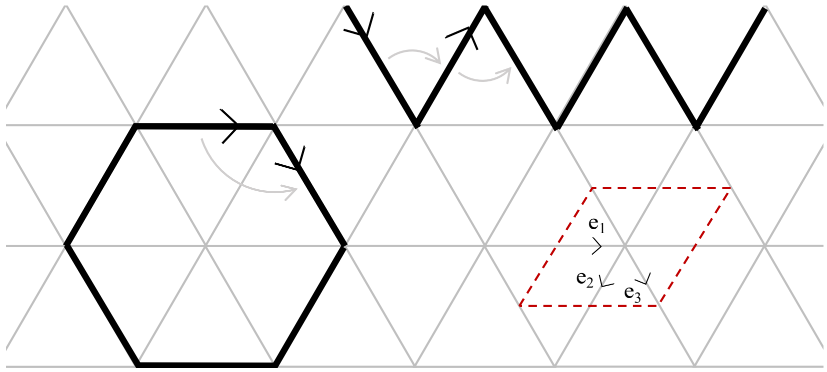



Figure 3.10: Triangular tiling. Left: A simple closed polygonal chain that would be covered by applying $(\Lambda, L) = (C_r, 2m)$. Top: A (part of a) simple open polygonal chain that would be covered by applying $(\Lambda, L) = (B_r, 2m+1)$. Right: A periodic cell with the vertex and edges labeled [22].

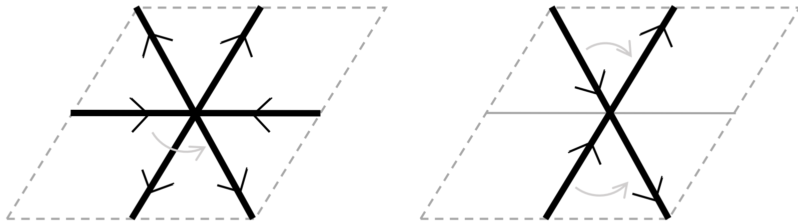
This second edge has itself to be glued to the strand covering its second counterclockwise adjacent edge in $v''$, and so forth. This means that each element of $\Delta_{(C_r,s)}$ is therefore the second clockwise adjacent edge to the previous one at their common vertex, Therefore, for all $i \in (1, ..., k, k+1 = 0)$, we have in Definition 3.10

$$\delta_i = (\delta_{i-1})_2,$$

where each $\delta_i$ is an edge that would be covered by a single line when applying the polygonal link method $(C_r, s)$ to $U_{\mathcal{T}}$ [22].

### 3.3.2 Crossed curves and $m$-twisted double line covering: $(\Lambda, L) = (C_r, m)$

The approach for the crossed curves and $m$-twisted double line covering method is similar to that of a single covering and we will use the same notations. Here, the main difference concerns the covering of each edge by an $m$-twisted double line. For this method, we observed that the parity of the positive integer $m$ representing the number of twists between the two strands that replace each edge of $U_{\mathcal{T}}$, can influence the construction of closed curve components.



$\Delta_{(Cr,\,2m)} = (e_1^+, e_3^+, e_2^+, e_1^-, e_3^-, e_2^-)$ $\Delta_{(Br,\,2m+1)} = (e_3^+, e_2^-)$

Figure 3.11: Periodic cell of a triangular tiling. Left: Characteristic loop corresponding to Figure 3.10 (left). Right: Characteristic loop corresponding to Figure 3.10 (top) [22].

Therefore, we can state that [22],

- if $m$ is even, each curve created by $(C_r, m)$ is an alternating union of a right strand, followed by a left strand, and so forth which cover edges which are consecutive second counterclockwise adjacent ones, at each vertex they cross.

  This means that for all $i \in (1, ..., k, k+1 = 0)$, we have in Definition 3.10,

$$\delta_i = (\delta_{i-1})_2,$$

- if $m$ is odd, each curve created by $(C_r, m)$ is an alternating union of two consecutive right strands, followed by two consecutive left strands, and so forth. In this case, the consecutive edges forming the polygonal chain alternate between the second counterclockwise adjacent edge and the second clockwise adjacent edge from the previous one. This means that there exists a positive integer $i$ such that in Definition 3.10, for all $i \in (1, ..., k, k+1 = 0)$,

$$\delta_{2i} = (\delta_{2i-1})_2 \quad \text{and} \quad \delta_{2i+1} = (\delta_{2i})_{-2}.$$

### 3.3.3 Branched curves and $m$-twisted double line covering: $(\Lambda, L) = (B_r, m)$

Finally, to study the characteristic loops for the branched curves and $m$-twisted double line covering method, we apply a similar strategy as done for $(\Lambda, L) = (C_r, m)$. The only difference consists in replacing the second clockwise or counterclockwise adjacent edge with the first one. The polygonal loop can therefore be written in the same fashion [22],

- if $m$ is even, then in Definition 3.10, for all $i \in (1, ..., k, k+1 = 0)$,

$$\delta_i = (\delta_{i-1})_1,$$

- if $m$ is odd, then there exists a positive integer $i$ such that in Definition 3.10, for all $i \in (1, ..., k, k+1 = 0)$,

$$\delta_{2i} = (\delta_{2i-1})_1 \text{ and } \delta_{2i+1} = (\delta_{2i})_{-1}.$$

## 3.4 Weaving motifs, polycatenanes motifs or mixed motifs

Let $\mathcal{T}$ be an edge-to-edge doubly periodic tiling of $\mathbb{E}^2$ and denote by $U_{\mathcal{T}}$ a generating cell of $\mathcal{T}$, namely a $\mathcal{T}$-cell. Then, as seen previously, a pair $\big(U_{\mathcal{T}}, (\Lambda, L)\big)$ define an entangled motif. In particular, to characterize a weaving motif, none of the characteristic loops $\Delta_{(\Lambda,L)}$ associated with the chosen polygonal link method $(\Lambda, L)$ must be null-homotopic in the torus graph. Moreover, recall that we need to identify in the set of these polygonal loops at least two distinct sets of threads. More generally, we can characterize a polycatenanes motif and a mixed motif by the type of their components.

**Proposition 3.11.** [22] *A pair $\big(U_{\mathcal{T}}, (\Lambda, L)\big)$ will generate*

- *a polycatenane motif if and only if all the characteristic loops $\delta_{(\Lambda,L)}$ in $U_{\mathcal{T}}$ are null-homotopic.*
- *a mixed motif if and only if the set of all the characteristic loops $\delta_{(\Lambda,L)}$ in $U_{\mathcal{T}}$ contains at least a null-homotopic closed component and an essential closed curve.*

- *a weaving motif if and only if all the characteristic loops $\delta_{(\Lambda,L)}$ in $U_{\mathcal{T}}$ are homotopic to essential closed curves, and such that at least two of them are non-parallel, up to isotopy.*

*Proof.* The proof of this proposition can be done without specific difficulties. By isotopy, any weaving, polycatenane, or mixed motif can have its components covering the edges of $U_{\mathcal{T}}$, and can therefore be decomposed into a set of essential, null-homotopic closed curves or both types, respectively. The particularity of a weaving motif is that its components are organized in at least two disjoint sets of threads by definition. Conversely, if one characteristic loop is null-homotopic, then the curve component created by the method $(\Lambda, L)$ that covers it on the torus will lift as a closed component in the universal covering $\mathbb{E}^2$ and the result cannot be a weaving motif. It is therefore either a polycatenanes motif if all the characteristic loops are null-homotopic, or a mixed motif otherwise. Finally, if all the components are essential closed curves on the torus, then there are isotopic to geodesic curves, each of them characterized by a set of straight line segments on a flat torus. Therefore, it suffices to verify the existence of at least two such segments with different slopes, which implies the existence of at least two distinct axes of direction, and confirm the case of a weaving motif. □

Then, the polygonal description of characteristic loops defined previously will be useful now to predict if a characteristic loop will be null-homotopic or essential on a torus, which leads to our main theorem. Indeed, a characteristic loop, being a closed curve on a torus, can be characterized by a reduced cyclic word using the fundamental group of the torus $\pi_1((T)^2)$, [19]. Let $\alpha$ and $\beta$ be the two generators of $\pi_1((T)^2)$, representing respectively a meridian and a longitude of the torus along which we cut to obtain a flat torus. The identity element is therefore given by the word $\alpha\beta\alpha^{-1}\beta^{-1}$, by commutativity of $\pi_1((T)^2)$, which also characterizes a null-homotopic curve.

**Theorem 3.12.** [22] *Let $U_{\mathcal{T}}$ be a $\mathcal{T}$-cell. Then, a pair $\big(U_{\mathcal{T}}, (\Lambda, L)\big)$ will generate,*

- *a polycatenane motif if and only if all the characteristic loops $\delta_{(\Lambda,L)}$ in $U_{\mathcal{T}}$ are trivial polygonal chains.*
- *a mixed motif if and only if the set of all the characteristic loops $\delta_{(\Lambda,L)}$ in $U_{\mathcal{T}}$ contains at least a trivial polygonal chain and a nontrivial one.*

- *a weaving motif if and only if all the characteristic loops $\delta_{(\Lambda,L)}$ in $U_{\mathcal{T}}$ are nontrivial polygonal chains, and such that at least two of them are non-parallel.*

*Proof.* Let $U_{\mathcal{T}}$ be an arbitrary $\mathcal{T}$-cell embedded on a flat torus. First, label all the edges and vertices of $U_{\mathcal{T}}$, and assign to each edge an arbitrary orientation. For a chosen method $(\Lambda, L)$, define a first characteristic loop as described in Definition 3.10 and below, denoted by $\Delta_{(\Lambda,L)}$, such that the polygonal loop encodes the orientation of the edges with a sign $+$ or $-$, if the orientation of the loop follows the given orientation of the edge or not, respectively,

$$\Delta_{(\Lambda,L)} = (\delta_0^{\pm}, \delta_1^{\pm}, \cdots, \delta_k^{\pm}).$$

Therefore, given the presentation of the torus considered above, it becomes possible to associate to each characteristic loop a word $w(\Delta_{(\Lambda,L)})$, and deduce that it represents an essential closed curve, if and only if its reduced word represents a nontrivial element of $\pi_1((T)^2)$. Let $\alpha\beta\alpha^{-1}\beta^{-1}$ be the oriented flat torus, with opposite sides characterized by the same symbol and oriented in opposite direction, in which the given labelled $\mathcal{T}$-cell is embedded. Let $i$ and $j$ be two positive integers used to label the edges. Then, if an edge $\delta_i^{\pm}$ of $\Delta_{(\Lambda,L)}$ intersects a boundary of the flat torus denoted by $\alpha$, and such that the orientation of the edge matches with (resp. is opposite to) the orientation given to $\beta$, then $\delta_i^{\pm}$ will be assigned the value $\beta$ (resp. $\beta^{-1}$) in the word $w(\Delta_{(\Lambda,L)})$. The same holds conversely, by exchanging the roles of $\alpha$ and $\beta$. However, an edge $\delta_j^{\pm}$ of $\Delta_{(\Lambda,L)}$ that does not intersect any boundary of the flat torus will not contribute to $w(\Delta_{(\Lambda,L)})$. More generally, if between two oriented edges of $\Delta_{(\Lambda,L)}$ that intersect a boundary of the flat torus, there exists a finite subsequence of edges that do not intersect any such boundary and such that the vertices of this subsequence are also all distinct, then its word is the identity $I$. In other words, if there is a simple arc in $\Delta_{(\Lambda,L)}$ that does not intersect the boundary of $U_{\mathcal{T}}$, then this arc is contractible and therefore does not contribute to the word of the characteristic loop. The last case to consider is when none of the edges of the characteristic loop $\Delta_{(\Lambda,L)}$ intersect a boundary of the flat torus, which characterizes immediately a null-homotopic component. Notice that a word containing as many elements $\alpha$ as elements $\alpha^{-1}$, as well as as many elements $\beta$ as elements $\beta^{-1}$ is trivial. This implies that in a characteristic loop, if for all indices $i$, the number of edges $\delta_i^{+}$ is equal to the number of edges $\delta_i^{-}$, then this loop is null-homotopic.

We say in this case that the polygonal chain is trivial, or nontrivial otherwise. Therefore, the polygonal chain of a characteristic loop, described by an ordered sequence of vertices and oriented edges as described above, characterizes the type of the closed component of the motif that will cover it. By defining the polygonal chains of all the characteristic loops of the $\mathcal{T}$-cell, it is thus possible to predict the type of motif that can be generated by a pair $\left(U_{\mathcal{T}}, (\Lambda, L)\right)$. Note that in the case of a set of essential components, the additional argument that there must exist at least two characteristic loops with distinct words is necessary to conclude about a weaving motif, where two such loops are said to be *non-parallel*. Finally, recall that even if the words of the characteristic loops depend on the choice of the generators of $\pi_1((T)^2)$ and the orientation and label of the elements $U_{\mathcal{T}}$, the fact that these words are trivial or not is independent of such a choice, by definition of the mapping class group of the torus [19], and therefore does not affect our result, which ends the proof of the theorem. □

We end this chapter with an example of application of our theorem with a tiling by hexagons.

**Example 3.13.** [22] Let $U_{\mathcal{T}}$ be a periodic cell of a hexagonal lattice. Label and orient its edges in an arbitrary way, as shown in Figure 3.12.

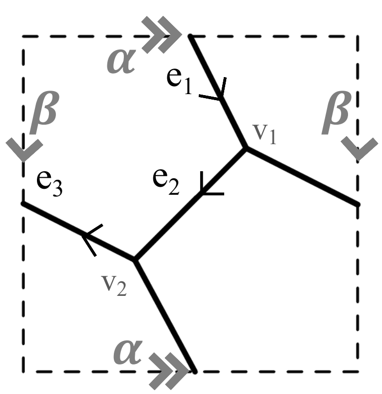



Figure 3.12: Periodic cell of an hexagonal tiling [22].

- If we want to predict the type of motif that would be created by applying $(\Lambda, L) = (C_r, 2)$ to $U_{\mathcal{T}}$, then we must list the characteristic loops that would be covered for each strand after transformation, using the polygonal chain description given above. In this case, all the characteristic loops are equivalent, up to cyclic permutation and reversing of all orientations, to the following polygonal chain,

  $$\Delta_{(C_r,2)} = (e_1^+, e_3^-, e_2^-, e_1^-, e_3^+, e_2^+)$$

  Here the symbols $\pm$ relate to the orientation of the edges given in Figure 3.12, with a sign $+$ for the natural direction and $-$ for the reversed one. For simpler notations, we can also borrow from homotopy theory the notation of loop composition,

  $$\Delta_{(C_r,2)} = e_1^+.e_3^-.e_2^-.e_1^-.e_3^+.e_2^+$$

  We can read that this loop is null-homotopic by Theorem 3.12. Therefore, the crossed curves and 2-twisted double line covering method applied to any hexagonal tiling will always generate a polycatenane. The same conclusion actually for any even number of twists.

- for $(\Lambda, L) = (B_r, 3)$, we can describe all the characteristic loops by the polygonal chains, up to cyclic permutation and reversing all orientations,

  $$\Delta^1_{(B_r,3)} = e_1^+.e_3^-,$$
  $$\Delta^2_{(B_r,3)} = e_1^+.e_2^+,$$
  $$\Delta^3_{(B_r,3)} = e_3^+.e_2^+,$$

  We can read that these loops are essential by Theorem 3.12. Their words are also all distinct. Therefore, the branched curves and 3-twisted double line covering method applied to any hexagonal tiling will always generate a twisted weave. The same conclusion actually holds for any odd number of twists.

Finally, we can end this section with some properties that can be proved immediately [22] .

- For any $\mathcal{T}$-cell, $(\Lambda, L) = (B_r, 2m)$ always generates a polycatenane, with the particular case of a trivial structure without crossing if $m = 0$. This result generalizes to non-periodic tilings. Indeed, each tile boundary is isotopic to a characteristic loop itself.

- Similarly, for any $\mathcal{T}$-cell such that all the vertices have degree 3, $(\Lambda, L) = (C_r, 2m)$ always generates a polycatenane by definition. This result generalizes to non-periodic tilings. Indeed, once again, each tile boundary is isotopic to a characteristic loop itself.

- A characteristic loop isotopic to a polygonal chain containing edges of the same slope is essential by definition. Therefore, for any $\mathcal{T}$-cell such that all the vertices have degree 4, $(\Lambda, L) = (C_r, 2m)$ always generates a polycatenane by definition. This result generalizes to non-periodic tilings. Indeed, once again, each tile boundary is isotopic to a characteristic loop itself.

# Chapter 4

# TAIT'S FIRST AND SECOND CONJECTURES FOR ALTERNATING WEAVING MOTIFS

To study the classification of weaves, one must first consider focusing on simple structures before targeting more complex objects. In the history of knot theory, alternating links seem to have fascinated mathematicians and still do. Even though they can simply be described by entanglements alternating cyclically between undercrossing and overcrossing, as one travels along each of its components, they are far from being trivial objects. This inspired us to start approaching the classification of weaves which are both periodic and alternating. One of the main criteria to classify entangled objects is the number of crossings, which can describe their complexity. In the late nineteenth century, P.G. Tait [65] stated famous conjectures on alternating link diagrams, and one of the main purposes of this chapter is to extend and prove two of his main results for alternating weaving motifs of $\mathbb{E}^2$ or $\mathbb{H}^2$. We will indeed prove that a reduced alternating minimal motif is a minimum diagram of its alternating periodic weave and that two reduced alternating minimal motifs of an oriented periodic weave have the same writhe. The *writhe* is defined as the sum of the signs of all the crossings, where each crossing is given a sign $\pm 1$. Then by *minimal motif*, we imply that the generating cell is a unit cell, while the definition of *reduced* motif has been detailed at the end of the second chapter. The first result, well-known as Tait's First Conjecture has been proved independently by M.B Thistlethwaite [66], K. Murasugi [55], and L.H. Kauffman [39] for general links. We will prove this theorem for periodic weaves following the strategy used by Kauffman, which requires an extension of the bracket polynomial to $\mathbb{H}^2$, since it has already been defined for doubly periodic structures in $\mathbb{E}^2$ [25], [24]. Similarly, the second result, known as Tait's Second Conjecture, has also originally been demonstrated independently by Thistlethwaite [66] and Murasugi [56] for links. We will however follow the approach of R. Stong [64] for links, the main difference being that the planar projection of any component of a weaving motif should not cross itself unless it makes a simple twist that can be easily undone. These results are based on our paper [48].

