博士論文

The category of modules of the triplet W-algebras associated to the Virasoro minimal models

(ヴィラソロ極小モデルに付随する
トリプレットW代数の加群の圏)

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Dissertation

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Abstract

We study the structure of the category of modules over the triplet W-algebras $\mathcal{W}_{p_+,p_-}$ and $\mathcal{SW}(m)$, where the former was defined by Feigin, Gainutdinov, Semikhatov and Tipunin [22], and the latter by Adamović and Milas [2]. Since $\mathcal{W}_{p_+,p_-}$ and $\mathcal{SW}(m)$ satisfy the C_2-cofinite condition, by a series of papers by Huang, Lepowsky and Zhang [36, 37, 38, 39, 40, 41, 42, 43, 44], every simple module has the projective cover and the module categories have the structure of a braided tensor category. In the case of the triplet W-algebra $\mathcal{W}_{p_+,p_-}$, we determine the structure of the projective covers of all simple $\mathcal{W}_{p_+,p_-}$-modules and determine certain non-semisimple fusion rules conjectured by Rasmussen [60] and Gaberdiel, Runkel and Wood [32]. In the case of the triplet W-algebra $\mathcal{SW}(m)$, we determine the structure of the projective covers of all simple $\mathcal{SW}(m)$-modules and prove that, as a tensor category, $\mathcal{SW}(m)$ is rigid. Furthermore we show that a certain non-semisimple fusion ring of $\mathcal{SW}(m)$ can be derived from the non-semisimple fusion ring of the triplet W-algebra $\mathcal{W}_p$ [64].

Contents

Chapter 1

Introduction

We have studied the structure of the category of modules of the family of vertex operator algebras called triplet W-algebras. In this thesis, we mainly discuss the triplet W-algebras $\mathcal{W}_{p_+,p_-}$ constructed by Feigin, Gainutdinov, Semikhatov and Tipunin and the super triplet W-algebras $\mathcal{SW}(m)$ constructed by Adamović and Milas. These triplet W-algebras are one of the few examples of non-rational vertex operator algebras satisfying the C_2-cofinite condition. In general, for any rational vertex operator algebra, the abelian category of modules is semisimple, but for any non-rational vertex operator algebra, the abelian category of modules of the vertex operator algebra is not semisimple and contains logarithmic modules whose L_0 nilpotent rank $n \geq 2$, where L_0 is the zero mode of the Virasoro algebra. Furthermore, if the C_2-cofinite condition is satisfied, the number of simple modules is finite, and the category of modules has braided tensor category structure as developed in the series of papers by Huang, Lepowsky and Zhang [37, 38, 39, 40, 41, 42, 43, 44]. Thus, the triplet W-algebras are mathematically tractable among the non-rational vertex operator algebras, but specific aspects such as the structure of logarithmic modules and tensor products among logarithmic modules are still not fully understood.

In the following, we will give a brief description of the research background and problems related to triplet W-algebras.

First let us review triplet W-algebras associated to Virasoro minimal models. Let $p \in \mathbb{Z}_{\geq 1}$ and let $p_- > p_+ \geq 2$ be coprime integers. Let

$$c_p = 1 - 6\frac{(1-p)^2}{p}, \qquad c_{p_+,p_-} = 1 - 6\frac{(p_+ - p_-)^2}{p_+p_-}$$

be the minimal central charges for the Virasoro algebra. A well-known example of a irrational C_2-cofinite vertex operator algebra with these central charges c_p and c_{p_+,p_-} are the triplet W-algebras $\mathcal{W}_p$ and $\mathcal{W}_{p_+,p_-}$, respectively. The former was defined by Kausch [52] (see also [1],[24],[25], [58]), and the latter by Feigin, Gainutdinov, Semikhatov and Tipunin [22]. Let $\mathcal{C}_p$ and $\mathcal{C}_{p_+,p_-}$ be the category of modules of $\mathcal{W}_p$ and $\mathcal{W}_{p_+,p_-}$, respectively. The structure of the category $\mathcal{C}_p$ has been studied in detail in recent studies [3], [56], [58], [64] and is known to be rigid as tensor category. Furthermore $\mathcal{W}_p$-mod is shown to be ribbon tensor equivalent to the category of modules of the restricted quantum group $\overline{U}_q(sl_2)$ [34]. On the other hand, mathematical studies on the category $\mathcal{C}_{p_+,p_-}$ were limited to basic results such as the classification of simple modules [4],[5],[65]. From the physics side, Rasmussen [60] and Gaberdiel, Runkel, Wood [32],[33] used the methods of lattice models and Nahm-Gaberdiel-Kausch algorithm [35],[51], respectively, to conjecture the structure of the projective covers of simple modules and certain non-semisimple fusion rules.

Next let us review $N=1$ super triplet W-algebras associated to super Virasoro minimal models. In the papers [2] and [3], Adamović and Milas introduced the $N=1$ super triplet W-algebras $\mathcal{SW}(p,q)$, where p and q are positive integers such that $q > p$ and $(q, \frac{q-p}{2}) = 1$. The vertex operator superalgebras $\mathcal{SW}(p,q)$ are extensions of the minimal super Virasoro models

$$L^{\mathfrak{ns}}(c_{p,q}^{N=1},0) \subset \mathcal{SW}(p,q)$$

where $L^{\mathfrak{ns}}(c_{p,q}^{N=1},0)$ is the Neveu-Schwarz vertex operator superalgebra of central charge

$$c_{p,q}^{N=1} = \frac{3}{2}(1-2\frac{(p-q)^2}{pq}),$$

and are natural super analogs of the triplet W-algebras $\mathcal{W}_p$ and $\mathcal{W}_{p_+,p_-}$. Let $\mathcal{SW}(m) = \mathcal{SW}(1,2m+1)$. Adamović and Milas showed that $\mathcal{SW}(m)$ satisfies C_2-cofinite condition, classified all simple $\mathcal{SW}(m)$-modules and conjectured that the category of $\mathcal{SW}(m)$-modules are equivalent to the category of modules of the small quantum group $U_q^{small}(sl_2)$, $q = e^{\frac{2\pi i}{2m+1}}$. Furthermore they showed that the characters of the simple $\mathcal{SW}(m)$-modules can be expressed in the characters of the simple $\mathcal{W}_p$-modules. Thus, the super triplet W-algebra $\mathcal{SW}(m)$ was expected to have the same interesting properties as $\mathcal{W}_p$, but study on the structure of the projective modules and tensor category remained as problems.

The main results of this thesis are as follows:

- We determine the structure of the projective covers of all simple $\mathcal{W}_{p_+,p_-}$-modules.
- We prove that the projective covers of all simple $\mathcal{W}_{p_+,p_-}$-modules except for minimal simple modules $L(h_{r,s})$ are self-dual, and determine certain non-semisimple fusion rules conjectured and computed in [32],[60],[66].
- We show the rigidity of the quotient category $C^0_{p_+,p_-}$, where $C^0_{p_+,p_-}$ is the quotient of $\mathcal{C}_{p_+,p_-}$ by the Serre subcategory consisting of all minimal simple $\mathcal{W}_{p_+,p_-}$-modules $L(h_{r,s})$.
- We determine the tensor structure of the $\mathcal{SW}(m)$-module category and show that this tensor category is rigid.
- We determine the structure of a certain non-semisimple fusion ring of $\mathcal{SW}(m)$ which is a commutative ring defined on the set of all simple and projective $\mathcal{SW}(m)$-modules, and show that this non-semisimple fusion ring can be derived from the non-semisimple fusion ring of the triplet W-algebra $\mathcal{W}_p$ [64] by specializing one variable.

From the last result, we can expect a deep relationship between the triplet W-algebras and the $N=1$ super triplet W-algebras.

This thesis is organized as follows, where Chapters 3 through 9 are about the triplet W-algebra $\mathcal{W}_{p_+,p_-}$ and Chapters 10 through 12 are about the super triplet W-algebra $\mathcal{SW}(m)$.

In Chapter 2, we review the definitions of vertex operator (super)algebras and concepts such as vertex algebra modules and intertwining operators used in later chapters.

In Chapter 3, we review the structure of Fock modules and the Felder complex in accordance with [65]. The basic facts in this chapter are frequently used in later chapters.

In Chapter 4, we introduce the vertex operator algebra $\mathcal{W}_{p_+,p_-}$ and review some results in [4],[5],[65] briefly. In Section 4.3, we introduce the block decomposition of $\mathcal{C}_{p_+,p_-}$.

Each block of $\mathcal{C}_{p_+,p_-}$ is assigned to one of three groups: $\frac{(p_+-1)(p_--1)}{2}$ thick blocks $C^{thick}_{r,s}$, p_++p_--2 thin blocks $C^{thin}_{r,p_-}, C^{thin}_{p_+,s}$ and two semisimple blocks $C^{\pm}_{p_+,p_-}$. The most complex groups are the thick blocks and each thick block $C^{thick}_{r,s}$ contains five simple modules $\mathcal{X}^+_{r,s}, \mathcal{X}^+_{r^\vee,s^\vee}, \mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r,s^\vee}$ and $L(h_{r,s})$, where $L(h_{r,s})$ is the minimal simple module module of the Virasoro algebra. The thick blocks and the thin blocks contain logarithmic $\mathcal{W}_{p_+,p_-}$-modules on which the Virasoro zero-mode L_0 acts non-semisimply.

In Chapter 5, by gluing lattice irreducible modules $\mathcal{V}^{\pm}_{r,s}$ using the logarithmic deformation by J. Fjeistad et al.[27], we construct logarithmic $\mathcal{W}_{p_+,p_-}$-modules $\mathcal{P}^{\pm}_{r,s}$ and $\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ whose L_0 nilpotent rank three and two, respectively.

In Chapter 6, we determine the structure of certain logarithmic Virasoro modules, which have L_0 nilpotent rank two, and certain Ext^1-groups, by using the results for logarithmic Virasoro modules in [55] and certain logarithmic modules $F(\tau)$ which can be constructed by gluing Fock modules.

In Chapter 7, we determine the projective covers of all $\mathcal{W}_{p_+,p_-}$-simple modules. In this chapter, we study mainly the thick blocks $C^{thick}_{r,s}$. Based on the structure of the logarithmic Virasoro modules determined in Chapter 6, we compute Ext^1 groups between certain indecomposable $\mathcal{W}_{p_+,p_-}$-modules and the simple modules, and show that the logarithmic modules $\mathcal{P}^+_{r,s}$, $\mathcal{P}^+_{r^\vee,s^\vee}$, $\mathcal{P}^-_{r^\vee,s}$ and $\mathcal{P}^-_{r,s^\vee}$ are the projective covers of the simple modules $\mathcal{X}^+_{r,s}$, $\mathcal{X}^+_{r^\vee,s^\vee}$, $\mathcal{X}^-_{r^\vee,s}$ and $\mathcal{X}^-_{r,s^\vee}$, respectively. In Section 7.4, we determine the structure of the projective covers of the minimal simple modules by using the structure of the center of the Zhu-algebra $A(\mathcal{W}_{p_+,p_-})$ determined in [4],[5],[65].

In Chapter 8, we study the structure of the braided tensor category on $\mathcal{C}_{p_+,p_-}$. We introduce indocomposable modules $\mathcal{K}_{r,s}$ and, using methods in [15] and [56], prove the rigidity of $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$ in Theorems 8.3.7 and 8.3.15. Using the rigidity of $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$, we show that the indecomposable modules $\mathcal{K}_{r,s}$, $\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ and $\mathcal{P}^{\pm}_{r,s}$ can be obtained by repeatedly multiplying $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$. As a result we see that all indecomposable modules of $\mathcal{K}_{r,s}$, $\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ and $\mathcal{P}^{\pm}_{r,s}$ are rigid objects. We also determine the tensor product between all simple modules in the process of these proofs.

In Chapter 9, we introduce two commutative rings $P^0(\mathcal{C}_{p_+,p_-})$ and $K^0(\mathcal{C}_{p_+,p_-})$ in accordance with [64], and study the structure of these rings. The latter commutative ring $K^0(\mathcal{C}_{p_+,p_-})$ is the quotient ring of the Grothendieck ring of $\mathcal{C}_{p_+,p_-}$ quotiented by all minimal simple modules. The structure of the quotient ring $K^0(\mathcal{C}_{p_+,p_-})$ is determined in [61] (cf. [22],[60],[66]). The former commutative ring $P^0(\mathcal{C}_{p_+,p_-})$ is defined on the set of all simple modules and all indecomposable modules $\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ and $\mathcal{P}^{\pm}_{r,s}$. Using the structure of $P^0(\mathcal{C}_{p_+,p_-})$, we can compute the tensor product between indecomposable modules $\mathcal{X}_{r,s}$, $\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ and $\mathcal{P}^{\pm}_{r,s}$. For the simple modules in the thick blocks, however, we need to multiply by a factor $\mathcal{K}^*_{1,1}$, as shown in Proposition 8.4.4. In Section 9.3, we introduce a certain quotient category $\mathcal{C}^0_{p_+,p_-}$ of $\mathcal{C}_{p_+,p_-}$ and show that two categories $\mathcal{C}_{p_+}$ and $\mathcal{C}_{p_-}$ are embedded in $\mathcal{C}^0_{p_+,p_-}$.

In Chapter 10, we review facts of the representation theory of $N=1$ Neveu-Schwarz algebra in accordance with [11],[46],[47].

In Chapter 11, we review some basic results on the super triplet W-algebra $\mathcal{SW}(m)$ by Admović and Milas in [2] briefly. Let $\mathcal{SC}_m$ be the abelian category of the super triplet W-algebra $\mathcal{SW}(m)$. Similar to $\mathcal{W}_{p_+,p_-}$, $\mathcal{SC}_m$ has the block decomposition. Each block of $\mathcal{SC}_m$ is assigned to m non-semisimple blocks and one semisimple block. In Section 11.3, we will construct logarithmic $\mathcal{SW}(m)$-modules $\mathcal{SP}^{\pm}_{\bullet}$ in the non-semisimple blocks by using the logarithmic deformation by J. Fjeistad et al.[27].

In Chapter 12, we determine the non-semisimple fusion rules between all simple and projective modules. By using self-duality of $\mathcal{SX}_1^-$, we show that the simple modules and the indecomposable modules $\mathcal{SP}_\bullet^\pm$ can be obtained by repeatedly multiplying $\mathcal{X}_1^-$. As a result, we can determine the structure of all projective modules and show that $\mathcal{SC}_m$ is rigid as a tensor category and equivalent to $U_q^{small}(sl_2)$ as abelian categories. In Section 12.5, we introduce a commutative ring $P(\mathcal{SC}_m)$ and determine the structure of $P(\mathcal{SC}_m)$. Furthermore we show that $P(\mathcal{SC}_m)$ can be obtained from the non-semisimple fusion ring $P(\mathcal{C}_{2m+1})$ of the triplet W-algebra $\mathcal{W}_{2m+1}$ by specializing one variable.

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Chapter 2

Basic definitions and notations of vertex operator algebras

In this chapter we briefly review the definitions of vertex operator (super)algebras and concepts such as vertex algebra modules and intertwining operators used in later chapters. See [14],[29],[30],[49], [50] for details.

2.1 Vertex operator algebras

Definition 2.1.1. *A tuple $(V, |0\rangle, T, Y)$ is called a vertex operator algebra where*

1. *V is a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-vector space*

$$V = \bigoplus_{n=0}^{\infty} V[n].$$

2. *$|0\rangle \in V[0]$ is called* the vacuum vector.

3. *$T \in V[2]$ is called* the conformal vector.

4. *Y is a $\mathbb{C}$-linear map*

$$Y : V \to \mathrm{End}_{\mathbb{C}}(V)[[z, z^{-1}]].$$

These data are subject to the following axioms:

1. $\dim_{\mathbb{C}} V[0] = 1$ *and* $0 < \dim_{\mathbb{C}} V[n] < \infty$ *for any* $n \in \mathbb{Z}_{\geq 0}$.

2. *For each $A \in V[h]$ there exists a* field

$$Y(A; z) = \sum_{n \in \mathbb{Z}} A[n] z^{-n-h}$$

and each field satisfies

$$Y(A; z) |0\rangle - A \in V[[z]]z, \qquad Y(|0\rangle ; z) = \mathrm{id}_V.$$

3. *The field*

$$Y(T; z) = T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

of modes define the commutation relations of the Virasoro algebra *with fixed central charge $c = c_V$*

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c_V}{12}(m^3 - m)\delta_{m+n,0}.$$

The field $T(z)$ is called the energy-momentum tensor.

4. *The zero mode L_0 of $T(z)$ acts semisimply on V and*

$$V[h] = \{\ A \in V \mid L_0 A = hA\ \}.$$

5. *For all $A \in V$*

$$Y(L_{-1}A; z) = \frac{\mathrm{d}}{\mathrm{d}z} Y(A; z).$$

6. *For any fields are local, i.e., there exists $N \geq 0$ (depending on A, B) such that*

$$(z-w)^N [Y(A; z), Y(B; w)] = 0.$$

7. *For homogeneous elements $A \in V[h_A]$ and $B \in V[h_B]$, the fields $Y(A; z)$ and $Y(B; w)$ satisfy* the operator product expansion

$$\begin{aligned} Y(A; z) Y(B; w) &= Y(Y(A; z-w)B; w) \\ &= \sum_{n \in \mathbb{Z}} Y(A[n]B; w)(z-w)^{-n-h_A}. \end{aligned}$$

Definition 2.1.2. *Let V be a vertex operator algebra and let $A \in V[h_A]$ and $B \in V[h_B]$ be homogeneous elements. The holomorphic part of $Y(A; z)Y(B; w)$ at $z = w$ is given by the following* the normal ordered product

$$\begin{aligned} &: Y(A; z) Y(B; w) : \\ &:= \sum_{n \in \mathbb{Z}} \Big\{ \sum_{p + h_A - 1 < 0} A[p] z^{-p-h_A} B[n] z^{-p-h_A} + \sum_{p + h_A - 1 \geq 0} B[n] A[p] z^{-p-h_A} \Big\} w^{-n-h_B}. \end{aligned}$$

We abbreviate the holomorphic part $: Y(A; z)Y(B; w) :$ of the operator product expansion of $Y(A; z)Y(B; w)$ as $\cdots$.

Remark 2.1.3. *The operator product expansion and the normal ordered product can be defined similarly for field not belonging to vertex operator algebras. See [29] and [50] for more details.*

Definition 2.1.4. *Given a vertex operator algebra $(V, |0\rangle, T, Y)$, a weak V-module is a pair (M, Y_M) of a vector space M and a linear map Y_M from V to $\mathrm{End} M[[z, z^{-1}]]$ satisfying the following conditions*

1. *$Y_M(|0\rangle; z) = \mathrm{Id}_M$ and the Fourier modes of $Y_M(T; z) = \sum_{n \in \mathbb{Z}} L_n^M z^{-n-2}$ satisfy the commutation relations of the Virasoro algebra with the central charge c_V.*

2. *For all $A \in V$,*

$$Y_M(L_{-1}A; z) = \frac{\mathrm{d}}{\mathrm{d}z} Y_M(A; z).$$

3. *For $A, B \in V$, the following Jacobi identity holds*

$$\begin{aligned} z_0^{-1} \delta\Big(\frac{z_1 - z_2}{z_0}\Big) Y_M(A; z_1) Y_M(B; z_2) - z_0^{-1} \delta\Big(\frac{z_2 - z_1}{-z_0}\Big) Y_M(B; z_2) Y_M(A; z_1) \\ = z_2^{-1} \delta\Big(\frac{z_1 - z_0}{z_2}\Big) Y_M(Y(A; z_0)B; z_2), \end{aligned}$$

where $\delta(z)$ is the formal delta function $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$.

4. M has the following decomposition $M = \sum_{h \in H(M)} M[h]$:

- *For some finite subset $H_0(M)$ of $\mathbb{C}$, $H(M) = H_0(M) + \mathbb{Z}_{\geq 0}$.*
- *For $h \in H(M)$, $M[h] = \{\psi \in M : \exists n \geq 0 \text{ s.t. } (L_0 - h)^n \psi = 0\}$.*
- *$0 < \dim_{\mathbb{C}} M[h] < \infty$.*
- *For all $A \in V$, $A[h]M[h'] \subset M[h + h']$.*

Definition 2.1.5. *Given a vertex operator algebra $(V, |0\rangle, T, Y)$ and a V-module M, we call M* ordinary *or* non-logarithmic *V-module if L_0 acts semisimply on M, and call M* logarithmic *V-module if M has L_0-nilpotent rank $n \geq 2$.*

Let us define contragredient modules.

Definition 2.1.6. *Let V be a vertex operator algebra and M be a weak V-module. Let*

$$M^* = \bigoplus_{h \in H(M)} M^*[h]$$

be the graded dual space of M, where $M^*[h] = \mathrm{Hom}_{\mathbb{C}}(M[h], \mathbb{C})$, and let $\langle\ ,\ \rangle$ be the natural dual pairing between M^* and M. Then we can define the V-module structure Y_{M^*} as follows

$$\langle Y_{M^*}(A; z)\psi^*, \psi\rangle := \langle \psi^*, Y_M(e^{zL_1}(-z^{-2})^{L_0} A; z^{-1})\psi\rangle,$$

where $\psi^* \in M^*$, $\psi \in M$ and $A \in V$.

In the following let us introduce the Zhu-algebra.

Definition 2.1.7. *Let $(V, |0\rangle, T, Y)$ be a vertex operator algebra.*

1. For a homogeneous vectors $A, B \in V$ whose L_0 weights h_A and h_B, set

$$A * B := \mathrm{Res}_z \Big(Y(A; z) \frac{(1+z)^{h_A}}{z} B \Big).$$

2. Let $O(V) \subset V$ be the vector subspace spanned by

$$\mathrm{Res}_z \Big(Y(A; z) \frac{(1+z)^{h_A}}{z^2} B \Big)$$

for homogeneous vectors $A, B \in V$, and set $A(V) = V/O(V)$. Let $[A] \in A(V)$ denote the classes represented by $A \in V$.

Theorem 2.1.8 ([68])**.** *Let $(V, |0\rangle, T, Y)$ be a vertex operator algebra. $O(V) \subset V$ is a two-sided ideal with respect to the multiplication $*$, and thus $*$ defines a multiplication on $A(V)$. Moreover, the following holds:*

1. $*$ is associative on $A(V)$

2. $[|0\rangle] \in A(V)$ is the unit element

3. $[T] \in A(V)$ belongs to the center.

4. *For any weak V-module M, let $\overline{M}$ be the highest weight space of M. Then one can introduce an $A(V)$-module structure on $\overline{M}$ as follows:*

$$[A]|_{\overline{M}} := A[0].$$

In this thesis, we study the category of modules over the triplet W-algebras. The triplet W-algebras satisfies strong finiteness, which is called the C_2-cofinite condition. Let us review the definition of the C_2-cofinite condition and the theorem that follows from it.

Definition 2.1.9. *Given a vertex operator algebra V. Let $C_2(V)$ be the subspace of V given by*

$$C_2(V) := \mathrm{span}\{\ A[-h_A - n]B|\ A \in V[h_A],\ B \in V,\ n \geq 1\ \}.$$

The vertex operator algebra V is said to satisfy the Zhu's C_2-cofinite condition *if the quotient vector space*

$$V/C_2(V)$$

is finite dimensional.

The following theorem is due to Huang-Lepowsky-Zhang [36, 37, 38, 39, 40, 41, 42, 43, 44].

Theorem 2.1.10. *Given a vertex operator algebra V satisfying the C_2-cofinite condition. Then the following holds:*

1. *The number of simple V-modules is finite.*

2. *Any V-module has finite length.*

3. *All simple modules have projective covers.*

4. *The category of V-modules has the structure of a braided tensor category.*

The following notation is used frequently in this thesis.

Definition 2.1.11. *Let V be a vertex operator algebra and let M be a finite length V-module. Let $\mathrm{Soc}(M)$ be the socle of M, that is $\mathrm{Soc}(M)$ is the maximal semisimple submodule of M. Since M is finite length, we have the sequence of the submodule*

$$0 \leq \mathrm{Soc}_1(M) \leq \mathrm{Soc}_2(M) \leq \cdots \leq \mathrm{Soc}_n(M) = M$$

such that $\mathrm{Soc}_1(M) = \mathrm{Soc}(M)$ and $\mathrm{Soc}_{i+1}(M)/\mathrm{Soc}_i(M) = \mathrm{Soc}(M/\mathrm{Soc}_i(M))$. We call such a sequence of the submodules of M the socle series *of M.*

In this thesis, we do not go into the detailed theory of logarithmic intertwining operators, however review the definition because it is important concept. See [38],[39] for a more detailed definition and properties of logarithmic intertwining operators.

Definition 2.1.12. *Let V be a vertex operator algebra and M_1, M_2 and M_3 a triple of V-module. Denote by $M_3\{z\}[\log z]$ the space of formal power series in z and $\log z$ with coefficient in M_3, where the exponents of z can be arbitrary complex numbers and with only finitely many $\log z$ terms. An intertwining operator $\mathcal{Y}(\cdot, z)$ of type $\binom{M_3}{M_1\ M_2}$ is a linear map*

$$\begin{aligned}\mathcal{Y} :&M_1 \to \mathrm{End}(M_2, M_3)\{z\}[\log z],\\ &\psi_1 \mapsto \mathcal{Y}(\psi_1, z) = \sum_{t\in\mathbb{C}}\sum_{s\geq 0}(\psi_1)_{t,s}z^{-t-1}(\log z)^s,\end{aligned}$$

satisfying the following conditions for $\psi_i \in M_i$, $i = 1, 2$ and $A \in V$:

1. $\mathcal{Y}(L_{-1}\psi_1, z) = \frac{\mathrm{d}}{\mathrm{d}z}\mathcal{Y}(\psi_1, z)$.
2. $(\psi_1)_{t,s}\psi_2 = 0$ *for* $\mathrm{Re(t)}$ *sufficiently large.*
3. *The following Jacobi identity holds*

$$\begin{aligned}&z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y_{M_3}(A; z_1)\mathcal{Y}(\psi_1, z_2) - z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)\mathcal{Y}(\psi_1, z_2)Y_{M_2}(A; z_1)\\ &= z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)\mathcal{Y}(Y_{M_1}(A; z_0)\psi_1, z_2).\end{aligned}$$

We call intertwining operators without a $\log z$ term *ordinary* or *non-logarithmic* intertwining operators.

2.2 $N = 1$ vertex operator superalgebras

The notions introduced in this section will be used from Chapter 10.

Definition 2.2.1. *A five pairs $(V, |0\rangle, T, G, Y)$ is called a $N = 1$ vertex operator superalgebra where*

1. *V is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-vector space*

$$V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_{\geq 0}} V[n].$$

For $\bar{0}, \bar{1} \in \mathbb{Z}/2\mathbb{Z}$, let

$$V^{\bar{0}} := \bigoplus_{n\in\mathbb{Z}_{\geq 0}} V[n], \qquad V^{\bar{1}} := \bigoplus_{n\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} V[n].$$

$V^{\bar{0}}$ is called the even *part of V and $V^{\bar{1}}$ is called the* odd *part of V.*

2. $|0\rangle \in V[0]$ *is called* the vacuum vector.
3. $T \in V[2]$ *is called* the conformal vector.
4. $G \in V[\frac{3}{2}]$ *is called* the super partner of T.

5. Y is a $\mathbb{C}$-linear map

$$Y : V \to \mathrm{End}_{\mathbb{C}}(V)[[z, z^{-1}]].$$

These data are subject to the following axioms:

1. $\dim_{\mathbb{C}} V[0] = 1$ *and* $0 < \dim_{\mathbb{C}} V[n] < \infty$ *for any* $n \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

2. For each $A \in V^{\bar{i}}[h]$ *there exists a* field

$$Y(A; z) = \sum_{n \in \mathbb{Z} + \frac{i}{2}} A[n] z^{-n-h}.$$

3. $Y(|0\rangle\,; z) = \mathrm{id}_V$ *and*

$$Y(A; z)\,|0\rangle - A \in V[[z]]z$$

for all $A \in V$.

4. The fields

$$Y(T; z) = T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \qquad Y(G; z) = G(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} G_r z^{-r-\frac{3}{2}},$$

of modes define the commutation relations of the Neveu-Schwarz algebra *with fixed central charge* $c = c_V$

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3 - m}{12} c_V, \\ [L_m, G_r] &= (\frac{1}{2}m - r)G_{m+r}, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{1}{3}(r^2 - \frac{1}{4})\delta_{r+s,0} c_V, \end{aligned}$$

where $\{,\}$ *is the anti-commutator.*

5. The zero mode L_0 *of* $T(z)$ *acts semisimply on* V *and*

$$V[h] = \{\ A \in V \mid L_0 A = hA\ \}.$$

6. For all $A \in V$

$$Y(L_{-1}A; z) = \frac{\mathrm{d}}{\mathrm{d}z} Y(A; z).$$

7. For $A \in V^{\bar{i}}$ *and* $B \in V^{\bar{j}}$*, the following super Jacobi identity holds*

$$\begin{aligned} z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big) Y(A; z_1) Y(B; z_2) - (-1)^{ij} z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big) Y(B; z_2) Y(A; z_1) \\ = z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big) Y(Y(A; z_0)B; z_2). \end{aligned}$$

Definition 2.2.2. *Given a vertex operator superalgebra* $(V, |0\rangle, T, G, Y)$*, a weak* V*-module is a pair* (M, Y_M) *of a vector space* M *and a linear map* Y_M *from* V *to* $\mathrm{End}M[[z, z^{-1}]]$ *satisfying the following conditions*

1. $Y_M(|0\rangle; z) = \mathrm{Id}_M$ *and the Fourier modes of*

$$Y_M(T; z) = \sum_{n\in\mathbb{Z}} L_n^M z^{-n-2}, \qquad Y_M(G; z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r^M z^{-r-\frac{3}{2}}$$

satisfy the commutation relations of the Neveu-Schwarz algebra with the central charge c_V*.*

2. *For all* $A \in V$*,*

$$Y_M(L_{-1}A; z) = \frac{\mathrm{d}}{\mathrm{d}z} Y_M(A; z).$$

3. *For* $A \in V^{\bar{i}}$ *and* $B \in V^{\bar{j}}$*, the following super Jacobi identity holds*

$$z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big) Y_M(A; z_1) Y_M(B; z_2) - (-1)^{ij} z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big) Y_M(B; z_2) Y_M(A; z_1)$$
$$= z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big) Y_M(Y(A; z_0)B; z_2),$$

where $\delta(z)$ *is the formal delta function* $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$*.*

4. M *is a* $\mathbb{C}$*-graded superspace*

$$M = \bigoplus_{\bar{i}\in\mathbb{Z}/2\mathbb{Z}} M^{\bar{i}} = \bigoplus_{h\in H(M)} M[h]$$

such that

- *For some finite subset* $H_0(M)$ *of* $\mathbb{C}$*,* $H(M) = H_0(M) + \frac{1}{2}\mathbb{Z}_{\geq 0}$*.*
- *For* $h \in H(M)$*,* $M[h] = \{\psi \in M : \exists n \geq 0 \text{ s.t. } (L_0 - h)^n \psi = 0\}$*.*
- $0 < \dim_{\mathbb{C}} M[h] < \infty$*.*
- *For all* $A \in V$*,* $A[h]M[h'] \subset M[h + h']$*.*
- *For* $\bar{i} = \bar{0}, \bar{1}$*,* $M^{\bar{i}} = \bigoplus_{h\in H(M)} M^{\bar{i}}$*, where* $M^{\bar{i}}[h] = M^{\bar{i}} \cap M[h]$*.*
- *For any* $A \in V^{\bar{i}}$ *and* $\psi \in M^{\bar{i}}$*,* $\bar{i}, \bar{j} \in \mathbb{Z}/2\mathbb{Z}$*,* $Y_M(A; z)\psi \in M^{\bar{i}+\bar{j}}[[z, z^{-1}]]$*.*

Definition 2.2.3. *Given a* $N = 1$ *vertex operator superalgebra* $(V, |0\rangle, T, G, Y)$ *and a* V*-module* M*, we call* M ordinary *or* non-logarithmic V*-module if* L_0 *acts semisimply on* M*, and call* M logarithmic V*-module if* M *has* L_0*-nilpotent rank* $n \geq 2$*.*

Let us define contragredient modules for vertex operator superalgebras.

Definition 2.2.4. *Let V be a vertex operator superalgebra and M be a weak V-module. Let*

$$M^* = \bigoplus_{h\in H(M)} M^*[h]$$

be the graded dual space of M, where $M^*[h] = \mathrm{Hom}_{\mathbb{C}}(M[h],\mathbb{C})$ with parity decomposition

$$(M^*)^{\bar{i}} = \bigoplus_{h\in H(M)} (M^*[h])^{\bar{i}}, \qquad (M^*[h])^{\bar{i}} = \mathrm{Hom}_{\mathbb{C}}(M^{\bar{i}}[h],\mathbb{C}).$$

Let $\langle\ ,\ \rangle$ be the natural dual pairing between M^* and M. Then we can define the V-module structure Y_{M^*} as follows

$$\langle Y_{M^*}(A;z)\psi^*,\psi\rangle := (-1)^{ij}\langle \psi^*, Y_M(e^{zL_1}(-z^{-2})^{L_0}A;z^{-1})\psi\rangle,$$

where $\psi^*\in (M^*)^{\bar{i}}$, $\psi\in M$ and $A\in V^{\bar{j}}$, for $\bar{i},\bar{j}\in\mathbb{Z}/2\mathbb{Z}$.

Definition 2.2.5. *Let V be a $N=1$ vertex operator superalgebra and M_1, M_2 and M_3 a triple of ordinary V-module. Denote by $M_3\{z\}[\log z]$ the space of formal power series in z and $\log z$ with coefficient in M_3, where the exponents of z can be arbitrary complex numbers and with only finitely many $\log z$ terms. An intertwining operator $\mathcal{Y}(\cdot,z)$ of type $\binom{M_3}{M_1\ M_2}$ is a linear map*

$$\begin{aligned}\mathcal{Y}:&M_1\to \mathrm{End}(M_2,M_3)\{z\}[\log z],\\ &\psi_1\mapsto \mathcal{Y}(\psi_1,z)=\sum_{t\in\mathbb{C}}\sum_{s\geq 0}(\psi_1)_{t,s}z^{-t-1}(\log z)^s,\end{aligned}$$

satisfying the following conditions for $\psi_1\in M_1^{\bar{i}}$, $\psi_2\in M_2$, $\bar{i}\in\mathbb{Z}/2\mathbb{Z}$ and $A\in V^{\bar{j}}$, $\bar{j}\in\mathbb{Z}/2\mathbb{Z}$:

1. $\mathcal{Y}(L_{-1}\psi_1,z)=\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{Y}(\psi_1,z)$.
2. $(\psi_1)_{t,s}\psi_2=0$ *for* $\mathrm{Re(t)}$ *sufficiently large.*
3. *The following super Jacobi identity holds*

$$\begin{aligned}z_0^{-1}\delta\Big(\frac{z_1-z_2}{z_0}\Big)Y_{M_3}(A;z_1)\mathcal{Y}(\psi_1,z_2)-(-1)^{ij}z_0^{-1}\delta\Big(\frac{z_2-z_1}{-z_0}\Big)\mathcal{Y}(\psi_1,z_2)Y_{M_2}(A;z_1)\\ =z_2^{-1}\delta\Big(\frac{z_1-z_0}{z_2}\Big)\mathcal{Y}(Y_{M_1}(A;z_0)\psi_1,z_2).\end{aligned}$$

4. *For $\psi_1\in M_1^{\bar{i}}$, $\psi_2\in M_2^{\bar{j}}$, $\bar{i},\bar{j}\in\mathbb{Z}/2\mathbb{Z}$,*

$$\mathcal{Y}(\psi_1,z)\psi_2\in M_3^{\bar{i}+\bar{j}}\{z\}[\log z].$$

Remark 2.2.6. *The Zhu algebra and the C_2-cofinite condition as in Definitions 2.1.7, 2.1.9 can be defined in the case of vertex operator superalgebras, in simlar ways (see [2],[49]). Furthermore Theorem 2.1.10 holds in the case of C_2-cofinite vertex operator superalgebras. We omit details in this thesis.*

Chapter 3

Bosonic Fock modules

Fix two coprime integers p_+, p_- such that $p_- > p_+ \geq 2$. In this chapter, we briefly review theories of Fock modules whose central charges are

$$c_{p_+,p_-} := 1 - 6\frac{(p_+ - p_-)^2}{p_+p_-}$$

in accordance with [65]. As for the representation theory of the Virasoro algebra, see [21] and [48]. For the terminology of vertex operator algebras such as operator expansions and normal order products, refer to [29].

3.1 Free field theory

The Heisenberg Lie algebra

$$\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}a_n \oplus \mathbb{C}K_{\mathcal{H}}$$

is the Lie algebra whose commutation is given by

$$[a_m, a_n] = m\delta_{m+n,0}K_{\mathcal{H}}, \qquad [K_{\mathcal{H}}, \mathcal{H}] = 0.$$

Let

$$\mathcal{H}^{\pm} = \bigoplus_{n>0} \mathbb{C}a_{\pm n}, \qquad \mathcal{H}^0 = \mathbb{C}a_0 \oplus \mathbb{C}K_{\mathcal{H}}, \qquad \mathcal{H}^{\geq} = \mathcal{H}^+ \oplus \mathcal{H}^0.$$

For any $\alpha \in \mathbb{C}$, let $\mathbb{C}|\alpha\rangle$ be the one dimensional $\mathcal{H}^{\geq}$-module defined by

$$a_n|\alpha\rangle = \delta_{n,0}\alpha|\alpha\rangle \ (n \geq 0), \qquad K_{\mathcal{H}}|\alpha\rangle = |\alpha\rangle.$$

For any $\alpha \in \mathbb{C}$, the bosonic Fock module is defined by

$$F_\alpha = \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}^{\geq}}\mathbb{C}|\alpha\rangle.$$

Let

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$$

be the bosonic current. Then we have the following operator expansion

$$a(z)a(w) = \frac{1}{(z-w)^2} + \cdots,$$

where $\cdots$ denotes the regular part in $z = w$. We define the energy-momentum tensor

$$T(z) := \frac{1}{2} : a(z)a(z) : + \frac{\alpha_0}{2}\partial a(z), \qquad \alpha_0 := \sqrt{\frac{2p_-}{p_+}} - \sqrt{\frac{2p_+}{p_-}}.$$

where : : is the normal ordered product. The energy-momentum tensor satisfies the following operator expansion

$$T(z)T(w) = \frac{c_{p_+,p_-}}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots .$$

The Fourier modes of $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ generate the Virasoro algebra whose central charge is c_{p_+,p_-}.

By the energy-momentum tensor $T(z)$, each Fock module F_α has the structure of a Virasoro module whose central charge is c_{p_+,p_-}. Note that

$$L_0|\alpha\rangle = \frac{1}{2}\alpha(\alpha-\alpha_0)|\alpha\rangle.$$

Let us denote

$$h_\alpha := \frac{1}{2}\alpha(\alpha-\alpha_0). \tag{3.1.1}$$

We define the following conformal vector in F_0

$$T = \frac{1}{2}(a_{-1}^2 + \alpha_0 a_{-2})\,|0\rangle\,.$$

Definition 3.1.1. *The Fock module F_0 carries the structure of a $\mathbb{Z}_{\geq 0}$-graded vertex operator algebra, with*

$$Y(|0\rangle\,; z) = \mathrm{id}, \qquad Y(a_{-1}\,|0\rangle\,; z) = a(z), \qquad Y(T; z) = T(z).$$

We denote this vertex operator algebra by $\mathcal{F}_{\alpha_0}$.

3.2 The structure of Fock modules

We set

$$\alpha_+ = \sqrt{\frac{2p_-}{p_+}}, \qquad \alpha_- = -\sqrt{\frac{2p_+}{p_-}}.$$

For $r, s, n \in \mathbb{Z}$ we introduce the following symbols

$$\alpha_{r,s;n} = \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_- + \frac{\sqrt{2p_+p_-}}{2}n, \qquad \alpha_{r,s} = \alpha_{r,s;0}. \tag{3.2.1}$$

For $r, s, n \in \mathbb{Z}$, let

$$F_{r,s;n} = F_{\alpha_{r,s;n}}, \qquad F_{r,s} = F_{\alpha_{r,s}}$$

For $r, s, n \in \mathbb{Z}$, we set

$$h_{r,s;n} := \frac{1}{2}\alpha_{r,s;n}(\alpha_{r,s;n}-\alpha_0), \qquad h_{r,s} := \frac{1}{2}\alpha_{r,s}(\alpha_{r,s}-\alpha_0).$$

Note that

$$h_{r,s;n} = h_{r-np_+,s} = h_{r,s+np_-}$$

and

$$h_{r,s;n} = h_{-r,-s;-n}$$

for $r, s, n \in \mathbb{Z}$. For each $r, s, n \in \mathbb{Z}$, let $L(h_{r,s;n})$ be the irreducible Virasoro module whose highest weight is $h_{r,s;n}$ and the central charge $C = c_{p_+,p_-} \cdot \mathrm{id.}$ be the maximal semisimple Virasoro submodules of $F_{r,s;n}$. The following proposition is due to Feigin and Fuchs [21].

Proposition 3.2.1 ([21])**.** *As the Virasoro module, there are four cases of socle series for the Fock modules $F_{r,s;n} \in \mathcal{F}_{\alpha_0}$-Mod:*

1. *For each $1 \leq r \leq p_+ - 1,\ 1 \leq s \leq p_- - 1,\ n \in \mathbb{Z}$, we have*

$$0 \leq \mathrm{Soc}_1(F_{r,s;n}) \leq \mathrm{Soc}_2(F_{r,s;n}) \leq \mathrm{Soc}_3(F_{r,s;n}) = F_{r,s;n}$$

with

$$\begin{aligned}
\mathrm{Soc}_1(F_{r,s;n}) = \mathrm{Soc}(F_{r,s;n}) &= \bigoplus_{k\geq 0} L(h_{r,p_- - s;|n|+2k+1}),\\
\mathrm{Soc}_2(F_{r,s;n})/\mathrm{Soc}_1(F_{r,s;n}) &= \mathrm{Soc}(F_{r,s;n}/\mathrm{Soc}_1(F_{r,s;n}))\\
&= \bigoplus_{k\geq a} L(h_{r,s;|n|+2k}) \oplus \bigoplus_{k\geq 1-a} L(h_{p_+-r,p_--s;|n|+2k}),\\
\mathrm{Soc}_3(F_{r,s;n})/\mathrm{Soc}_2(F_{r,s;n}) &= \mathrm{Soc}(F_{r,s;n}/\mathrm{Soc}_2(F_{r,s;n})) = \bigoplus_{k\geq 0} L(h_{p_+-r,s;|n|+2k+1}),
\end{aligned}$$

where $a = 0$ if $n \geq 0$ and $a = 1$ if $n < 0$.

2. *For each $1 \leq s \leq p_- - 1,\ n \in \mathbb{Z}$, we have*

$$0 \leq \mathrm{Soc}_1(F_{p_+,s;n}) \leq \mathrm{Soc}_2(F_{p_+,s;n}) = F_{p_+,s;n}$$

with

$$\begin{aligned}
\mathrm{Soc}_1(F_{p_+,s;n}) = \mathrm{Soc}(F_{p_+,s;n}) &= \bigoplus_{k\geq 0} L(h_{p_+,p_--s;|n|+2k+1}),\\
\mathrm{Soc}_2(F_{p_+,s;n})/\mathrm{Soc}_1(F_{p_+,s;n}) &= \bigoplus_{k\geq a} L(h_{p_+,s;|n|+2k})
\end{aligned}$$

where $a = 0$ if $n \geq 1$ and $a = 1$ if $n < 1$.

3. *For each $1 \leq r \leq p_+ - 1,\ n \in$, we have*

$$0 \leq \mathrm{Soc}_1(F_{r,p_-;n}) \leq \mathrm{Soc}_2(F_{r,p_-;n}) = F_{r,p_-;n}$$

with

$$\begin{aligned}
\mathrm{Soc}_1(F_{r,p_-;n}) = \mathrm{Soc}(F_{r,p_-;n}) &= \bigoplus_{k\geq 0} L(h_{r,p_-;|n|+2k}),\\
\mathrm{Soc}_2(F_{r,p_-;n})/\mathrm{Soc}_1(F_{r,p_-;n}) &= \bigoplus_{k\geq a} L(h_{p_+-r,p_-;|n|+2k-1})
\end{aligned}$$

where $a = 1$ if $n \geq 0$ and $a = 0$ if $n < 0$.

4. For each $n \in \mathbb{Z}$, the Fock module $F_{p_+,p_-;n}$ is semi-simple as a Virasoro module:

$$F_{p_+,p_-;n} = \mathrm{Soc}(F_{p_+,p_-;n}) = \bigoplus_{k\geq 0} L(h_{p_+,p_-;|n|+2k}).$$

Let the Fock modules, whose socle length are three, be denoted by braided type, and let the Fock modules, whose length are two, be denoted by chain type.

3.3 Screening currents and Felder complex

We introduce a free scalar field $\phi(z)$, which is a formal primitive of $a(z)$

$$\phi(z) = \hat{a} + a_0 \log z - \sum_{n\neq 0} \frac{a_n}{n} z^{-n}$$

where $\hat{a}$ is defined by

$$[a_m, \hat{a}] = \delta_{m,0}\mathrm{id}. \tag{3.3.1}$$

The scalar field $\phi(z)$ satisfies the operator product expansion

$$\phi(z)\phi(w) = \log(z-w) + \cdots .$$

For any $\alpha \in \mathbb{C}$ we introduce the field $V_\alpha(z)$

$$V_\alpha(z) =: e^{\alpha\phi(z)} := e^{\alpha\hat{a}} z^{\alpha a_0} \overline{V}_\alpha(z),\ \ z^{\alpha a_0} = e^{\alpha a_0 \log z} \ ,$$
$$\overline{V}_\alpha(z) = e^{\alpha\sum_{n\geq 1}\frac{a_{-n}}{n}z^n} e^{-\alpha\sum_{n\geq 1}\frac{a_n}{n}z^{-n}}.$$

The fields $V_\alpha(z)$ satisfy the following operator product expansion

$$V_\alpha(z)V_\beta(w) = (z-w)^{\alpha\beta} : V_\alpha(z)V_\beta(w) : .$$

We introduce the following two screening currents $Q_+(z), Q_-(z)$

$$Q_\pm(z) = V_{\alpha_\pm}(z)$$

whose conformal weights are $h_{\alpha_\pm} = 1$:

$$T(z)Q_\pm(w) = \frac{Q_\pm(w)}{(z-w)^2} + \frac{\partial_w Q_\pm(w)}{z-w} + \cdots$$
$$= \partial_w\Big(\frac{Q_\pm(w)}{z-w}\Big) + \cdots .$$

Therefore the zero modes of the fields $Q_\pm(z)$

$$\mathrm{Res}_{z=0}Q_+(z)\mathrm{d}z = Q_+ \ : \ F_{1,k} \to F_{-1,k}, \quad k \in \mathbb{Z}$$
$$\mathrm{Res}_{z=0}Q_-(z)\mathrm{d}z = Q_- \ : \ F_{k,1} \to F_{k,-1}, \quad k \in \mathbb{Z}$$

commute with every Virasoro mode.

For $r, s \geq 1$, we introduce more complicated screening currents

$$Q_+^{[r]}(z) \in \mathrm{Hom}_{\mathbb{C}}(F_{r,k}, F_{-r,k})[[z, z^{-1}]], \quad r \geq 1, k \in \mathbb{Z},$$
$$Q_-^{[s]}(z) \in \mathrm{Hom}_{\mathbb{C}}(F_{k,s}, F_{k,-s})[[z, z^{-1}]], \quad s \geq 1, k \in \mathbb{Z},$$

constructed by Tsuchiya-Kanie ([63],[48]) as follows

$$\begin{aligned} Q_+^{[r]}(z) &= \int_{\overline{\Gamma}_r(\kappa_+)} Q_+(z)Q_+(zx_1)Q_+(zx_2)\cdots Q_+(zx_{r-1})z^{r-1}\mathrm{d}x_1\cdots \mathrm{d}x_{r-1}, \\ Q_-^{[s]}(z) &= \int_{\overline{\Gamma}_s(\kappa_-)} Q_-(z)Q_-(zx_1)Q_-(zx_2)\cdots Q_-(zx_{s-1})z^{s-1}\mathrm{d}x_1\cdots \mathrm{d}x_{s-1}, \end{aligned} \tag{3.3.2}$$

where $\overline{\Gamma}_n(\kappa_\pm)$ is a certain regularized cycle constructed from the simplex

$$\Delta_{n-1} = \{\ (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \mid 1 > x_1 > \cdots > x_{n-1} > 0\ \}.$$

These fields satisfy the following operator product expansion

$$T(z)Q_+^{[r]}(w) = \frac{Q_+^{[r]}(w)}{(z-w)^2} + \frac{\partial_w Q_+^{[r]}(w)}{z-w} + \cdots$$
$$T(z)Q_-^{[s]}(w) = \frac{Q_-^{[s]}(w)}{(z-w)^2} + \frac{\partial_w Q_-^{[s]}(w)}{z-w} + \cdots .$$

In particular the following proposition holds

Proposition 3.3.1. *The zero modes*

$$\mathrm{Res}_{z=0}Q_+^{[r]}(z)\mathrm{d}z = Q_+^{[r]} \in \mathrm{Hom}_{\mathbb{C}}(F_{r,k}, F_{-r,k}), \quad r \geq 1, k \in \mathbb{Z},$$
$$\mathrm{Res}_{z=0}Q_-^{[s]}(z)\mathrm{d}z = Q_-^{[s]} \in \mathrm{Hom}_{\mathbb{C}}(F_{k,s}, F_{k,-s}), \quad s \geq 1, k \in \mathbb{Z}$$

commute with every Virasoro mode of $\mathcal{F}_{\alpha_0}$-Mod. *These zero modes are called* screening operators.

For $1 \leq r \leq p_+ - 1$, $1 \leq s \leq p_- - 1$, we set

$$r^\vee := p_+ - r, \quad s^\vee := p_- - s.$$

For $1 \leq r \leq p_+, 1 \leq s \leq p_-$ and $n \in \mathbb{Z}$, we define the following Virasoro modules :

1. For $1 \leq r < p_+,\ 1 \leq s \leq p_-,\ n \in \mathbb{Z}$

$$K_{r,s;n;+} = \mathrm{ker}Q_+^{[r]} : F_{r,s;n} \to F_{r^\vee,s;n+1}$$
$$X_{r^\vee,s;n+1;+} = \mathrm{im}Q_+^{[r]} : F_{r,s;n} \to F_{r^\vee,s;n+1}.$$

2. For $1 \leq r \leq p_+,\ 1 \leq s < p_-,\ n \in \mathbb{Z}$

$$K_{r,s;n;-} = \mathrm{ker}Q_-^{[s]} : F_{r,s;n} \to F_{r,s^\vee;n-1}$$
$$X_{r,s^\vee;n-1;-} = \mathrm{im}Q_-^{[s]} : F_{r,s;n} \to F_{r,s^\vee;n-1}.$$

The following propositions are due to Felder [26].