## 4.1 The Bracket Polynomial

One of the main challenges in weaving theory will be to detect equivalent weaves. However, playing with the Reidemeister moves to identify if two weaving motifs are equivalent has obvious limitations. Therefore, with similar reasoning as in knot theory, the strategy is to define *weaving invariants*, which can be understood as 'properties' assigned to a weaving motif that remain invariant under the Reidemeister moves. Indeed, we have seen in the Reidemeister theorem for periodic weaves that if we can transform one weaving motif into another weaving motif via the three Reidemeister moves, then they are equivalent. Thus, to prove that such an object is a weaving invariant, one must verify independently that it remains unchanged under the action of each Reidemeister move. At the beginning of the 1980s, the mathematician V. Jones discovered a new invariant, which is a Laurent polynomial that has been a key element to prove the two Tait's conjectures for general links. In this section, we will generalize the definition of a useful polynomial associated to link diagrams, that has also been defined for weaving motifs on a torus in [25], [24]. This polynomial is called the *bracket polynomial* and has been discovered by Kauffman in 1987 to study the Jones polynomial [39].

### 4.1.1 A Kauffman-type weaving invariant

The purpose of this subsection is to recall the results stated by Ghrishanov et al. [25], [24] regarding the bracket polynomial of a doubly periodic weaving diagram of $\mathbb{E}^2$ and to extend them to periodic hyperbolic weaving diagrams of $\mathbb{H}^2$, so that we define a polynomial for any weaving motif of $\mathbb{X}^2$, lying on a surface $\Sigma_g$ of genus $g \geq 1$. First, we impose the value 1 to this polynomial for a null-homotopic simple closed curve. Moreover, if such a null-homotopic curve is added in a 'split' weaving motif without crossing the other components, we will just multiply the entire polynomial by a variable $d$. Finally, to obtain the bracket polynomial of a weaving motif in terms of the bracket polynomials of simpler motifs, we will split each crossing vertically and horizontally to obtain two new motifs, each of which has one fewer crossing. This makes it possible to write the bracket polynomial of a weaving motif as a linear combination of the bracket polynomials of the two new motifs.

**Definition 4.1.** [39] *Let $D_{\mathrm{W}}$ be an unoriented weaving motif embedded on a surface $\Sigma_g$ of genus $g \geq 1$, of a periodic weaving diagram of $\mathbb{X}^2 = \mathbb{E}^2$ or $\mathbb{H}^2$. Let $\langle D_{\mathrm{W}} \rangle$ be the element of the ring $\mathbb{Z}[A, B, d]$ defined recursively by the following identities,*

(1) $\langle O \rangle = 1$, *with $O$ a trivial simple closed curve on $D_{\mathrm{W}}$.*

(2) $\langle D_{\mathrm{W}} \cup O \rangle = d \langle D_{\mathrm{W}} \rangle$, *when adding an isolated circle $O$ to a diagram $D_{\mathrm{W}}$.*

(3) $\langle \times \rangle = A \langle \asymp \rangle + B \langle )( \rangle$, *for diagrams that differ locally around a single crossing.*

*This last relation is called the skein relation and $\langle D_{\mathrm{W}} \rangle$ denotes the bracket polynomial.*

This Laurent polynomial is well defined on weaving motifs, but we will see that, as for links, it is not invariant under the Reidemeister moves. The first step is to find a relation between the variables $A$, $B$, and $d$. To any weaving motif $D_{\mathrm{W}}$ of a given weaving diagram of $\mathbb{X}^2$, we have seen previously that we can assign an *orientation* to each crossing such that the upper thread passes from bottom left to top right. When all the crossings are oriented, we say that $D_{\mathrm{W}}$ is *oriented.* The second step is to split each crossing of an oriented $D_{\mathrm{W}}$ via an operation of type $A$ or $B$, also called *smoothing*, as illustrated in Figure 4.1. We can express the overall operations as a *state* $S$ of $D_{\mathrm{W}}$, consisting of symbols $A$ and $B$, and which has for length the number of crossings $C$ in $D_{\mathrm{W}}$.

$$S = ABAABB \ldots ABBA.$$

More specifically, let $D_{\mathrm{W}}$ be a weaving motif with $C$ crossings indexed $1, 2, 3, \cdots, C$. Then, a state $S$ for $D_{\mathrm{W}}$ is a function $S : \{1, 2, 3, ..., C\} \mapsto \{A, B\}$. Note that any weaving motif with $C$ crossings admits therefore $2^C$ states. Then, given a motif $D_{\mathrm{W}}$ and a state $S$ for $D_{\mathrm{W}}$, we replace each crossing with two arcs that do not cross. in one of the two ways illustrated on Figure 4.1.

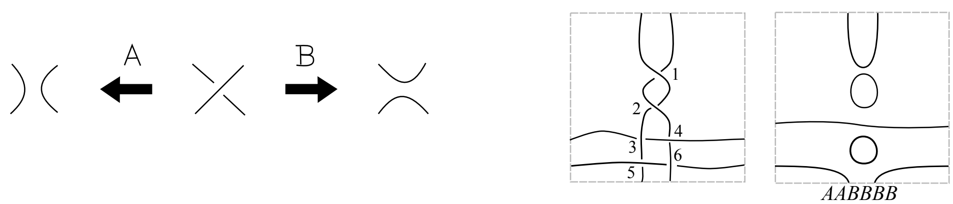



Figure 4.1: On the left, the two types of splitting. On the right, an example of a state $S = AABBBB$ of a weaving motif [48].

In knot theory, a general link diagram $D_S$ in a state $S$ can be described as a disjoint union of $c_S$ simple closed curves embedded in the Euclidean plane without any crossing nor intersection,

$$D_S = OOOO \ldots O.$$

In other words, each component is isotopic to a circle $O$ and is thus said to be null-homotopic. Using Definition 4.1 (2), we can write the bracket polynomial of $D_S$ as,

$$\langle D_S \rangle = d^{\,c_{S-1}}.$$

Next, let $i$ be the number of splits of type $A$ and $j$ be the number of splits of type $B$ in $D_S$, Then, if we apply the skein relation recursively, we obtain,

$$\langle D_S/S \rangle = A^i B^{\,j}.$$

Therefore, the total contribution of a single given state $S$ to the bracket polynomial of a classic link diagram is given by,

$$P_S = \langle D_S/S \rangle \langle D_S \rangle = A^i B^{\,j} d^{\,c_{S-1}}.$$

To apply similar reasoning to a periodic cell $D_{\mathrm{W}}$ of a weaving diagram of the Euclidean or the hyperbolic plane, we have to notice that a trivial weaving motif in a given state may contain both essential and null homotopic simple closed curves on $\Sigma_g$. Recall that by essential curves, we mean curves that are wound up around the surface $\Sigma_g$. For any state of a diagram, the set of essential simple closed curves has been called a *winding* in [25], [24], and it is denoted by a pair of two positive integers $(m', n')$ in the case of a torus, where $m'$ and $n'$ are the number of intersections of the winding with a torus meridian and longitude, respectively. For example, in the case of the diagram on the right of Figure 4.1, we have for the state $S = AABBBB$ a winding given by the pair of integers $(m', n') = (0, 2)$.

We can naturally extend the concept of winding to the general case of higher genus surfaces $\Sigma_g$, with $g \geq 1$, where the winding is denoted by a sequence of positive integers,

$$(m^1, \dots, m^g, n^1, \dots, n^g),$$

where $m^1, \dots, m^g, n^1, \dots, n^g$ represent the number of intersections of the winding with the curves $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ of $\Sigma_g$ respectively, see Figure 2.14 and [19], [43] for more details. Then, we can define the state $S$ of a weaving motif $D_{\mathrm{W}}$ by,

$$D_{\mathrm{W},S} = OOOO\dots O \cup (m_s^1, \dots, m_s^g, n_s^1, \dots, n_s^g), \text{ for every } g \geq 1.$$

Recall that an essential simple closed curve in $\Sigma_g$ is closed because it characterizes the periodicity of the associated infinite weaving diagram, and is therefore an open simple curve in $\mathbb{X}^2$, which is not the case for classic links, as in [4], [5], [20]. It is thus necessary to define the bracket polynomial of a winding, as an extension of the torus case,

$$\langle (m_s^1, \dots, m_s^g, n_s^1, \dots, n_s^g) \rangle \;=\; (m_s^1, \dots, m_s^g, n_s^1, \dots, n_s^g), \text{ for every } g \geq 1.$$

We are now ready to define the bracket polynomial of a weaving motif of $\mathbb{X}^2 = \mathbb{E}^2$ or $\mathbb{H}^2$.

**Proposition 4.2.** [48] *The bracket polynomial $\langle D_{\mathrm{W}} \rangle$ is uniquely determined on weaving motifs of $\mathbb{X}^2$ by the identities (1), (2), (3) of Definition 4.1, and is expressed by summation over all states of the diagram,*

$$\langle D_{\mathrm{W}} \rangle = \sum_S A^i B^j d^{\,c_{S-1}} (m_s^1, \dots, m_s^g, n_s^1, \dots, n_s^g), \text{ for every } g \geq 1. \tag{4.1}$$

The bracket polynomial is now well-defined for weaving motifs embedded on any surface $\Sigma_g$ of genus $g \geq 1$. The next step to make it a weaving invariant is to study its invariance for the Reidemeister moves.

**Lemma 4.3.** [48] *If the three diagrams represent the same projection except in the area indicated, we have* $\langle \asymp\!\!\!\times \rangle = AB\langle\, )( \,\rangle + (ABd + A^2 + B^2)\langle \asymp \rangle$.

Thus, the bracket is invariant for the Reidemeister move $\Omega_2$ for all weaving motifs if

$$AB = 1 \text{ and } d = -A^2 - A^{-2}.$$

At this point, we obtain a Laurent polynomial of a single variable $A$, and this lemma also implies its invariance for the Reidemeister moves of type $\Omega_3$, which allows a partial conclusion about the invariance of the polynomial.

**Lemma 4.4.** [48] *The invariance of the bracket polynomial for the Reidemeister move $\Omega_2$ implies its invariance for Reidemeister move $\Omega_3$. Thus, the bracket polynomial is an invariant of regular isotopy for unoriented weaving motifs.*

Here, Lemma 4.4 is a consequence of the generalization of Kauffman's observation for general link diagrams to weaving motifs: if we can obtain a weaving motif $D'_{\mathrm{W}}$ by applying these Reidemeister moves $\Omega_2$ and $\Omega_3$ a finite number of times to a weaving motif $D_{\mathrm{W}}$ of a given periodic weaving diagram, then the two are regular equivalent. However, it is not possible to conclude at this point, that the bracket polynomial is an invariant of ambient isotopy, since we do not have the invariance for the Reidemeister move $\Omega_1$.

**Proposition 4.5.** [48] *If $AB = 1$ and $d = -A^2 - A^{-2}$, then, for the Reidemeister move $\Omega_1$, we have*

$$\begin{cases} \langle \text{curl} \rangle = (-A^3)\langle \frown \rangle, \\ \langle \text{curl} \rangle = (-A^{-3})\langle \frown \rangle. \end{cases}$$

The idea used to define an ambient isotopy invariant from the bracket polynomial is to consider the *writhe* $w_r(D_{\mathrm{W}})$ of a weaving motif $D_{\mathrm{W}}$, which is the sum of the signs of all the crossings where each crossing is given a sign $\pm 1$, as in Figure 4.2.

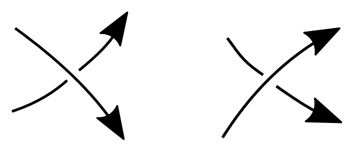



Figure 4.2: Sign convention.

For any periodic weaving diagram, a weaving motif $D_{\mathrm{W}}$ consists of $c_S$ components, each denoted by $t_i$, that can be oriented in an arbitrary way. We call $D^i_{\mathrm{W}}$ the part of diagram $D_{\mathrm{W}}$ that corresponds to the component $t_i$. Then we have in $D_{\mathrm{W}}$,

$$w_r(D_{\mathrm{W}}) = \sum_{i=1}^{c_S} w_r(D^i_{\mathrm{W}}). \tag{4.2}$$

By using the writhe, we are now able to define an extension of the bracket polynomial for every $g \geq 1$,

$$\begin{aligned} f[D_{\mathrm{W}}] &= (-A)^{-3w_r(D_{\mathrm{W}})}\langle D_{\mathrm{W}}\rangle, \\ &= (-A)^{-3w_r(D_{\mathrm{W}})}\Big(\sum_S A^{i-j}(-A^2-A^{-2})^{c_S-1}(m^1_s,\dots,m^g_s,n^1_s,\dots,n^g_s)\Big). \end{aligned} \tag{4.3}$$

**Theorem 4.6.** [48] *The polynomial $f[D_{\mathrm{W}}] \in \mathbb{Z}[A]$ defined above is an ambient isotopic invariant for oriented weaving motifs.*

*Proof.* From Lemma 4.4, we already have the invariance of $f[D_{\mathrm{W}}]$ for the Reidemeister moves $\Omega_2$ and $\Omega_3$. Then, by combining the behavior of the writhe defined above under the Reidemeister move $\Omega_1$ with the previous relation of the bracket for $\Omega_1$ in Proposition 4.5, it follows that $f[D_{\mathrm{W}}]$ is invariant under $\Omega_1$ type moves. Thus, $f[D_{\mathrm{W}}]$ is invariant under all three moves, and is therefore an invariant of ambient isotopy. □

However, the limits of this polynomial is that it still depends on the choice of the generating cell since the multipliers $(m^1_s,\dots,m^g_s,n^1_s,\dots,n^g_s)$, associated with the windings, and thus on the Dehn twists. A Dehn twist can indeed change the slope of an essential simple closed curve, which implies a change of these multipliers by definition of a winding. Moreover, we have seen that for periodic weaving diagrams we must also consider the surface deformation, which implies the independence of the polynomial for the Dehn twists of $\Sigma_g$ to have a weaving invariant. Once again, the particular case of the torus is described in [24] and is extended below to $g \geq 1$.

**Theorem 4.7.** [48] *The polynomial $f[D_{\mathrm{W}}]$, when $(m_s^1, \ldots, m_s^g, n_s^1, \ldots, n_s^g)$ is independent of the Dehn twists, or in canonical form, defines a Kauffman-type weaving invariant.*

*Proof.* To construct an invariant independent of the twists of $\Sigma_g$, it is necessary to consider the set $\{v_s\} = \big\{(m_s^1, \ldots, m_s^g, n_s^1, \ldots, n_s^g)\big\}$ of windings for every state $S$. Indeed, since this set $\{v_s\}$ depends on the twists of $\Sigma_g$, one must transform it to the canonical form by using the Dehn twists, in order to make it invariant. Dehn twists are known to be elements of the Mapping Class Group of a surface of genus $g$ and the *Dehn-Lickorish theorem* states that it is sufficient to select a finite number of Dehn twists to generate the Mapping Class Group $MCG(\Sigma_g)$ of a surface $\Sigma_g$ of genus $g$. Moreover, since the symplectic representation $\psi : MCG(\Sigma_g) \to Sp(2g, \mathbb{Z})$ is surjective for $g \geq 1$ ([19]), then it means that the images of the Dehn twists generate $Sp(2g, \mathbb{Z})$. Besides, recall that the determinant of every symplectic matrix $A \in Sp(2g, \mathbb{Z})$ is one and that for $g = 1$, $Sp(2g, \mathbb{Z}) = SL_2(\mathbb{Z})$. Thus, it is possible to use the same reasoning as in [24] and to represent the transformation of any winding $v_s = (m_s^1, \ldots, m_s^g, n_s^1, \ldots, n_s^g)$ by a sequence of Dehn twists of $\Sigma_g$ as a product of $v_s$ by a matrix $U \in Sp(2g, \mathbb{Z})$, $v_{s'} = v_s U$, considering the canonical matrix multiplication on $Sp(2g, \mathbb{Z})$.

To define the canonical form of a set $V = \{v_s\}$, we associate a quadratic functional $Q$,

$$Q(V) := \sum_{s}^{N} |v_s|^2. \tag{4.4}$$

A sequence of twists, defined by a symplectic matrix $U$, converts the set $V = \{v_s\}$ to a set $V' = \{v_{s'}\}$, with $v_{s'} = v_s U$ and the value of $Q$ changes to:

$$Q(V') = \sum_{s}^{N} |v_{s'}|^2 = \sum_{s}^{N} v_s U U^T v_s^T. \tag{4.5}$$

So for a given set $V = \{v_s\}$, the idea is to find a sequence of twists represented by a matrix $U$ of determinant 1, that minimizes the value of $Q$,

$$Q(V') = \sum_{s}^{N} |v_{s'}|^2 = \sum_{s}^{N} v_s U U^T v_s^T \longrightarrow \min, U \in Sp(2g, \mathbb{Z}). \tag{4.6}$$

This equation always has a unique non-trivial solution $U_0$, which can be shown considering the following setting: let $M = UU^T$, then $M$ is a symmetric definite positive matrix, and let $x = v_s$ and let $f(x) = xMx^T$, then $f(0) = 0$ and for all $x \neq 0$, $f(x) > 0$. Thus, there is an orthonormal basis $\{e_1, \dots e_d\}$ such that for all $i$ in $\{1, ..., d\}$, $e_i$ is an eigenvector of $M$. We denote the corresponding eigenvalue $\lambda_i$ and we show that $f$ is strictly convex.