Proposition 3.3.2 ([26]). *The socle series of $K_{r,s;n;\pm}$ and $X_{r,s;n;\pm}$ are given by :*

1. *For $1 \le r \le p_+ - 1,\ 1 \le s \le p_- - 1$, we have*

$$0 \le S_1(K_{r,s;n;\pm}) = \mathrm{Soc}(K_{r,s;n;\pm}) \le K_{r,s;n;\pm}$$
$$0 \le S_1(X_{r,s;n;\pm}) = \mathrm{Soc}(X_{r,s;n;\pm}) \le X_{r,s;n;\pm}$$

such that

$$n \ge 0 \qquad\qquad n \le -1$$
$$S_1(K_{r,s;n;+}) = \bigoplus_{k\ge1} L(h_{r,s^\vee;n+2k-1}), \qquad S_1(K_{r,s;n;+}) = \bigoplus_{k\ge1} L(h_{r,s^\vee;-n+2k-1}),$$
$$K_{r,s;n;+}/S_1 = \bigoplus_{k\ge1} L(h_{r,s;n+2(k-1)}), \qquad K_{r,s;n;+}/S_1 = \bigoplus_{k\ge1} L(h_{r,s;-n+2k}),$$
$$S_1(X_{r,s;n+1;+}) = \bigoplus_{k\ge1} L(h_{r,s^\vee;n+2k}), \qquad S_1(X_{r,s;n+1;+}) = \bigoplus_{k\ge1} L(h_{r,s^\vee;-n+2(k-1)}),$$
$$X_{r,s;n+1;+}/S_1 = \bigoplus_{k\ge1} L(h_{r,s;n+2k-1}), \qquad X_{r,s;n+1;+}/S_1 = \bigoplus_{k\ge1} L(h_{r,s;-n+2k-1}).$$

$$n \ge 1 \qquad\qquad n \le 0$$
$$S_1(K_{r,s;n;-}) = \bigoplus_{k\ge1} L(h_{r,s^\vee;n+2k-1}), \qquad S_1(K_{r,s;n;-}) = \bigoplus_{k\ge1} L(h_{r,s^\vee;-n+2k-1}),$$
$$K_{r,s;n;-}/S_1 = \bigoplus_{k\ge1} L(h_{r,s;n+2(k-1)}), \qquad K_{r,s;n;-}/S_1 = \bigoplus_{k\ge1} L(h_{r,s;-n+2k}),$$
$$S_1(X_{r,s;n+1;-}) = \bigoplus_{k\ge1} L(h_{r,s^\vee;n+2(k-1)}), \qquad S_1(X_{r,s;n+1;-}) = \bigoplus_{k\ge1} L(h_{r,s^\vee;-n+2k}),$$
$$X_{r,s;n+1;-}/S_1 = \bigoplus_{k\ge1} L(h_{r^\vee,s^\vee;n+2k-1}), \qquad X_{r,s;n+1;-}/S_1 = \bigoplus_{k\ge1} L(h_{r^\vee,s^\vee;-n+2k-1}).$$

2. *For $1 \le r \le p_+ - 1,\ s = p_-,\ n \in \mathbb{Z}$, we have*

$$X_{r,p_-;n} = \mathrm{Soc}(F_{r,p_-;n}).$$

3. *For $r = p_+,\ 1 \le s \le p_- - 1,\ n \in \mathbb{Z}$, we have*

$$X_{p_+,s;n} = \mathrm{Soc}(F_{p_+,s;n}).$$

Proposition 3.3.3 ([26]).

1. *For $1 \le r < p_+,\ 1 \le s < p_-$ and $n \in \mathbb{Z}$ the screening operators $Q_+^{[r]}$ and $Q_+^{[r^\vee]}$ define the* Felder complex

$$\cdots \xrightarrow{Q_+^{[r]}} F_{r^\vee,s;n-1} \xrightarrow{Q_+^{[r^\vee]}} F_{r,s;n} \xrightarrow{Q_+^{[r]}} F_{r^\vee,s;n+1} \xrightarrow{Q_+^{[r^\vee]}} \cdots .$$

This complex is exact everywhere except in $F_{r,s} = F_{r,s;0}$ where the cohomology is given by

$$\mathrm{ker} Q_+^{[r]}/\mathrm{im} Q_+^{[r^\vee]} \simeq L(h_{r,s;0}).$$

2. *For* $1 \le r < p_+,\ 1 \le s < p_-$ *and* $n \in \mathbb{Z}$ *the screening operators* $Q_-^{[s]}$ *and* $Q_-^{[s^\vee]}$ *define the* Felder complex

$$\cdots \xrightarrow{Q_-^{[s]}} F_{r,s^\vee;n+1} \xrightarrow{Q_-^{[s^\vee]}} F_{r,s;n} \xrightarrow{Q_-^{[s]}} F_{r,s^\vee;n-1} \xrightarrow{Q_-^{[s^\vee]}} \cdots .$$

This complex is exact everywhere except in $F_{r,s} = F_{r,s;0}$ *where the cohomology is given by*

$$\mathrm{ker} Q_-^{[s]} / \mathrm{im} Q_-^{[s^\vee]} \simeq L(h_{r,s;0}).$$

3. *For* $1 \le r < p_+$ *and* $n \in \mathbb{Z}$ *the screening operators* $Q_+^{[r]}$ *and* $Q_+^{[r^\vee]}$ *define the* Felder complex

$$\cdots \xrightarrow{Q_+^{[r]}} F_{r^\vee,p_-;n-1} \xrightarrow{Q_+^{[r^\vee]}} F_{r,p_-;n} \xrightarrow{Q_+^{[r]}} F_{r^\vee,p_-;n+1} \xrightarrow{Q_+^{[r^\vee]}} \cdots$$

and this complex is exact.

4. *For* $1 \le s < p_-$ *and* $n \in \mathbb{Z}$ *the screening operators* $Q_-^{[s]}$ *and* $Q_-^{[s^\vee]}$ *define the* Felder complex

$$\cdots \xrightarrow{Q_-^{[s]}} F_{p_+,s^\vee;n+1} \xrightarrow{Q_-^{[s^\vee]}} F_{p_+,s;n} \xrightarrow{Q_-^{[s]}} F_{p_+,s^\vee;n-1} \xrightarrow{Q_-^{[s^\vee]}} \cdots .$$

and this complex is exact.

Chapter 4

The triplet W-algebra $\mathcal{W}_{p_+,p_-}$

In this chapter, we introduce a vertex operator algebra $\mathcal{W}_{p_+,p_-}$ which is called the triplet W-algebra of type (p_+,p_-) and review some results in [4],[5],[65] briefly. In Section 4.3, we introduce the abelian category of $\mathcal{W}_{p_+,p_-}$-modules and the block decomposition of this abelian category.

4.1 The lattice vertex operator algebra and the vertex operator algebra $\mathcal{W}_{p_+,p_-}$

Definition 4.1.1.
The lattice vertex operator algebra $\mathcal{V}_{[p_+,p_-]}$ *is the tuple*

$$(\mathcal{V}^+_{1,1}, |0\rangle, \frac{1}{2}(a^2_{-1} - \alpha_0 a_{-2})|0\rangle, Y),$$

where underlying vector space of $\mathcal{V}_{[p_+,p_-]}$ *is given by*

$$\mathcal{V}^+_{1,1} = \bigoplus_{n\in\mathbb{Z}} F_{1,1;2n} = \bigoplus_{n\in\mathbb{Z}} F_{n\sqrt{2p_+p_-}},$$

and $Y(|\alpha_{1,1;2n}\rangle; z) = V_{\alpha_{1,1;2n}}(z)$ *for* $n \in \mathbb{Z}$.

It is a known fact that simple $\mathcal{V}_{[p_+,p_-]}$-modules are given by the following $2p_+p_-$ direct sum of Fock modules

$$\mathcal{V}^+_{r,s} = \bigoplus_{n\in\mathbb{Z}} F_{r,s;2n}, \qquad \mathcal{V}^-_{r,s} = \bigoplus_{n\in\mathbb{Z}} F_{r,s;2n+1},$$

where $1 \le r \le p_+, 1 \le s \le p_-$.

Note that the two screening operators Q_+ and Q_- act on $\mathcal{V}^+_{1,1}$. We define the following vector subspace of $\mathcal{V}^+_{1,1}$:

$$\mathcal{K}_{1,1} = \mathrm{ker}Q_+ \cap \mathrm{ker}Q_- \subset \mathcal{V}^+_{1,1}.$$

Definition 4.1.2. *The* triplet W-algebra

$$\mathcal{W}_{p_+,p_-} = (\mathcal{K}_{1,1}, |0\rangle, T, Y)$$

is a sub vertex operator algebra of $\mathcal{V}_{[p_+,p_-]}$*, where the vacuum vector, conformal vector and vertex operator map are those of* $\mathcal{V}_{[p_+,p_-]}$.

Definition 4.1.3. *Let* $W^{\pm}, W^0$ *be the following singular vectors*

$$W^+ = Q_-^{[p_--1]}|\alpha_{1,p_--1;3}\rangle, \quad W^- = Q_+^{[p_+-1]}|\alpha_{p_+-1,1;-3}\rangle, \quad W^0 = Q_+^{[2p_+-1]}|\alpha_{p_+-1,1;-3}\rangle.$$

Proposition 4.1.4. $\mathcal{W}_{p_+,p_-}$ *is strongly generated by the fields* $T(z), Y(W^{\pm}; z), Y(W^0; z)$.

Theorem 4.1.5 ([4, 5, 65]). $\mathcal{W}_{p_+,p_-}$ *is* C_2*-cofinite.*

4.2 Simple $\mathcal{W}_{p_+,p_-}$-modules

For each $1 \le r \le p_+,\ 1 \le s \le p_-$, let $\mathcal{X}^{\pm}_{r,s}$ be the following vector subspace of $\mathcal{V}^{\pm}_{r,s}$:

1. For $1 \le r \le p_+ - 1,\ 1 \le s \le p_- - 1$,
$$\mathcal{X}^{+}_{r,s} = Q_+^{[r^\vee]}(\mathcal{V}^-_{r^\vee,s}) \cap Q_-^{[s^\vee]}(\mathcal{V}^-_{r,s^\vee}), \qquad \mathcal{X}^{-}_{r,s} = Q_+^{[r^\vee]}(\mathcal{V}^+_{r^\vee,s}) \cap Q_-^{[s^\vee]}(\mathcal{V}^+_{r,s^\vee}).$$
2. For $1 \le r \le p_+ - 1,\ s = p_-$,
$$\mathcal{X}^{+}_{r,p_-} = Q_+^{[r^\vee]}(\mathcal{V}^-_{r^\vee,p_-}), \qquad \mathcal{X}^{-}_{r,p_-} = Q_+^{[r^\vee]}(\mathcal{V}^+_{r^\vee,p_-}).$$
3. For $r = p_+,\ 1 \le s \le p_- - 1$,
$$\mathcal{X}^{+}_{p_+,s} = Q_-^{[s^\vee]}(\mathcal{V}^-_{p_+,s^\vee}), \qquad \mathcal{X}^{-}_{p_+,s} = Q_-^{[s^\vee]}(\mathcal{V}^+_{p_+,s^\vee}).$$
4. $r = p_+,\ s = p_-$,
$$\mathcal{X}^{+}_{p_+,p_-} = \mathcal{V}^+_{p_+,p_-}, \qquad \mathcal{X}^{-}_{p_+,p_-} = \mathcal{V}^-_{p_+,p_-}.$$

Definition 4.2.1.

1. *We define the interior Kac table $\mathcal{T}$ as the following quotient set*
$$\mathcal{T} = \{(r,s)|\ 1 \le r < p_+, 1 \le s < p_-\}/\sim$$
where $(r,s) \sim (r',s')$ if and only if $r' = p_+ - r, s' = p_- - s$. Note that $\#\mathcal{T} = \frac{(p_+-1)(p_--1)}{2}$.

2. *For each $1 \le r \le p_+$, $1 \le s \le p_-$, $n \ge 0$, we define the following symbols*
$$\Delta^+_{r,s;n} = \begin{cases} h_{r^\vee,s;-2n-1} & r \ne p_+, s \ne p_- \\ h_{p_+,s;-2n} & r = p_+, s \ne p_- \\ h_{r,p_-;2n} & r \ne p_+, s = p_- \\ h_{p_+,p_-;-2n} & r = p_+, s = p_- \end{cases}, \quad \Delta^-_{r,s;n} = \begin{cases} h_{r^\vee,s;-2n-2} & r \ne p_+, s \ne p_- \\ h_{p_+,s;-2n-1} & r = p_+, s \ne p_- \\ h_{r,p_-;2n+1} & r \ne p_+, s = p_- \\ h_{p_+,p_-;-2n-1} & r = p_+, s = p_-. \end{cases}.$$

Proposition 4.2.2 ([4, 5, 65])**.** *For each $\mathcal{X}^{\pm}_{r,s}(1 \le r \le p_+,\ 1 \le s \le p_-)$, we have the following decompositions as the Virasoro modules*
$$\mathcal{X}^+_{r,s} = \bigoplus_{n\ge0}(2n+1)L(\Delta^+_{r,s;n}), \qquad \mathcal{X}^-_{r,s} = \bigoplus_{n\ge0}(2n+2)L(\Delta^-_{r,s;n}).$$

Theorem 4.2.3 ([4, 5, 65])**.** *The $\frac{(p_+-1)(p_--1)}{2} + 2p_+p_-$ vector spaces*
$$L(h_{r,s}),\ (r,s) \in \mathcal{T}, \qquad \mathcal{X}^{\pm}_{r,s},\ 1 \le r \le p_+,\ 1 \le s \le p_-$$
become simple $\mathcal{W}_{p_+,p_-}$-modules and give all simple $\mathcal{W}_{p_+,p_-}$-modules.

Proposition 4.2.4 ([4, 5, 65]). *Each $2p_+p_-$ simple $\mathcal{V}_{[p_+,p_-]}$-module becomes $\mathcal{W}_{p_+,p_-}$-module and has the following socle series:*

1. *For each $1 \le r < p_+,\ 1 \le s < p_-$, $\mathcal{V}^+_{r,s}$ has the following socle series*

$$0 \le \mathrm{Soc}_1(\mathcal{V}^+_{r,s}) \le \mathrm{Soc}_2(\mathcal{V}^+_{r,s}) \le \mathrm{Soc}_3(\mathcal{V}^+_{r,s}) = \mathcal{V}^+_{r,s}$$

   with

$$\begin{aligned}
&\mathrm{Soc}_1(\mathcal{V}^+_{r,s}) = \mathrm{Soc}(\mathcal{V}^+_{r,s}) = \mathcal{X}^+_{r,s},\\
&\mathrm{Soc}(\mathcal{V}^+_{r,s}/\mathrm{Soc}_1(\mathcal{V}^+_{r,s})) = \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r,s^\vee} \oplus L(h_{r,s}),\\
&\mathrm{Soc}(\mathcal{V}^+_{r,s}/\mathrm{Soc}_2(\mathcal{V}^+_{r,s})) = \mathcal{X}^+_{r^\vee,s^\vee}.
\end{aligned}$$

2. *For each $1 \le r < p_+,\ 1 \le s < p_-$, $\mathcal{V}^-_{r,s}$ has the following socle series*

$$0 \le \mathrm{Soc}_1(\mathcal{V}^-_{r,s}) \le \mathrm{Soc}_2(\mathcal{V}^-_{r,s}) \le \mathrm{Soc}_3(\mathcal{V}^-_{r,s}) = \mathcal{V}^-_{r,s}$$

   with

$$\begin{aligned}
&\mathrm{Soc}_1(\mathcal{V}^-_{r,s}) = \mathrm{Soc}(\mathcal{V}^-_{r,s}) = \mathcal{X}^-_{r,s},\\
&\mathrm{Soc}(\mathcal{V}^-_{r,s}/\mathrm{Soc}_1(\mathcal{V}^-_{r,s})) = \mathcal{X}^+_{r^\vee,s} \oplus \mathcal{X}^+_{r,s^\vee},\\
&\mathrm{Soc}(\mathcal{V}^-_{r,s}/\mathrm{Soc}_2(\mathcal{V}^-_{r,s})) = \mathcal{X}^-_{r^\vee,s^\vee}.
\end{aligned}$$

3. *For each $1 \le r < p_+$, $\mathcal{V}^+_{r,p_-}$ and $\mathcal{V}^-_{r^\vee,p_-}$ have the following socle series*

$$\mathcal{V}^+_{r,p_-}/\mathcal{X}^+_{r,p_-} \simeq \mathcal{X}^-_{r^\vee,p_-}, \qquad \mathcal{V}^-_{r^\vee,p_-}/\mathcal{X}^-_{r^\vee,p_-} \simeq \mathcal{X}^+_{r,p_-}.$$

4. *For each $1 \le s < p_-$, $\mathcal{V}^+_{p_+,s}$ and $\mathcal{V}^-_{p_-,s^\vee}$ have the following socle series*

$$\mathcal{V}^+_{p_+,s}/\mathcal{X}^+_{p_+,s} \simeq \mathcal{X}^-_{p_+,s^\vee}, \qquad \mathcal{V}^-_{p_+,s^\vee}/\mathcal{X}^-_{p_+,s^\vee} \simeq \mathcal{X}^+_{p_+,s}.$$

5. *For $r = p_+,\ s = p_-$,*

$$\mathcal{V}^+_{p_+,p_-} = \mathcal{X}^+_{p_+,p_-}, \qquad \mathcal{V}^-_{p_+,p_-} = \mathcal{X}^-_{p_+,p_-}.$$

Let $A(\mathcal{W}_{p_+,p_-})$ be the Zhu-algebra [68] of $\mathcal{W}_{p_+,p_-}$.

Proposition 4.2.5 ([4, 5, 65]). *In $A(\mathcal{W}_{p_+,p_-})$, the following relations hold*

$$\begin{aligned}
[W^0]*[W^-] - [W^-]*[W^0] &= -2f([T])[W^-],\\
[W^0]*[W^+] - [W^+]*[W^0] &= 2f([T])[W^+],\\
[W^+]*[W^-] - [W^-]*[W^+] &= 2f([T])[W^0],\\
[W^0]*[W^0] &= g([T]),\\
[W^+]*[W^+] &= 0,\\
[W^-]*[W^-] &= 0,
\end{aligned}$$

where $f([T])$ and $g([T])$ are non-trivial polynomials of $[T]$.

Proposition 4.2.6 ([4, 5, 65])**.**

1. *$\mathcal{X}^+_{1,1}$ acts trivially on $L(h_{r,s})$, $(1 \le r \le p_+ - 1,\ 1 \le s \le p_- - 1)$.*

2. *For each $1 \le r \le p_+$, $1 \le s \le p_-$, the highest weight space of $\mathcal{X}^+_{r,s}$ is a one dimensional $A(\mathcal{W}_{p_+,p_-})$-module.*

3. *For each $1 \le r \le p_+$, $1 \le s \le p_-$, the highest weight space of $\mathcal{X}^-_{r,s}$ is a two dimensional irreducible $A(\mathcal{W}_{p_+,p_-})$-module.*

Proposition 4.2.7 ([4, 5, 65])**.** *For any $1 \le r \le p_+$, $1 \le s \le p_-$,*

$$f(\Delta^-_{r,s;0}) \neq 0.$$

In particular, the highest weight space of $\mathcal{X}^-_{r,s}$ has the structure of a two dimensional irreducible sl_2-module with respect to the following elements

$$E = \frac{1}{\sqrt{2}f(\Delta^-_{r,s;0})}[W^+], \qquad F = -\frac{1}{\sqrt{2}f(\Delta^-_{r,s;0})}[W^-], \qquad H = \frac{1}{f(\Delta^-_{r,s;0})}[W^0].$$

For $1 \le r \le p_+, 1 \le s \le p_-$, we define

$$G(\Delta^+_{r,s;n}) := \begin{cases} \bigoplus_{n-1 \ge i \ge 0}(2i+1)L(\Delta^+_{r,s;i}) & n \ge 1 \\ 0 & n = 0, \end{cases}$$
$$G(\Delta^-_{r,s;n}) := \begin{cases} \bigoplus_{n-1 \ge i \ge 0}(2i+2)L(\Delta^-_{r,s;i}) & n \ge 1 \\ 0 & n = 0. \end{cases}$$

As an extension of Proposition 4.2.6, the following propositions holds (see the proof of Proposition 5.6 in [65]).

Proposition 4.2.8 ([4, 5, 65])**.**

1. *With respect to the actions of the zero-modes of the fields $Y(W^+;z)$, $Y(W^-;z)$ and $Y(W^0;z)$, the Virasoro highest weight space of the vector subspace $(2n+1)L(\Delta^+_{r,s;n}) \subset \mathcal{X}^+_{r,s}$ becomes a $(2n+1)$-dimensional irreducible sl_2-module modulo $G(\Delta^+_{r,s;n})$.*

2. *With respect to the actions of the zero-modes of the fields $Y(W^+;z)$, $Y(W^-;z)$ and $Y(W^0;z)$, the Virasoro highest weight space of the vector subspace $(2n+2)L(\Delta^-_{r,s;n}) \subset \mathcal{X}^-_{r,s}$ becomes a $(2n+2)$-dimensional irreducible sl_2-module modulo $G(\Delta^-_{r,s;n})$.*

For $W = W^{\pm}, W^0$, let $W[n]$ be the n-th mode of the field $Y(W;z)$ defined by

$$W[n] = \oint_{z=0} Y(W;z) z^{h_{4p_+-1,1}+n-1} \mathrm{d}z.$$

Proposition 4.2.9 ([4, 5, 65])**.**

1. *For $n \geq 0$, let $\{w_i^{(n)}\}_{i=-n}^{n}$ be the basis of the Virasoro highest weight space of the vector subspace $(2n+1)L(\Delta^+_{r,s;n}) \subset \mathcal{X}^+_{r,s}$ such that*

$$W^{\pm}[0]w_i^{(n)} \in \mathbb{C}^{\times} w_{i\pm 1}^{(n)} + G(\Delta^+_{r,s;n}), \quad \text{for } -n \leq i \leq n,$$

where $w_{-n-1}^{(n)} = w_{n+1}^{(n)} = 0$ and $W^{\pm}[0]$ is the zero mode of the field $Y(W^{\pm};z)$. Then we have

$$\begin{aligned}
W^{\pm}[\Delta^+_{r,s;n} - \Delta^+_{r,s;n-1}]w_i^{(n)} &\in \mathbb{C}^{\times} w_{i\pm1}^{(n-1)} + G(\Delta^+_{r,s;n-1}),\\
W^{\pm}[\Delta^+_{r,s;n} - \Delta^+_{r,s;n+1}]w_i^{(n)} &\in \mathbb{C}^{\times} w_{i\pm1}^{(n+1)} + G(\Delta^+_{r,s;n+1}),\\
W^{0}[\Delta^+_{r,s;n} - \Delta^+_{r,s;n-1}]w_i^{(n)} &\in \mathbb{C}^{\times} w_{i}^{(n-1)} + G(\Delta^+_{r,s;n-1}),\\
W^{0}[\Delta^+_{r,s;n} - \Delta^+_{r,s;n+1}]w_i^{(n)} &\in \mathbb{C}^{\times} w_{i}^{(n+1)} + G(\Delta^+_{r,s;n+1}),
\end{aligned}$$

where $w_i^{(-1)} = 0$.

2. *For $n \geq 0$, let $\{v_{\frac{i}{2}}^{(n)}, v_{\frac{-i}{2}}^{(n)}\}_{i=1}^{n+1}$ be the basis of the Virasoro highest weight space of the vector subspace $(2n+2)L(\Delta^-_{r,s;n}) \subset \mathcal{X}^-_{r,s}$ such that*

$$W^{\pm}[0]v_{\frac{i}{2}}^{(n)} \in \mathbb{C}^{\times} v_{\frac{i}{2}\pm 1}^{(n)} + G(\Delta^-_{r,s;n}), \quad \text{for } -n-1 \leq i \leq n+1 \wedge i \neq 0,$$

where $v_{\frac{-n-2}{2}}^{(n)} = v_{\frac{n+2}{2}}^{(n)} = 0$. Then we have

$$\begin{aligned}
W^{\pm}[\Delta^-_{r,s;n} - \Delta^-_{r,s;n-1}]v_{\frac{i}{2}}^{(n)} &\in \mathbb{C}^{\times} v_{\frac{i}{2}\pm1}^{(n-1)} + G(\Delta^-_{r,s;n-1}),\\
W^{\pm}[\Delta^-_{r,s;n} - \Delta^-_{r,s;n+1}]v_{\frac{i}{2}}^{(n)} &\in \mathbb{C}^{\times} v_{\frac{i}{2}\pm1}^{(n+1)} + G(\Delta^-_{r,s;n+1}),\\
W^{0}[\Delta^-_{r,s;n} - \Delta^-_{r,s;n-1}]v_{\frac{i}{2}}^{(n)} &\in \mathbb{C}^{\times} v_{\frac{i}{2}}^{(n-1)} + G(\Delta^-_{r,s;n-1}),\\
W^{0}[\Delta^-_{r,s;n} - \Delta^-_{r,s;n+1}]v_{\frac{i}{2}}^{(n)} &\in \mathbb{C}^{\times} v_{\frac{i}{2}}^{(n+1)} + G(\Delta^-_{r,s;n+1}),
\end{aligned}$$

where $v_i^{(-1)} = 0$.

The following results for the Zhu-algebra $A(\mathcal{W}_{p_+,p_-})$ will be used to determine the structure of the projective covers of the minimal simple modules in Section 7.4.

Theorem 4.2.10 ([4, 5, 65])**.** *The center of the Zhu-algebra $A(\mathcal{W}_{p_+,p_-})$ is generated by $[T]$ and isomorphic to*

$$\mathbb{C}[x]/f_{p_+,p_-}(x),$$

where

$$
\begin{aligned}
f_{p_+,p_-}(x) = &\prod_{(i,j)\in\mathcal{T}} (x-h_{r,s})^3 \\
&\times \prod_{i=1}^{p_+-1}\prod_{j=1}^{p_--1}(x-\Delta^+_{i,j;0})^2 \prod_{i=1}^{p_+-1}\prod_{j=1}^{p_--1}(x-\Delta^-_{i,j;0}) \\
&\times \prod_{i=1}^{p_+-1}(x-\Delta^+_{i,p_-;0})^2 \prod_{i=1}^{p_+-1}(x-\Delta^-_{i,p_-;0}) \\
&\times \prod_{j=1}^{p_--1}(x-\Delta^+_{p_+,j;0})^2 \prod_{j=1}^{p_--1}(x-\Delta^-_{p_+,j;0}) \\
&\times (x-\Delta^+_{p_+,p_-;0})(x-\Delta^-_{p_+,p_-;0}).
\end{aligned}
$$

Corollary 4.2.11. *The Zhu algebra $A(\mathcal{W}_{p_+,p_-})$ has three dimensional indecomposable modules on which $[T]$ acts as*

$$
\begin{pmatrix} h_{r,s} & 1 & 0 \\ 0 & h_{r,s} & 1 \\ 0 & 0 & h_{r,s} \end{pmatrix},
$$

where $(r,s)\in\mathcal{T}$.

4.3 The block decomposition of $\mathcal{C}_{p_+,p_-}$

Definition 4.3.1. *Let $\mathcal{C}_{p_+,p_-}$ be the abelian category of weak $\mathcal{W}_{p_+,p_-}$-modules.*

Since $\mathcal{W}_{p_+,p_-}$ is C_2-cofinite, any M in $\mathcal{C}_{p_+,p_-}$ has finite length. For any M in $\mathcal{C}_{p_+,p_-}$, let M^* be the contragredient of M. Note that $\mathcal{C}_{p_+,p_-}$ is closed under contragredient.