Let $0 < \mu < 1$ and consider $f\big(\mu x + (1-\mu)y\big)$, with $x \neq y$. Then,

$$\begin{aligned} f\big(\mu x + (1-\mu)y\big) &= \langle \mu x + (1-\mu)y, M(\mu x + (1-\mu)y) \rangle, \\ &= \langle \sum_{i=1}^{d} \mu x_i e_i + (1-\mu) y_i e_i, M \sum_{i=1}^{d} \mu x_i e_i + (1-\mu) y_i e_i \rangle, \\ &= \sum_{i=1}^{d} \lambda_i \big(\mu x_i + (1-\mu) y_i\big)^2. \end{aligned} \tag{4.7}$$

Moreover, $x^2$ is strictly convex and for some $i$, $x_i \neq y_i$, thus,

$$\sum_{i=1}^{d} \lambda_i \big(\mu x_i + (1-\mu) y_i\big)^2 < \sum_{i=1}^{d} \lambda_i \big(\mu x_i^2 + (1-\mu) y_i^2\big). \tag{4.8}$$

Therefore, since $f$ is strictly convex and as a limit at infinity, it has a unique minimizer, which concludes our proof. So, for every state $S$, the canonical form of $V = \{v_s\}$, with the winding as coordinates $(m_s^1, \dots, m_s^g, n_s^1, \dots, n_s^g)$ is an invariant and thus, $f[D_{\mathrm{W}}]$ too. □

### 4.1.2 The case of alternating weaving diagrams

As mentioned earlier, the definition of an alternating weaving motif, given in Definition 2.24, does not seem complicated as we are dealing with crossings that alternate between over and under as one travels around the components in a fixed direction. However such alternating structures are far from being trivial objects. Moreover, as for a general reduced alternating link, a $\Sigma_g$-reduced alternating weaving motif can be seen as a motif that cannot be changed into one with a fewer number of crossings. Nevertheless, Tait's conjectures on alternating knots took more than a century to be proved, and need some adjustments to include them in our weaving theory. To prove these conjectures, we first need to study the bracket polynomial for the case of alternating periodic weaving diagrams of $\mathbb{X}^2 = \mathbb{E}^2$ or $\mathbb{H}^2$. It is known [57] that the *degree of a polynomial* is the most important aspect of the polynomial as an invariant. The following proposition and its proof follow the strategy of a similar result in classic knot theory, but depends strongly on the definition of reduced and proper diagrams stated in the second chapter, in order to count the number of disjoint regions around a crossing. Indeed, the idea here is to think about the bracket polynomial as a tool to connect or disconnect regions on a motif after the split of the crossings. In the skein relation, four regions of a motif are adjacent to a crossing, two labelled $A$ and two labelled $B$. Therefore, to compute the bracket polynomial at the given crossing, we split open a crossing in two different ways, connecting either the two regions labeled $A$ or the two regions labeled $B$, which is called an $A$-split or a $B$-split, respectively.

**Proposition 4.8.** [48] *Let $D_{\mathrm{W}}$ be an alternating weaving motif $\mathbb{X}^2$ that is connected and $\Sigma_g$-reduced. Let $D_{\mathrm{W}}$ be colored so that all the regions labeled $A$ are white and all the regions labeled $B$ are black. Let $C$ be the number of crossings, $W$ be the number of white regions and $B$ be the number of black ones. Then,*

$$\begin{aligned} \max\deg(\langle D_{\mathrm{W}}\rangle) &= C + 2W - 2, \\ \min\deg(\langle D_{\mathrm{W}}\rangle) &= -C - 2B + 2, \end{aligned} \tag{4.9}$$

*with* $\max\deg(P)$ *and* $\min\deg(P)$ *are respectively the maximal and the minimal degree of any polynomial $P$ in $\mathbb{Z}[A, B, d]$.*

*Proof.* Since $D_{\mathrm{W}}$ is alternating, it has a canonical checkerboard coloring, which means that two edge-adjacent regions always have different colors. Let $S = S_A$ be the state obtained by splitting every crossing in the diagram $D_{\mathrm{W}}$ in the $A$-direction. Then, we have $\langle D_{\mathrm{W}}/S\rangle = A^C$, and since the number of components $c_S$ is equal to W, thus as seen earlier, the total contribution of the state $S$ to the bracket polynomial is given by,

$$\begin{aligned} P_S &= \langle D_{\mathrm{W}}/S \rangle d^{\,c_S - 1} \langle m_s^1, \ldots, m_s^g, n_s^1, \ldots, n_s^g \rangle \\ &= A^C d^{\,W-1} (m_s^1, \ldots, m_s^g, n_s^1, \ldots, n_s^g), \; g \geq 1. \end{aligned} \tag{4.10}$$

And since $d = -A^2 - A^{-2}$, then $\max\deg(P_S) = C + 2W - 2$, which is the desired relation.

Now let $S' \neq S_A$, be any another state and verify that $\deg(P_{S'}) \not\geq \deg(P_S)$. Then $S'$ can be obtained from $S = S_A$ by switching some splittings of $S$. Thus, a sequence of states can be defined by $S(0), S(1), \ldots, S(n)$ such that $S = S(0)$, $S' = S(1)$, and for every positive integer $i$, $S(i+1)$ is obtained from $S(i)$ by switching one splitting from type $A$ to type $B = A^{-1}$. Then, since a splitting of type $B = A^{-1}$ contributes a factor of $A^{-1}$ in the polynomial

$$\langle D_{\mathrm{W}}/S(i+1) \rangle = A^{-2} \langle D_{\mathrm{W}}/S(i) \rangle. \tag{4.11}$$

We need now to distinguish two cases.

Firstly, the weaving motif $D_{\mathrm{W}}$ is $\Sigma_g$-reduced and proper. Then, $c_{S(i+1)} \leq c_{S(i)} - 1$, since switching one splitting can change the component number by at most one. Thus, $\max\deg(P_{S(i+1)}) \leq \max\deg(P_{S(i)})$. Moreover, let $c$ be the crossing point for which we change the $A$-splice into the $B$-splice from $S(0)$ to $S(1)$. Since $D_{\mathrm{W}}$ is proper, the crossing $c$ is proper. Thus, we can extend the following lemma, stated in [37].

**Lemma 4.9.** [48] *Let $D_{\mathrm{W}}$ be an alternating weaving motif, and let $S_A$ (or $S_B$, resp.) be the state of $D_{\mathrm{W}}$ obtained from $D_{\mathrm{W}}$ by doing an $A$-splice (resp. $B$-splice) for every crossing. For a crossing $c$ of $D_{\mathrm{W}}$, let $R_1(c)$ and $R_2(c)$ be the closed regions of $S_A$ (or $R'_1(c)$ and $R'_2(c)$ be the closed regions of $S_B$) around $c$. If $c$ is a proper crossing, then*

$$R_1(c) \neq R_2(c) \text{ and } R'_1(c) \neq R'_2(c).$$

*Proof.* Since $c$ is a proper crossing, the four closed regions of $D_{\mathrm{W}}$ appearing around $c$ are all distinct. Next, $D_{\mathrm{W}}$ is alternating, so it admits a canonical checkerboard coloring and there is a one-to-one correspondence,

$$\{\text{closed regions of } S_A\} \cup \{\text{closed regions of } S_B\} \to \{\text{closed regions of } D_{\mathrm{W}}\}$$

Then $R_1(c), R_2(c), R'_1(c)$ *and* $R'_2(c)$ correspond to the four distinct closed regions of $D_{\mathrm{W}}$ around $c$. This concludes the proof. □

Thus, from this lemma, since $S(1)$ is obtained from $S(0)$ by changing an $A$-splice to a $B$-splice at $c$, two distinct regions $R_1(c)$ and $R_2(c)$ become a single region. Hence $c_{S(1)} = c_S - 1$. To conclude, the term of maximal degree in the entire bracket polynomial is contributed by the state $S = S_A$, and is not canceled by terms from any other state, so we arrive at

$$\max\deg(\langle D_{\mathrm{W}}\rangle) = C + 2W - 2.$$

The proof is similar for,

$$\min\deg(\langle D_{\mathrm{W}}\rangle) \ = -C - 2B + 2.$$

Secondly, the weaving motif is $\Sigma_g$-reduced but not proper. Then, there exists at least one crossing which is not proper. If we change an $A$-splice to a $B$-splice at a crossing $c$ that is proper, then the conclusion is the same than before. Now, if we change an $A$-splice to a $B$-splice at a crossing $c'$ that is not proper, then some white regions would touch both sides of a crossing. In this case, the number of split components does not decrease from $S(0)$ to $S(1)$: $c_{S(1)} = c_{S(0)}$. But, as seen before, $\langle D_{\mathrm{W}}/S(1)\rangle = A^{-2}\langle D_{\mathrm{W}}/S(0)\rangle$ and there is no isthmus in the diagram, so the number of components either decreases or is constant. Therefore,

$$\max\deg(P_{S(1)}) \leq \max\deg(P_{S(0)}). \tag{4.12}$$

Thus, once again, the term of maximal degree in the entire bracket polynomial is contributed by the state $S = S_A$, and is not cancelled by terms from any other state and thus,

$$\max\deg(\langle D_{\mathrm{W}}\rangle) = C + 2W - 2.$$

The proof is similar for,

$$\min\deg(\langle D_{\mathrm{W}}\rangle) \ = -C - 2B + 2.$$

□

We are now able to define a relation between the closed regions of $D_{\mathrm{W}}$ and the regions of the motif after splitting as in [37]. Let $S_A$ (resp. $S_B$) be again the state obtained by splitting every crossing of the weaving motif in the $A$ (resp. $B$)-direction, and $D_{\mathrm{W}}$ be colored so that all the regions labeled "$A$" are white (or grey) and all the regions labeled "$B$" are black, as illustrated in Figure 4.3.

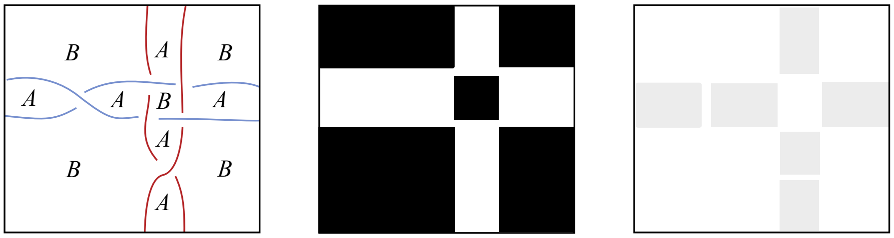



Figure 4.3: Example of $D_{\mathrm{W}}$ (left), $S_A = A...A$ (middle) and $S_B = B...B$ (right) [48].

Therefore, we have the following correspondences,

$$\{\text{closed regions of } S_A\} \to \{\text{closed regions of } D_{\mathrm{W}} \text{ in black regions } B\},$$
$$\{\text{closed regions of } S_B\} \to \{\text{closed regions of } D_{\mathrm{W}} \text{ in white regions } W\}.$$

This implies the following bijection,

$$\{\text{closed regions of } S_A\} \cup \{\text{closed regions of } S_B\} \to \{\text{closed regions of } D_{\mathrm{W}}\}.$$

And when considering a weaving motif on a surface of genus $g$, using the Euler characteristic and the fact that such a diagram represents a quadrivalent graph, we conclude that for every $g \geq 1$,

*there is* $C + 2 - 2g$ *closed regions in* $D_{\mathrm{W}}$*.*

We can now state that the number of crossings is also a topological invariant for periodic weaves, under the condition that this number is compared between motifs with the same topology.

**Theorem 4.10.** [48] *The number of crossings $C$ in a simple alternating projection of a weaving motif $D_{\mathrm{W}}$ of $\mathbb{X}^2$ is a topological weaving invariant of its associated periodic weaving diagram. Therefore any two $\Sigma_g$-reduced connected alternating weaving motifs of a given periodic weaving diagram have the same number of crossings if they have the same topology.*

*Proof.* Let $\mathrm{span}(D_{\mathrm{W}})$ defined by

$$\mathrm{span}(D_{\mathrm{W}}) = \max\deg(\langle D_{\mathrm{W}}\rangle) - \min\deg(\langle D_{\mathrm{W}}\rangle).$$

Then we have, $\mathrm{span}(D_{\mathrm{W}}) = 2C + 2(W+B) - 4 = 2C + 2(C+2-2g) - 4$. So finally, $\mathrm{span}(D_{\mathrm{W}}) = 4C - 4g$, for every $g \geq 1$. □

**Remark 4.11.** Recall that throughout this manuscript, we defined a weaving motif as being a generating cell of an infinite periodic weaving diagram. However depending of the choice of a *scale*, we can obtain periodic cells with different number of crossings. Therefore, we assume such a scale being fixed so that we compare weaving motifs of the same scale. Finding the scale such that a weaving motif has the minimum number of crossings will be the object of the next chapter, and here, we will only assume its existence.

## 4.2 Tait's First Conjecture for Periodic Alternating Weaves

We are now ready to discuss the connections between the bracket polynomial and the Jones polynomial, as well as their implication in the proof of Tait's first conjecture.

### 4.2.1 The Jones Polynomial

The Jones polynomial has been defined by the following identities in [39],

- $V_O = 1$,
- $t^{-1} V_{\times} - t V_{\times} = (\sqrt{t} - \frac{1}{\sqrt{t}}) V_{\asymp}$.

And it is related to the weaving invariant defined above with the bracket polynomial by the following relation.

**Theorem 4.12.** [48] *The Jones polynomial $V_{\mathrm{W}}$ of a weaving motif is related to its bracket-type polynomial, for every $g \geq 1$, by the expression,*

$$V_{\mathrm{W}}(t) = f[D_{\mathrm{W}}](t^{-1/4}) =$$

$$(-1)^{-3w_r(D_{\mathrm{W}})} t^{-(1/4)-3w_r(D_{\mathrm{W}})} \Big( \sum_S t^{\,i-j} (-t^{\,2} - t^{\,-2})^{c_S - 1} \langle m_s^1, \ldots, m_s^g, n_s^1, \ldots, n_s^g \rangle \Big).$$

*Proof.* By the skein relation:

$$\begin{cases} \langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle\, )(\, \rangle, \\ \langle \times \rangle = A^{-1}\langle \asymp \rangle + A\langle\, )(\, \rangle. \end{cases}$$

Thus, we have $A\langle \times \rangle - A^{-1}\langle \times \rangle = (A^2 - A^{-2})\langle \asymp \rangle$.

If we consider the writhe $w_r(D_\mathrm{W})$ of the weaving motif in the bracket on the right side of the equation, then the other two motifs on the left have writhes $(w_r(D_\mathrm{W}) + 1)$ and $(w_r(D_\mathrm{W}) - 1)$ respectively. Therefore, by multiplying the previous equation by the appropriate writhe, we obtain

$$A^4 f[\times] - A^{-4} f[\times] = (A^{-2} - A^2) f[\asymp].$$

□

### 4.2.2 Tait's First Conjecture

Before stating the first main theorem of this chapter, we must define the notion of *crossing number* of periodic weaving diagram, which must be computed on a weaving motif. However, since we saw that depending on the choice of the topology or scale of the unit, we can obtain generating cells with different number of crossings, which cannot change under the Reidemeister theorem for periodic weaves. This particularity makes the definition of crossing number different from that of general links.

**Definition 4.13.** [48] *The crossing number of a weaving diagram is defined as the minimum number of crossings that can possibly be found in an associated minimal motif $D_\mathrm{W}$, for a fixed $g \geq 1$,*

$$C(\mathrm{W}) = \min\big\{ C(D_\mathrm{W}) \text{: } D_\mathrm{W} \text{ is minimal} \big\}.$$

*Any such weaving motif which has exactly $C(\mathrm{W})$ crossings is said to be minimum.*

It is important to recall at this point that any weaving motif $D_\mathrm{W}$ must encode all the crossing information of the different sets of threads and the periodicity of its associated weaving diagram. For example, in Figure 4.4, the picture on the right is a weaving motif of an alternating weaving diagram that is different from the one on the left. Moreover, finding a minimal diagram to apply Tait's first and second conjectures to alternating periodic weaves is a complex problem. We will introduce a solution for the particular case of untwisted weaves in the next chapter.

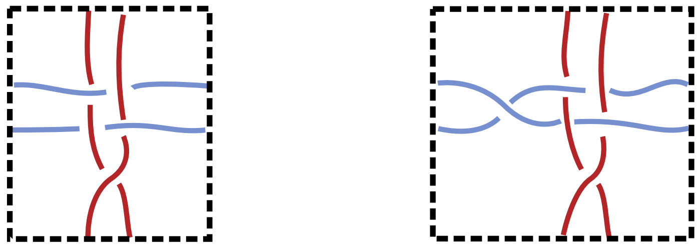


Figure 4.4: two weaving motifs representing distinct weaving diagrams [48].