Definition 4.3.2. *In the following, we define $\frac{(p_+-1)(p_--1)}{2}$ thick blocks, p_++p_--2 thin blocks and two semi-simple blocks.*

1. *For each $(r,s)\in\mathcal{T}$, we denote by $C^{thick}_{r,s}=C^{thick}_{p_+-r,p_--s}$ the full abelian subcategory of $\mathcal{C}_{p_+,p_-}$ such that*

$$
\begin{aligned}
&M\in C^{thick}_{r,s} \\
&\Leftrightarrow \text{ all composition factors of } M \text{ are given by } \mathcal{X}^+_{r,s}, \mathcal{X}^+_{r^\vee,s^\vee}, \\
&\quad\ \mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r,s^\vee} \text{ and } L(h_{r,s}).
\end{aligned}
$$

2. *For each $1\le s\le p_--1$, we denote by $C^{thin}_{p_+,s}$ the full abelian subcategory of $\mathcal{C}_{p_+,p_-}$ such that*

$$
\begin{aligned}
&M\in C^{thin}_{p_+,s} \\
&\Leftrightarrow \text{ all composition factors of } M \text{ are given by } \mathcal{X}^+_{p_+,s} \text{ and } \mathcal{X}^-_{p_+,s^\vee}.
\end{aligned}
$$

3. *For each* $1 \le r \le p_+ - 1$, *we denote by* C^{thin}_{r,p_-} *the full abelian subcategory of* $\mathcal{C}_{p_+,p_-}$ *such that*

$$\begin{aligned} & M \in C^{thin}_{r,p_-} \\ \Leftrightarrow \ & \text{all composition factors of } M \text{ are given by } \mathcal{X}^{+}_{r,p_-} \text{ and } \mathcal{X}^{-}_{r^\vee,p_-}. \end{aligned}$$

4. *We denote by* $C^{\pm}_{p_+,p_-}$ *the full abelian subcategory of* $\mathcal{C}_{p_+,p_-}$ *such that*

$$\begin{aligned} & M \in C^{\pm}_{p_+,p_-} \\ \Leftrightarrow \ & \text{all composition factors of } M \text{ are given by } \mathcal{X}^{\pm}_{p_+,p_-}. \end{aligned}$$

By using Theorem 6.1.6 in Section 6, we can prove the block decomposition of $\mathcal{C}_{p_+,p_-}$ in the same way as Theorem 4.4 in [1]. We omit the proof and state only the result.

Theorem 4.3.3. *The abelian category* $\mathcal{C}_{p_+,p_-}$ *has the following block decomposition*

$$\mathcal{C}_{p_+,p_-} = \bigoplus_{(r,s)\in\mathcal{T}} C^{thick}_{r,s} \oplus \bigoplus_{r=1}^{p_+-1} C^{thin}_{r,p_-} \oplus \bigoplus_{s=1}^{p_--1} C^{thin}_{p_+,s} \oplus C^{+}_{p_+,p_-} \oplus C^{-}_{p_+,p_-}.$$

Chapter 5

Logarithmic $\mathcal{W}_{p_+,p_-}$ modules

In this chapter, by using the logarithmic deformation by J. Fjeistad et al.[27], we construct certain logarithmic $\mathcal{W}_{p_+,p_-}$-modules which correspond to the projective covers of all simple $\mathcal{W}_{p_+,p_-}$-modules $\mathcal{X}^{\pm}_{\bullet,\bullet}$ in the thick blocks and the thin blocks, and we introduce indecomposable modules $\mathcal{Q}(\mathcal{X}^{\pm}_{\bullet,\bullet})_{\bullet,\bullet}$ which become important after this chapter. These logarithmic modules are closely related to certain indecomposable modules of the quantum group $\mathfrak{g}_{p_+,p_-}$ at roots of unity [9],[23].

5.1 Logarithmic deformation

Proposition 5.1.1. *For $r,s\geq 1$, we have the following relation*

$$\alpha_-[Q^{[r]}_+,Q^{[s]}_-(z)]=\alpha_+[Q^{[s]}_-,Q^{[r]}_+(z)].$$

Proof. Recall the definition of the screening currents $Q^{[\bullet]}_{\pm}$ of (3.3.2).

$$\begin{aligned}
&\mathrm{Res}_{z=w}Q^{[r]}_+(z)Q^{[s]}_-(w)\\
&=\mathrm{Res}_{z=w}\int_{\overline{\Gamma}_r(\kappa_+)}\mathrm{d}x_1\cdots\mathrm{d}x_{r-1}\int_{\overline{\Gamma}_s(\kappa_-)}\mathrm{d}y_1\cdots\mathrm{d}y_{s-1}\\
&\qquad\times\frac{1}{(z-w)^2}:e^{\alpha_+\phi(z)+\alpha_-\phi(w)}:Q(zx_1)\cdots Q_+(zx_{r-1})Q_-(wy_1)\cdots Q_-(wy_{s-1})z^{r-1}w^{s-1}\\
&=\int_{\overline{\Gamma}_r(\kappa_+)}\mathrm{d}x_1\cdots\mathrm{d}x_{r-1}\int_{\overline{\Gamma}_s(\kappa_-)}\mathrm{d}y_1\cdots\mathrm{d}y_{s-1}\\
&\qquad\frac{\alpha_+}{\alpha_++\alpha_-}\Big(\frac{\partial}{\partial w}V_{\alpha_++\alpha_-}(w)\Big)Q(wx_1)\cdots Q_+(wx_{r-1})Q_-(wy_1)\cdots Q_-(wy_{s-1})w^{r+s-2}\\
&+\int_{\overline{\Gamma}_r(\kappa_+)}\mathrm{d}x_1\cdots\mathrm{d}x_{r-1}\int_{\overline{\Gamma}_s(\kappa_-)}\mathrm{d}y_1\cdots\mathrm{d}y_{s-1}z^{r-1}V_{\alpha_++\alpha_-}(z)\\
&\qquad\times\frac{\partial}{\partial z}\big(Q_+(zx_1)\cdots Q_+(zx_{r-1})\big)Q_-(wy_1)\cdots Q_-(wy_{s-1})w^{s-1}\Big|_{z=w}\\
&+(r-1)\int_{\overline{\Gamma}_r(\kappa_+)}\mathrm{d}x_1\cdots\mathrm{d}x_{r-1}\int_{\overline{\Gamma}_s(\kappa_-)}\mathrm{d}y_1\cdots\mathrm{d}y_{s-1}\\
&\qquad V_{\alpha_++\alpha_-}(w)Q_+(wx_1)\cdots Q_+(wx_{r-1})Q_-(wy_1)\cdots Q_-(wy_{s-1})w^{r+s-3}.
\end{aligned}\tag{5.1.1}$$

Since

$$\begin{aligned}
&\frac{\partial}{\partial z}\big(Q_+(zx_1)\cdots Q_+(zx_{r-1})\big)\\
&=\sum_{i=1}^{r-1} Q_+(zx_1)\cdots \frac{\partial}{\partial z}Q_+(zx_i)\cdots Q_+(zx_{r-1})\\
&=\sum_{i=1}^{r-1} Q_+(zx_1)\cdots \Big(\frac{1}{z}x_i\frac{\partial}{\partial x_i}Q_+(zx_i)\Big)\cdots Q_+(zx_{r-1})\\
&=\frac{1}{z}\sum_{i=1}^{r-1} Q_+(zx_1)\cdots \Big(\frac{\partial}{\partial x_i}x_iQ_+(zx_i)\Big)\cdots Q_+(zx_{r-1})-\frac{r-1}{z}Q_+(zx_1)\cdots Q_+(zx_{r-1}),
\end{aligned}$$

the second term of (5.1.1) becomes

$$\begin{aligned}
&-(r-1)\int_{\overline{\Gamma}_r(\kappa_+)}\mathrm{d}x_1\cdots\mathrm{d}x_{r-1}\int_{\overline{\Gamma}_s(\kappa_-)}\mathrm{d}y_1\cdots\mathrm{d}y_{s-1}\\
&\qquad V_{\alpha_++\alpha_-}(w)Q_+(wx_1)\cdots Q_+(wx_{r-1})Q_-(wy_1)\cdots Q_-(wy_{s-1})w^{r+s-3}\\
&+\int_{\overline{\Gamma}_s(\kappa_-)}\mathrm{d}y_1\cdots\mathrm{d}y_{s-1}\int_{\overline{\Gamma}_r(\kappa_+)}V_{\alpha_++\alpha_-}(w)\\
&\quad\times \mathrm{d}_x\Big(\sum_{i=1}^{r-1}x_iQ_+(wx_1)\cdots Q_+(wx_{r-1})\mathrm{d}x_1\cdots\widehat{\mathrm{d}x_i}\cdots\mathrm{d}x_{r-1}\Big)Q_-(wy_1)\cdots Q_-(wy_{s-1})
\end{aligned}\tag{5.1.2}$$

The first term of (5.1.2) cancels with the third term of (5.1.1) and the second term of this equation becomes zero because $\overline{\Gamma}_r(\kappa_+)$ is the twisted cycle. Thus $[Q_+^{[r]},Q_-^{[s]}(w)]$ becomes

$$\begin{aligned}
&\int_{\overline{\Gamma}_r(\kappa_+)}\mathrm{d}x_1\cdots\mathrm{d}x_{r-1}\int_{\overline{\Gamma}_s(\kappa_-)}\mathrm{d}y_1\cdots\mathrm{d}y_{s-1}\\
&\qquad\frac{\alpha_+}{\alpha_++\alpha_-}\Big(\frac{\partial}{\partial w}V_{\alpha_++\alpha_-}(w)\Big)Q(wx_1)\cdots Q_+(wx_{r-1})Q_-(wy_1)\cdots Q_-(wy_{s-1})w^{r+s-2}.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
&[Q_-^{[s]},Q_+^{[r]}(w)]\\
&=\int_{\overline{\Gamma}_r(\kappa_+)}\mathrm{d}x_1\cdots\mathrm{d}x_{r-1}\int_{\overline{\Gamma}_s(\kappa_-)}\mathrm{d}y_1\cdots\mathrm{d}y_{s-1}\\
&\qquad\frac{\alpha_-}{\alpha_++\alpha_-}\Big(\frac{\partial}{\partial w}V_{\alpha_++\alpha_-}(w)\Big)Q(wx_1)\cdots Q_+(wx_{r-1})Q_-(wy_1)\cdots Q_-(wy_{s-1})w^{r+s-2}.
\end{aligned}$$

Therefore we obtain

$$\alpha_-[Q_+^{[r]},Q_-^{[s]}(z)]=\alpha_+[Q_-^{[s]},Q_+^{[r]}(z)].$$

□

Proposition 5.1.2. *For $r,s\geq 1$ the screening operators $Q_+^{[r]}$ and $Q_-^{[s]}$ are $\mathcal{W}_{p_+,p_-}$-homomorphism, that is, for $A\in\mathcal{W}_{p_+,p_-}$ we have*

$$[Q^{[r]},Y(A;z)]=0,\qquad\qquad [Q_-^{[s]},Y(A;z)]=0.$$

Proof. For each generator of $\mathcal{W}_{p_+,p_-}$, we have the following two expressions

$$\begin{aligned} W^+ &= Q_-^{[p_--1]} |\alpha_{1,p_--1;3}\rangle = Q_+^{[3p_+-1]} |\alpha_{p_+-1,1;-3}\rangle\,, \\ W^- &= Q_+^{[p_+-1]} |\alpha_{p_+-1,1;-3}\rangle = Q_-^{[3p_--1]} |\alpha_{1,p_--1;3}\rangle\,, \\ W^0 &= Q_+^{[2p_+-1]} |\alpha_{p_+-1,1;-3}\rangle = Q_-^{[2p_--1]} |\alpha_{1,p_--1;3}\rangle\,, \end{aligned}$$

up to non-zero constants. Thus, by the proof of Proposition 5.1.1, we obtain

$$[Q^{[r]}, Y(A;z)] = [Q_-^{[s]}, Y(A;z)] = 0.$$

□

We introduce the following logarithmic deformation introduced by J. Fjeistad et al.

Definition 5.1.3 ([27]). *1. Let $E(z)$ and $A(z)$ be any mutually local fields. We define the logarithmic deformation of $A(z)$ by $E(z)$ as follows*

$$\Delta_E(A(z)) = \log z (E[0]A)(z) + \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \frac{(E[n]A)(z)}{z^n},$$

where

$$(E[n]A)(w) = \oint_{z=w} (z-w)^n E(z)A(w)\mathrm{d}z.$$

2. Let $E(z)$, $A(z)$ and $B(z)$ be any mutually local fields. We define

$$\Delta_E(A(z)B(w)) = \sum_{n\in\mathbb{Z}} \frac{\Delta_E((A[n]B)(w))}{(z-w)^{n+1}}.$$

Theorem 5.1.4 ([27]). *Let $E(z)$, $A(z)$ and $B(z)$ be any mutually local fields. Then the operator Δ_E satisfies the following derivation property*

$$\Delta_E(A(z)B(w)) = \Delta_E(A(z))B(w) + \mu A(z)\Delta_E(B(w)),$$

where μ is the mutual locality index of E with A.

In our case, we define the logarithmic deformations by the screening currents $Q_\pm^{[\bullet]}(z)$ and set

$$\Delta_+^{[r]} := \Delta_{Q_+^{[r]}}, \qquad\qquad \Delta_-^{[s]} := \Delta_{Q_-^{[s]}}.$$

Note that, for the energy-momentum tensor, we have

$$\Delta_+^{[r]}(T(z)) = T(z) + \frac{Q_+^{[r]}(z)}{z}, \qquad \Delta_-^{[s]}(T(z)) = T(z) + \frac{Q_-^{[s]}(z)}{z}. \tag{5.1.3}$$

By Proposition 5.1.2, each $\Delta_\pm^{[\bullet]}(Y(A;z)),\ A \in \mathcal{W}_{p_+,p_-}$ do not contain log terms in z.

5.2 Logarithmic modules in the thick block

For each $1 \le r \le p_+ - 1,\ 1 \le s \le p_- - 1$ we set

$$\mathcal{P}_{r,s} = \mathcal{V}^+_{r,s} \oplus \mathcal{V}^+_{r^\vee,s^\vee} \oplus \mathcal{V}^-_{r,s^\vee} \oplus \mathcal{V}^-_{r^\vee,s},$$

where $r^\vee = p_+ - r$ and $s^\vee = p_- - s$. Note that $\mathcal{V}^+_{r,s}, \mathcal{V}^+_{r^\vee,s^\vee}, \mathcal{V}^-_{r,s^\vee}, \mathcal{V}^-_{r^\vee,s} \in C^{thick}_{r,s}$. Let $(\mathcal{P}_{r,s}, Y_{\mathcal{P}_{r,s}})$ be the ordinary $\mathcal{W}_{p_+,p_-}$-module. Fix any element $\tau = (a,b,\epsilon)$ in

$$\{(a,b,\epsilon)\} = \{(r,s,+),(r^\vee,s^\vee,+),(r^\vee,s,-),(r,s^\vee,-)\}.$$

For $A \in \mathcal{W}_{p_+,p_-}$, we define the following operators on $\mathcal{P}_{r,s}$:

$$\begin{aligned}
&\Delta^{[a,b]}_\tau(Y_{\mathcal{P}_{r,s}}(A;z)) \\
&= \begin{cases} (\alpha_- - \alpha_+)\big(\Delta^{[a]}_+ + \Delta^{[b]}_-\big)(Y_{\mathcal{P}_{r,s}}(A;z)) & \\ +\big(-\alpha_+\Delta^{[b]}_- \circ \Delta^{[a]}_+ + \alpha_-\Delta^{[a]}_+ \circ \Delta^{[b]}_-\big)(Y_{\mathcal{P}_{r,s}}(A;z)) & \text{on } \mathcal{V}^\epsilon_{a,b} \\ 0 & \text{on } \mathcal{P}_{r,s} \setminus \mathcal{V}^\epsilon_{a,b}, \end{cases}
\end{aligned}$$

$$\Delta^{[b]}_\tau(Y_{\mathcal{P}_{r,s}}(A;z)) = \begin{cases} \Delta^{[b]}_-(Y_{\mathcal{P}_{r,s}}(A;z)) & \text{on } \mathcal{V}^{-\epsilon}_{a^\vee,b} \\ 0 & \text{on } \mathcal{P}_{r,s} \setminus \mathcal{V}^{-\epsilon}_{a^\vee,b}, \end{cases}$$

$$\Delta^{[a]}_\tau(Y_{\mathcal{P}_{r,s}}(A;z)) = \begin{cases} \Delta^{[a]}_+(Y_{\mathcal{P}_{r,s}}(A;z)) & \text{on } \mathcal{V}^{-\epsilon}_{a,b^\vee} \\ 0 & \text{on } \mathcal{P}_{r,s} \setminus \mathcal{V}^{-\epsilon}_{a,b^\vee}. \end{cases}$$

By the following lemma, we can see that above operators does not contain a $\log z$ terms.

Lemma 5.2.1. *For each* $1 \le r \le p_+ - 1,\ 1 \le s \le p_- - 1$ *and* $A \in \mathcal{W}_{p_+,p_-}$,

$$-\alpha_+\Delta^{[s]}_-\big(\Delta^{[r]}_+(Y(A;z))\big) + \alpha_-\Delta^{[r]}_+\big(\Delta^{[s]}_-(Y(A;z))\big)$$

does not contain log *terms in* z.

Proof. The $\log z$ terms of $\Delta^{[s]}_-\big(\Delta^{[r]}_+(Y(A;z))\big)$ and $\Delta^{[r]}_+\big(\Delta^{[s]}_-(Y(A;z))\big)$ are given by

$$[Q^{[s]}_-, \Delta^{[r]}_+(Y(A;z))]\log z, \qquad [Q^{[r]}_+, \Delta^{[s]}_-(Y(A;z))]\log z.$$

By using Proposition 5.1.1 we have

$$\begin{aligned}
&[Q^{[s]}_-, \Delta^{[r]}_+(Y(A;z))] \\
&= [Q^{[s]}_-, \sum_{n\ge1} \frac{(-1)^{n+1}}{n} \oint_{w=z} \frac{(w-z)^n}{z^n} Q^{[r]}_+(w)Y(A;z)] \\
&= \frac{\alpha_-}{\alpha_+}[Q^{[r]}_+, \sum_{n\ge1} \frac{(-1)^{n+1}}{n} \oint_{w=z} \frac{(w-z)^n}{z^n} Q^{[s]}_-(w)Y(A;z)] \\
&= \frac{\alpha_-}{\alpha_+}[Q^{[r]}_+, \Delta^{[s]}_-(Y(A;z))].
\end{aligned}$$

Therefore

$$-\alpha_+\Delta^{[s]}_-\big(\Delta^{[r]}_+(Y(A;z))\big) + \alpha_-\Delta^{[r]}_+\big(\Delta^{[s]}_-(Y(A;z))\big)$$

does not contain log terms in z. □

Using Theorem 5.1.4, we can define logarithmic defomations of the ordinary $\mathcal{W}_{p_+,p_-}$-module $(\mathcal{P}_{r,s}, Y_{\mathcal{P}_{r,s}})$ as follows.

Theorem 5.2.2. *Fix any* $\tau = (a, b, \epsilon)$ *in*

$$\{(a,b,\epsilon)\} = \{(r,s,+),(r^\vee,s^\vee,+),(r^\vee,s,-),(r,s^\vee,-)\}.$$

We can define the logarithmic $\mathcal{W}_{p_+,p_-}$*-module* $(\mathcal{P}^{\epsilon}_{a^\vee,b^\vee}, J^{\epsilon}_{a^\vee,b^\vee})$ *that have* L_0 *nilpotent rank three as follows. As the vector space* $\mathcal{P}^{\epsilon}_{a^\vee,b^\vee} = \mathcal{P}_{r,s}$ *and the module actions is defined by*

$$J^{\epsilon}_{a^\vee,b^\vee}(A;z) = Y_{\mathcal{P}_{r,s}}(A;z) + \big(\Delta^{[a,b]}_{\tau} + \Delta^{[a]}_{\tau} + \Delta^{[b]}_{\tau}\big)(Y_{\mathcal{P}_{r,s}}(A;z)),$$

for any $A \in \mathcal{W}_{p_+,p_-}$.

Proof. By Lemma 5.2.1, we have $J^{\epsilon}_{a^\vee,b^\vee} : \mathcal{W}_{p_+,p_-} \to \mathrm{End}\mathcal{P}^{\epsilon}_{a^\vee,b^\vee}[[z,z^{-1}]]$. $J^{\epsilon}_{a^\vee,b^\vee}(|0\rangle\,;z) = \mathrm{id}_{\mathcal{P}^{\epsilon}_{a^\vee,b^\vee}}$ is trivial from the definition of logarithmic deformation. In the following we prove the compatibility condition

$$J^{\epsilon}_{a^\vee,b^\vee}(A;z)J^{\epsilon}_{a^\vee,b^\vee}(B;w) = J^{\epsilon}_{a^\vee,b^\vee}(Y(A;z-w)B;w)$$

for $A, B \in \mathcal{W}_{p_+,p_-}$. Fix any non-zero vector $v \in \mathcal{P}_{r,s}$ and write v be as follows

$$v = v^{\epsilon}_{a,b} + v^{-\epsilon}_{a^\vee,b} + v^{-\epsilon}_{a,b^\vee} + v^{\epsilon}_{a^\vee,b^\vee},$$

where $v^{\epsilon}_{a,b} \in \mathcal{V}^{\epsilon}_{a,b}$, $v^{-\epsilon}_{a^\vee,b} \in \mathcal{V}^{-\epsilon}_{a^\vee,b}$, $v^{-\epsilon}_{a,b^\vee} \in \mathcal{V}^{-\epsilon}_{a,b^\vee}$, $v^{\epsilon}_{a^\vee,b^\vee} \in \mathcal{V}^{\epsilon}_{a^\vee,b^\vee}$. By using Theorem 5.1.4 we have

$$\begin{aligned}
&J^{\epsilon}_{a^\vee,b^\vee}(A;z)J^{\epsilon}_{a^\vee,b^\vee}(B;z)v^{\epsilon}_{a,b}\\
&= Y(A;z)Y(B;w)v^{\epsilon}_{a,b}\\
&\quad + (\alpha_- - \alpha_+)\big[\Delta^{[a]}_{+}(Y(A;z)) + \Delta^{[b]}_{-}(Y(A;z))\big]Y(B;w)v^{\epsilon}_{a,b}\\
&\quad + \big[-\alpha_+\Delta^{[b]}_{-}(\Delta^{[a]}_{+}(Y(A;z))) + \alpha_-\Delta^{[a]}_{+}(\Delta^{[b]}_{-}(Y(A;z)))\big]Y(B;w)v^{\epsilon}_{a,b}\\
&\quad + (\alpha_- - \alpha_+)Y(A;z)\big[\Delta^{[a]}_{+}(Y(B;w)) + \Delta^{[b]}_{-}(Y(B;w))\big]v^{\epsilon}_{a,b}\\
&\quad + (\alpha_- - \alpha_+)\big[\Delta^{[b]}_{-}(Y(A;z))\Delta^{[a]}_{+}(Y(B;w)) + \Delta^{[a]}_{+}(Y(A;z))\Delta^{[b]}_{-}(Y(B;w))\big]v^{\epsilon}_{a,b}\\
&\quad + Y(A;z)\big[-\alpha_+\Delta^{[b]}_{-}(\Delta^{[a]}_{+}(Y(B;w))) + \alpha_-\Delta^{[a]}_{+}(\Delta^{[b]}_{-}(Y(B;w)))\big]v^{\epsilon}_{a,b}\\
&= Y(A;z)Y(B;w)v^{\epsilon}_{a,b}\\
&\quad + (\alpha_- - \alpha_+)\big[\Delta^{[a]}_{+}(Y(A;z)Y(B;w)) + \Delta^{[b]}_{-}(Y(A;z)Y(B;w))\big]v^{\epsilon}_{a,b}\\
&\quad - \alpha_+\Delta^{[b]}_{-}\circ\Delta^{[a]}_{+}(Y(A;z)Y(B;w))v^{\epsilon}_{a,b} + \alpha_-\Delta^{[a]}_{+}\circ\Delta^{[b]}_{-}(Y(A;z)Y(B;w))v^{\epsilon}_{a,b}\\
&= Y(Y(A;z-w)B;w)v^{\epsilon}_{a,b} + \big[\big(\Delta^{[a,b]}_{\tau} + \Delta^{[a]}_{\tau} + \Delta^{[b]}_{\tau}\big)(Y(Y(A;z-w)B;w))\big]v^{\epsilon}_{a,b}\\
&= J^{\epsilon}_{a^\vee,b^\vee}(Y(A;z-w)B;w)v^{\epsilon}_{a,b}.
\end{aligned}$$

In the same way, we can prove

$$\begin{aligned}
J^{\epsilon}_{a^\vee,b^\vee}(A;z)J^{\epsilon}_{a^\vee,b^\vee}(B;z)v^{-\epsilon}_{a^\vee,b} &= J^{\epsilon}_{a^\vee,b^\vee}(Y(A;z-w)B;w)v^{-\epsilon}_{a^\vee,b},\\
J^{\epsilon}_{a^\vee,b^\vee}(A;z)J^{\epsilon}_{a^\vee,b^\vee}(B;z)v^{-\epsilon}_{a,b^\vee} &= J^{\epsilon}_{a^\vee,b^\vee}(Y(A;z-w)B;w)v^{-\epsilon}_{a,b^\vee},\\
J^{\epsilon}_{a^\vee,b^\vee}(A;z)J^{\epsilon}_{a^\vee,b^\vee}(B;z)v^{\epsilon}_{a^\vee,b^\vee} &= J^{\epsilon}_{a^\vee,b^\vee}(Y(A;z-w)B;w)v^{\epsilon}_{a^\vee,b^\vee}.
\end{aligned}$$

Therefore we obtain

$$J^{\epsilon}_{a^\vee,b^\vee}(A;z)J^{\epsilon}_{a^\vee,b^\vee}(B;z)v = J^{\epsilon}_{a^\vee,b^\vee}(Y(A;z-w)B;w)v.$$

□

By (5.1.3), we can see that the four logarithmic modules $\mathcal{P}^{\pm}_{\bullet,\bullet} \in C^{thick}_{r,s}$ have L_0 nilpotent rank three.