**Theorem 4.14.** ***(Tait's First Conjecture for periodic alternating weaves)*** [48] *A connected minimal $\Sigma_g$-reduced alternating weaving motif is a minimum diagram of its alternating weaving diagram. Moreover, a minimum weaving motif of a prime alternating weaving diagram can only be alternating. Thus, a non-alternating weaving motif can never be the minimum diagram of a prime alternating weaving diagram.*

*Proof.* Since $V_{\mathrm{W}}(t) = f[D_{\mathrm{W}}](t^{-1/4})$ and $\mathrm{span}(D_{\mathrm{W}}) = 4C - 4g$, for every $g \geq 1$, thus,

$$\begin{aligned} \mathrm{span}\big(V_{\mathrm{W}}(t)\big) &= \max\deg\big(\langle V_{\mathrm{W}}(t)\rangle\big) - \min\deg\big(\langle V_{\mathrm{W}}(t)\rangle\big). \\ &= C - g. \end{aligned} \tag{4.13}$$

Besides, the number of crossings is an invariant so it is fixed here for a minimal $\Sigma_g$-reduced alternating weaving motif. Moreover, we have a generalization of the previous result for the general case, not necessary alternating, that can be proven in a similar way than in the proof of the generalization of Proposition 4.8 in [39],

$$\mathrm{span}(D_{\mathrm{W}}) \leq 4C - 4g, \text{ for every } g \geq 1. \tag{4.14}$$

Thus, the number of crossings cannot decrease below $\mathrm{span}\big(V_{\mathrm{W}}(t)\big)$. We conclude that $D_{\mathrm{W}}$ must be a minimum diagram. Then $\mathrm{span}\big(V_{\mathrm{W}}(t)\big) = C - g$ is not true for a non-alternating weaving motif of a prime alternating weaving diagram; this can be demonstrated in a similar way as done in [66]. Thus, such a diagram cannot be a minimum diagram. □

## 4.3 Tait's Second Conjecture for Periodic Alternating Weaves

The natural following step is to prove Tait's second conjecture, which provides a second invariant for alternating weaving motifs, namely the writhe. Here our approach follows one of the proofs for general links [64], considering that by definition a thread in a weave does not entangle with itself in the sense of a knot. We will only admit *self-crossing* of a thread, which intuitively consists in a simple twist of a part of a thread, that can be done and undone by a Reidemeister move $\Omega_1$.

### 4.3.1 Writhe, linking number and adequacy of weaving diagrams

Earlier in this chapter, we have seen that the writhe is not invariant under Reidemeister moves of type $\Omega_1$, which only concerns the cases of self-crossing of a thread, as discussed above. Therefore, we will start with the study of crossings between two distinct threads of a weaving motif.

**Definition 4.15.** [48] *Let $D_{\mathrm{W}}$ be a weaving motif of an oriented weaving diagram and $W_{\mathrm{U}}$ be the lift of $D_{\mathrm{W}}$ to the thickened surface $\Sigma_g \times [-1, 1]$. Select any two components of $W_{\mathrm{U}}$; denote them as $t_i$ and $t_j$, and let $D^i_{\mathrm{W}}$ and $D^j_{\mathrm{W}}$ denote the pieces of $D_{\mathrm{W}}$ to which they correspond. The linking number of $D^i_{\mathrm{W}}$ and $D^j_{\mathrm{W}}$, denoted $lk(D^i_{\mathrm{W}}, D^j_{\mathrm{W}})$, is the sum, taken over crossings involving only threads from both $D^i_{\mathrm{W}}$ and $D^j_{\mathrm{W}}$, where each term is either $+1$ or $-1$, depending on whether the crossing is of $+1$ type or $-1$ type.*

We will now prove that the linking number, which is defined for a pair of threads, is a weaving invariant.

**Proposition 4.16.** [48] *Let $D_{\mathrm{W}_1}$ and $D_{\mathrm{W}_2}$ be two weaving motifs of a same oriented weaving diagram which differ by a Reidemeister move $\Omega_2$, or $\Omega_3$. Then, with the same notation as before, we have,*

$$lk(D^i_{\mathrm{W}_1}, D^j_{\mathrm{W}_1}) = lk(D^i_{\mathrm{W}_2}, D^j_{\mathrm{W}_2}).$$

As for general links, the proof of this proposition follows immediately from the invariance of the writhe under Reidemeister moves of type $\Omega_2$ and

$\Omega_3$, as well as with the assumption that surface isotopies and Dehn twists on $\Sigma_g$ do not affect the linking number by definition. Moreover, note that the linking number is defined independently from the choice of the periodic generating cell and the continuous deformation of the weaving motifs, for a given periodic scale. Therefore, we are able to extend the definition of the linking number from $\mathbb{X}^2$ to $\mathbb{X}^3 = \mathbb{X}^2 \times [-1, 1]$.

**Definition 4.17.** [48] *Let $D_{\mathrm{W}}$ be a weaving motif of an oriented weaving diagram. With the same notation as before, for any two components $t_i$ and $t_j$ of $W_{\mathrm{U}}$, the linking number of $D^i_{\mathrm{W}}$ and $D^j_{\mathrm{W}}$ , denoted $lk(D^i_{\mathrm{W}}, D^j_{\mathrm{W}})$, is given by*

$$lk(t_i, t_j) := lk(D^i_{\mathrm{W}}, D^j_{\mathrm{W}}).$$

Now the next step is to define the notion of *adequacy* of a weaving motif. For this purpose, we will use once again the notion of states described for the definition of the bracket polynomial earlier. However from now on, when we mention the number of components of a weaving motif in a state $S$, we will only consider null-homotopic curves.

**Definition 4.18.** [48] *Let $D_{\mathrm{W}}$ be a weaving motif of an oriented weaving diagram. Let $S_A$ denote the state of $D_{\mathrm{W}}$ in which all crossings are $A$ split, and $S_B$ if all crossings are $B$ split. Let $c_S$ denote the number of components in the state $S$ of $D_{\mathrm{W}}$. If, for all states $S$ that have exactly one $B$ split, we have that $c_{S_A} > c_S$, then $D_{\mathrm{W}}$ is said to be plus-adequate. If, for all states $S$ that have exactly one $A$ split, we have that $c_{S_B} > c_S$, then $D_{\mathrm{W}}$ is said to be minus-adequate. If $D_{\mathrm{W}}$ is both plus-adequate and minus-adequate, $D_{\mathrm{W}}$ is said to be adequate.*

Here, we will use a well-known approach to characterize the adequacy of a weaving motif $D_{\mathrm{W}}$, and we refer to the proof of Proposition 4.8 and to [64] for more details. Considering the state $S_A$ of $D_{\mathrm{W}}$, we start by selecting an arbitrary crossing that we switch from an $A$ split to a $B$ split. Such an operation will always decrease the number of components, unless if the chosen crossing was a self-crossing of positive type since it would transform one thread into two. So if each component of $S_A$ never forms such a self-crossing at a former crossing of $D_{\mathrm{W}}$, then it is plus-adequate. Similarly, if each component of $S_B$ never forms a negative self-crossing at a former crossing of $D_{\mathrm{W}}$, then it is minus-adequate. The fact that a $\Sigma_g$-reduced weaving motif is always adequate follows directly from this observation since it never contains any self-crossing by definition.

In the proof of Tait's second conjecture for classic links by Stong [64], one of the most interesting point is the notion of parallels of a diagram to study their adequacy.

**Definition 4.19.** [48] *Let $D_{\mathrm{W}}$ be a weaving motif of an oriented weaving diagram and let $r$ be a positive integer. The $r$-parallel of $D_{\mathrm{W}}$, denoted $(D_{\mathrm{W}})^r$, is a weaving motif in which each thread of $D_{\mathrm{W}}$ is replaced by $r$ parallel copies of that thread that follow the same crossing sequence as the original thread.*

The notion of parallels is defined such that for a fixed weaving motif $D_{\mathrm{W}}$ of $\mathbb{X}^2$, each thread is replaced by a set of $r$ copies of the thread, all related by a translation of the plane.

**Lemma 4.20.** [48] *Let $D_{\mathrm{W}}$ be a weaving motif of an oriented weaving diagram and $(D_{\mathrm{W}})^r$, the $r$-parallel of $D_{\mathrm{W}}$. If $D_{\mathrm{W}}$ is plus-adequate (resp. minus-adequate), then $(D_{\mathrm{W}})^r$ is plus-adequate (resp. minus-adequate).*

This lemma can be proved directly by observing that the state $S'_A$ of $(D_{\mathrm{W}})^r$ consists of $r$ parallel copies of each curve of the state $S_A$ of $D_{\mathrm{W}}$. Therefore, if each component of $S_A$ never forms a self-crossing at a former crossing of $D_{\mathrm{W}}$, then the same applies for $S'_A$.

### 4.3.2 Relation between the number of crossings and the writhe

In this section, we start by generalizing Proposition 4.8 to study the degree of the bracket polynomial for non-alternating structures.

**Lemma 4.21.** [48] *Let* $\max\deg(\langle D_{\mathrm{W}}\rangle$ *and* $\min\deg(\langle D_{\mathrm{W}}\rangle)$ *be respectively the maximal and the minimal degree of any polynomial $\langle D_{\mathrm{W}}\rangle$. Let $S_A$ denote the state of $D_{\mathrm{W}}$ in which all crossings are $A$ split, and let $S_B$ denote the state of $D_{\mathrm{W}}$ in which all crossings are $B$ split. Let $C$ be the number of crossings. Then,*

$$\begin{aligned} &\max\deg(\langle D_{\mathrm{W}}\rangle) \leq C + 2c_{S_A} - 2, \text{ with equality if } D_{\mathrm{W}} \text{ is plus-adequate,} \\ &\min\deg(\langle D_{\mathrm{W}}\rangle) \geq -C - 2c_{S_B} + 2, \text{ with equality if } D_{\mathrm{W}} \text{ is minus-adequate.} \end{aligned} \tag{4.15}$$

Then, we can also relate the number of crossings in a plus-adequate $\Sigma_g$-diagram to its writhe, and connect together the concepts of linking numbers and $r$-parallels.

**Lemma 4.22.** [48] *Let $D_{\mathrm{W}_1}$ and $D_{\mathrm{W}_2}$ be two weaving motifs of a same oriented weaving diagram. Suppose that $D_{\mathrm{W}_1}$ is plus-adequate. Let $C_1$ and $C_2$ denote, respectively, the number of crossings in $D_{\mathrm{W}_1}$ and $D_{\mathrm{W}_2}$. Let $w_r(D_{\mathrm{W}_1})$ and $w_r(D_{\mathrm{W}_2})$ denote, respectively, the writhes of $D_{\mathrm{W}_1}$ and $D_{\mathrm{W}_2}$. Then,*

$$C_1 - w_r(D_{\mathrm{W}_1}) \leq C_2 - w_r(D_{\mathrm{W}_2}).$$

*Proof.* We start by indexing each of the threads of the periodic cell $W_{\mathrm{U}}$ as $t_i$, which corresponds to $D^i_{\mathrm{W}_1}$ and $D^i_{\mathrm{W}_2}$ in $D_{\mathrm{W}_1}$ and $D_{\mathrm{W}_2}$ respectively. Since $D_{\mathrm{W}_1}$ is plus-adequate, it does not admit a 'self-crossing'. However, $D_{\mathrm{W}_2}$ can contain finitely many self-crossings of type $+1$ or $-1$. If we consider each thread individually, then we can choose an integer $k_i$ that cancels the writhe of the component $D^i_{\mathrm{W}_2}$ containing self-crossings. This means that, by performing appropriate Reidemeister moves $\Omega_1$, we add $k_i$ twists of type $+1$ if the original writhe of $D^i_{\mathrm{W}_2}$ is negative, or of type $-1$ if it is positive. So, for all integer $i$, we have,

$$w_r(D^i_{\mathrm{W}_1}) = w_r(D^i_{\mathrm{W}_2}) + k_i = 0.$$

By performing these deformations to each $D^i_{\mathrm{W}_2}$, we have added $\sum_{i\geq 1} k_i$ self-crossings to $D_{\mathrm{W}_2}$. We call this new motif $D'_{\mathrm{W}_2}$.

Now to compare $w_r(D'\mathrm{W}_1)$ with $w_r(D'_{\mathrm{W}_2})$, we need also to consider the crossings which involved distinct threads, and thus their linking numbers.

Therefore, we have,

$$\begin{aligned} w_r(D_{\mathrm{W}_1}) &= \sum_{i\geq j} lk(t_i, t_j) \\ w_r(D'_{\mathrm{W}_2}) &= \sum_{i\geq 1} w_r(D^i_{\mathrm{W}_2}) + \sum_{i\geq 1} k_i + \sum_{i\geq j} lk(t_i, t_j) = \sum_{i\geq j} lk(t_i, t_j) \end{aligned} \tag{4.16}$$

Indeed, since the linking numbers are invariant, we have,

$$w_r(D_{\mathrm{W}_1}) = w_r(D'_{\mathrm{W}_2}).$$

We now consider the $r$-parallel $(D_{\mathrm{W}_1})^r$ and $(D'_{\mathrm{W}_2})^r$. Clearly, they are both projections of the same weaving motif (with $r-1$ parallel components added) and thus, are equivalent. Therefore, they have the same bracket polynomial. Moreover, since every crossing of $D_{\mathrm{W}_1}$ and $D'_{\mathrm{W}_2}$ corresponds to $r^2$ crossings of $(D_{\mathrm{W}_1})^r$ and $(D'_{\mathrm{W}_2})^r$, we see that

$$w_r((D_{\mathrm{W}_1})^r) = r^2 w_r(D_{\mathrm{W}_1}) = r^2 w_r(D'_{\mathrm{W}_2}) = w_r((D'_{\mathrm{W}_2})^r).$$

By the definition of the bracket polynomial, it follows that,

$$\langle (D'_{\mathrm{W}_1})^r \rangle = \langle (D'_{\mathrm{W}_2})^r \rangle.$$

Let $c_{S_A^1}$ denote the number of connected components in an all-$A$ splitting of $D_{\mathrm{W}_1}$, and let $c_{S_A^2}$ denote the number of connected components in all-$A$ splitting of $D_{\mathrm{W}_2}$. Adding self-crossings to $D_{\mathrm{W}_2}$ means that the number of connected components in the all-$A$ splitting of $D'_{\mathrm{W}_2}$ becomes $c_{S_A^2} + \sum_{i\geq 1} k_i$. Then, when we pass to the $r$-parallels, we find that the number of connected components in the all-$A$ splitting of $(D_{\mathrm{W}_1})^r$ and $(D'_{\mathrm{W}_2})^r$ becomes $r(c_{S_A^1})$ and $r(c_{S_A^2} + \sum_{i\geq 1} k_i)$, respectively. Moreover, adding the sum of self-crossings in $D_{\mathrm{W}_2}$ means that we increase the number of crossings in $D'_{\mathrm{W}_2}$, which becomes $C_2 + \sum_{i\geq 1} k_i$. Furthermore, making $r-$parallels means that the number of crossings in $(D_{\mathrm{W}_1})^r$ and $(D'_{\mathrm{W}_2})^r$ becomes $r^2 C_1$ and $r^2(C_2 + \sum_{i\geq 1} k_i)$ respectively. Since $D_{\mathrm{W}_1}$ is plus-adequate, we have, by Lemma 4.20, that $(D_{\mathrm{W}_1})^r$ is also plus-adequate. Thus, from Lemma 4.21, we conclude that,

$$\max\deg(\langle (D'_{\mathrm{W}_1})^r \rangle) = r^2 C_1 + 2r c_{S_A^1} - 2$$

and,

$$\max\deg(\langle (D'_{\mathrm{W}_2})^r \rangle) = r^2 (C_2 + \sum_{i\geq 1} k_i) + 2r(c_{S_A^2} + \sum_{i\geq 1} k_i) - 2$$

Since $\langle (D_{\mathrm{W}_1})^r \rangle = \langle (D'_{\mathrm{W}_2})^r \rangle$, then, $\max\deg(\langle (D_{\mathrm{W}_1})^r \rangle) = \max\deg(\langle (D'_{\mathrm{W}_2})^r \rangle)$, and thus, for all positive integers $r$, we have

$$r\, C_1 + 2c_{S_A^1} \leq r(C_2 + \sum_{i\geq 1} k_i) + 2(c_{S_A^2} + \sum_{i\geq 1} k_i)$$

Therefore,

$$C_1 \leq C_2 + \sum_{i\geq 1} k_i.$$

And since for all positive integer $i$, we have $w_r(D^i_{\mathrm{W}_2}) + k_i = 0$, then,

$$C_1 \leq C_2 - \sum_{i \geq 1} w_r(D^i_{\mathrm{W}_2}).$$

Again, since the linking number is an invariant, we have

$$lk(D^i_{\mathrm{W}_1}, D^j_{\mathrm{W}_1}) = lk(D^i_{\mathrm{W}_2}, D^j_{\mathrm{W}_2})$$

So as desired, we finally have from (5.2) that,

$$C_1 - w_r(D_{\mathrm{W}_1}) \leq C_2 - w_r(D_{\mathrm{W}_2}).$$

□

### 4.3.3 Tait's Second Conjecture

Let $D_{\mathrm{W}_1}$ and $D_{\mathrm{W}_2}$ be two minimal weaving motifs of the same oriented weaving diagram, which are both alternating, $\Sigma_g$-reduced, and therefore adequate. Let $C_1$ and $C_2$ denote, respectively, the number of crossings in $D_{\mathrm{W}_1}$ and $D_{\mathrm{W}_2}$. Then, by the previous Lemma, we have $C_1 - w_r(D_{\mathrm{W}_1}) \leq C_2 - w_r(D_{\mathrm{W}_2})$, and $C_2 - w_r(D_{\mathrm{W}_2}) \leq C_1 - w_r(D_{\mathrm{W}_1})$, and thus, $C_1 - w_r(D_{\mathrm{W}_1}) = C_2 - w_r(D_{\mathrm{W}_2})$. Moreover, these two weaving motifs have the same crossing number according to Tait's First Conjecture, so $C_1 = C_2$, and thus, $w_r(D_{\mathrm{W}_1}) = w_r(D_{\mathrm{W}_2})$. This finally proves Tait's Second Conjecture for periodic alternating weaves.

**Theorem 4.23.** ***(Tait's Second Conjecture for weaves)*** [48] *Any two connected minimal $\Sigma_g$-reduced alternating weaving motifs of an oriented periodic alternating weave have the same writhe.*

# Chapter 5

# CLASSIFICATION OF UNTWISTED $(p, q)$-WEAVING MOTIFS

In this chapter, we will continue to explore the classification of weaves according to their number of crossings and their spatial configuration at the diagrammatic scale. In particular, we will investigate the particular class of doubly periodic untwisted $(p, q)$-weaves and will define new results for weaving motifs in the Euclidean plane for this class of weaves. Recall that such a weaving diagram is isotopic to a graph $\Gamma$, that can be decomposed into at least two disjoint sets of parallel lines characterized by different slopes, or axes of direction, together with an over or under information at each vertex encoded in a set of crossing sequences $\Sigma$. However, we observed situations where two distinct infinite weaving diagrams could be reconstructed from the same pair $(\Gamma, \Sigma)$, which implies that the attribution of crossing information is not unique. A very simple example that illustrates this fact is shown in Figure 5.1, where we can compare weaving motifs of a *basket weave* $(+2, -2)$ and a *twill weave* $(+2, -2)$. These two weaving motifs can be reconstructed from a *topological* square tiling, that encodes the two sets of threads, to which we assign for each thread two consecutive overcrossings followed by two consecutive undercrossings, cyclically. By topological tiling, we mean that we only consider the degree of the vertices and the number of adjacent neighbors of each tile, and not geometric features. This notion is detailed in [31]. Observing that a pair $(\Gamma, \Sigma)$ could possibly encode two different weaves motivated the investigation of equivalence classes of weaves and the construction of a new weaving invariant $\Pi$, such that any weaving diagram generated from a triple $(\Gamma, \Sigma, \Pi)$ would define a *unique* equivalence class. This topological invariant $\Pi$, will be defined as a set of *crossing matrices* whose elements are symbols $\pm 1$, and will characterize the organization of the crossings on a weaving motif of a doubly periodic untwisted $(p, q)$-weaves. This matrices generalizes the notion of 'design' described in [28], [29], [30], [32], to the case of weaves with more than two sets of threads. Then, the second main result of this chapter relates to the computation of the minimal number of crossings for this class of weaves, which is an important parameter characterizing the complexity of an entangled structure and a nontrivial open problem, as discussed in the previous chapter. Here, we will use a combinatorial description of untwisted $(p, q)$-weaving motifs to define a formula which depends on $(\Gamma, \Sigma)$ and also gives a solution of a minimal periodic cell, called a *minimal motif*. Therefore, using the results stated in this chapter, which are based on our paper [21], we are now able to classify doubly periodic untwisted $(p, q)$-weaves according to two new weaving invariants, namely the set of crossing matrices and the crossing number. An interesting observation is that two weaving diagrams characterized by the same pair $(\Gamma, \Sigma)$ but that differ from their parameter $\Pi$ seem to have different crossing number.

## 5.1 Equivalence classes of doubly periodic untwisted $(p, q)$-weaves

We start this chapter with the study of the equivalence classes of doubly periodic untwisted $(p, q)$-weaves. By a $(p, q)$-weave, we mean that given any pair of sets of threads, all the threads of one set have the same crossing sequence with the second set, and this sequence consists of only two positive integers.

**Definition 5.1.** [21] *Let $i$, $j$, and $k$ be strictly positive integers. A $(p, q)$-weave* $\mathrm{W}$ *is defined such that all its crossing sequences, possibly distinct, can be described by two positive integers $p_k$ and $q_k$. This means that if $C_{i,j} = (+p_k, -q_k)$ is the crossing sequence associated to the disjoint sets of threads $T_i$ and $T_j$ of* $\mathrm{W}$*, each thread of $T_i$ is cyclically $p_k$ consecutive times over the threads of $T_j$, followed by $q_k$ consecutive times under. Moreover, the* *complementary crossing sequence of $C_{i,j}$ is given by $C_{j,i} = (+q_k, -p_k)$.*

In other words, the crossing sequence $C_{i,j} = (+p, -q)$ of minimal length associated with the disjoint sets of threads $T_i$ and $T_j$ is defined such that if one walks along any thread $t_i \in T_i$, there exists a crossing $c_{i,j} = \pi(t_i) \cap \pi(t_j)$ having at least one of its two neighboring crossings $c'_{i,j}$ with a different over or under information, where $c'_{i,j}$ is a crossing between $t_i$ and another thread of $T_j$. Then, by walking on $t_i$ in the opposite direction of $c'_{i,j}$, there are cyclically $p$ crossings for which $t_i$ is over the other threads of $T_j$, followed by $q$ crossings for which it is under. Moreover, by definition of the complementary crossing sequence, we can encode all these crossing information with only one of these two crossing sequences, including non-crossing sets of threads with crossing sequences $(+1, 0)$ or $(0, -1)$ in an entangled weave with $N > 2$ sets of threads.

Recall that from the Reidemeister Theorem for weaves, we can characterize the equivalence classes of doubly periodic untwisted $(p, q)$-weaves using their diagrammatic representation. This means that the parameter $\Pi$ that we will construct in this section, which consists of a set of crossing matrices must be invariant by the three Reidemeister moves, as well as the torus deformation. We will construct these matrices as an extension of the well-known concept of bicolored *design* existing only for the case of two sets of threads, which can also be interpreted as binary matrices [32]. An example of designs is given in Figure 5.1 for the basket and twill weaves mentioned above.

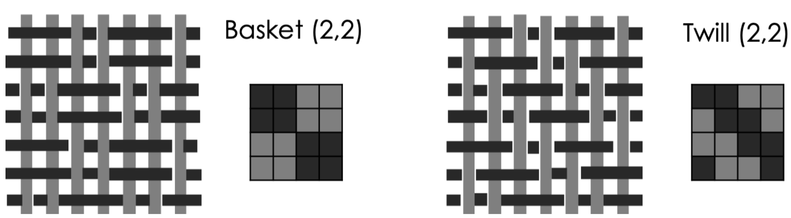



Figure 5.1: Twill and Basket $(2, 2)$ square weaving diagram with their associate design [21].

### 5.1.1 Background on the case of weaves with two sets of threads

Our new weaving invariant $\Pi$ has been constructed as an extension of the concept of *design*, also called *armor* in the literature, used for the classification of weaves in the textile industry that considers only the case of doubly periodic untwisted weave with $N = 2$ sets of threads [32]. A design characterizes on a bicolored square grid the organization of crossings on a weaving motif, and is associated with the crossing sequence of the given weave. In this setting, these two sets of threads, called *warp* and *weft*, are seen as perpendicular to each other, following existing weavers' techniques, and are represented in a design by the columns and by the rows, respectively. On such a regular square tiling, each tile represents therefore the crossing between a weft thread and a warp strand, and by usual convention, a square is colored white if the weft thread passes over the warp thread, or black conversely. Notice in Figure 5.1 that for the twill weave, the gray and black squares have a *diagonal* organization, however, the basket weave they are organized into *blocks*. This illustrates the different possibilities of specifying the over or under information at each vertex of a graph $\Gamma$, given a set of crossing sequences.

In 1867, the mathematician E. Lucas [14] used arithmetic arguments to study the specific class of doubly periodic untwisted weaves, called *regular satins*, containing two sets of threads. He noticed that the structure of the design of a regular satin can be represented on a square chessboard, whose dimension is a positive integer $m$ called the *module* of the design. This chessboard, assuming its invariance by translation in all directions, must satisfy the condition that the black squares are arranged in a regular fashion such that two of them are not in the same horizontal or vertical row, and so that, with respect to any one of these black squares, the other black squares are always placed in the same way. The distance between two such black squares, which is counted vertically, is a multiple of an integer $a$, called *shift*, such that, $a < m$ and $gcd(a, m) = 1$, which allows the construction of two sequences,

- the sequence of column indices of the matrix $x = 0, 1, 2, \cdots, m - 1$,
- the sequence of row indices of the matrix $y = 0, a, 2a, \cdots, (m - 1)a$, calculated in $\mathbb{Z}_m$.

This situation relates to a classic theorem of Gauss, which characterizes a systematic way to construct regular satins, as mentioned by Lucas. Indeed, for a design of module $m$ the black squares appear in each column depending on the shift $a$ modulo $m$, therefore if $m$ and $a$ are coprime, the condition to build a regular satin is satisfied. In other words, the remainders (mod. $m$) of the numbers $y$ thus obtained constitute a permutation of the numbers $0, 1, 2, \cdots, m-1$. The twill $(+2, -2)$ illustrated in Figure 5.1 is an example of a weave belonging to the twill family, which is a particular class of regular satin with module $m \geq 2$ and shift $a = \pm 1 mod(m)$. The black boxes are therefore colored on the chessboard at the intersection of the column $x = k$ and the row $y = ka \ (modm)$. Note that for a given module $m$, there are as many regular satin weaves, as there are integers $a$ coprime with $m$, with $a < m$. Lucas also proved that the regular satin weaves with shift $a$, $m - a$, $a'$ and $m - a'$ are equivalent, with $aa' = 1 \ mod(m)$.

More generally, it is said that two weaves are essentially distinct if their designs are not equivalent, up to rigid motions and an interchange of the colors black and white.

Our purpose is to generalize this concept to the general case of $N \geq 2$ sets of threads, so that we can encode the organization of crossings for each pair of sets of threads on a periodic cell and distinguish non-equivalent weaving motifs with such a topological invariant. Grünbaum and Shephard used geometric restriction to consider such a generalization and made an attempt using tilings by polygons, which are isotopic to a regular projection of a weave, and labeled each vertex to indicate the crossing information [30]. However, they mentioned that such a pattern can become very complex and unintelligible.

### 5.1.2 Our generalization

To generalize the notion of design considering more sets of threads, we choose to define a set of *crossing matrices* associated with a weaving motif such that it would become possible to distinguish doubly periodic untwisted $(p, q)$-weaves characterized by the same pair $(\Gamma, \Sigma)$. The key point of our strategy relates to the fact that we defined the crossing sequences of a weave in a pairwise fashion, in the sense that each matrix is associated with a pair of distinct sets of threads of the diagrams. A crossing matrix is a square matrix made of symbols $+1$ or or $-1$, representing an overcrossing or an undercrossing, respectively. More specifically, if a pair of sets of threads $(T_i, T_j)$ has for crossing sequence $C_{i,j} = (+p, -q)$, then the crossing matrix will have size $m = p + q$, which naturally generalized the results of Lucas. At this point, it is important to also note that the size of a crossing matrix is connected with the number of strand components in a weaving motif. It particular, the crossing matrix of size $m$ associated with a pair $(T_i, T_j)$ of a weave characterizes the crossing information of a weaving motif containing $m$ strands of $T_i$ as well as $m$ strands of $T_j$. Moreover, as observed for the crossing sequences, we will see that only one crossing matrix for each pair of sets of threads is enough to characterize the full structures. This means that for a weave with $N$ sets of threads, we can define $\frac{N(N-1)}{2}$ matrices for a corresponding weaving motif $D_W$, where each matrix describes the crossing organization for a given sets of threads, as illustrated in Figure 5.2. In other words, for each crossing between two strands $s_i \in T_i$ and $s_j \in T_j$, we say that $s_i$ is over (resp. under) $s_j$, if we look at the position of the strands of $T_i$ according to the position of the strands of $T_j$, or conversely that $s_j$ is under (resp. over) $s_i$, if we consider the position of the strands of $T_j$ according to the position of the strands of $T_i$.

**Definition 5.2.** [21] *Let $i, j \in \{1, \cdots, N\}$ and $C_{i,j} = (+p, -q)$ be the crossing sequence of the two disjoint sets of threads $T_i$ and $T_j$ of a doubly periodic untwisted $(p, q)$-weaving diagram $D_{W_0}$ with $N \geq 2$. Let $M_{i,j}$ be a square $m \times m$ $(-1, +1)$-matrix, where $m = p + q$ is called the module of $M_{i,j}$. Then, $M_{i,j}$ is called the crossing matrix of $D_{W_0}$ associated with $C_{i,j}$ if each row and each column of $M_{i,j}$ simultaneously contains $p$ consecutive symbol $+1$ followed by $q$ consecutive symbols $-1$, considering cyclic and countercyclical permutations of rows and columns of the matrix.*

**Remark 5.3.** [21] To construct the crossing matrices associated to a fixed weaving motif $D_W$ of $D_{W_0}$ with $N \geq 2$ sets of threads, one must consider a flat torus cut from any preferred meridian-longitude pair, or equivalently fix any square unit cell of $D_{W_0}$ containing $m^2$ crossings and $m$ strands of each set. The strands $s_{k,i}$ of a set $T_i$ are oriented and indexed by a strictly positive integer $k$ from left to right and top to bottom, starting from the top left crossing. Then, for any crossing sequence $C_{i,j}$ of $D_{W_0}$, one must use the following convention to fill the associated crossing matrix $M_{i,j} = (m_{x,y})_{0 \leq x, y \leq m-1}$,

(1) $m_{1,1} = +1$ (resp. $m_{1,1} = -1$) if the most top left crossing $c_{1,1} = s_{1,i} \cap s_{1,j}$ of $D_W$ is such that the strand $s_{1,i}$ of $T_i$ is over (resp. under) the strand $s_{1,j}$ of $T_j$;

(2) fill the first row of the matrix by walking on $s_{1,i}$ such that the element $m_{1,k}$ gives the information of the crossing $c_{1,k} = s_{1,i} \cap s_{k,j}$, with $k$ the index of the strand of $T_j$;

(3) fill the $k^{th}$ row of the matrix by walking on $s_{k,i}$, starting with the element $m_{k,1} = m_{1,k}$ given by the information of the crossing $c_{k,1} = s_{k,i} \cap s_{1,j}$. Note that the $k^{th}$ column of the matrix is equal to its $k^{th}$ row.

Furthermore, a cyclic or countercyclical permutation of the rows (resp. the columns) of such a crossing matrix, also called a translation of the design in [32], corresponds to a vertical (resp. horizontal) translation of the unit cell on $D_{W_0}$.

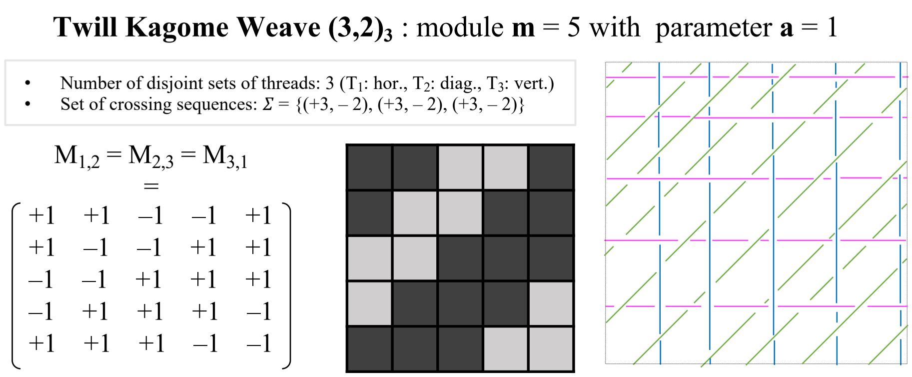



Figure 5.2: Kagome Matrices

**Remark 5.4.** [21] The over or under information at the crossing between the threads $s_i$ and $s_j$ must be opposite on the two crossing matrices $M_{i,j}$ and $M_{j,i}$, describing the position of the two sets of threads considered, since if $s_i$ is over $s_j$ at a crossing, then the corresponding element of the matrix from the viewpoint of $T_i$ will be $+1$, while it will be $-1$ from the point of view of $T_j$. This justifies why only one of the two matrices associated with a pair of sets of threads is enough to characterize this class of weaves. Moreover, one can deduce the crossing matrix $M_{j,i}$ from $M_{i,j}$ by transposition and by exchanging the symbols $+1$ and $-1$. In other words, $M_{j,i} = -(M_{i,j})^T$.

Following Lucas' strategy on satins [14], we formalized construction rules for crossing matrices for specific subclasses of doubly periodic untwisted $(p, q)$-weaves in the following examples. Here we also reformulate with our notations the case of satins, which is also considered in the case of a weave with more than two sets of threads and such that one of its crossing sequences can be written as $(+p, -1)$. An interesting challenge would be the enumeration of all the possibilities to construct a crossing matrix for a given crossing sequence of type $(+p, -q)$ and then extending this enumeration to more complex periodic weaves.

**Example 5.5.** [21] Consider the crossing matrix $M = (m_{x,y})_{0\leq x,y\leq m-1}$ such that $x$ represents the column indices of the matrix and $y$ its row indices. Then, if $M$ is associated with a crossing sequence $(+p, -1)$ such that it satisfies Lucas' condition on regular satin [14], the symbols $-1$ can be positioned at the element $m_{x,y}$ satisfying the system,

$$\begin{cases} x &= k, \text{ with } k \in 0, 1, \cdots, p+q-1 \\ y &= ak \ mod(m), \text{ with } a < m \text{ fixed and } gcd(a, m) = 1. \end{cases}$$

In this case, the elements that do not satisfy this system are symbols $+1$.