Remark 5.2.3. *These logarithmic modules $\mathcal{P}^+_{r,s}$, $\mathcal{P}^+_{r^\vee,s^\vee}$, $\mathcal{P}^-_{r^\vee,s}$ and $\mathcal{P}^-_{r,s^\vee}$ correspond to the projective covers of $\mathcal{X}^+_{r,s}$, $\mathcal{X}^+_{r^\vee,s^\vee}$, $\mathcal{X}^-_{r^\vee,s}$ and $\mathcal{X}^-_{r,s^\vee}$, respectively (see Subsection 7.3).*

Remark 5.2.4. *The structure of these logarithmic modules were conjectured in* [32],[33] *in the case $(p_+,p_-)=(2,3)$ and explicit realizations were given by* [7] *in the case of $C^{thick}_{1,1}$ by using lattice constructions (cf.* [3]*). In their notation*

$$\mathcal{P}(1)=\mathcal{P}^+_{1,2},\qquad \mathcal{P}(2)=\mathcal{P}^+_{1,1},\qquad \mathcal{P}(5)=\mathcal{P}^-_{1,2},\qquad \mathcal{P}(7)=\mathcal{P}^-_{1,1}.$$

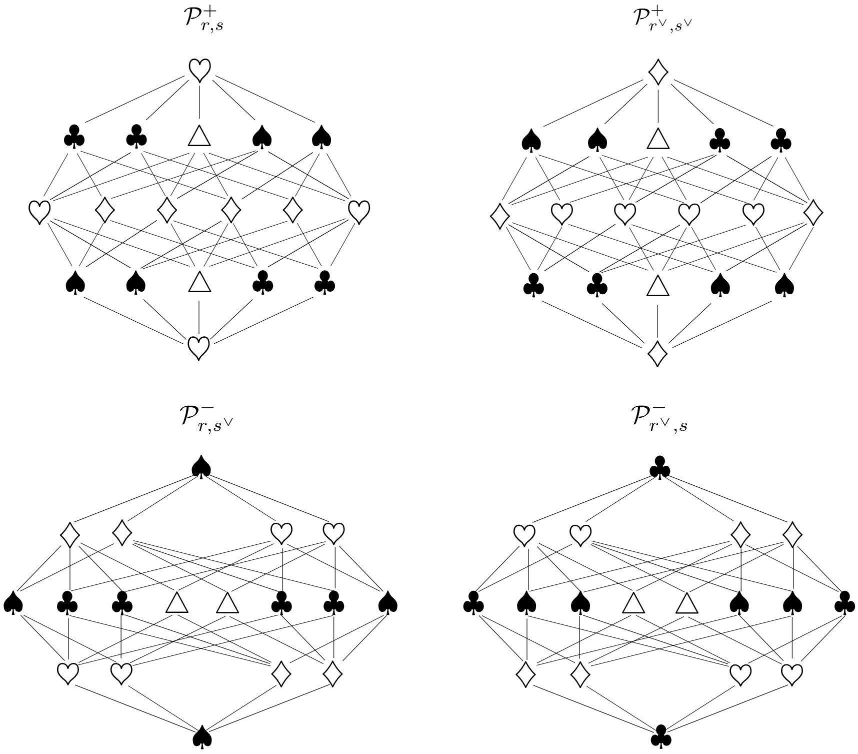


Figure 5.1: The embedding structure of logarithmic $\mathcal{W}_{p_+,p_-}$-modules $\mathcal{P}^{\pm}_{\bullet,\bullet}$. The triangle $\triangle$ corresponds to the simple module $L(h_{r,s})$, $\heartsuit$ to $\mathcal{X}^+_{r,s}$, $\diamondsuit$ to $\mathcal{X}^+_{r^\vee,s^\vee}$, $\spadesuit$ to $\mathcal{X}^-_{r,s^\vee}$ and $\clubsuit$ to $\mathcal{X}^-_{r^\vee,s}$.

Remark 5.2.5. *Figure 5.1 is the embedding structure of the logarithmic $\mathcal{W}_{p_+,p_-}$-modules defined in Theorem 5.2.2.*

Theorem 5.2.6. *By taking quotients of $\mathcal{P}^+_{r,s}$, $\mathcal{P}^+_{r^\vee,s^\vee}$, $\mathcal{P}^-_{r^\vee,s}$ and $\mathcal{P}^-_{r,s^\vee}$, we obtain eight logarithmic modules $\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{b,c}$ where*

$$\begin{aligned}\{(\epsilon,a,b,c,d)\}=\big\{&(+,r,s,r^\vee,s),(+,r,s,r,s^\vee),(+,r^\vee,s^\vee,r^\vee,s),(+,r^\vee,s^\vee,r,s^\vee),\\&(-,r^\vee,s,r,s),(-,r^\vee,s,r^\vee,s^\vee),(-,r,s^\vee,r,s),(-,r,s^\vee,r^\vee,s^\vee)\big\},\end{aligned}$$

and each composition series is given by:

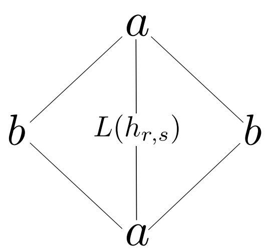
1. *For* $\mathcal{Q}(\mathcal{X}^+_{a,b})_{c,d}$,

$$
\begin{aligned}
&G_1 = \mathcal{X}^+_{a,b},\\
&G_2/G_1 \oplus G_3/G_2 = \mathcal{X}^-_{c,d} \oplus L(h_{a,b}) \oplus \mathcal{X}^-_{c,d},\\
&\mathcal{Q}(\mathcal{X}^+_{a,b})_{c,d}/G_3 = \mathcal{X}^+_{a,b}.
\end{aligned}
$$

2. *For* $\mathcal{Q}(\mathcal{X}^-_{a,b})_{c,d}$,

$$
\begin{aligned}
&G_1 = \mathcal{X}^-_{a,b},\\
&G_2/G_1 \oplus G_3/G_2 = \mathcal{X}^+_{c,d} \oplus \mathcal{X}^+_{c,d},\\
&\mathcal{Q}(\mathcal{X}^-_{a,b})_{c,d}/G_3 = \mathcal{X}^-_{a,b}.
\end{aligned}
$$

Remark 5.2.7. *Figure 5.2 is the embedding structure of the logarithmic $\mathcal{W}_{p_+,p_-}$-modules defined in Theorem 5.2.6.*

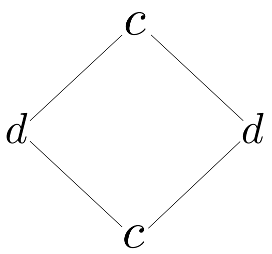


$\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}: \quad a = \mathcal{X}^+_{r,s},\ b = \mathcal{X}^-_{r^\vee,s}$

$\mathcal{Q}(\mathcal{X}^+_{r,s})_{r,s^\vee}: \quad a = \mathcal{X}^+_{r,s},\ b = \mathcal{X}^-_{r,s^\vee}$

$\mathcal{Q}(\mathcal{X}^+_{r^\vee,s^\vee})_{r^\vee,s}: \quad a = \mathcal{X}^+_{r^\vee,s^\vee},\ b = \mathcal{X}^-_{r^\vee,s}$

$\mathcal{Q}(\mathcal{X}^+_{r^\vee,s^\vee})_{r,s^\vee}: \quad a = \mathcal{X}^+_{r^\vee,s^\vee},\ b = \mathcal{X}^-_{r,s^\vee}$



$\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}: \quad c = \mathcal{X}^-_{r^\vee,s},\ d = \mathcal{X}^+_{r,s}$

$\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r^\vee,s^\vee}: \quad c = \mathcal{X}^-_{r^\vee,s},\ d = \mathcal{X}^+_{r^\vee,s^\vee}$

$\mathcal{Q}(\mathcal{X}^-_{r,s^\vee})_{r,s}: \quad c = \mathcal{X}^-_{r,s^\vee},\ d = \mathcal{X}^+_{r,s}$

$\mathcal{Q}(\mathcal{X}^-_{r,s^\vee})_{r^\vee,s^\vee}: \quad c = \mathcal{X}^-_{r,s^\vee},\ d = \mathcal{X}^+_{r^\vee,s^\vee}$

Figure 5.2: The embedding structure of logarithmic $\mathcal{W}_{p_+,p_-}$-modules $\mathcal{Q}(\bullet)_{\bullet,\bullet}$.

5.3 Logarithmic modules in the thin blocks

For each $1 \le r \le p_+ - 1,\ 1 \le s \le p_- - 1$, we set

$$\mathcal{P}_{r,p_-} = \mathcal{V}^+_{r,p_-} \oplus \mathcal{V}^-_{r^\vee,p_-} \in C^{thin}_{r,p_-}, \qquad \mathcal{P}_{p_+,s} = \mathcal{V}^+_{p_+,s} \oplus \mathcal{V}^-_{p_+,s^\vee} \in C^{thin}_{p_+,s}.$$

Let $(\mathcal{P}_{r,p_-}, Y_{\mathcal{P}_{r,p_-}})$ and $(\mathcal{P}_{p_+,s}, Y_{\mathcal{P}_{p_+,s}})$ be the ordinary $\mathcal{W}_{p_+,p_-}$-module. Similar to Theorem 5.2.2, we can construct the following logarithmic modules.

Theorem 5.3.1.

1. *For $1 \leq r \leq p_+ - 1$, we can define the logarithmic modules $(\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}, J^+_{r,p_-})$ and $(\mathcal{Q}(\mathcal{X}^-_{r^\vee,p_-})_{r,p_-}, J^-_{r^\vee,p_-})$ as follows. As the vector spaces*

$$\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-} = \mathcal{Q}(\mathcal{X}^-_{r^\vee,p_-})_{r,p_-} = \mathcal{P}_{r,p_-}$$

and the module actions are defined by

$$J^+_{r,p_-}(A;z) = \begin{cases} Y_{\mathcal{P}_{r,p_-}}(A;z) + \Delta^{[r^\vee]}_+(Y_{\mathcal{P}_{r,p_-}}(A;z)) & \text{on } \mathcal{V}^-_{r^\vee,p_-} \\ Y_{\mathcal{P}_{r,p_-}}(A;z) & \text{on } \mathcal{V}^+_{r,p_-}, \end{cases}$$

$$J^-_{r^\vee,p_-}(A;z) = \begin{cases} Y_{\mathcal{P}_{r,p_-}}(A;z) + \Delta^{[r]}_+(Y_{\mathcal{P}_{r,p_-}}(A;z)) & \text{on } \mathcal{V}^+_{r,p_-} \\ Y_{\mathcal{P}_{r,p_-}}(A;z) & \text{on } \mathcal{V}^-_{r^\vee,p_-}, \end{cases}$$

for $A \in \mathcal{W}_{p_+,p_-}$.

2. *For $1 \leq s \leq p_- - 1$, we can define the logarithmic modules $(\mathcal{Q}(\mathcal{X}^+_{p_+,s})_{p_+,s^\vee}, J^+_{p_+,s})$ and $(\mathcal{Q}(\mathcal{X}^-_{p_+,s^\vee})_{p_+,s}, J^-_{p_+,s^\vee})$ as follows. As the vector spaces*

$$\mathcal{Q}(\mathcal{X}^+_{p_+,s})_{p_+,s^\vee} = \mathcal{Q}(\mathcal{X}^-_{p_+,s^\vee})_{p_+,s} = \mathcal{P}_{p_+,s}$$

and the module actions are defined by

$$J^+_{p_+,s}(A;z) = \begin{cases} Y_{\mathcal{P}_{p_+,s}}(A;z) + \Delta^{[s^\vee]}_-(Y_{\mathcal{P}_{p_+,s}}(A;z)) & \text{on } \mathcal{V}^-_{p_+,s^\vee} \\ Y_{\mathcal{P}_{p_+,s}}(A;z) & \text{on } \mathcal{V}^+_{p_+,s}, \end{cases}$$

$$J^-_{p_+,s^\vee}(A;z) = \begin{cases} Y_{\mathcal{P}_{p_+,s}}(A;z) + \Delta^{[s]}_-(Y_{\mathcal{P}_{p_+,s}}(A;z)) & \text{on } \mathcal{V}^+_{p_+,s} \\ Y_{\mathcal{P}_{p_+,s}}(A;z) & \text{on } \mathcal{V}^-_{p_+,s^\vee}. \end{cases}$$

for $A \in \mathcal{W}_{p_+,p_-}$.

Proposition 5.3.2.

1. *The composition series of $\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}$ is given by*

$$\begin{aligned} &G_1 = \mathcal{X}^+_{r,p_-}, \\ &G_2/G_1 \oplus G_3/G_2 = \mathcal{X}^-_{r^\vee,p_-} \oplus \mathcal{X}^-_{r^\vee,p_-}, \\ &\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}/G_3 = \mathcal{X}^+_{r,p_-}. \end{aligned}$$

2. *The composition series of $\mathcal{Q}(\mathcal{X}^-_{r^\vee,p_-})_{r,p_-}$ is given by*

$$\begin{aligned} &G_1 = \mathcal{X}^-_{r^\vee,p_-}, \\ &G_2/G_1 \oplus G_3/G_2 = \mathcal{X}^+_{r,p_-} \oplus \mathcal{X}^+_{r,p_-}, \\ &\mathcal{Q}(\mathcal{X}^-_{r^\vee,p_-})_{r,p_-}/G_3 = \mathcal{X}^-_{r^\vee,p_-}. \end{aligned}$$

3. *The composition series of* $\mathcal{Q}(\mathcal{X}^+_{p_+,s})_{p_+,s^\vee}$ *is given by*

$$
\begin{aligned}
&G_1 = \mathcal{X}^+_{p_+,s},\\
&G_2/G_1 \oplus G_3/G_2 = \mathcal{X}^-_{p_+,s^\vee} \oplus \mathcal{X}^-_{p_+,s^\vee},\\
&\mathcal{Q}(\mathcal{X}^+_{p_+,s})_{p_+,s^\vee}/G_3 = \mathcal{X}^+_{p_+,s}.
\end{aligned}
$$

4. *The composition series of* $\mathcal{Q}(\mathcal{X}^-_{p_+,s^\vee})_{p_+,s}$ *is given by*

$$
\begin{aligned}
&G_1 = \mathcal{X}^-_{p_+,s^\vee},\\
&G_2/G_1 \oplus G_3/G_2 = \mathcal{X}^+_{p_+,s} \oplus \mathcal{X}^+_{p_+,s},\\
&\mathcal{Q}(\mathcal{X}^-_{p_+,s^\vee})_{p_+,s}/G_3 = \mathcal{X}^-_{p_+,s^\vee}.
\end{aligned}
$$

Remark 5.3.3. *These logarithmic modules* $\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}$*,* $\mathcal{Q}(\mathcal{X}^-_{r^\vee,p_-})_{r,p_-}$*,* $\mathcal{Q}(\mathcal{X}^+_{p_+,s})_{p_+,s^\vee}$ *and* $\mathcal{Q}(\mathcal{X}^-_{p_+,s^\vee})_{p_+,s}$ *correspond to the projective covers of* $\mathcal{X}^+_{r,p_-}$*,* $\mathcal{X}^-_{r^\vee,p_-}$*,* $\mathcal{X}^+_{p_+,s}$ *and* $\mathcal{X}^-_{p_+,s^\vee}$*, respectively (see Subsection 7.5).*

Chapter 6

Logarithmic extension of Virasoro modules

In this chapter, we determine Ext^1-groups between simple Virasoro modules and certain indecomposable modules in the abelian category of generalized Virasoro modules, by using the results in [55] and the structure of Fock modules. The results of this chapter will be crucial in analyzing the complex structure of logarithmic $\mathcal{W}_{p_+,p_-}$-modules. From this chapter, we identify any Virasoro modules that are isomorphic to each other, unless otherwise stated.

6.1 Ext^1-groups between simple Virasoro modules

We set

$$A_{p_+,p_-} := \{\ \alpha_{r,s;n} \mid r,s,n \in \mathbb{Z}\ \},$$
$$H_{p_+,p_-} := \left\{\ h_\alpha \ \middle|\ \alpha \in A_{p_+,p_-}\ \right\}$$

(for the definition of symbols $\alpha_{r,s;n}$ and h_α, see (3.2.1) and (3.1.1), respectively). Let $U(\mathcal{L})$ be the universal enveloping algebra of the Virasoro algebra.

Definition 6.1.1. *Let $\mathcal{L}_{c_{p_+,p_-}}$-Mod be the abelian category of left generalized $U(\mathcal{L})$-modules whose morphisms are Virasoro-homomorphisms and whose objects are left $U(\mathcal{L})$-modules that satisfy the following conditions:*

1. *For the central charge, $C = c_{p_+,p_-} \cdot \mathrm{id}$ on M.*
2. *Every object M has the following decomposition $M = \sum_{h\in H(M)} M[h]$:*
   - *For some finite subset $H_0(M)$ of $\mathbb{C}$, $H(M) = H_0(M) + \mathbb{Z}_{\geq 0}$.*
   - *For $h \in H(M)$, $M[h] = \{m \in M : \exists n \geq 0 \text{ s.t. } (L_0 - h)^n m = 0\}$.*
   - *$0 < \dim_{\mathbb{C}} M[h] < \infty$.*
3. *For every object $M \in \mathcal{L}_{c_{p_+,p_-}}$-Mod, there exists the contragredient object $M^* \in \mathcal{L}_{c_{p_+,p_-}}$-Mod on which the anti-involution $\sigma(L_n) = L_{-n}$ induces the structure of a left $U(\mathcal{L})$-module by*

$$\langle L_n\phi, u\rangle = \langle \phi, \sigma(L_n)u\rangle, \quad \phi \in M^*,\ u \in M.$$

Definition 6.1.2. *We define $\mathcal{L}_{c_{p_+,p_-}}$-mod to be the full subcategory of $\mathcal{L}_{c_{p_+,p_-}}$-Mod such that all objects in $\mathcal{L}_{c_{p_+,p_-}}$-mod satisfy the following conditions:*

1. *The socle series of M has finite length.*

2. *The highest weights h of the simple modules $L(h)$, appearing in the composition factors of M, are elements of H_{p_+,p_-}.*

We denote the n-th Ext-groups in $\mathcal{L}_{c_{p_+,p_-}}$-mod as $\mathrm{Ext}^n_{\mathcal{L}}(\bullet,\bullet)$.

For each $a,b\geq 1$, let $M(h_{a,b},c_{p_+,p_-})$ be the Verma module of the Virasoro algebra whose highest weight is $h_{a,b}$ and the central charge $C=c_{p_+,p_-}\cdot\mathrm{id}$. Note that $M(h_{a,b},c_{p_+,p_-})$ has the singular vector whose L_0-weight is $h_{a,b}+ab$. Let $S_{a,b}\in U(\mathcal{L})$ be the Shapovalov element corresponding to this singular vector, normalized as

$$S_{a,b}\left|h_{a,b}\right\rangle = (L_{-1}^{ab}+\cdots)\left|h_{a,b}\right\rangle, \tag{6.1.1}$$

and let $S^*_{a,b}=\sigma(S_{a,b})$ be the anti-involution of $S_{a,b}$ where $\sigma(L_n)=L_{-n},\ n\in\mathbb{Z}$.

The following theorem is due to [67].

Theorem 6.1.3. *For $r,s\geq 1$, let us consider $S^*_{r,s}S_{r,s}$ in $U(\mathcal{L})$ transformed as*

$$S^*_{r,s}S_{r,s}-f(L_0,C)\in\mathcal{L}_-U(\mathcal{L}_-)\otimes_{\mathbb{C}}U(\mathcal{L}_0)\otimes_{\mathbb{C}}U(\mathcal{L}_+)\mathcal{L}_+,$$

where $f(X,Y)$ is a non-zero polynomial of X,Y, $\mathcal{L}_\pm:=\bigoplus_{\pm n>0}\mathbb{C}L_n$ and $\mathcal{L}_0:=\mathbb{C}C\oplus\mathbb{C}L_0$. Then, for the polynomial $f(X,Y)$, we have

$$f(h,c_{p_+,p_-})=R_{r,s}(h-h_{r,s})+O((h-h_{r,s})^2),$$

where $R_{r,s}$ is given by

$$R_{r,s}=2\prod_{\substack{(k,l)\in\mathbb{Z}^2,\\ 1-r\leq k\leq r,1-s\leq l\leq s,\\ (k,l)\neq(0,0),(r,s)}}\Big(k\big(\frac{p_+}{p_-}\big)^{-\frac12}+l\big(\frac{p_+}{p_-}\big)^{\frac12}\Big).$$

Remark 6.1.4. *In this thesis, it is important that $R_{r,s}$ be non-zero, specific value is not necessary. In fact, the non-triviality of $R_{r,s}$ can be shown using the Jantzen-filtration of the Fock module $F_{r,s}$.*

By using Theorem 6.1.3, we obtain the following theorem (cf.[35]).

Theorem 6.1.5. *For $h\in H_{p_+,p_-}$, we have*

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h),L(h))=0.$$

Proof. We prove only for case $h=h_{r,s},(r,s)\in\mathcal{T}$. The other cases can be proved in the same way.

Assume $\mathrm{Ext}^1_{\mathcal{L}}(L(h_{r,s}),L(h_{r,s}))\neq 0$. Fix a non-trivial extension

$$0\to L(h_{r,s})\xrightarrow{\iota}E\xrightarrow{\pi}L(h_{r,s})\to 0.$$

Let $\{u_0,u_1\}$ be a basis of the highest weight space of E such that

$$\pi(u_0)=\left|h_{r,s}\right\rangle,\qquad \iota(\left|h_{r,s}\right\rangle)=u_1,$$
$$(L_0-h_{r,s})u_0=cu_1,$$

where c is a non-zero constant and $|h_{r,s}\rangle$ is the highest weight vector of $L(h_{r,s})$. Then, by Theorem 6.1.3, we have

$$S^*_{r,s}S_{r,s}u_0 = f(c)u_1,$$
$$\frac{f(c)}{c}\Big|_{c=0} \neq 0,$$

where $f(c)$ is a polynomial of c. Thus, we see that $S_{r,s}u_0$ is non-zero and

$$S_{r,s}u_0 \in \iota(L(h_{r,s})).$$

On the other hand, by the irreducibility of $L(h_{r,s})$, we have

$$S^*_{r,s}S_{r,s}u_0 = 0.$$

But this is a contradiction. □

For $h, h' \in H_{p_+,p_-}$, $h \neq h'$, let us consider a extension $[E] \in \mathrm{Ext}^1_{\mathcal{L}}(L(h), L(h'))$. Since $h \neq h'$, we see that the Virasoro zero mode L_0 acts semisimply on E. Thus, according to [12],[48], and by Theorem 6.1.5, we have the following theorem for the Ext^1-groups between the irreducible modules for the different highest weights.

Theorem 6.1.6. *For $\mathrm{Ext}^1_{\mathcal{L}}(L(h_{r,s;n}), L(h))$, $h_{r,s;n} \neq h$, $h \in H_{p_+,p_-}$, we have:*

1. *For $1 \leq r < p_+, 1 \leq s < p_-$ and $n = 0$, we have*

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h_{r,s}), L(h)) = \begin{cases} \mathbb{C} & \text{for } h = h_{r^\vee,s;-1} \text{ or } h_{r^\vee,s;1} \\ 0 & \text{otherwise} \end{cases}.$$

2. *For $1 \leq r < p_+, 1 \leq s < p_-$ and $n \geq 1$, we have*

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h_{r,s;n}), L(h)) = \begin{cases} \mathbb{C} & \text{for } h = h_{r^\vee,s;n-1}, h_{r,s^\vee;n-1}, h_{r^\vee,s;n+1} \text{ or } h_{r,s^\vee;n+1} \\ 0 & \text{otherwise} \end{cases}.$$

3. *For $1 \leq r < p_+, s = p_-$ and $n = 0$, we have*

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h_{r,p_-}), L(h)) = \begin{cases} \mathbb{C} & \text{for } h = h_{r^\vee,p_-;1} \\ 0 & \text{otherwise} \end{cases}.$$

4. *For $1 \leq r < p_+, s = p_-$ and $n \geq 1$, we have*

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h_{r,p_-;n}), L(h)) = \begin{cases} \mathbb{C} & \text{for } h = h_{r^\vee,p_-;n+1} \text{ or } h_{r^\vee,p_-;n-1} \\ 0 & \text{otherwise} \end{cases}.$$

5. *For $r = p_+, 1 \leq s < p_-$ and $n = 0$, we have*

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h_{p_+,s}), L(h)) = \begin{cases} \mathbb{C} & \text{for } h = h_{p_+,s^\vee;-1} \\ 0 & \text{otherwise} \end{cases}.$$

6. *For $r = p_+, 1 \leq s < p_-$ and $n \leq -1$, we have*

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h_{p_+,s;n}), L(h)) = \begin{cases} \mathbb{C} & \text{for } h = h_{p_+,s^\vee;n-1} \text{ or } h_{p_+,s^\vee;n+1} \\ 0 & \text{otherwise} \end{cases}.$$

7. *For $r = p_+, s = p_-, n \in \mathbb{Z}$, we have*

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h_{p_+,p_-;n}), L(h)) = 0.$$

6.2 Logarithmic extensions

Let us define the following indecomposable modules in $\mathcal{L}_{p_+,p_-}$-mod as quotient modules of certain Virasoro Verma modules.

Definition 6.2.1. *For $h, h' \in H_{p_+,p_-}$ such that $\mathrm{Ext}^1_{\mathcal{L}}(L(h), L(h')) \simeq \mathbb{C}$ and $h < h'$, we define the following indecomposable module*

$$[L(h,h')] \in \mathrm{Ext}^1_{\mathcal{L}}(L(h), L(h')) \setminus \{0\}.$$

Definition 6.2.2. *For $h, h', h'' \in H_{p_+,p_-}$ such that $\mathrm{Ext}^1_{\mathcal{L}}(L(h), L(h')) \simeq \mathbb{C}$, $\mathrm{Ext}^1_{\mathcal{L}}(L(h), L(h'')) \simeq \mathbb{C}$ and $h > h'$, $h > h''$, we define $L^d(h) \in \mathcal{L}_{p_+,p_-}$-mod as a unique indecomposable module satisfying the following exact sequence*

$$0 \to L(h', h) \to L^d(h) \to L(h'') \to 0.$$

Definition 6.2.3. *For $h, h', h'' \in H_{p_+,p_-}$ such that $\mathrm{Ext}^1_{\mathcal{L}}(L(h), L(h')) \simeq \mathbb{C}$, $\mathrm{Ext}^1_{\mathcal{L}}(L(h), L(h'')) \simeq \mathbb{C}$ and $h < h'$, $h < h''$, we define $L^u(h) \in \mathcal{L}_{p_+,p_-}$-mod as a unique indecomposable module satisfying the following exact sequence*

$$0 \to L(h'') \to L^u(h) \to L(h, h') \to 0.$$

The following theorems are due to [55] (see also [16]).