**Example 5.6.** [21] Consider the crossing matrix $M = (m_{x,y})_{0\leq x,y\leq m-1}$ such that $x$ represents the column indices of the matrix and $y$ its row indices. Then, if $M$ is associated with a crossing sequence $(+p, -p)$ such that at least two rows or two columns are equal, the symbols $+1$ can be positioned at the element $m_{x,y}$ satisfying the system,

$$\begin{cases} x_k &= k, \text{ with } k \in \{0, 1, ..., 2p-1\} \\ y_{k,l} &= y_{k,l-1} \pm 1, \text{ if } k \neq p \ mod(m), \text{ or} \\ y_{k,j} &= y_{k,j-1} \pm p, \text{ otherwise}, j \in \{1, \cdots, p-1\}. \end{cases}$$

with the first column given by,

$$\begin{cases} x_0 &= 0 \\ y_{0,0} &= 0 \text{ and } y_{0,j} = y_{0,j-1} \pm 1, j \in \{1, \cdots, p-1\}. \end{cases}$$

In this case, the elements that do not satisfy this system are symbols $-1$.

**Example 5.7.** [21] Consider the crossing matrix $M = (m_{x,y})_{0\leq x,y\leq m-1}$ such that $x$ represents the column indices of the matrix and $y$ its row indices. Then, if $M$ is associated with a crossing sequence $(+p, -q)$, with $p, q \neq 1$

and such that there are not two equal rows or columns, the symbols $+1$ can be positioned at the element $m_{x,y}$ satisfying the system,

$$\begin{cases} x = k, \text{ with } k \in 0, 1, \cdots, p+q-1 \\ y = ak \pm i \ mod(m), \text{ with } a = \pm 1 \text{ fixed}, i \in \{1, \cdots, p-1\}. \end{cases}$$

In this case, the elements that do not satisfy this system are symbols $-1$.

**Remark 5.8.** [21] Note that the basket weave of Figure 5.1 has a crossing matrix of rank 1, while the twill weave has a crossing matrix of rank 2, which confirms the non-equivalence of the two structures. This result can be generalize for any crossing matrix corresponding to a crossing sequence $(+p, -p)$, whose rank would be equal to 1 if it is constructed as in Example 5.6, or equal to $p$ if it is constructed as in Example 5.7.

We will now prove that the set of crossing matrices defined above is a weaving invariant that allows one to distinguish two doubly periodic untwisted $(p, q)$-weaves characterized by the same pair $(\Gamma, \Sigma)$. This set of crossing matrices is denoted by $\Pi = \{M_{i,j} \setminus i, j \in \{1, \cdots, N\}\}$, where each $M_{i,j}$ is a crossing matrix. Moreover, when comparing two such weaves, we should consider identical regular projection, up to isotopy, and such that the sets of threads are identically labeled to compare strands from the same set on the two weaving motifs.

**Theorem 5.9.** ***(Equivalence Classes of doubly periodic untwisted*** $(p, q)$***-weaves)*** [21] *Let* $\mathrm{W}_1$ *and* $\mathrm{W}_2$ *be two doubly periodic untwisted* $(p, q)$*-weaves with* $N \geq 2$ *sets of threads, such that their corresponding regular projections are equivalent, up to isotopy of* $\mathbb{E}^2$*, and with the same set of crossing sequences. Let* $D_{W_1}$ *and* $D_{W_2}$ *be two weaving motifs of same area of* $\mathrm{W}_1$ *and* $\mathrm{W}_2$*, respectively. Then,* $D_{W_1}$ *and* $D_{W_2}$ *are equivalent if and only if their crossing matrices are pairwise equivalent, meaning that all the matrices of* $D_{W_2}$ *can simultaneously be obtained from the respective matrices of* $D_{W_1}$ *from at least one of two conditions,*

- *a same cyclic or countercyclical permutations of rows and columns,*
- *a same clockwise or counterclockwise rotation of* $\pi$*, or of* $\frac{\pi}{2}$ *together with an inversion of all its symbols.*

*Proof.* First, we prove that the equivalence of the weaving motifs implies the equivalence of the crossing matrices, using the Reidemeister Theorem for weaves. We start by studying the invariance of the Reidemeister moves. Without loss of generality, we assume that our weaving diagrams are geodesic. Then, by definition, Reidemeister moves of types $\Omega_1$ and $\Omega_2$ do not occur. For a Reidemeister move of type $\Omega_3$, we consider three threads from three different sets. However, recall that each crossing matrix is obtained from the crossing information of a pair of sets of threads only. This means that an $\Omega_3$ move does not change any of the crossing matrices. Now, it suffice to show the invariance of the matrices of a weaving motif $D_W$ for the torus twists. We fix a lattice $\mathbb{Z}^2$ in $\mathbb{E}^2$ where the infinite weaving diagrams are embedded. Let $p$ be the intersection point of preferred meridian-longitude pair $(\mu, \lambda)$ on $D_W$, such that a flat weaving motif is obtained by cutting along this pair. Notice at this point that if we reverse the meridian with the longitude, meaning that we replace the pair $(\mu, \lambda)$ into $(\lambda, \mu)$, then the cut along this reversed pair is realized by rotating the original weaving motif in $\mathbb{E}^2$ by $\frac{\pi}{2}$. Let $p' \neq p$ be a point on the longitude $\lambda$ and $p'' \neq p$ be a point on the meridian $\mu$ of $D_W$. Then, we obtain new preferred meridian-longitude pairs $(\mu, \lambda')$ and $(\mu', \lambda)$ on $D_W$. Cutting $D_W$ along $(\mu', \lambda)$ instead of $(\mu, \lambda)$ corresponds to a cyclical or countercyclical permutations of the rows of the crossing matrices of $D_W$. Similarly, cutting $D_W$ along $(\mu, \lambda')$ instead of $(\mu, \lambda)$ corresponds to a cyclical or countercyclical permutations of the colums of the crossing matrices of $D_W$. In particular, these transformations correspond to a vertical and horizontal translation of the unit cell in the infinite weaving diagram respectively. Besides, recall that a torus twist represents a modular transformation that preserves the fixed lattice [19], and that two periodic cells with the same area of an infinite weaving diagram have the same number of crossings, by Theorem 4.10. Therefore, two weaving motifs related by a sequence of torus twists would correspond to two distinct parallelogram unit cells of the same infinite diagram. We can conclude using Remark 5.3 and the definition of the crossing sequence of a $(p, q)$-weaves, that the crossing matrices of two such diagrams are equivalent up to a sequence of cyclical or countercyclical permutations of the rows and columns, which conclude the first part of our proof. Then, to prove the reverse implication, we start from a set of crossing matrices to which we apply the same transformation. If we apply a same cyclic or countercyclical permutation of the rows and the columns applied simultaneously to all the crossing matrices associated to a weaving motif, we have seen above that it corresponds to a translation of the periodic unit cell in the infinite diagram, which implies the equivalence of the corresponding weaving motifs. Furthermore, if we apply to the given set of crossing matrices the same clockwise or counterclockwise rotation of $\pi$, or of $\frac{\pi}{2}$ together with an inversion of all its symbols, then we would obtain pairwise equivalent matrices, up to cyclic or countercyclical permutations of the rows and the columns, from which we can construct once again equivalent weaving motifs. □

**Remark 5.10.** [21] Observe that if one of the crossing matrices of $D_{W_2}$ has a different permutation or rotation than the others matrices, then $D_{W_1}$ and $D_{W_2}$ are not equivalent. Indeed, there would exist a thread $t_i$ of $D_{W_2}$ for which the order of all its crossings will be different than its representative in $D_{W_1}$. This means that there would exist a crossing $c_{i,j}$, in $D_{W_2}$, with $T_i$ and $T_j$ two sets of threads, such that by walking on the thread $t_i$, one of the nearest neighboring crossing $c_{i,k}$, of $c_{i,j}$,, with a different set of threads $T_k$ will have a different type than the corresponding one in $D_{W_1}$. Therefore, the two weaving motifs cannot be superimposed, which means in other words that they are not equivalent, as in Figure 5.3.

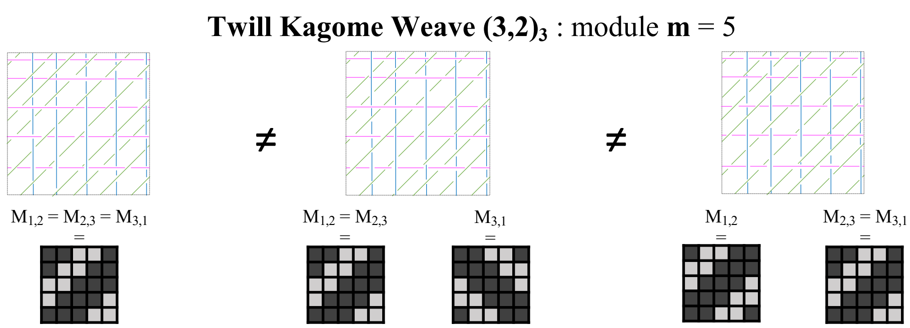



Figure 5.3: Non Equivalent Kagome Matrices [21].

We can deduce from this main first theorem of this chapter an easy way to construct new weaving diagrams. Indeed, starting with a triple $(\Gamma, \Sigma, \Pi)$, it is possible to obtain nonequivalent structures having the same pair $(\Gamma, \Sigma)$ just by doing a permutation or rotation of some of the matrices of $\Pi$, but not all, with respect the related crossing sequences.

## 5.2 The crossing number of a doubly periodic untwisted $(p, q)$-weave

In knot theory, the main classification criterion is the number of crossings in the diagram of a link, and more specifically the *crossing number*, which is the minimum number of crossings in any diagram of the link. It is therefore natural for us to orient our study towards the analysis of the number of crossings in a weaving motif. However, there exist many different weaving motifs of the same weave, so the idea is to identify and eliminate duplicates. The other particularity for doubly periodic structures is that the crossing number is define for a *unit cell* and not any periodic cell. Any such motif is called *minimal*, and its construction is not a trivial problem. Indeed, the weaving motifs corresponding to the crossing matrices (or the simpler case of design for the case of two sets of threads) mentioned in the previous section give traditional representations of the corresponding weave. Nevertheless, these motifs are usually not minimal in terms of the number of crossings because they contain several unit cells. Our final objective is the enumeration of our class of weaves in order of increasing crossing number. However, it has been noticed that there are no general methods for computing the crossing number of a given weave.

In this section, we will introduce a solution to this problem of finding the crossing number and the minimal weaving motif of a doubly periodic untwisted $(p, q)$-weave, using an approach based on combinatorial arguments. We will also end this chapter with a note on the entanglement of our weaving motifs, discussing the invariance of the number of A-triangles, related to the notion of blocking crossing defined in Chapter 2 .

### 5.2.1 The crossing number

Let W be a doubly periodic untwisted $(p, q)$-weave in the Euclidean thickened plane and consider its regular projection to the plane. We observed that this projection can also be seen as the skeleton of a specific type of periodic topological quadrivalent tiling by convex polygons. We call such a tiling a *thread-tiling*.

**Definition 5.11.** [21] *A periodic thread-tiling, composed of $N \geq 2$ sets of threads, is a planar edge-to-edge quadrivalent topological tiling by convex polygons, such that each edge of these polygons belongs to a single thread and two adjacent edges belong to two threads of different sets.*

The next step toward the computation of the crossing number is to describe such a thread-tiling in a combinatorial way. As for doubly periodic weaves, any generating cell of a doubly periodic thread-tiling can be seen as a graph embedded in a torus that can be decomposed into a set of finite number of essential simple closed curves, denoted by $\Gamma'$. Recall that any nontrivial simple closed embedded on a torus can be characterized by a pair of coprime integers $(a, b)$ [19], which encodes the slope of the axis of direction of the corresponding thread in the Euclidean plane. Moreover, note that the pairs $(a, b)$ and $(-a, -b)$ represent axes of the same slope, as well as the pairs $(-a, b)$ and $(a, -b)$. We thus use by convention the pairs $(a, b)$ and $(-a, b)$ for these two cases, respectively. Another important property of such essential simple closed curves is their number of intersections on the torus. More specifically, the number of intersections between two such curves $(a, b)$ and $(a', b')$, denoted by $|v|$, is given by the *geometric intersection number* [19], via the equation $|a\ b' - a'\ b| = |v|$. This number is well-defined for free homotopy classes of simple closed curves as the minimum number of intersections between a representative curve of each set of threads. However, note that the same thread-tiling can be characterized by non-equivalent sets of curves, and the challenge is to find a unit cell, meaning a generating cell with the fewest vertices. This is the case of the basic square tiling which can for example be recovered from a generating cell containing one curve $(1, 0)$ and one curve $(0, 1)$ with only one vertex, and that also can be reconstructed from a periodic cell containing one curve $(1, 1)$ and one curve $(-1, 1)$ with this time two vertices. Moreover, by the Reidemeister theorem, a weave can also admit different thread-tilings as regular projection. The classification of doubly periodic thread-tilings in terms of sets of essential simple closed curves on a unit cell is an interesting problem, and in this chapter, we assume that for a given number of set of threads, a periodic cell containing curves given by pairs of coprime minimized in absolute value will have the fewest vertices.

The idea behind the computation of the crossing number of doubly periodic untwisted $(p, q)$-weave, is to use the argument of periodicity at each step, both for encoding the thread-tiling unit cell $\Gamma'$ and the set of crossing sequence $\Sigma$. In other words, for each pair of disjoint sets of threads of such weaves, we can deduce from the associated geometric intersection number of two representative curves and their crossing sequence the minimum number of crossings required to ensure the periodicity associated with these two sets.

**Lemma 5.12.** ***(Pairwise crossing number)*** [21] *Let $D_{W_0}$ be a weaving diagram with $N$ sets of threads characterized by the pair $(\Gamma', \Sigma)$. Let $i, j \in \{1, \cdots, N\}$ be distinct integers and $T_i$, $T_j$ be two sets of threads of $D_{W_0}$. Then, to ensure the periodicity on a weaving motif of $D_{W_0}$, the minimum number of necessary crossings $c_{i,j}$ on a thread $t_i \in T_i$ and $t_j \in T_j$ is given by,*

$$\mathcal{C}_{i,j} = lcm(\zeta^i_{i,j}; \zeta^j_{i,j})$$

*where $\zeta^i_{i,j} = |v_{i,j}| \times lcm(\{|C_{i,k}| \ / \ k \in \{1, \cdots, N\}\})$, with $|C_{i,k}|$ the sum of the two positive integers of the crossing sequence $C_{i,k}$, for all $k \in \{1, \cdots, N\}$. We call $\mathcal{C}_{i,j}$ the $(i,j)$-pairwise crossing number.*

*Proof.* Let $s_i \in T_i$ and $s_j \in T_j$ be the two curves representative of a unit cell of the thread-tiling corresponding to $D_{W_0}$. Then, to ensure the periodicity of the crossing sequence $C_{i,j}$ in a weaving motif of $D_{W_0}$ we must first consider the minimum number of vertices $v_{i,j}$ which is necessary on $s_i$ and $s_j$ to satisfy the definition of the thread-tiling given by $\Gamma'$. This number $|v_{i,j}|$ is the geometric intersection number between $s_i$ and $s_j$ by definition. Then, we must multiply this number by the minimum number of crossings necessary to read the crossing sequence $C_{i,j} = (+p, -q)$, which is its module $m = |C_{i,j}| = p + q$, which is also minimal by definition. However, on a strand $s_i$ (resp. $s_j$) there are not only crossings of type $c_{i,j}$ when $N > 2$, we must therefore also consider the minimum number of necessary crossings of type $c_{i,k}$ on $s_i$ (resp. $c_{j,k}$ on $s_j$) to read the other crossing sequences $C_{i,k}$ (resp. $C_{j,k}$) and ensure the periodicity. We must therefore consider the least common multiple of their modules $|C_{i,k}|$ (resp. $|C_{j,k}|$) as the multiplier of $|v_{i,j}|$. Finally, the global minimality is ensured by taking the least common multiple of these two products, considering that we must obtain the same number of crossings on each thread of $T_i$ and $T_j$ on the weaving motif. □

At this stage, we however do not encode yet a weaving motif in its whole. Each $(i,j)$-pairwise crossing number $\mathcal{C}_{i,j}$ represents the number of crossings in each simple closed curve of both sets $T_i$ and $T_j$, unless these sets do not cross. However, this condition must be satisfied simultaneously for all pairs of sets of threads of the weaving motifs. Recall that each curve of a given set of threads can be given by a pair of coprime integers, that not equivalent to the pair of coprime representing another set of threads by definition of a weave. Therefore, computing the crossing number equals solving a system of equations, in which one can simultaneously satisfy the geometric intersection number equations for each pair of sets of threads. Note that the possibilities of choosing the integers are restricted and a minimal motif might be made of more than one strand for each set of threads.

**Theorem 5.13.** ***(Total Crossing Number)*** [21] *Let $i, j \in \{1, \cdots, N\}$ distinct and $\mathcal{C}_{i,j}$ be the $(i, j)$-pairwise crossing numbers of a weaving diagram $D_{W_0}$ with $N$ sets of threads, characterized by the pair $(\Gamma', \Sigma)$. Let $(\mathcal{S}_{min})$, be the system of geometric intersection number equations, defined for integers $a_i$ and $b_i$, either coprime or such that one of them equals zero, satisfying for each equation that we can multiply both parts by a same even number $k_l$ if the two sets of threads implied on the equation cross ($k_l = 1$ otherwise), with $l \in \{1, \cdots, \frac{N(N-1)}{2}\}$ being the index of the equation in the system.*

$$(\mathcal{S}_{min}) \begin{cases} k_1 \times |a_1\ b_2 - a_2\ b_1| & = \quad k_1 \times \mathcal{C}_{1,2} \\ \qquad \vdots & \\ k_l \times |a_i\ b_j - a_j\ b_i| & = \quad k_l \times \mathcal{C}_{i,j} \\ \qquad \vdots & \end{cases}$$

*Then, from the solution of $(\mathcal{S}_{min})$ with smallest multipliers $k_l$ which minimizes each integers $a_m$ and $b_m$ in absolute value, and such that every two pairs $(a_m, b_m)$ are distinct, we can deduce the total crossing number of $D_{W_0}$ given by,*

$$\mathcal{C} = \sum_{i<j=1}^{N} \mathcal{C'}_{i,j}$$

*with each $\mathcal{C'}_{i,j} = k_l\ \mathcal{C}_{i,j}$ in $(\mathcal{S}_{min})$.*

*Proof.* A unit cell of a periodic weaving diagram is an embedding of essential simple closed curves crossing on the surface of a torus. Thus, the geometric number of intersection between a simple closed curve of a set of thread $T_i$ and a simple closed curve of a set $T_j$ defines the minimum number of crossings between these two curves on a weaving motif, given by the $(i,j)$-pairwise crossing number $\mathcal{C}_{i,j}$. From Lemma 5.12, we therefore have to find two pairs integers, either coprime or such that one of them equals zero, whose geometric number of intersection is $\mathcal{C}_{i,j}$, for each pair $(i,j)$.

$$|a_i\ b_j - a_j\ b_i| = \mathcal{C}_{i,j}$$

These four integers are chosen by convention such that their absolute value is minimal. Moreover, since this condition must be satisfied simultaneously for all distinct pairs of sets of threads $(T_i, T_j)$, we conclude that to find the minimal number of crossings on the weaving motif, we have to solve the system of equations $(\mathcal{S}_{min})$ of the theorem statement. □

**Remark 5.14.** [21] Considering that two periodic weaving motifs with the same scale or topology contain the same number of crossings, we can construct a minimal diagram on a square unit cell given the fact that $(|a|+|b|)$ parallel segments that do not intersect any corner of the square, correspond to a $(a,b)$-simple closed curve on the torus after identification of the opposite sides of the square. Indeed, by taking $k_l$ (or $\frac{k_l}{2}$ if $k_l$ is an even number divided to be distributed for the four integers of the geometric intersection number equation) simple closed curves of the same slope for each set of threads $T_i$, represented by the pairs $(a_i, b_i)$ and satisfying Theorem 5.13, we can construct a minimal diagram on a square unit cell, whose associated infinite weaving diagrams can be built from the pair $(\Gamma', \Sigma)$, up to isotopy.

The classification tables of the last section of this chapter show examples of minimal diagrams for some simple cases of weaving diagrams. However, as discussed in the previous subsection, non-equivalent weaving diagrams can be constructed from the same pair $(\Gamma', \Sigma)$. At this point, we can conclude about their crossing number and their minimal diagram.

**Remark 5.15.** [21] The total crossing number of a doubly periodic untwisted $(p, q)$-weave is given by the following,

- if a set of crossing matrices, corresponding to a minimal diagram constructed by Remark 5.14 is unique up to equivalence, then the associated weaving diagram built from the pair $(\Gamma', \Sigma)$ is unique and has its crossing number given by Theorem 5.13,

- if a non-equivalent set of crossing matrices is found, it means that there exist at least two non-equivalent weaving diagrams generated from a same couple $(\Gamma', \Sigma)$. One of these diagrams has its crossing number given by Theorem 5.13 and a minimal (unit) cell constructed using Remark 5.14. The crossing numbers of the other non-equivalent weaving diagrams are given by solving the system $(\mathcal{S}_{min})$ of Theorem 5.13 with the next smallest solutions, such that the corresponding weaving diagrams generated by Remark 5.14 have associated crossing-matrices that are not equivalent to the ones given for the previous smallest solutions of $(\mathcal{S}_{min})$.

We can illustrate this method with the same example of basket and twill square $(+2, -2)$ weaving diagrams of Figure 5.1.

**Example 5.16.** [21] The minimal diagram for the twill case is obtained with one representative simple closed curve for each set of thread. Recall that each of these curves must have four crossings, two over and two under, so its crossing number is $\mathcal{C} = 4$. This can be done with a $(2, 1)$-curve for one set and a $(-2, 1)$-curve for the other set. For the basket case, we find that $k_{1,2} = 2$ is the next smallest solution of $(\mathcal{S}_{min})$, with the four integers satisfying this relation and such that their absolute value is minimal, given by two $(1, 1)$-curve for one set and two $(-1, 1)$-curve for the other set. Finally since we can organize the crossings on an associated unit cell with eight crossings, such that the corresponding crossing matrix is not equivalent to the one of the twill case, we can confirm that the crossing number of the plain square $(+2, -2)$ weaving diagram is 8.

A sub-classification for each regular projection associated to a set of crossing sequences can be done by solving the system $(\mathcal{S}_{min})$ in Theorem 5.13, with different values for the multipliers $k_l$, starting from the smallest possible, and by studying the equivalence class of the set of crossing matrices.

### 5.2.2 The A-triangle number

Finally, we take the opportunity to introduce a last weaving invariant for doubly periodic untwisted $(p, q)$-weave in this chapter. We have seen in the second chapter that we can characterize the notion of entanglement at the diagrammatic scale by considering the existence of blocking crossings. We can therefore study the invariance of the number of such blocking crossings in equivalent minimal diagrams. In the case of a weave with $N \geq 3$ sets of threads, we notice that it implies the existence of *alternating triangles*, which we call *A-triangle*, in the weaving diagram. Such an A-triangle is locally defined by three threads $t_i \in T_i$, $t_j \in T_j$ and $t_k \in T_k$, with $i, j, k \in \{1, \cdots, N\}$ distinct, and three closest neighboring crossings of different types, such that $t_i$ is over $t_j$ and under $t_k$, and $t_j$ is over $t_k$, or conversely up to the indexes. Recall that two minimal diagrams of the same doubly periodic untwisted $(p, q)$-weave have the same number of crossings. Therefore, we can prove the following result with the same strategy used to prove Theorem 5.9.

**Proposition 5.17** (The A-triangles Number)**.** [21] *Let $D_{W_1}$ and $D_{W_2}$ be two minimal weaving motifs of doubly periodic untwisted $(p, q)$-weaves with $N \geq 3$ sets of threads. Then, they have the same number of A-triangles if and only if every Reidemeister moves of type $\Omega_3$ that occur at a crossing belonging to a A-triangle cross the complete A-triangle.*

*Proof.* The proof follows the same strategy used to prove Theorem 5.9. The difference concerns Reidemeister moves of type $\Omega_3$. We denote by $K$ the number of A-triangles in $D_{W_1}$. If an $\Omega_3$ move happens at a crossing that does not belong to a A-triangle, then $K$ is obviously not affected. However, if it occurs at a crossing belonging to a $A - triangle$, there are two possibilities. Firstly, the thread lies in the interior of the A-triangle after the $\Omega_3$ move. It generates a new triangle that is not alternating, meaning that $K$ decreases, which is a contradiction. Secondly, the thread crosses the A-triangle after a $\Omega_3$ move, meaning that it is in the exterior of the A-triangle. In this case, $K$ remains unchanged. □

## 5.3 Examples of Classification tables

We end this chapter with an illustration of our main results by constructing and classifying some simple examples of square and kagome weaving motifs by hand [21]. We hope that these tables could be filled with more complex structures in the future with the help of a computer program.

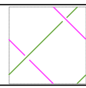
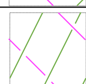
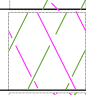
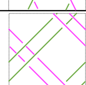
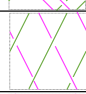
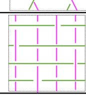
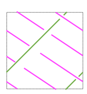
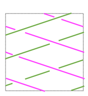
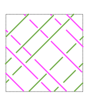
| CLASSIFICATION SQUARE WEAVING DIAGRAMS: N = 2 | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Set of Crossing Sequences** | **Crossing number (Writhe)** | **Minimal Diagram** | **Set of Crossing Matrices** | **Matrices** | **Number of Crossings by S.C.C.** | **Number of S.C.C. for Each Set on the Minimal Diagram** | **Name** |
| {(1,1)} | 2 (0) |  | $\left\{\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}\right\}$ | Rank = 1 | 2 | 1 *Ex: (1,1) and (-1,1)* | Plain Square Weaving (1,1) |
| {(2,1)} | 3 (1) |  | $\left\{\begin{pmatrix} +1 & +1 & -1 \\ +1 & -1 & +1 \\ -1 & +1 & +1 \end{pmatrix}\right\}$ | Rank = 3 "Diagonal configuration" | 3 | 1 *Ex: (2,1) and (-1,1)* | Twill Square Weaving (1,1) |
| {(2,2)} | 4 (0) |  | $\left\{\begin{pmatrix} +1 & +1 & -1 & -1 \\ -1 & +1 & +1 & -1 \\ -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 \end{pmatrix}\right\}$ | Rank = 2 "Diagonal configuration" | 4 | 1 *Ex: (2,1) and (-2,1)* | Twill Square Weaving (2,2) |
| {(2,2)} | 8 (0) |  | $\left\{\begin{pmatrix} +1 & +1 & -1 & -1 \\ +1 & +1 & -1 & -1 \\ -1 & -1 & +1 & +1 \\ -1 & -1 & +1 & +1 \end{pmatrix}\right\}$ | Rank = 1 | 4 | 2 *Ex: (1,1) and (-1,1)* | Plain Square Weaving (2,2) |
| {(3,1)} | 4 (2) |  | $\left\{\begin{pmatrix} +1 & +1 & +1 & -1 \\ +1 & +1 & -1 & +1 \\ +1 & -1 & +1 & +1 \\ -1 & +1 & +1 & +1 \end{pmatrix}\right\}$ | Rank = 4 "Diagonal configuration" | 4 | 1 *Ex: (2,1) and (-2,1)* | Twill Square Weaving (3,1) |
| {(3,1)} | 16 (8) |  | $\left\{\begin{pmatrix} +1 & +1 & -1 & +1 \\ -1 & +1 & +1 & +1 \\ +1 & +1 & +1 & -1 \\ +1 & -1 & +1 & +1 \end{pmatrix}\right\}$ | Rank = 4 | 4 | 4 *Ex: (1,1) and (-1,1)* | Satin Square Weaving (3,1) |
| {(3,2)} | 5 (1) |  | $\left\{\begin{pmatrix} +1 & +1 & +1 & -1 & -1 \\ +1 & +1 & -1 & -1 & +1 \\ +1 & -1 & -1 & +1 & +1 \\ -1 & -1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 & -1 \end{pmatrix}\right\}$ | Rank = 5 "Diagonal configuration" | 5 | 1 *Ex: (1,1) and (-3,2)* | Twill Square Weaving (3,2) |
| {(3,3)} | 6 (0) |  | $\left\{\begin{pmatrix} +1 & +1 & +1 & -1 & -1 & -1 \\ +1 & +1 & -1 & -1 & -1 & +1 \\ +1 & -1 & -1 & -1 & +1 & +1 \\ -1 & -1 & -1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 & +1 & -1 \\ -1 & +1 & +1 & +1 & -1 & -1 \end{pmatrix}\right\}$ | Rank = 3 "Diagonal configuration" | 6 | 1 *Ex: (3,1) and (-3,1)* | Twill Square Weaving (3,3) |
| {(3,3)} | 18 (0) |  | $\left\{\begin{pmatrix} +1 & +1 & +1 & -1 & -1 & -1 \\ +1 & +1 & +1 & -1 & -1 & -1 \\ +1 & +1 & +1 & -1 & -1 & -1 \\ -1 & -1 & -1 & +1 & +1 & +1 \\ -1 & -1 & -1 & +1 & +1 & +1 \\ -1 & -1 & -1 & +1 & +1 & +1 \end{pmatrix}\right\}$ | Rank = 1 | 6 | 3 *Ex: (1,1) and (-1,1)* | Plain Square Weaving (3,3) |

Figure 5.4: Classification table of square weaves (1/2) [21].

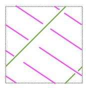
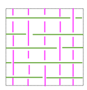
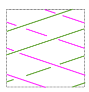
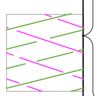
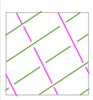
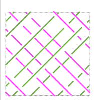
| CLASSIFICATION SQUARE WEAVING DIAGRAMS: N = 2 | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Set of Crossing Sequences** | **Crossing number (Writhe)** | **Minimal Diagram** | **Set of Crossing Matrices** | **Matrices** | **Number of Crossings by S.C.C.** | **Number of S.C.C. for Each Set on the Minimal Diagram** | **Name** |
| {(4,1)} | 5<br>(3) |  | $\left\{\begin{pmatrix} +1 & +1 & +1 & +1 & -1 \\ +1 & +1 & +1 & -1 & +1 \\ +1 & +1 & -1 & +1 & +1 \\ +1 & -1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 & +1 \end{pmatrix}\right\}$ | Rank = 5<br>"Diagonal configuration" | 5 | 1<br>*Ex: (1,1) and (-3,2)* | Twill Square Weaving (4,1) |
| {(4,1)} | 25<br>(15) |  | $\left\{\begin{pmatrix} +1 & +1 & +1 & +1 & -1 \\ +1 & -1 & +1 & +1 & +1 \\ +1 & +1 & +1 & -1 & +1 \\ -1 & +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & +1 & -1 \end{pmatrix}\right\}$ | Rank = 5 | 5 | 5<br>*Ex: (1,1) and (-1,1)* | Satin Square Weaving (4,1) |
| {(4,2)} | 6<br>(2) |  | $\left\{\begin{pmatrix} +1 & +1 & +1 & +1 & -1 & -1 \\ +1 & +1 & +1 & -1 & -1 & +1 \\ +1 & +1 & -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 & +1 & -1 \end{pmatrix}\right\}$ | Rank = 5<br>"Diagonal configuration" | 6 | 1<br>*Ex: (3,1) and (-3,1)* | Twill Square Weaving (4,2) |
| {(4,3)} | 7<br>(1) |  | $\left\{\begin{pmatrix} +1 & +1 & +1 & +1 & -1 & -1 & -1 \\ +1 & +1 & +1 & -1 & -1 & -1 & +1 \\ +1 & +1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & -1 & +1 & +1 & +1 \\ -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 & +1 & +1 & -1 \\ -1 & +1 & +1 & +1 & +1 & -1 & -1 \end{pmatrix}\right\}$ | Rank = 7<br>"Diagonal configuration" | 7 | 1<br>*Ex: (4,1) and (-3,1)* | Twill Square Weaving (4,3) |
| {(4,4)} | 8<br>(0) |  | $\left\{\begin{pmatrix} +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 \\ +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 \\ -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ -1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 \\ -1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 \\ -1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 \end{pmatrix}\right\}$ | Rank = 4<br>"Diagonal configuration" | 5 | 1<br>*Ex: (2,1) and (-3,2)* | Twill Square Weaving (4,4) |
| {(4,4)} | 32<br>(0) |  | $\left\{\begin{pmatrix} +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ -1 & -1 & -1 & -1 & +1 & +1 & +1 & -1 \\ -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \end{pmatrix}\right\}$ | Rank = 1 | 5 | 4<br>*Ex: (1,1) and (-1,1)* | Plain Square Weaving (4,4) |

Figure 5.5: Classification table of square weaves (2/2) [21].