Theorem 6.2.4 ([55]). *For any $h_1, h_2, h_3 \in H_{p_+,p_-}$ such that $h_1 < h_2 < h_3$, $\mathrm{Ext}^1_{\mathcal{L}}(L(h_1), L(h_2)) \neq 0$ and $\mathrm{Ext}^1_{\mathcal{L}}(L(h_2), L(h_3)) \neq 0$, let E be any logarithmic module satisfying the following exact sequence*

$$0 \to L(h_1, h_2) \to E \to L(h_2, h_3) \to 0.$$

Then the quotient module $E/L(h_2)$ is indecomposable.

Theorem 6.2.5 ([55]). *For any indecomposable modules $L^d(h)$ and $L^u(h)$, let E be any logarithmic module satisfying the following exact sequence*

$$0 \to L^d(h) \to E \to L^u(h) \to 0.$$

Then there is no injection from $L^u(h)$ to $E/L(h)$.

Remark 6.2.6. *The non-vanishingness of certain logarithmic Virasoro modules was proved in [55]. The two theorems above are their consequences.*

In the following, we introduce indecomposable modules $K(\tau)$ and $K(\Delta_{r,s;n})$, and determine the Ext^1-groups of types

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\tau), L(h_{\alpha_2})), \qquad \mathrm{Ext}^1_{\mathcal{L}}(K(\Delta_{r,s;n}), L(\Delta_{r,s;n})).$$

Definition 6.2.7. *We define $\mathcal{T}_{p_+,p_-}$ to be the subset of $A^3_{p_+,p_-}$ such that every element $(\alpha_1, \alpha_2, \alpha_3) \in A^3_{p_+,p_-}$ satisfies the following conditions:*

1. $h_{\alpha_1} \le h_{\alpha_2} < h_{\alpha_3}$.

2. *The three Fock modules F_{α_1}, F_{α_2} and F_{α_3} are contained in the same Felder complex in Proposition 3.3.3 and are adjacent to each other:*

$$\cdots \to F_{\alpha_1} \xrightarrow{Q_\epsilon^{[\bullet]}} F_{\alpha_2} \xrightarrow{Q_\epsilon^{[\bullet]}} F_{\alpha_3} \to \cdots .$$

For example, we have

$$\tau = (\alpha_{p_+,s^\vee;1}, \alpha_{p_+,s;0}, \alpha_{p_+,s^\vee;-1}) \in \mathcal{T}_{p_+,p_-}, \qquad h_{\alpha_{p_+,s^\vee;1}} = h_{\alpha_{p_+,s;0}}.$$

Definition 6.2.8.

1. *For any $\tau = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}_{p_+,p_-}$ such that $h_{\alpha_1} = h_{\alpha_2}$, we define $K(\tau) = L(h_{\alpha_2}, h_{\alpha_3})$.*
2. *For any $\tau = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}_{p_+,p_-}$ such that $h_{\alpha_1} \neq h_{\alpha_2}$, we define $K(\tau) \in \mathcal{L}_{p_+,p_-}$-mod as a unique indecomposable module satisfying the following exact sequence*

$$0 \to L(h_{\alpha_1}) \to K(\tau) \to L(h_{\alpha_2}, h_{\alpha_3}) \to 0.$$

Theorem 6.2.9. *For any $\tau = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}_{p_+,p_-}$, we have*

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\tau), L(h_{\alpha_2})) = \mathbb{C}.$$

Proof. Fix any $\tau = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}_{p_+,p_-}$. From the Virasoro module structure of the logarithmic $\mathcal{W}_{p_+,p_-}$-modules $\mathcal{Q}(\mathcal{X}^\pm_{r,s})_{\bullet,\bullet}$ defined in Chapter 5 (see also Remark 6.2.14), we have a non-trivial logarithmic Virasoro module in

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\tau), L(h_{\alpha_2})) \setminus \{0\}.$$

Fix any logarithmic Virasoro module in this Ext^1-group and denote it by $P(\tau)$. If $h_{\alpha_1} = h_{\alpha_2}$, then we obtain the claim of theorem claim by Theorems 6.1.5 and 6.1.6, and thus let $h_{\alpha_1} \neq h_{\alpha_2}$. It is sufficient to show that $\mathrm{Ext}^1_{\mathcal{L}}(P(\tau), L(h_{\alpha_2})) = 0$. Let us assume that

$$\mathrm{Ext}^1_{\mathcal{L}}(P(\tau), L(h_{\alpha_2})) \neq 0. \tag{6.2.1}$$

Note that, by Theorem 6.2.4, $P(\tau)$ has $L(h_{\alpha_1}, h_{\alpha_2})$ as a submodule. Then, by the exact sequence

$$0 \to L(h_{\alpha_1}, h_{\alpha_2}) \to P(\tau) \to L(h_{\alpha_2}, h_{\alpha_3}) \to 0$$

and by the assumption (6.2.1), we have

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h_{\alpha_2}, h_{\alpha_3}), L(h_{\alpha_2})) \neq 0.$$

Let E any non-trivial extension of $\mathrm{Ext}^1_{\mathcal{L}}(L(h_{\alpha_2}, h_{\alpha_3}), L(h_{\alpha_2}))$. Then, by Theorem 6.1.5, E must have $L(h_{\alpha_2}, h_{\alpha_3})^*$ as a submodule. By the exact sequence

$$0 \to L(h_{\alpha_2}, h_{\alpha_3})^* \to E \to L(h_{\alpha_2}) \to 0,$$

we have the following exact sequence

$$0 \to \mathbb{C} \to \mathrm{Ext}^1_{\mathcal{L}}(E, L(h_{\alpha_1})).$$

Thus we have $\mathrm{Ext}^1_{\mathcal{L}}(E, L(h_{\alpha_1})) \neq 0$. Let F be the non-trivial extension of $\mathrm{Ext}^1_{\mathcal{L}}(L(h_{\alpha_1}), E^*)$. By Theorem 6.1.3, we see that F is logarithmic, that is, F has L_0 nilpotent rank two. Note that

$$F/L(h_{\alpha_2}) = L(h_{\alpha_1}) \oplus L(h_{\alpha_2}, h_{\alpha_3}).$$

But this contradicts Theorem 6.2.4.

□

Remark 6.2.10. *Fix any* $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}_{p_+,p_-}$ *such that* $h_{\alpha_1} = h_{\alpha_2}$. *Then, by Theorem 6.1.5, the logarithmic module* $P(\tau)$ *(defined in Theorem 6.2.9) is self-contragredient.*

Definition 6.2.11. *For* $1 \leq r \leq p_+ - 1, 1 \leq s \leq p_- - 1$, $n \geq 1$, *we define* $K(\Delta^-_{r,s;n-1})$ *and* $K(\Delta^+_{r,s;n})$ *as unique indecomposable modules satisfying the following exact sequences*

$$\begin{aligned} &0 \to L(\Delta^+_{r,s^\vee;n}) \oplus L(\Delta^+_{r,s^\vee;n-1}) \oplus L(\Delta^+_{r^\vee,s;n}) \oplus L(\Delta^+_{r^\vee,s;n-1}) \\ &\quad \to K(\Delta^-_{r,s;n-1}) \to L(\Delta^-_{r,s;n-1}) \to 0, \\ &0 \to L(\Delta^-_{r,s^\vee;n}) \oplus L(\Delta^-_{r,s^\vee;n-1}) \oplus L(\Delta^-_{r^\vee,s;n}) \oplus L(\Delta^-_{r^\vee,s;n-1}) \\ &\quad \to K(\Delta^+_{r,s;n}) \to L(\Delta^+_{r,s;n}) \to 0. \end{aligned}$$

Theorem 6.2.12. *For* $1 \leq r \leq p_+ - 1, 1 \leq s \leq p_- - 1$, $n \geq 1$, $\epsilon = \pm$, *we have*

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}}), L(\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}})) = \mathbb{C}^2,$$

where $\delta_{-,-} = 1$ *and* $\delta_{+,-} = 0$.

Fix any non-trivial extension $[E] \in \mathrm{Ext}^1_{\mathcal{L}}(K(\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}}), L(\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}}))$ *and let* v *be a generator of* E *such that* $v \in E[\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}}]$. *Then we have*

$$S_{r^\vee,s^\vee+(2n-\delta_{\epsilon,-})p_-} S^*_{r^\vee,s^\vee+(2n-\delta_{\epsilon,-})p_-} v \neq 0,$$

or

$$S_{r^\vee+(2n-\delta_{\epsilon,-})p_+,s^\vee} S^*_{r^\vee+(2n-\delta_{\epsilon,-})p_+,s^\vee} v \neq 0.$$

Proof. By the Virasoro module structure of the logarithmic $\mathcal{W}_{p_+,p_-}$-module $\mathcal{P}^\epsilon_{r,s}$, we have a indecomposable module in

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}}), L(\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}}) \oplus L(\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}}))$$

(see the structure of $\mathcal{W}_{p_+,p_-}$-modules $\mathcal{P}^{+u}_{r,s}$ and $\mathcal{P}^{-u}_{r,s}$ in Definitions 7.3.17 and 7.3.22). We denote by $\widetilde{P}(\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}})$ this indecomposable module. Note that, by Theorem 6.2.9, $\widetilde{P}(\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}})$ has $P(\tau_1)$ and $P(\tau_2)$ as subquotients, where

$$\begin{aligned} \tau_1 &= (\alpha_{r^\vee,s^\vee;-2n+\delta_{\epsilon,-}}, \alpha_{r^\vee,s;-2n-1+\delta_{\epsilon,-}}, \alpha_{r^\vee,s^\vee;-2n-2+\delta_{\epsilon,-}}), \\ \tau_2 &= (\alpha_{r^\vee,s^\vee;2n-\delta_{\epsilon,-}}, \alpha_{r,s^\vee;2n+1-\delta_{\epsilon,-}}, \alpha_{r^\vee,s^\vee;2n+2-\delta_{\epsilon,-}}). \end{aligned}$$

Similar to the proof of Theorem 6.2.9, we can show

$$\mathrm{Ext}^1_{\mathcal{L}}(\widetilde{P}(\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}}), L(\Delta^\epsilon_{r,s;n-\delta_{\epsilon,-}})) = 0$$

by using Theorem 6.2.5.

□

Definition 6.2.13. *Let* $\mathcal{T}^{\mathrm{Min}}_{p_+,p_-}$ *be the subset of* $\mathcal{T}_{p_+,p_-}$ *defined by*

$$\mathcal{T}^{\mathrm{Min}}_{p_+,p_-} = \{\ (\alpha_1,\alpha_2,\alpha_3) \in \mathcal{T}_{p_+,p_-} |\ \alpha_1 = \alpha_{r,s},\ 1 \le r < p_+,\ 1 \le s < p_-\}.$$

Recall that $P(\tau)$ ($\tau = (\alpha_1,\alpha_2,\alpha_3) \in \mathcal{T}_{p_+,p_-}$) is the logarithmic module defined by the following exact sequence

$$0 \to L(h_{\alpha_2}) \to P(\tau) \to K(\tau) \to 0.$$

In the following, we will prove the following theorem.

Theorem 6.2.14. *For any* $\tau = (\alpha_1,\alpha_2,\alpha_3) \in \mathcal{T}^{\mathrm{Min}}_{p_+,p_-}$*, we have*

$$\mathrm{Socle}(P(\tau)) = L(h_{\alpha_2}).$$

Before the proof of Theorem 6.2.14, we will introduce some definitions and propositions as follows.

Definition 6.2.15. *For any* $r,s \in \mathbb{Z}_{\ge 1}$, $k,n \in \mathbb{Z}$*, we define the following* $\mathbb{C}$*-linear operators:*

1. *Let* $\Lambda^{[r]}_{+;k;n} :\ U(\mathcal{L}) \to \mathrm{Hom}_{\mathbb{C}}(F_{r,k;n}, F_{r^\vee,k;n+1})$ *be as follows*

$$\Lambda^{[r]}_{+;k;n}(A) = \lim_{t\to 0}\frac{1}{t}[Q^{[r]}_+, e^{-t\hat{a}}Ae^{t\hat{a}}], \qquad \text{for } A \in U(\mathcal{L}),$$

where $F_{\alpha_{r,k;n}+t}, F_{\alpha_{r^\vee,k;n+1}+t} \in \mathcal{L}_{p_+,p_-}$-Mod *for all* $t \in \mathbb{C}$.

2. *Let* $\Lambda^{[s]}_{-;k;n} :\ U(\mathcal{L}) \to \mathrm{Hom}_{\mathbb{C}}(F_{k,s;n}, F_{k,s^\vee;n-1})$ *be as follows*

$$\Lambda^{[s]}_{-;k;n}(A) = \lim_{t\to 0}\frac{1}{t}[Q^{[s]}_-, e^{-t\hat{a}}Ae^{t\hat{a}}], \qquad \text{for } A \in U(\mathcal{L}).$$

where $F_{\alpha_{k,s;n}+t}, F_{\alpha_{k,s^\vee;n-1}+t} \in \mathcal{L}_{p_+,p_-}$-Mod *for all* $t \in \mathbb{C}$.

From now on, we omit k,n and denote

$$\Lambda^{[r]}_+ = \Lambda^{[r]}_{+;k;n}, \qquad\qquad \Lambda^{[s]}_- = \Lambda^{[s]}_{-;k;n}.$$

Proposition 6.2.16. *The two operators* $\Lambda^{[r]}_+$ *and* $\Lambda^{[s]}_-$ *satisfy the following derivation property*

$$\begin{aligned}\Lambda^{[r]}_+(AB) &= \Lambda^{[r]}_+(A)B + A\Lambda^{[r]}_+(B), \quad A,B \in U(\mathcal{L}),\\ \Lambda^{[s]}_-(AB) &= \Lambda^{[s]}_-(A)B + A\Lambda^{[s]}_-(B), \quad A,B \in U(\mathcal{L}).\end{aligned}$$

Proof. For any $A,B \in U(\mathcal{L})$, we have

$$\begin{aligned}&[Q^{[r]}_+, e^{-t\hat{a}}ABe^{t\hat{a}}]\\ &= [Q^{[r]}_+, e^{-t\hat{a}}Ae^{t\hat{a}} \cdot e^{-t\hat{a}}Be^{t\hat{a}}]\\ &= [Q^{[r]}_+, e^{-t\hat{a}}Ae^{t\hat{a}}]e^{-t\hat{a}}Be^{t\hat{a}} + e^{-t\hat{a}}Ae^{t\hat{a}}[Q^{[r]}_+, e^{-t\hat{a}}Be^{t\hat{a}}]\\ &= [Q^{[r]}_+, e^{-t\hat{a}}Ae^{t\hat{a}}]B + A[Q^{[r]}_+, e^{-t\hat{a}}Be^{t\hat{a}}]\\ &\quad + [Q^{[r]}_+, e^{-t\hat{a}}Ae^{t\hat{a}}](e^{-t\hat{a}}Be^{t\hat{a}} - B) + (e^{-t\hat{a}}Ae^{t\hat{a}} - A)[Q^{[r]}_+, e^{-t\hat{a}}Be^{t\hat{a}}].\end{aligned}$$

Dividing both sides by t and taking the limit, we have the derivation property. □

Fix any $\tau = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}_{p_+,p_-}$ such that F_{α_1} is of braided type. Let $(k_\tau, l_\tau) \in \mathbb{Z}^2$ be a unique integer pair such that

$$(1 \le k_\tau < p_+ \wedge 1 \le l_\tau < p_-) \wedge (Q_+^{[k_\tau]} \text{ and } Q_-^{[l_\tau]} \text{ are screening operators on } F(\tau)).$$

Note that $F_{\alpha_2} = F_{\alpha_1 + k_\tau \alpha_+}$ or $F_{\alpha_1 + l_\tau \alpha_-}$. We set

$$F(\tau) = F_{\alpha_1} \oplus F_{\alpha_1 + k_\tau \alpha_+} \oplus F_{\alpha_1 + l_\tau \alpha_-}.$$

For $A \in U(\mathcal{L})$, we define the following operator $J_\tau(A)$ on $F(\tau)$:

$$J_\tau(A) = \begin{cases} A + \Lambda_+^{[k_\tau]}(A) + \Lambda_-^{[l_\tau]}(A) & \text{on } F_{\alpha_1}, \\ A & \text{on } F_{\alpha_1 + k_\tau \alpha_+} \oplus F_{\alpha_1 + l_\tau \alpha_-}. \end{cases}$$

Then, by Proposition 6.2.16, we have

$$J_\tau(AB) = J_\tau(A) J_\tau(B), \quad \text{for any } A, B \in U(\mathcal{L}).$$

Thus we see that J_τ defines a structure of Virasoro module on $F(\tau)$. In the following, we omit the action J_τ of the logarithmic module $(F(\tau), J_\tau)$, and simply denoted as $F(\tau)$. $F(\tau)$ has L_0-nilpotent rank two. In fact we have the following proposition.

Proposition 6.2.17. *Fix any $\tau = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}_{p_+,p_-}$ such that F_{α_1} is of braided type. Let v be any non-zero vector of F_{α_1} and let h_v be the L_0 weight of v. Then we have*

$$(J_\tau(L_0) - h_v)v = -k_\tau \alpha_+ Q_+^{[k_\tau]}(v) - l_\tau \alpha_- Q_-^{[l_\tau]}(v).$$

Proof. Note that the ordinary action of L_0 on the Fock modules is given by

$$L_0 = \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_m a_{-m} : -\frac{1}{2} \alpha_0 a_0. \tag{6.2.2}$$

Then, by (3.3.1), (6.2.2) and $[Q_\epsilon^{[k]}, L_0] = 0$, we have

$$\begin{aligned}
&(J_\tau(L_0) - L_0)v \\
&= \Lambda_+^{[k_\tau]}(L_0)v + \Lambda_-^{[l_\tau]}(L_0)v \\
&= \lim_{t \to 0} \frac{1}{t} (Q_+^{[k_\tau]} + Q_-^{[l_\tau]}) e^{-t\hat{a}} L_0 e^{t\hat{a}} v - \lim_{t \to 0} \frac{1}{t} e^{-t\hat{a}} L_0 e^{t\hat{a}} (Q_+^{[k_\tau]} + Q_-^{[l_\tau]}) v \\
&= \lim_{t \to 0} \frac{1}{t} (Q_+^{[k_\tau]} + Q_-^{[l_\tau]}) t a_0 v - \lim_{t \to 0} \frac{1}{t} t a_0 (Q_+^{[k_\tau]} + Q_-^{[l_\tau]}) v \\
&= (\alpha_1 - \alpha_1 - k_\tau \alpha_+) Q_+^{[k_\tau]}(v) + (\alpha_1 - \alpha_1 - l_\tau \alpha_-) Q_-^{[l_\tau]}(v) \\
&= -k_\tau \alpha_+ Q_+^{[k_\tau]}(v) - l_\tau \alpha_- Q_-^{[l_\tau]}(v).
\end{aligned}$$

□

Proof of Theorem 6.2.14. For $\tau = (\alpha_{r^\vee,s^\vee}, \alpha_{r,s^\vee;1}, \alpha_{r^\vee,s^\vee;2}) \in \mathcal{T}_{p_+,p_-}^{\mathrm{Min}}$, let us consider the logarithmic module $F(\tau)$. Let v be a cosingular vector in $F_{r^\vee,s^\vee}[r^\vee s^\vee]$ defined by

$$v = \lim_{t \to 0} \frac{1}{t} S_{r^\vee,s^\vee} \left| \alpha_{r^\vee,s^\vee} + t \right\rangle.$$

By Proposition 6.2.17, we have

$$(J_\tau(L_0) - h_{r,s^\vee;1})v = (\alpha_{r^\vee,s^\vee} - \alpha_{r,s^\vee;1})Q_+^{[r^\vee]}(v) + (\alpha_{r^\vee,s^\vee} - \alpha_{r^\vee,s;-1})Q_-^{[s^\vee]}(v). \tag{6.2.3}$$

in $F(\tau)$. By using Theorem 6.1.3 we have

$$J_\tau(S_{r^\vee,s^\vee}S^*_{r^\vee,s^\vee})v = \frac{1}{2}(2\alpha_{r^\vee,s^\vee} - \alpha_0)R_{r^\vee,s^\vee}(Q_+^{[r^\vee]}(v) + Q_-^{[s^\vee]}(v)) \tag{6.2.4}$$

in $F(\tau)$.

For $1 \le r < p_+$, $1 \le s < p_-$, let $K(\Delta^+_{r,s;0})$ be the indecomposable Virasoro module defined by the following exact sequence

$$0 \to L(\Delta^-_{r,s^\vee;0}) \to K(\Delta^+_{r,s;0}) \to K(\tau) \to 0.$$

By (6.2.3) and (6.2.4), as the quotient of $F(\tau)$, we can define the indecomposable module $\widetilde{P}(\Delta^+_{r,s;0})$ satisfying the following exact sequence

$$0 \to L(\Delta^+_{r,s;0}) \oplus L(\Delta^+_{r,s;0}) \to \widetilde{P}(\Delta^+_{r,s;0}) \to K(\Delta^+_{r,s;0}) \to 0.$$

Note that, by Theorem 6.2.9, $\mathrm{Ext}^1_{\mathcal{L}}(\widetilde{P}(\Delta^+_{r,s;0}), L(\Delta^+_{r,s;0})) = 0$ and $\widetilde{P}(\Delta^+_{r,s;0})$ has $P(\tau)$ and $P(\tau')$, $\tau' = (\alpha_{r^\vee,s^\vee}, \alpha_{r^\vee,s;-1}, \alpha_{r^\vee,s^\vee;-2})$, as subquotients. Let us consider the indecomposable module $R(\Delta^+_{r,s;0}) = \widetilde{P}(\Delta^+_{r,s;0})/L(h_{r,s}, \Delta^+_{r,s;0})$. By Theorems 6.1.5 and 6.2.9, we see that

$$\mathrm{Socle}(R(\Delta^+_{r,s;0})) = L(\Delta^+_{r,s;0}).$$

In particular we have

$$\mathrm{Socle}(P(\tau)) = \mathrm{Socle}(P(\tau')) = L(\Delta^+_{r,s;0}).$$

□

Corollary 6.2.18. *Fix any element $\tau = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}^{\mathrm{Min}}_{p_+,p_-}$. Then we have*

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\tau), L(h_{\alpha_2}, h_{\alpha_3})^*) = \mathbb{C}.$$

We fix any L_0-homogeneous vector $u_\tau \in K(\tau)[h_{\alpha_2}]$ such that $K(\tau) = U(\mathcal{L}).u_\tau$. Let $v_\tau \in K(\tau)$ be the highest weight vector of the submodule $L(h_{\alpha_3}) \subset K(\tau)$. Fix any non-zero L_0-homogeneous vectors $u'_\tau \in L(h_{\alpha_2}, h_{\alpha_3})^*[h_{\alpha_2}]$ and $v'_\tau \in L(h_{\alpha_2}, h_{\alpha_3})^*[h_{\alpha_3}]$ such that $u'_\tau \in U(\mathcal{L}).v'_\tau$. Let $S \in U(\mathcal{L})$ be the Shapovalov element such that $\sigma(S)v'_\tau \in \mathbb{C}^\times u'_\tau$. Let $\widetilde{K}(\tau)$ be the non-trivial extension

$$0 \to L(h_{\alpha_2}, h_{\alpha_3})^* \xrightarrow{\iota} \widetilde{K}(\tau) \xrightarrow{p} K(\tau) \to 0,$$

and let $\tilde{u}_\tau$ and $\tilde{v}_\tau$ be any L_0-homogeneous vectors of $\widetilde{K}(\tau)$ such that

$$p(\tilde{u}_\tau) = u_\tau, \qquad p(\tilde{v}_\tau) = v_\tau.$$

Then we have

$$S\tilde{u}_\tau \in \mathbb{C}^\times \tilde{v}_\tau + \mathbb{C}^\times \iota(v'_\tau) + U(\mathcal{L}).\iota(u'_\tau),$$

and

$$\sigma(S)\tilde{v}_\tau \in \mathbb{C}^\times \iota(u'_\tau).$$

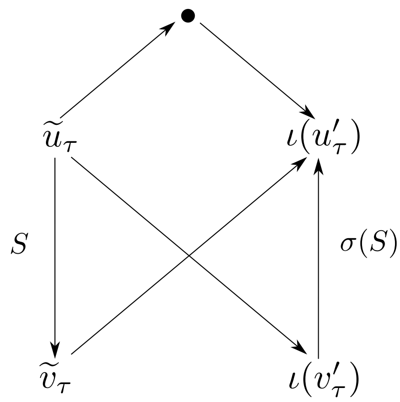


Figure 6.1: The embedding structure of the logarithmic module $\widetilde{K}(\tau)$.

Remark 6.2.19. *Figure 6.1 represents the embedding structure of the logarithmic module $\widetilde{K}(\tau)$ defined in Corollary 6.2.18. The black circle represents the highest weight vector of $\widetilde{K}(\tau)$.*

Chapter 7

The projective covers of simple modules $\mathcal{X}^{\pm}_{r,s}$

Since $\mathcal{W}_{p_+,p_-}$ is C_2-cofinite, so by [36], every simple $\mathcal{W}_{p_+,p_-}$-module has the projective cover. In this chapter, we determine some Ext^1-groups between certain indecomposable modules and simple modules. Based on these Ext^1 groups, we determine the structure of the projective covers of the simple modules in each thick block and thin block. From this chapter, we denote the n-th Ext-groups in $\mathcal{C}_{p_+,p_-}$ as $\mathrm{Ext}^n(\bullet,\bullet)$ simply and identify any $\mathcal{W}_{p_+,p_-}$-modules that are isomorphic to each other, unless otherwise stated.

7.1 The structure of the logarithmic modules $\mathcal{Q}(\mathcal{X}^{\pm}_{\bullet,\bullet})_{\bullet,\bullet}$ in the thick blocks $C^{thick}_{r,s}$

We fix any thick block $C^{thick}_{r,s}$. In this subsection we consider the structure of the indecomposable modules $\mathcal{Q}(\mathcal{X}^{\epsilon}_{a,b})_{c,d}$, where

$$\begin{aligned}\{(\epsilon,a,b,c,d)\} =& \big\{(+,r,s,r^\vee,s),(+,r,s,r,s^\vee),(+,r^\vee,s^\vee,r^\vee,s),(+,r^\vee,s^\vee,r,s^\vee),\\ &(-,r^\vee,s,r,s),(-,r^\vee,s,r^\vee,s^\vee),(-,r,s^\vee,r,s),(-,r,s^\vee,r^\vee,s^\vee)\big\}.\end{aligned}$$

First let us consider the structure of the logarithmic module $\mathcal{Q}(\mathcal{X}^{+}_{r,s})_{r^\vee,s}$. Recall that $\mathcal{Q}(\mathcal{X}^{+}_{r,s})_{r^\vee,s}$ is defined as the quotient of $\mathcal{P}^{+}_{r,s}$ and, as the vector space, $\mathcal{P}^{+}_{r,s}=\mathcal{V}^{+}_{r^\vee,s^\vee}\oplus\mathcal{V}^{-}_{r,s^\vee}\oplus\mathcal{V}^{-}_{r^\vee,s}\oplus\mathcal{V}^{+}_{r,s}$. Let u_0 and v^{+}_1 be cosingular vectors in $F_{r^\vee,s^\vee}[r^\vee s^\vee]$ and $F_{r,s^\vee;1}[r(s^\vee+p_-)]$, respectively. We define the following vectors in the ordinary $\mathcal{W}_{p_+,p_-}$-modules $\mathcal{V}^{+}_{r^\vee,s^\vee}$ and $\mathcal{V}^{-}_{r,s^\vee}$:

$$\begin{aligned}&u_1=|\alpha_{r,s^\vee;1}\rangle\in\mathcal{V}^{-}_{r,s^\vee},\\ &v^{-}_1=W^{-}[0]v^{+}_1\in F_{r,s^\vee;-1}[(r+p_+)s^\vee]\subset\mathcal{V}^{-}_{r,s^\vee},\\ &v^{-}_2=S_{r,s^\vee+p_-}u_0\in\mathcal{V}^{+}_{r^\vee,s^\vee},\\ &v^{+}_2=W^{+}[0]v^{-}_2\in\mathbb{C}^\times\,|\alpha_{r^\vee,s^\vee;2}\rangle\in\mathcal{V}^{+}_{r^\vee,s^\vee}.\end{aligned}$$

By the definition of $\mathcal{P}^{+}_{r,s}$, these vectors become highest weight vectors of the composition factors of the quotient module $\mathcal{Q}(\mathcal{X}^{+}_{r,s})_{r^\vee,s}$. The vector u_0 become the highest weight vector of the top composition factor $\mathcal{X}^{+}_{r,s}$ of $\mathcal{Q}(\mathcal{X}^{+}_{r,s})_{r^\vee,s}$. The sets $\{v^{+}_1,v^{-}_1\}$ and $\{v^{+}_2,v^{-}_2\}$ become basis of the highest weight space of the composition factors $2\mathcal{X}^{-}_{r^\vee,s}$ of $\mathcal{Q}(\mathcal{X}^{+}_{r,s})_{r^\vee,s}$. The vector u_1 become the highest weight vector of the submodule $\mathcal{X}^{+}_{r,s}\subset\mathcal{Q}(\mathcal{X}^{+}_{r,s})_{r^\vee,s}$. For these vectors, we use the same symbols in the quotient module $\mathcal{Q}(\mathcal{X}^{+}_{r,s})_{r^\vee,s}$. Note that we have

$$(L_0-\Delta^{+}_{r,s;0})u_0\in\mathbb{C}^\times u_1 \tag{7.1.1}$$

in $\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}$. From the Virasoro module structure of $\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}$, we define the following logarithmic Virasoro module

$$K = U(\mathcal{L}).u_0 + U(\mathcal{L}).v_1^+ \ \in \mathcal{L}_{p_+,p_-}\text{-mod}.$$

Note that K has the submodule $L(\Delta^+_{r,s;1})$. Then, by (7.1.1), we have

$$K/L(\Delta^+_{r,s;1}) \in \mathrm{Ext}^1_{\mathcal{L}}(K(\tau), L(\Delta^+_{r,s;0}, \Delta^-_{r^\vee,s;0})^*) \setminus \{0\},$$

where $\tau = (\alpha_{r^\vee,s^\vee}, \alpha_{r,s^\vee;1}, \alpha_{r^\vee,s^\vee;2})$. Thus, by Corollary 6.2.18, we have

$$\begin{aligned} &S_{r,s^\vee+p_-}u_0 \in \mathbb{C}^\times v_1^+ + \mathbb{C}^\times v_2^- \mod \mathcal{W}_{p_+,p_-}.u_1, \\ &S^*_{r,s^\vee+p_-}v_2^- \in \mathbb{C}^\times u_1, \end{aligned} \tag{7.1.2}$$

in $\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}$. By (7.1.2) we see that $\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}$ has two submodules

$$\begin{aligned} &\mathcal{W}_{p_+,p_-}.v_1^+ \in \mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^+_{r,s}) \setminus \{0\}, \\ &\mathcal{W}_{p_+,p_-}.v_2^- \in \mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^+_{r,s}) \setminus \{0\}. \end{aligned} \tag{7.1.3}$$

By (7.1.3), we see that $\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}$ is generated from the top composition factor $\mathcal{X}^+_{r,s}$ and $\mathrm{Socle}(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}) = \mathcal{X}^+_{r,s}$.

We have similar results for the other indecomposable modules of type $\mathcal{Q}(\mathcal{X}^+_{\bullet,\bullet})_{\bullet,\bullet}$ in $C^{thick}_{r,s}$. Thus we obtain the following theorem.

Theorem 7.1.1. *Let (a,b,c,d) be any element in*

$$\{(a,b,c,d)\} = \big\{(r,s,r^\vee,s), (r,s,r,s^\vee), (r^\vee,s^\vee,r^\vee,s), (r^\vee,s^\vee,r,s^\vee)\big\}.$$

Then the socle series of $\mathcal{Q}(\mathcal{X}^+_{a,b})_{c,d}$ is given by

$$\begin{aligned} &\mathrm{Soc}_1 = \mathrm{Socle} = \mathcal{X}^+_{a,b}, \\ &\mathrm{Soc}_2/\mathrm{Soc}_1 = \mathcal{X}^-_{c,d} \oplus L(h_{a,b}) \oplus \mathcal{X}^-_{c,d}, \\ &\mathcal{Q}(\mathcal{X}^+_{a,b})_{c,d}/\mathrm{Soc}_2 = \mathcal{X}^+_{a,b}. \end{aligned}$$

Moreover, $\mathcal{Q}(\mathcal{X}^+_{a,b})_{c,d}$ is generated from the top composition factor $\mathcal{X}^+_{a,b}$.

Remark 7.1.2. *Figure 7.1 represents the schematic diagram of the indecomposable module $\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}$.*

Next let us consider the logarithmic module $\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}$. Let $\{v_+, v_-\}$ be a basis of the highest weight space of $\mathcal{X}^-_{r^\vee,s}$ such that

$$W^\pm[0]v_\pm = 0, \qquad\qquad W^\pm[0]v_\mp \in \mathbb{C}^\times v_\pm.$$

For the surjection $\pi : \mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s} \to \mathcal{X}^-_{r^\vee,s}$, we fix L_0-homogeneous vectors $\tilde{v}_\pm \in \mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}$ such that $\pi(\tilde{v}_\pm) = v_\pm$. Note that

$$(L_0 - \Delta^-_{r^\vee,s;0})\tilde{v}_\pm \in \mathbb{C}^\times v_\pm,$$

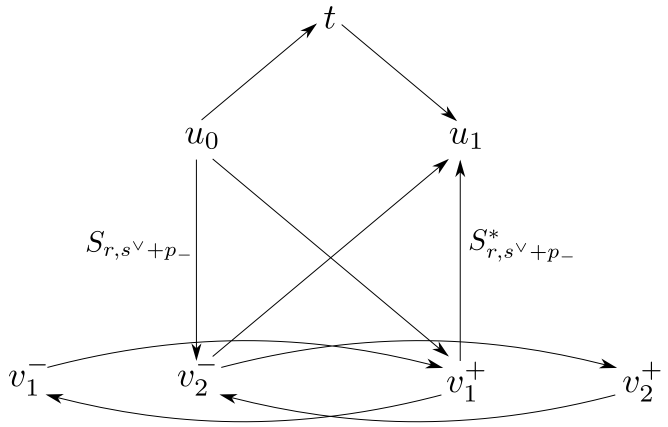


Figure 7.1: The schematic diagram of the indecomposable module $\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}$, where t is the highest weight vector whose L_0 weight is $h_{r,s;0}$.

in $\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}$(see the proof of Proposition 7.3.4). Then, by Theorem 6.2.9, we obtain

$$S_{r,s^\vee+p_-}S^*_{r,s^\vee+p_-}\tilde{v}_\pm \in \mathbb{C}^\times v_\pm. \tag{7.1.4}$$

By (7.1.4) we see that $\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}$ has two submodules

$$\mathcal{W}_{p_+,p_-}.S^*_{r,s^\vee+p_-}\tilde{v}_\pm \in \mathrm{Ext}^1(\mathcal{X}^+_{r,s},\mathcal{X}^-_{r^\vee,s})\setminus\{0\}.$$

In particular we see that $\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}$ is generated from the top composition factor $\mathcal{X}^-_{r^\vee,s}$. We have similar results for the other indecomposable modules of type $\mathcal{Q}(\mathcal{X}^-_{\bullet,\bullet})_{\bullet,\bullet}$ in $C^{thick}_{r,s}$. Thus we obtain the following theorem.

Theorem 7.1.3. *Let (a,b,c,d) be any element in*

$$\{(a,b,c,d)\} = \big\{(r^\vee,s,r,s),(r^\vee,s,r^\vee,s^\vee),(r,s^\vee,r,s),(r,s^\vee,r^\vee,s^\vee)\big\}.$$

Then the socle series of $\mathcal{Q}(\mathcal{X}^-_{a,b})_{c,d}$ is given by

$$\begin{aligned}&\mathrm{Soc}_1 = \mathrm{Socle} = \mathcal{X}^-_{a,b},\\ &\mathrm{Soc}_2/\mathrm{Soc}_1 = \mathcal{X}^+_{c,d}\oplus\mathcal{X}^+_{c,d},\\ &\mathcal{Q}(\mathcal{X}^-_{a,b})_{c,d}/\mathrm{Soc}_2 = \mathcal{X}^-_{a,b}.\end{aligned}$$

Moreover, $\mathcal{Q}(\mathcal{X}^-_{a,b})_{c,d}$ is generated from the top composition factor $\mathcal{X}^-_{a,b}$.

Remark 7.1.4. *Figure 7.2 represents the schematic diagram of $\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}$.*

7.2 The Ext^1-groups between all simple modules in the thick block $C^{thick}_{r,s}$

We fix any thick block $C^{thick}_{r,s}$. In this section, we determine the Ext^1-groups between all simple modules in the thick block $C^{thick}_{r,s}$. From this section, we identify any $\mathcal{W}_{p_+,p_-}$-modules that are isomorphic to each other.

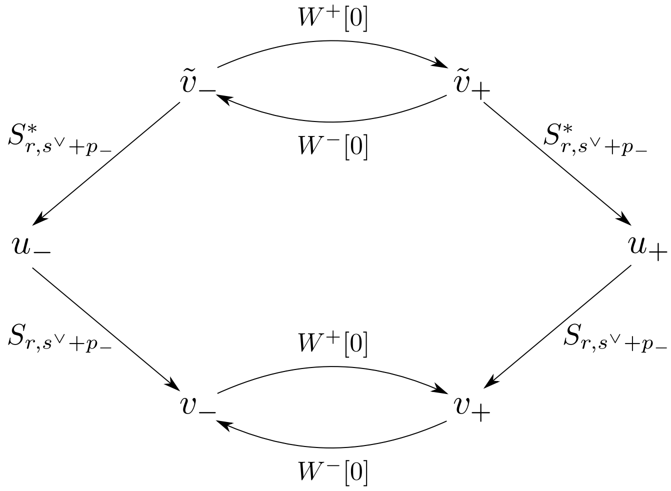


Figure 7.2: The schematic diagram of the structure of the indecomposable module $\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}$.

Definition 7.2.1. *Let us fix* (a,b,c,d) *in*

$$\{(a,b,c,d)\} = \{(r^\vee,s,r,s),(r^\vee,s,r^\vee,s^\vee),(r,s^\vee,r,s),(r,s^\vee,r^\vee,s^\vee)\}.$$

1. *For* $\mathcal{Q}(\mathcal{X}^-_{a,b})_{c,d}$*, let* $\{v_+,v_-\}$ *be a basis of the highest weight space of the submodule* $\mathcal{X}^-_{a,b} \subset \mathcal{Q}(\mathcal{X}^-_{a,b})_{c,d}$ *such that*

$$W^\pm[0]v_\pm = 0, \qquad\qquad W^\pm[0]v_\mp \in \mathbb{C}^\times v_\pm.$$

   and let $u_\pm$ *be the highest weight vectors of* $\mathcal{Q}(\mathcal{X}^-_{a,b})_{c,d}$ *such that* $v_\pm \in U(\mathcal{L}).u_\pm$*. Then we define*

$$\mathcal{E}^+(\mathcal{X}^+_{c,d})_{a,b} := \mathcal{W}_{p_+,p_-}.u_+, \qquad\qquad \mathcal{E}^-(\mathcal{X}^+_{c,d})_{a,b} := \mathcal{W}_{p_+,p_-}.u_-,$$

   which give different extensions in $\mathrm{Ext}^1(\mathcal{X}^+_{c,d},\mathcal{X}^-_{a,b}) \setminus \{0\}$*.*

2. *As the quotient of* $\mathcal{Q}(\mathcal{X}^+_{c,d})_{a,b}$*, we have a non-trivial extension in*

$$\mathrm{Ext}^1(\mathcal{E}^+(\mathcal{X}^+_{c,d})_{a,b},\mathcal{X}^-_{a,b}).$$

   We denote this quotient module by $\mathcal{E}(\mathcal{X}^+_{c,d})_{a,b}$*.*

Remark 7.2.2. *Figure 7.3 represents the schematic diagrams of the structure of the indecomposable modules* $\mathcal{E}^-(\mathcal{X}^+_{r,s})_{r^\vee,s}$ *and* $\mathcal{E}^+(\mathcal{X}^+_{r,s})_{r^\vee,s}$*.*

Definition 7.2.3. *We define*

$$\mathcal{K}_{r,s} := \mathcal{W}_{p_+,p_-}.\,|\alpha_{r,s}\rangle \qquad\qquad \mathcal{K}_{r^\vee,s^\vee} := \mathcal{W}_{p_+,p_-}.\,|\alpha_{r^\vee,s^\vee}\rangle$$

which are the non-trivial extensions of $\mathrm{Ext}^1(L(h_{r,s}),\mathcal{X}^+_{r,s}) \simeq \mathbb{C}$ *and* $\mathrm{Ext}^1(L(h_{r,s}),\mathcal{X}^+_{r^\vee,s^\vee}) \simeq \mathbb{C}$*, respectively.*

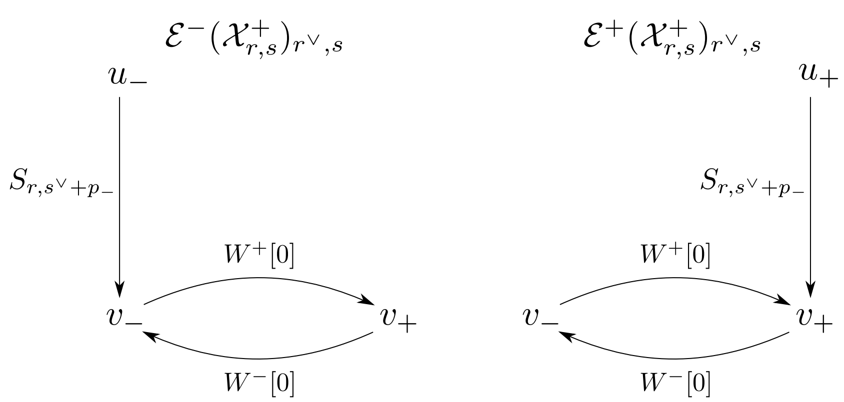


Figure 7.3: The schematic diagram of the structure of the indecomposable modules $\mathcal{E}^-(\mathcal{X}^+_{r,s})_{r^\vee,s}$ and $\mathcal{E}^+(\mathcal{X}^+_{r,s})_{r^\vee,s}$.

Definition 7.2.4. *Given a non-logarithmic Virasoro module M, any non-zeo vector $v \in M$ is called* primary *vector when the following satisfied*

$$L_n v = 0, \quad n \geq 1.$$

Similar to the arguments in Section 9.3 of [48], we can prove the following proposition (see also [57],[59]). We omit the proofs.

Proposition 7.2.5 ([21],[48])**.** *Let M_1, M_2 and M_3^* be non-logarithmic Virasoro modules which have primary vectors $v_1 \in M_1$, $v_2 \in M_2$ and $v_3^* \in M_3^*$ whose L_0 weights are h_{r_1,s_1}, h_{r_2,s_2} and h_{r_3,s_3}, respectively, where $r_i \geq 1$ and $s_i \geq 1$ $(i = 1,2,3)$. Assume that there exists a non-logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{M_3}{M_1\ M_2}$. Then we have*

$$\langle v_3^*, \mathcal{Y}(v_1,z) S_{r_2,s_2} v_2 \rangle = \prod_{i=1}^{r_2} \prod_{j=1}^{s_2} (h_{r_1,s_1} - h_{r_2+r_3-2i+1, s_2+s_3-2j+1}) \langle v_3^*, \mathcal{Y}(v_1,z) v_2 \rangle,$$

$$\langle v_3^*, S^*_{r_3,s_3} \mathcal{Y}(v_1,z) v_2 \rangle = \prod_{i=1}^{r_3} \prod_{j=1}^{s_3} (h_{r_1,s_1} - h_{r_2+r_3-2i+1, s_2+s_3-2j+1}) \langle v_3^*, \mathcal{Y}(v_1,z) v_2 \rangle.$$

Proposition 7.2.6 ([21],[48])**.** *For $h \in \mathbb{C}$, $1 \leq r_1, r_2 < p_+$, $1 \leq s_1, s_2 < p_-$ and $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathcal{N}^{L(h)}_{L(h_{r_1,s_1;n_1}), L(h_{r_2,s_2;n_2})} \leq 1,$$

where $\mathcal{N}^{L(h_3)}_{L(h_2),L(h_1)}$ is the dimension of the space of Virasoro intertwining operators of type $\binom{L(h_3)}{L(h_2)\ L(h_1)}$. If $\mathcal{N}^{L(h)}_{L(h_{r_1,s_1;n_1}), L(h_{r_2,s_2;n_2})} \neq 0$, then h is the common solution of the following *equations*

$$\prod_{i=1}^{r_1}\prod_{j=1}^{s_1+n_1p_-}(h-h_{r_1+r_2-2i+1,s_1+s_2-2j+1;n_1+n_2})=0,$$

$$\prod_{i=1}^{(n_1+1)p_+-r_1}\prod_{j=1}^{p_--s_1}(h-h_{2p_+-r_1-r_2-2i+1,2p_--s_1-s_2-2j+1;-n_1-n_2})=0,$$

$$\prod_{i=1}^{r_2}\prod_{j=1}^{s_2+n_2p_-}(h-h_{r_1+r_2-2i+1,s_1+s_2-2j+1;n_1+n_2})=0,$$

$$\prod_{i=1}^{(n_2+1)p_+-r_2}\prod_{j=1}^{p_--s_2}(h-h_{2p_+-r_1-r_2-2i+1,2p_--s_1-s_2-2j+1;-n_1-n_2})=0.$$

Lemma 7.2.7. *Let $n\geq 1$. Any extension in*

$$\mathrm{Ext}^1(\mathcal{X}^+_{r,s},n\mathcal{X}^-_{r^\vee,s})$$

splits if it decomposes as simple Virasoro modules.

Proof. We only prove in the case $n=1$. The $n\geq 2$ cases can be proved in the same way. Let E be any non-trivial extension

$$0\to\mathcal{X}^-_{r^\vee,s}\xrightarrow{\iota}E\to\mathcal{X}^+_{r,s}\to 0.$$

Let u be the highest weight vector in $E[\Delta^+_{r,s;0}]$. Assume that

$$S_{r,s^\vee+p_-}u=0. \tag{7.2.1}$$

Let $\{v_+,v_-\}$ be a basis of the highest weight space of $\mathcal{X}^-_{r^\vee,s}$ such that

$$W^\pm[0]v_\pm=0,\qquad\qquad W^\pm[0]v_\mp\in\mathbb{C}^\times v_\pm.$$

Let v^*_+ and v^*_- be L_0-homogeneous vectors of E^* such that $\langle v^*_\pm,\iota(v_\pm)\rangle\neq 0$, and $L_kv^*_\pm=0$ for $k\geq 1$. Assume that for any $W=W^\pm,W^0$

$$W[k]v^*_\pm=0,\quad k\geq 1.$$

Then the vector space $\mathbb{C}v^*_++\mathbb{C}v^*_-$ becomes a $A(\mathcal{W}_{p_+,p_-})$-module and this vector space is isomorphic to the highest weight space of $\mathcal{X}^-_{r^\vee,s}$ as a $A(\mathcal{W}_{p_+,p_-})$-module. Thus E^* has the submodule $\mathcal{W}_{p_+,p_-}.(\mathbb{C}v^*_++\mathbb{C}v^*_-)\simeq\mathcal{X}^-_{r^\vee,s}$ and thus $E^*=\mathcal{X}^+_{r,s}\oplus\mathcal{X}^-_{r^\vee,s}$. But this contradicts the assumption that E is non-trivial. Therefore we have

$$\langle v^*_\pm,Y_E(W^\bullet;z)u\rangle\neq 0,$$

where $W^\bullet$ is one of W^+, W^0 or W^-. On the other hand, using Proposition 7.2.5, we have

$$\begin{aligned}&\langle v^*_\pm,Y_E(W^\bullet;z)S_{r,s^\vee+p_-}u\rangle\\&=\prod_{i=1}^{r}\prod_{j=1}^{s^\vee+p_-}(h_{4p_+-1,1}-h_{r+r+2p_+-2i+1,s^\vee+p_-+s-2j+1})\langle v^*_\pm,Y_E(W^\bullet;z)u\rangle\\&\neq 0.\end{aligned}$$

But this contradicts (7.2.1). □

Proposition 7.2.8. *In the thick block $C^{thick}_{r,s}$, we have*

$$\mathrm{Ext}^1(\mathcal{X}^{\pm},\mathcal{X}^{\mp})=\mathbb{C}^2,\qquad \mathrm{Ext}^1(L(h_{r,s}),\mathcal{X}^{+})=\mathrm{Ext}^1(\mathcal{X}^{+},L(h_{r,s}))=\mathbb{C},$$

where $\mathcal{X}^+ = \mathcal{X}^+_{r,s}$ or $\mathcal{X}^+_{r^\vee,s^\vee}$ and $\mathcal{X}^- = \mathcal{X}^-_{r^\vee,s}$ or $\mathcal{X}^-_{r,s^\vee}$. The other extensions between the simple modules in $C^{thick}_{r,s}$ are trivial.

Proof. We will only prove

$$\begin{aligned} \mathrm{Ext}^1(L(h_{r,s}),\mathcal{X}^-_{r^\vee,s})&=0, & \mathrm{Ext}^1(L(h_{r,s}),\mathcal{X}^+_{r,s})&=\mathbb{C},\\ \mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,s},\mathcal{X}^-_{r^\vee,s})&=0, & \mathrm{Ext}^1(\mathcal{X}^+_{r,s},\mathcal{X}^-_{r^\vee,s})&=\mathbb{C}^2,\\ \mathrm{Ext}^1(\mathcal{X}^+_{r,s},\mathcal{X}^+_{r,s})&=0. \end{aligned}$$

The other Ext^1-groups can be proved in a similar way, so we omit the proofs.

First, let us prove $\mathrm{Ext}^1(L(h_{r,s}),\mathcal{X}^-_{r^\vee,s})=0$. Assume $\mathrm{Ext}^1(L(h_{r,s}),\mathcal{X}^-_{r^\vee,s})\neq 0$ and fix any non-trivial extension E_0 in this Ext^1-group. Note that E_0 is a direct sum of Virasoro simple modules. Then, from the $\mathcal{W}_{p_+,p_-}$-module action on E_0, we must have a non-trivial Virasoro intertwining operator of type

$$\begin{pmatrix} L(\Delta^-_{r^\vee,s;n}) \\ L(h_{4p_+-1,1})\ \ L(h_{r,s}) \end{pmatrix}$$

for some $n\geq 0$. But, by using Proposition 7.2.6, we can see the contradiction.

Next, we prove $\mathrm{Ext}^1(L(h_{r,s}),\mathcal{X}^+_{r,s})=\mathbb{C}$. Fix any extension

$$[E_1]\in \mathrm{Ext}^1(L(h_{r,s}),\mathcal{X}^+_{r,s}).$$

Let t be the highest weight vector of E_1 and assume $S_{r^\vee,s^\vee}t\neq 0$. Then, as a Virasoro module

$$E_1 = L(h_{r,s},\Delta^+_{r,s;0})\oplus\bigoplus_{n\geq 1}(2n+1)L(\Delta^+_{r,s;n}).$$

Since

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h_{r,s}),L(\Delta^+_{r,s;0}))=\mathbb{C},$$

as the Baer sum of extensions obtained from E_1 and $\mathcal{K}_{r,s}$, we have a extension $[E_1']\in \mathrm{Ext}^1(L(h_{r,s}),\mathcal{X}^+_{r,s})$ such that $S_{r^\vee,s^\vee}t'=0$, where t' is the highest weight vector of E_1'. Thus, by Theorem 6.1.6, we have the following decomposition as the Virasoro module

$$E_1' = L(h_{r,s})\oplus\bigoplus_{n\geq 0}(2n+1)L(\Delta^+_{r,s;n}).$$

Assume $[E_1']\neq 0$. Then, from the $\mathcal{W}_{p_+,p_-}$-module action on E_1', we must have a non-trivial Virasoro intertwining operator of type

$$\begin{pmatrix} L(\Delta^+_{r,s;n}) \\ L(h_{4p_+-1,1})\ \ L(h_{r,s}) \end{pmatrix}$$

for some $n \geq 0$. But, by using Proposition 7.2.6, we can see the contradiction. In case $S_{r^\vee,s^\vee}t = 0$, we see that $[E_1] = 0$ as shown above.