**CLASSIFICATION KAGOME WEAVING DIAGRAMS: N = 3** *(each set meet the 2 others)*

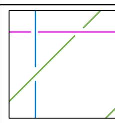
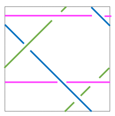
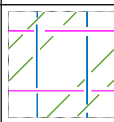
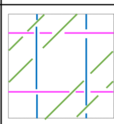
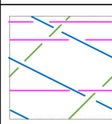
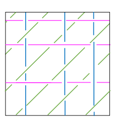
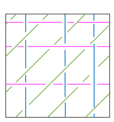
| Set of Crossing Sequences | Crossing number (Writhe) | Minimal Diagram | Set of Crossing Matrices | Matrices | Number of Crossings by S.C.C. (pair) | Number of S.C.C. for Each Set on the Minimal Diagram | Name |
|---|---|---|---|---|---|---|---|
| {(1,0) , (1,0) , (1,0)} | 3<br>(1) |  | [1] ; [1] ; [1] | Rank = 1<br>Rank = 1<br>Rank = 1 | ≥ 2<br>(1) | 1<br>*Ex: (1,0), (0,1) and (1,1)* | Kagome Weaving $(1,0)_3$ |
| {(1,0) , (1,0) , (1,1)} | 6<br>(0) |  | [1] ; [1] ; $\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix}$ | Rank = 1<br>Rank = 1<br>Rank = 1 | ≥ 2 for (1,0)<br>≥ 4 for (1,1)<br>(1 for (1,0) and 2 for (1,1)) | *1 for 2 sets and*<br>*2 for the last set*<br>*Ex: (1,0), (1,1) and (-1,1)* | Kagome Weaving $(1,0)_2$ , (1,1) |
| {(1,0) , (1,1) , (1,1)} | 12<br>(4) |  | [1] ; $\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix}$ ; $\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix}$ | Rank = 1<br>Rank = 1<br>Rank = 1 | ≥ 2 for (1,0)<br>≥ 4 for (1,1)<br>(1 for (1,0) and 2 for (1,1)) | 2<br>*Ex: (1,0), (0,1) and (1,1)* | Kagome Weaving (1,0) , $(1,1)_2$ |
| {(1,1) , (1,1) , (1,1)} | 12<br>(0) |  | $\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix}$ ; $\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix}$ ; $\begin{bmatrix}+1 & +1\\ +1 & +1\end{bmatrix}$ | Rank = 1<br>Rank = 1<br>Rank = 1 | ≥ 4<br>(2) | 2<br>*Ex: (1,0), (0,1) and (1,1)* | Kagome Weaving $(1,1)_3$ |
| {(1,0) , (1,0) , (2,1)} | 9<br>(1) |  | [1] ; [1] ; $\begin{pmatrix}+1 & +1 & -1\\ +1 & -1 & +1\\ -1 & +1 & +1\end{pmatrix}$ | Rank = 1<br>Rank = 1<br>Rank = 3<br>"Diagonal" | ≥ 2 for (1,0)<br>≥ 6 for (1,1)and(-2,1)<br>(1 for (1,0) and 3 for (1,1)and(-2,1)) | *1 for 2 sets and*<br>*3 for the last set*<br>*Ex: (1,0), (1,1) and (-2,1)* | Kagome Weaving $(1,0)_2$ , (2,1) |
| {(1,0) , (2,1) , (2,1)} | 27<br>(9) |  | [1] ; $\begin{pmatrix}+1 & +1 & -1\\ +1 & -1 & +1\\ -1 & +1 & +1\end{pmatrix}$ ; $\begin{pmatrix}+1 & -1 & +1\\ +1 & +1 & -1\\ -1 & +1 & +1\end{pmatrix}$ | Rank = 1<br>Rank = 3<br>Rank = 3<br>"Diagonal" | ≥ 2 for (1,0)<br>≥ 4 for (1,1)<br>(1 for (1,0) and 2 for (1,1)) | 3<br>*Ex: (1,0), (0,1) and (1,1)* | Kagome Weaving (1,0) , $(2,1)_2$ |
| {(2,1) , (2,1) , (2,1)} | 27<br>(3) |  | $\begin{pmatrix}+1 & +1 & -1\\ -1 & +1 & +1\\ +1 & -1 & +1\end{pmatrix}$ ; $\begin{pmatrix}+1 & +1 & -1\\ +1 & -1 & +1\\ -1 & +1 & +1\end{pmatrix}$ ; $\begin{pmatrix}+1 & -1 & +1\\ +1 & +1 & -1\\ -1 & +1 & +1\end{pmatrix}$ | Rank = 3<br>Rank = 3<br>Rank = 3<br>"Diagonal" | ≥ 4<br>(2) | 3<br>*Ex: (1,0), (0,1) and (1,1)* | Kagome Weaving $(2,1)_3$ |

Figure 5.6: Classification table of kagome weaves [21].

# Perspectives

In this thesis, we introduced an original topological description of *weaves*, which are well-known complex entangled structures for materials scientists but need further development in mathematics. We used arguments from many fields such as knot theory, tiling theory, or combinatorics to describe, construct and classify different families of weaves. In particular, we benefited from diagrammatical representations in the plane of these three-dimensional objects to facilitate their study.

First, we began this thesis with a new topological *definition* of a weave as the lift to the thickened Euclidean plane of a quadrivalent planar connected graph made of infinite colored straight lines, with an over or under information at each vertex. This defines our first class of *untwisted weaves*. Then, by applying local surgeries to make two closest neighboring curves twist, we defined the class of *twisted weaves*. Continuous deformations of these structures are considered to study *equivalence classes* of weaves. However, we are more interested in working with their planar version, called a *weaving diagram*. In the particular case of doubly periodic weaves, a generating cell of a corresponding weaving diagram is called a *weaving motif* and is a particular type of link diagram embedded on a torus.

Further, we define a new systematic method to *construct* weaving motifs, by applying *polygonal link* methods to periodic tilings of the plane, which consist in covering the vertices and edges of a generating cell of a tiling by polygons with strands that cross each according to specific rules. However, since these methods do not always generate weaving motifs, we developed a strategy to *predict* whether a polygonal link method applied on a given tiling will generate a weaving motif or not, using algebraic and combinatorial arguments.

Next, we started to consider the *classification* of doubly periodic weaves according to *topological invariant*. We began with the generalization of Tait's first and second conjectures for alternating periodic weaves, which gave us two classic important invariants to classify not only weaving motifs on a torus but also on higher genus surfaces, namely the *crossing number* and the *writhe*. Then, we focused on the specific class of doubly periodic untwisted $(p, q)$-weaves, which is characterized by the simplest type of crossing information. We first noticed that the combinatorial description of a such a weave from a graph with crossing information was not enough to define a weave uniquely, which motivated the construction of a new weaving invariant using binary *crossing matrices* to characterize the different *equivalence classes*. Then, we challenged the open problem of computing the minimal number of crossings, which is also called *crossing number* for this class of weaves, and found an interesting formula. However, the computation by hand is still not trivial.

From now there are still many open questions that we already started to consider, more or less deeply.

- Study the symmetries of doubly periodic weaves.
- Construct more complex weaves and study their algebraic structure using braid theory.
- Defining a group structure on weaving motifs, under particular boundary conditions, to construct more complex weaves.
- Generalize other knot invariants to weaves, such as the stick number, and explore new invariants that have a physical meaning, such as the linking number in [59].
- Extend the theory of doubly periodic weaves to triply periodic weaves.

# Bibliography


[1] C. Adams. The knot book: an elementary introduction to the mathematical theory of knots. (American Mathematical Society, Providence, 2004).

[2] C. Adams, C. Albors-Riera, B. Haddock, Z. Li, D. Nishida, B. Reinoso, L. Wang. Hyperbolicity of links in thickened surfaces. *Topology and Its Applications.* **256** (2019), 262–278.

[3] C. Adams, A. Calderon, N. Mayer. Generalized bipyramids and hyperbolic volumes of alternating k-uniform tiling links. *Topology and Its Applications.* **271** (2020),107045.

[4] C. Adams, T. Fleming, M. Levin, and A. Turner. Crossing number of alternating knots in S x I. *Pac. J. Math.* **203** (2002), 1–22.

[5] H.U. Boden and H. Karimi. The Jones-Krushkal polynomial and minimal diagrams of surface links. *arXiv:1908.06453v1.* (2019).

[6] M. Bright, V. Kurlin. Encoding and topological computation on textile structures. *Computers and Graphics.* **90** (2020) 51–61.

[7] G. Brinkmann, M. Deza. Lists of face-regular polyhedra. *J. Chem. Inf. Comput. Sci.* **40** (2000), 530–541.

[8] T. Castle, M.E. Evans, S.T. Hyde. Entanglement of embedded graphs. *Prog. Theor. Phys. Suppl.* **191** (2011), 235–244.

[9] A. Champanerkar, I. Kofman, J.S. Purcell. Volume bounds for weaving knots. *Algebr. Geom. Topol.* **16** (2016), 3301–3323.

[10] A. Champanerkar, I. Kofman, J.S. Purcell. Geometrically and diagrammatically maximal knots. *J. London Math. Soc.* **94** (2016), 883–908.

[11] J.H. Conway, H. Burgiel. C. Goodman-Strauss. The symmetries of things. (A K Peters/CRC Press, 2016).

[12] P.R. Cromwell. Knots and Links, 1st ed. (Cambridge University Press, 2004).

[13] S. De Toffoli, V. Giardino. Forms and Roles of Diagrams in Knot Theory. *Erkenn.* **79** (2014), 829–842.

[14] A.-M. Décaillot. Géométrie des tissus. Mosaïques. Échiquiers. Mathématiques curieuses et utiles. *Revue d'hist. Math.* **8** (2002), 145–206.

[15] T. Endo, T. Itoh, K. Taniyama. A graph-theoretic approach to a partial order of knots and links. *Topology and Its Applications.* **157** (2010), 1002–1010.

[16] M.E. Evans, V. Robins and S.T. Hyde. Periodic entanglement I: networks from hyperbolic reticulations. *Acta Cryst.* A**69** (2013), 241–261.

[17] M.E. Evans, V. Robins and S.T. Hyde. Periodic entanglement II: weavings from hyperbolic line patterns. *Acta Cryst.* A**69** (1984), 262–275.

[18] M.E. Evans and S.T. Hyde. Periodic entanglement III: tangled degree-3 finite and layer net intergrowths from rare forests. *Acta Cryst.* A**71** (2015),599–611.

[19] B. Farb and D. Margalit. A primer on mapping class groups. (Princeton University Press, Princeton, 2012).

[20] T. Fleming. Strict Minimality of Alternating Knots in S × I. *J. Knot Theory Its Ramif.* **12** (2003),445–462.

[21] M. Fukuda, M. Kotani and S. Mahmoudi. Classification of doubly periodic untwisted $(p, q)$-weaves by their crossing number. *arXiv:2202.01755* (2022).

[22] M. Fukuda, M. Kotani and S. Mahmoudi. Construction of weaving and polycatenanes motifs from periodic tilings of the plane. *arXiv:2206.12168* (2022).

[23] S. Grishanov, V. Meshkov and A. Omelchenko. A topological study of textile structures. Part I: an introduction to topological methods. *Text. Res. J.* **79** (2009),702–713.

[24] S. Grishanov, V. Meshkov and A. Omelchenko. A topological study of textile structures. Part II: topological invariants in application to textile structures. *Text. Res. J.* **79** (2009),822–836.

[25] S.A. Grishanov, V.R. Meshkov, A.V. Omel'Chenko. Kauffman-type polynomial invariants for doubly periodic structures. *J. Knot Theory Ramifications.* **16** (2007), 779–788.

[26] S.A. Grishanov, V.R. Meshkov, V.A. Vassiliev. Recognizing textile structures by finite type knot invariants. *J. Knot Theory Ramifications.* **18** (2009), 209–235.

[27] S.A. Grishanov, V.A. Vassiliev. Invariants of links in 3-manifolds and splitting problem of textile structures. *J. Knot Theory Ramifications.* **20** (2011), 345–370.

[28] B. Grünbaum and G.C. Shephard. A catalogue of isonemal fabrics. *in Discrete Geometry and Convexity (J. E. Goodman et al., eds); Ann. New York Acad. Sci.* **440** (1985), 279–298.

[29] B. Grünbaum and G.C. Shephard. An extension to the catalogue of isonemal fabrics. *Discrete Math.* **60** (1986), 155–192.

[30] B. Grünbaum and G.C. Shephard. Isonemal fabrics. *The American Math. Monthly.* **95** (1988), 5–30.

[31] B. Grünbaum and G.C. Shephard. Tilings and Patterns (second edition). (Dover Publications, New York, 2016).

[32] B. Grünbaum and G.C. Shephard. Satins and Twills: an introduction to the geometry of fabrics. *Math. Mag.* **53** (1980), 139–161.

[33] G. Hu, X.-D. Zhai, D. Lu and W.-Y. Qiu. The architecture of Platonic polyhedral links. *J. Math. Chem.* **46** (2009), 592–603.

[34] S.T. Hyde, B. Chen and M. O'Keeffe. Some equivalent two-dimensional weavings at the molecular scale in 2D and 3D metal–organic frameworks. *CrystEngComm.* **18** (2016), 7607–7613.

[35] S.T. Hyde, M.E. Evans. Symmetric tangled Platonic polyhedra. *Proc. Natl. Acad. Sci. U.S.A.* **119** (2022), e2110345118.

[36] S. Ikeda, M. Kotani. A New Direction in Mathematics for Materials Science. (Springer Japan, Tokyo, 2015).

[37] N. Kamada. Span of the Jones polynomial of an alternating virtual link. *Algebr. Geom. Topol.* **4** (2004), 1083–1101.

[38] L. H. Kauffman. Introduction to virtual knot theory. *J. Knot Theory Ramifications.* **21** (2012), 1240007.

[39] L. H. Kauffman. State models and the jones polynomial. *Topology.* **26** (1987), 395–407.

[40] A. Kawauchi. Complexities of a knitting pattern. *Reactive and Functional Polymers.* **131** (2018) 230–236.

[41] K. Kobata, T. Tanaka. A circular embedding of a graph in Euclidean 3-space. *Topology and Its Applications.* 157 (2010), 213–219.

[42] S. Lambropoulou. Diagrammatic Representations of Knots and Links as closed braids. *ArXiv:1811.11701.* (2018).

[43] A. Lavasani, G. Zhu and M. Barkeshli. Universal logical gates with constant overhead: instantaneous Dehn twists for hyperbolic quantum codes. *Quantum.* **3** (2019), 180.

[44] W.B.R. Lickorish. An introduction to knot theory. (Springer New York, New York, 1997).

[45] Y. Liu, M. O'Keeffe, M.M.J. Treacy and O.M. Yaghi. The geometry of periodic knots, polycatenanes and weaving from a chemical perspective: a library for reticular chemistry. *Chem. Soc. Rev.* **47** (2018), 4642–4664.

[46] P.R. Lord and M.H. Mohamed. An historical introduction to weaving. in: Weaving, (Elsevier, 1982), 1–16.

[47] D. Lu, G. Hu, Y.-Y. Qiu and W.-Y. Qiu. Topological transformation of dual polyhedral links. *MATCH Commun. Math. Comput. Chem.* **63** (2010), 67–78.

[48] S. Mahmoudi. The Tait First and Second Conjecture for Alternating Weaving Diagrams. *arXiv:2009.13896* (2021).

[49] W.W. Menasco, M.B. Thistlethwaite. The Classification of Alternating Links. *The Annals of Mathematics* **138** (1993), 113.

[50] W.W. Menasco, M.B. Thistlethwaite. Handbook of knot theory. (Elsevier, Amsterdam Boston, 2005).

[51] W.W. Menasco, M.B. Thistlethwaite. The Tait flyping conjecture. *Bull. Amer. Math. Soc.* **25** (1991), 403–413.

[52] E.D. Miro, A. Garciano and A. Zambrano. From colorings to weavings. *Acta Phys. Pol. A.* **126** (2014), 560–563.

[53] E.D. Miro, A. Zambrano and A. Garciano. Construction of weavings in the plane. *Acta Cryst.* A**74** (2018), 25–35.

[54] H.R. Morton, S.A. Grishanov. Doubly periodic textile structures. *J. Knot Theory Ramifications.* **18** (2009), 1597–1622.

[55] K. Murasugi. Jones polynomials and classical conjectures in knot theory. *Topology.* **26** (1987), 187–194.

[56] K. Murasugi. Jones polynomials and classical conjectures in knot theory. II. *Math. Proc. Camb. Philos. Soc.* **102** (1987) 317–318.

[57] K. Murasugi. Knot theory and its applications (Birkhäuser, Boston 2008).

[58] V.M. Ngo, S. Helmer, N.-A. Le-Khac, M.-T. Kechadi. Structural textile pattern recognition and processing based on hypergraphs. *Inf Retrieval J.* **24** (2021) 137–173.

[59] E. Panagiotou, K.C. Millett, S. Lambropoulou, The linking number and the writhe of uniform random walks and polygons in confined spaces. *J. Phys. A: Math. Theor.* **43** (2010), 045208.

[60] W.-Y. Qiu, X.-D. Zhai and Y.-Y. Qiu. Architecture of Platonic and Archimedean polyhedral links. *J. Math. Chem.* **51** (2008), 13–18.

[61] A. Schrijver. Tait's flyping conjecture for well-connected links. *J. of Combinatorial Theory, Series B.* **58** (1993) 65–146.

[62] S. Sekhri. Textbook of Fabric Science : Fundamentals to Finishing (third edition). (PHI Learning Pvt. Ltd., 2019).

[63] K. Shimokawa, K. Ishihara, Y. Tezuka. Topology of polymers. (Springer Nature, Tokyo, 2019).

[64] R. Stong, The Jones polynomial of parallels and applications to crossing number, Pac. J. Math. 164 (1994) 383–395.

[65] P.G. Tait. On Knots I, II, III. in: Sci. Pap. Vol I, (Cambridge University Press, London, 1898), 273–347.

[66] M.B. Thistlethwaite. Kauffman's polynomial and alternating links. *Topology.* **27** (1988), 311–318.

[67] B. Thompson and S.T. Hyde. A theoretical schema for building weavings of nets via colored tilings of two-dimensional spaces and some simple polyhedral, planar and three-periodic examples. *Isr. J. Chem.* **58** (2018), 1144–1156.

[68] P. Wadekar, C. Amanatides, L. Kapllani, G. Dion, R. Kamien, D.E. Breen. Geometric modeling of complex knitting stitches using a bicontinuous surface and its offsets. *Comput. Aided Geometric Design.* **89** (2021) 102024.

[69] P. Wadekar, P. Goel, C. Amanatides, G. Dion, R.D. Kamien, D.E. Breen, Geometric modeling of knitted fabrics using helicoid scaffolds, *J. of Engineered Fibers and Fabrics.* **15** (2020) 155892502091387.

[70] N. Wakayama, K. Shimokawa. On the classification of polyhedral links. *Submitted.* (2022).

[71] Y. Wu. Jones polynomial and the crossing number of links. in: Differ. Geom. Topol., (Springer Berlin Heidelberg, Berlin, 1989), 286–288.

[72] A. Zambrano, E.D. Miro and A. Garciano. Equivalent weavings. *J. Phys.: Conf. Ser.* **893** (2017), 012007.