Next, we prove $\mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0$. Fix any extension $[E_2] \in \mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s})$. By Theorem 6.1.5, we see that L_0 acts semisimply on E_2. Let $\overline{E}_2$ be the highest weight space of E_2. Note that E_2 is generated from $\overline{E}_2$. Let $\widetilde{E}_2$ be the $\mathcal{W}_{p_+,p_-}$-module induced from $\overline{E}_2$. Then we have $\widetilde{E}_2 = E_2$. By Proposition 4.2.7, we see that as a $A(\mathcal{W}_{p_+,p_-})$-module

$$\overline{E}_2 \simeq \overline{\mathcal{X}^-_{r^\vee,s}} \oplus \overline{\mathcal{X}^-_{r^\vee,s}},$$

where $\overline{\mathcal{X}^-_{r^\vee,s}}$ is the highest weight space of $\mathcal{X}^-_{r^\vee,s}$. Note that the $\mathcal{W}_{p_+,p_-}$-module induced from $\overline{\mathcal{X}^-_{r^\vee,s}}$ is isomorphic to $\mathcal{X}^-_{r^\vee,s}$. Thus we have $\widetilde{E}_2 \simeq \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s}$.

Next, we prove $\mathrm{Ext}^1(\mathcal{X}^+_{r,s}, \mathcal{X}^-_{r^\vee,s}) = \mathbb{C}^2$. Let us show

$$\mathrm{Ext}^1(\mathcal{E}^\pm(\mathcal{X}^+_{r,s})^*_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0. \tag{7.2.2}$$

We will only prove $\mathrm{Ext}^1(\mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0$. The other case can be proved in the same way. Assume that

$$\mathrm{Ext}^1(\mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) \neq 0$$

and fix any non-trivial extension

$$0 \to \mathcal{X}^-_{r^\vee,s} \xrightarrow{\iota} F \to \mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s} \to 0.$$

Let $\{v_+, v_-\}$ be the basis of the highest weight space of $\mathcal{X}^-_{r^\vee,s}$ such that

$$W^\pm[0]v_\pm = 0, \qquad\qquad W^\pm[0]v_\mp \in \mathbb{C}^\times v_\pm.$$

For the surjection $\pi : F \to \mathcal{X}^-_{r^\vee,s}$, let $\tilde{v}_\pm$ be any L_0 homogeneous vectors of F such that $\pi(\tilde{v}_\pm) = v_\pm$. Then, by Theorem 6.2.9 and $\mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0$, we must have

$$(L_0 - \Delta^-_{r^\vee,s;0})\tilde{v}_+ = k_+\iota(v_+) + k_-\iota(v_-), \tag{7.2.3}$$
$$(L_0 - \Delta^-_{r^\vee,s;0})\tilde{v}_- = 0, \tag{7.2.4}$$

where $(k_+, k_-) \neq (0,0)$. Assume $k_+ \neq 0$. Then, multiplying both sides of (7.2.3) by $W^-[0]$, we have

$$(L_0 - \Delta^-_{r^\vee,s;0})W^-[0]\tilde{v}_+ \in \mathbb{C}^\times \iota(v_-).$$

But this contradicts (7.2.4). Next assume $k_- \neq 0$. Then, multiplying both sides of (7.2.3) by $W^+[0]$, we have

$$(L_0 - \Delta^-_{r^\vee,s;0})W^+[0]\tilde{v}_+ \in \mathbb{C}^\times \iota(v_+).$$

On the other hand, by the definition of $\tilde{v}_+$, we have $(L_0 - \Delta^-_{r^\vee,s;0})W^+[0]\tilde{v}_+ = 0$. Thus we have a contradiction. Thus we obtain (7.2.2). Next let us show

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0. \tag{7.2.5}$$

Assume that $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) \neq 0$ and fix any non-trivial extension G in this Ext^1-group. Assume that $\mathrm{Socle}(G) = \mathcal{X}^-_{r^\vee,s}$. Then, by Lemma 7.2.7, we see that G has a indecomposable submodule in

$$\mathrm{Ext}^1(\mathcal{E}^\pm(\mathcal{X}^+_{r,s})^*_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}).$$

But this contradicts (7.2.2). Thus, by $\mathrm{Ext}^1(L(h_{r,s}), \mathcal{X}^-_{r^\vee,s}) = 0$, we see that G has the submodule $\mathcal{K}_{r,s}$ and $G/\mathcal{K}_{r,s}$ is indecomposable. Let $E_3 = G/\mathcal{K}_{r,s}$. Then we have

$$[E_3] \in \mathrm{Ext}^1(\mathcal{E}(\mathcal{X}^+_{r,s})_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) \setminus \{0\}.$$

By $\mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0$, we see that

$$\mathrm{Socle}(E_3) = \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s}. \tag{7.2.6}$$

Let u be the highest weight vector of E_3 and let us consider the submodule $\mathcal{W}_{p_+,p_-}.S_{r,s^\vee+p_-}u$ of E_3. By Theorems 4.2.5 and 4.2.6, we can see that

$$\mathcal{W}_{p_+,p_-}.S_{r,s^\vee+p_-}u = \mathcal{X}^-_{r^\vee,s} \text{ or } \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s}. \tag{7.2.7}$$

Thus by (7.2.6) and (7.2.7), we have

$$[E_3/\mathcal{W}_{p_+,p_-}.S_{r,s^\vee+p_-}u] \in \mathrm{Ext}^1(\mathcal{X}^+_{r,s}, n\mathcal{X}^-_{r^\vee}) \setminus \{0\},$$

where n is 1 or 2. But this contradicts Lemma 7.2.7. Thus we obtain (7.2.5). Therefore by (7.2.2), (7.2.5) and $\mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0$, we obtain

$$\mathrm{Ext}^1(\mathcal{X}^+_{r,s}, \mathcal{X}^-_{r^\vee,s}) = \mathbb{C}^2.$$

Finally let us prove $\mathrm{Ext}^1(\mathcal{X}^+_{r,s}, \mathcal{X}^+_{r,s}) = 0$. Let $\overline{\mathcal{X}^+_{r,s}}$ be the highest weight space of $\mathcal{X}^+_{r,s}$ and let $\mathcal{E}(\mathcal{X}^+_{r,s})$ be the induced $\mathcal{W}_{p_+,p_-}$-module from $\overline{\mathcal{X}^+_{r,s}}$. Since

$$\mathrm{Ext}^1(\mathcal{X}^+_{r,s}, \mathcal{X}^-_{r^\vee,s}) = \mathbb{C}^2, \qquad \mathrm{Ext}^1(\mathcal{X}^+_{r,s}, \mathcal{X}^-_{r,s^\vee}) = \mathbb{C}^2,$$
$$\mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0, \qquad \mathrm{Ext}^1(\mathcal{X}^-_{r,s^\vee}, \mathcal{X}^-_{r,s^\vee}) = 0,$$

we can see that the indecomposable module $\mathcal{E}(\mathcal{X}^+_{r,s})$ satisfies the following exact sequence

$$0 \to 2\mathcal{X}^-_{r^\vee,s} \oplus 2\mathcal{X}^-_{r,s^\vee} \to \mathcal{E}(\mathcal{X}^+_{r,s}) \to \mathcal{X}^+_{r,s} \to 0.$$

Let E_4 be any extension in $\mathrm{Ext}^1(\mathcal{X}^+_{r,s}, \mathcal{X}^+_{r,s})$. By Theorem 6.1.5, we see that L_0 acts semisimply on E_4. Let $\overline{E}_4$ be the highest weight space of E_4. Let us assume $\overline{E}_4 \not\simeq \overline{\mathcal{X}^+_{r,s}} \oplus \overline{\mathcal{X}^+_{r,s}}$ as a $A(\mathcal{W}_{p_+,p_-})$-module. Then, from the $\mathcal{W}_{p_+,p_-}$-module action on E_4, we have a non-trivial non-logarithmic Virasoro intertwining operator of type

$$\begin{pmatrix} & L(\Delta^+_{r,s;0}) & \\ L(h_{4p_+-1,1}) & & L(\Delta^+_{r,s;0}) \end{pmatrix}.$$

But we can see the contradiction by using Proposition 7.2.6. Thus, as a $A(\mathcal{W}_{p_+,p_-})$-module, $\overline{E}_4 \simeq \overline{\mathcal{X}^+_{r,s}} \oplus \overline{\mathcal{X}^+_{r,s}}$. Let $\widetilde{E}_4$ be the induced module from $\overline{E}_4$. Then we have

$$\widetilde{E}_4 \simeq \mathcal{E}(\mathcal{X}^+_{r,s}) \oplus \mathcal{E}(\mathcal{X}^+_{r,s}).$$

Therefore we obtain

$$E_4 \simeq \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s}.$$

□

7.3 The projective covers of the simple modules $\mathcal{X}^{\pm}_{\bullet,\bullet}$ in the thick blocks

In this section, we fix any thick block $C^{thick}_{r,s}$ and compute Ext^1 groups between certain indecomposable $\mathcal{W}_{p_+,p_-}$-modules and the simple modules in this block. Based on these Ext^1 groups, we prove that the logarithmic modules $\mathcal{P}^{\pm}_{\bullet,\bullet}$ are projective $\mathcal{W}_{p_+,p_-}$-modules.

First we will determine all trivial Ext^1-groups between the indecomposable modules $\mathcal{Q}(\mathcal{X}^{\pm}_{\bullet,\bullet})_{\bullet,\bullet} \in C^{thick}_{r,s}$ and simple modules $L(h_{r,s})$, $\mathcal{X}^{\pm}_{\bullet,\bullet} \in C^{thick}_{r,s}$.

Proposition 7.3.1. *Let (a,b) be (r,s) or $(r^\vee, s^\vee)$. Then we have*

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{a,b})_{a^\vee,b}, \mathcal{X}^+_{a,b}) = \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{a,b})_{a,b^\vee}, \mathcal{X}^+_{a,b}) = 0.$$

Proof. We will only prove $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{a,b})_{a^\vee,b}, \mathcal{X}^+_{a,b}) = 0$ in the case $(a,b) = (r,s)$. The other cases can be proved in the same way, so we omit the proofs.

Assume $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}, \mathcal{X}^+_{r,s}) \neq 0$. Then, by Theorem 7.2.8, we have

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}/\mathcal{K}_{r,s}, \mathcal{X}^+_{r,s}) \neq 0.$$

Fix any non-trivial extension $[E] \in \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}/\mathcal{K}_{r,s}, \mathcal{X}^+_{r,s})$. By Theorem 6.2.9, we see that L_0 acts semisimply on the highest weight space of E. Thus, by Propositions 4.2.8 and 4.2.9, we see that L_0 acts semisimply on E. We fix any L_0-homogeneous vector $u_0 \in E$ such that, for the surjection $\pi : E \to \mathcal{X}^+_{r,s}$, $\pi(u_0)$ gives the highest weight vector of $\mathcal{X}^+_{r,s}$. Let u_1 be the highest weight vector of the submodule $\mathcal{X}^+_{r,s} \subset E$ and fix any homogeneous vector $u_1^* \in E^*$ such that $\langle u_1^*, u_1 \rangle \neq 0$. Since $[E] \neq 0$, E has at least one of $\mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s}$ or $\mathcal{E}^-(\mathcal{X}^+_{r,s})^*_{r^\vee,s}$, as a submodule. Thus, by the structure of $\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}/\mathcal{K}_{r,s}$ and $\mathcal{E}^{\pm}(\mathcal{X}^+_{r,s})_{r^\vee,s}$, we see that

$$\langle u_1^*, S^*_{r,s^\vee+p_-} Y_E(W^\bullet; z) S_{r,s^\vee+p_-} u_0 \rangle \neq 0,$$

where $W^\bullet$ is one of W^+, W^0 or W^-. In particular, we have

$$\langle u_1^*, Y_E(W^\bullet; z) u_0 \rangle \neq 0. \tag{7.3.1}$$

Note that $S_{r^\vee+p_+,s} u_0 = 0$. Thus, by Proposition 7.2.5, we have

$$\begin{aligned} 0 &= \langle u_1^*, Y_E(W^\bullet; z) S_{r^\vee+p_+,s} u_0 \rangle \\ &= \prod_{i=1}^{r^\vee+p_+} \prod_{j=1}^{s} (h_{4p_+-1,1} - h_{2r^\vee+2p_+-2i+1,2s-2j+1}) \langle u_1^*, Y_E(W^\bullet; z) u_0 \rangle. \end{aligned}$$

The coefficient in the above equation is nonzero, so we have $\langle u_1^*, Y_E(W^\bullet; z) u_0 \rangle = 0$. But this contradicts (7.3.1). $\square$

Proposition 7.3.2. *Let (a,b) be $(r^\vee, s)$ or $(r, s^\vee)$. Then we have*

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{a,b})_{a^\vee,b}, \mathcal{X}^+_{a^\vee,b}) = \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{a,b})_{a,b^\vee}, \mathcal{X}^+_{a,b^\vee}) = 0.$$

Proof. We will only prove $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{a,b})_{a^\vee,b}, \mathcal{X}^+_{a^\vee,b}) = 0$ in the case $(a,b) = (r^\vee, s)$. The other cases can be proved in the same way, so we omit the proofs.

By Proposition 7.3.1, we have

$$\mathrm{Ext}^1(\mathcal{E}^+(\mathcal{X}^+_{r,s})_{r^\vee,s}, \mathcal{X}^+_{r,s}) = 0. \tag{7.3.2}$$

From the structure of $\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}$, we have the following exact sequence

$$0 \to \mathcal{E}^+(\mathcal{X}^+_{r,s})_{r^\vee,s} \to \mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s} \to \mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s} \to 0.$$

By this exact sequence and (7.3.2), we have the following exact sequence

$$0 \to \mathbb{C} \to \mathrm{Ext}^1(\mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s}, \mathcal{X}^+_{r,s}) \to \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}, \mathcal{X}^+_{r,s}) \to 0.$$

By Proposition 7.2.8, we have $\mathrm{Ext}^1(\mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s}, \mathcal{X}^+_{r,s}) \simeq \mathbb{C}$. Therefore we obtain

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}, \mathcal{X}^+_{r,s}) = 0.$$

□

Lemma 7.3.3. *Let (a,b) be $(r^\vee, s)$ or $(r, s^\vee)$. Then we have*

$$\mathrm{Ext}^1(\mathcal{E}^\pm(\mathcal{X}^+_{a^\vee,b})^*_{a,b}, \mathcal{X}^-_{a,b}) = \mathrm{Ext}^1(\mathcal{E}^\pm(\mathcal{X}^+_{a,b^\vee})^*_{a,b}, \mathcal{X}^-_{a,b}) = 0.$$

Proof. It can be proved in the same way as (7.2.2) in Proposition 7.2.8. □

Proposition 7.3.4. *Let (a,b) be $(r^\vee, s)$ or $(r, s^\vee)$. Then we have*

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{a,b})_{a^\vee,b}, \mathcal{X}^-_{a,b}) = \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{a,b})_{a,b^\vee}, \mathcal{X}^-_{a,b}) = 0.$$

Proof. We will only prove $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{a,b})_{a^\vee,b}, \mathcal{X}^-_{a,b}) = 0$ in the case $(a,b) = (r^\vee, s)$. The other cases can be proved in the same way, so we omit the proofs. Assume that

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}, \mathcal{X}^-_{r^\vee,s}) \neq 0$$

and fix any non-trivial extension

$$0 \to \mathcal{X}^-_{r^\vee,s} \xrightarrow{\iota} E \xrightarrow{p} \mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s} \to 0.$$

By Proposition 7.2.8 and Lemma 7.3.3, we see that the following sequence of submodules holds

$$\iota(\mathcal{X}^+_{r,s}) \subset \mathcal{E}(\mathcal{X}^+_{r,s})_{r^\vee,s} \subset E. \tag{7.3.3}$$

Let $\{v_+, v_-\}$ be the basis of the highest weight space of $\mathcal{X}^-_{r^\vee,s}$ such that

$$W^\pm[0]v_\pm = 0, \qquad W^\pm[0]v_\mp \in \mathbb{C}^\times v_\pm.$$

Let $\{v^0_+, v^0_-\}$ be the basis of the highest weight space of the submodule $\mathcal{X}^-_{r^\vee,s} \subset E$ such that $p(v^0_\pm) \neq 0$. For the canonical surjection $\pi : E \to \mathcal{X}^-_{r^\vee,s}$, we fix any L_0-homogeneous vectors $\tilde{v}_-, \tilde{v}_+ \in E$ such that $\pi(\tilde{v}_\pm) = v_\pm$. Note that, as a quotient module of $U(\mathcal{L}).\tilde{v}_\pm$, we have the logarithmic Virasoro module $P(\tau)$, where $\tau = (\alpha_{r,s^\vee;1}, \alpha_{r^\vee,s^\vee;2}, \alpha_{r,s^\vee;3})$ (for the definition of the logarithmic modules $P(\tau)$, see the proof of Theorem 6.2.9). Then, by Theorem 6.2.9, Proposition 7.2.8 and (7.3.3), we see that one of the followings holds

$$(L_0 - \Delta^-_{r^\vee,s;0})\tilde{v}_- = k_+\iota(v_+) + k_-\iota(v_-) + \mathbb{C}^\times v^0_-, \quad k_+ \neq 0,$$
$$(L_0 - \Delta^-_{r^\vee,s;0})\tilde{v}_+ = l_-\iota(v_-) + l_+\iota(v_+) + \mathbb{C}^\times v^0_+, \quad l_- \neq 0.$$

Assume that the first statement is true. Multiplying the first equation by $W^-[0]$, we have

$$(L_0 - \Delta^-_{r^\vee,s;0})W^-[0]\tilde{v}_- = k_+\iota(v_-).$$

By the definition of $\tilde{v}_-$, the left hand side becomes zero. But this is a contradiction. Similarly, assuming the second statement, we can show the contradiction. □

Proposition 7.3.5. *Let (a, b) be (r, s) or $(r^\vee, s^\vee)$. Then we have*

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{a,b})_{a^\vee,b}, \mathcal{X}^-_{a^\vee,b}) = \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{a,b})_{a,b^\vee}, \mathcal{X}^-_{a,b^\vee}) = 0.$$

Proof. We proved this proposition in the proof of Proposition 7.2.8, but we prove it again. We will only prove $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{a,b})_{a^\vee,b}, \mathcal{X}^-_{a^\vee,b}) = 0$ in the case $(a, b) = (r, s)$. The other cases can be proved in the same way, so we omit the proofs.

By the exact sequence

$$0 \to \mathcal{E}^+(\mathcal{X}^+_{r,s})_{r^\vee,s} \to \mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r^\vee,s} \to \mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s} \to 0,$$

and Proposition 7.3.4, we have

$$\mathrm{Ext}^1(\mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0.$$

Thus, by the exact sequence

$$0 \to \mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s} \to \mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s} \to \mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}/\mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s} \to 0,$$

we have the following exact sequence

$$0 \to \mathbb{C} \to \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}/\mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) \to \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) \to 0.$$

By Proposition 7.2.8 we have $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}/\mathcal{E}^+(\mathcal{X}^+_{r,s})^*_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) \simeq \mathbb{C}$. Therefore we obtain $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0$. □

Proposition 7.3.6. *Let (a, b, ϵ) be any element in*

$$\{(r, s, +), (r^\vee, s^\vee, +), (r^\vee, s, -), (r, s^\vee, -)\}.$$

Then we have

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{a^\vee,b}, \mathcal{X}^\epsilon_{a^\vee,b^\vee}) = \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{a,b^\vee}, \mathcal{X}^\epsilon_{a^\vee,b^\vee}) = 0.$$

Proof. We will prove only

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s},\mathcal{X}^+_{r^\vee,s^\vee})=0,\qquad\qquad \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s},\mathcal{X}^-_{r,s^\vee})=0.$$

The other equations can be proved in the same way, so we omit the proofs.

First we prove $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s},\mathcal{X}^+_{r^\vee,s^\vee})=0$. Let $K(\Delta^+_{r,s;0})$ be the indecomposable Virasoro module defined by the following exact sequence

$$0\to L(\Delta^-_{r,s^\vee;0})\to K(\Delta^+_{r,s;0})\to K(\tau)\to 0,$$

where $\tau=(\alpha_{r^\vee,s^\vee},\alpha_{r,s^\vee;1},\alpha_{r^\vee,s^\vee;2})$. By the structure of Virasoro Verma modules ([12],[21],[48]), we see that

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\Delta^+_{r,s;0})/L(h_{r,s}),L(\Delta^+_{r^\vee,s^\vee;0}))=0.$$

Thus, by the structure of the Fock module $F_{r^\vee,s^\vee}$, we have

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\Delta^+_{r,s;0}),L(\Delta^+_{r^\vee,s^\vee;0}))\simeq\mathbb{C}. \tag{7.3.4}$$

Let $V(\Delta^+_{r,s;0})$ be the non-trivial extension of this Ext^1-group. By (7.3.4), we have

$$\mathrm{Ext}^1_{\mathcal{L}}(V(\Delta^+_{r,s;0}),L(\Delta^+_{r^\vee,s^\vee;0}))=0.$$

Then, by the exact sequence

$$0\to L(\Delta^+_{r^\vee,s^\vee;0},\Delta^-_{r,s^\vee;0})^*\to V(\Delta^+_{r,s;0})\to K(\tau)\to 0,$$

we obtain

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\tau),L(\Delta^+_{r^\vee,s^\vee;0}))=0. \tag{7.3.5}$$

Assume $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s},\mathcal{X}^+_{r^\vee,s^\vee})\neq 0$. Then, by Proposition 7.2.8, we have

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}/\mathcal{X}^+_{r,s},\mathcal{X}^+_{r^\vee,s^\vee})\neq 0.$$

Fix a non-trivial extension $[E]\in\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}/\mathcal{X}^+_{r,s},\mathcal{X}^+_{r^\vee,s^\vee})$. Note that the Virasoro zero-mode L_0 acts semisimply on E. Let u_1 be the highest weight vector of the submodule $\mathcal{X}^+_{r^\vee,s^\vee}\subset E$. We fix any L_0-homogeneous vector $u_0\in E$ such that, for the surjection $\pi:E\to\mathcal{X}^+_{r,s}$, $\pi(u_0)$ gives the highest weight vector of $\mathcal{X}^+_{r,s}$ and fix a homogeneous vector $u_1^*\in E^*$ such that $\langle u_1^*,u_1\rangle\neq 0$. Then, by (7.3.5), we have $U(\mathcal{L}).u_0\simeq K(\tau)$, and thus

$$L_nu_1^*=0,\quad \text{for } n\geq 1. \tag{7.3.6}$$

Since $[E]\neq 0$, by the structure of $\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}/\mathcal{X}^+_{r,s}$, $\mathcal{E}^\pm(\mathcal{X}^+_{r,s})_{r^\vee,s}$ and $\mathcal{E}^\pm(\mathcal{X}^+_{r^\vee,s^\vee})_{r^\vee,s}$, we have

$$\langle u_1^*,Y_E(W^\bullet;z)u_0\rangle\neq 0, \tag{7.3.7}$$

where $W^\bullet$ is one of $W^\pm$ or W^0. Note that

$$S_{r^\vee+p_+,s}u_0\in L(h_{r,s}).$$

Then by Proposition 7.2.5 and (7.3.6), we have

$$\begin{aligned} 0 &= \langle u_1^*, Y_E(W^\bullet; z) S_{r^\vee+p_+,s} u_0 \rangle \\ &= \prod_{i=1}^{r^\vee+p_+} \prod_{j=1}^{s} (h_{4p_+-1,1} - h_{r^\vee+r+2p_+-2i+1, s+s^\vee-2j+1}) \langle u_1^*, Y_E(W^\bullet; z) u_0 \rangle. \end{aligned}$$

The coefficient in the above equation is nonzero, so we have $\langle u_1^*, Y_E(W^\bullet; z) u_0 \rangle = 0$. But this contradicts (7.3.7).

Next we prove $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}, \mathcal{X}^-_{r,s^\vee}) = 0$. Note that, by the structure of Virasoro Verma modules and by the structure of the Fock module $F_{r,s^\vee;1}$,

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\Delta^-_{r^\vee,s;0})/L(\Delta^+_{r^\vee,s^\vee;0}), L(\Delta^-_{r,s^\vee;0})) \simeq \mathbb{C} \tag{7.3.8}$$

(see Definitions 6.2.11 for the definitions of Virasoro module $K(\Delta^-_{r^\vee,s;0})$). Let $V(\Delta^-_{r^\vee,s;0})$ be the non-trivial extension of this Ext^1-group. By (7.3.8), we have

$$\mathrm{Ext}^1_{\mathcal{L}}(V(\Delta^-_{r^\vee,s;0}), L(\Delta^-_{r,s^\vee;0})) = 0.$$

Then, by the exact sequence

$$0 \to L(\Delta^-_{r,s^\vee;0}, \Delta^+_{r^\vee,s^\vee;1})^* \to V(\Delta^-_{r^\vee,s;0}) \to K(\tau') \to 0,$$

we obtain

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\tau'), L(\Delta^-_{r,s^\vee;0})) = 0, \tag{7.3.9}$$

where $\tau' = (\alpha_{r,s^\vee;1}, \alpha_{r^\vee,s^\vee;2}, \alpha_{r,s^\vee;3})$. Let us assume that

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}, \mathcal{X}^-_{r,s^\vee}) \neq 0.$$

Then, since $\mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r,s^\vee}) = 0$, we have

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}/\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r,s^\vee}) \neq 0.$$

Note that L_0 acts semisimply on any extensions of this Ext^1-group. Fix any non-trivial extension $[E] \in \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}/\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r,s^\vee})$. Then, noting Proposition 7.2.8, by the Virasoro module structure of E we have

$$\mathrm{Ext}^1_{\mathcal{L}}(K(\tau'), L(\Delta^-_{r,s^\vee;0})) \neq 0.$$

But this contradicts (7.3.9). □

Proposition 7.3.7. *Let (a,b) be (r,s) or $(r^\vee,s^\vee)$. Then we have*

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{a,b})_{a^\vee,b}, L(h_{a,b})) = \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{a,b})_{a,b^\vee}, L(h_{a,b})) = 0.$$

Proof. By the exact sequence

$$0 \to \mathcal{K}_{r,s} \to \mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s} \to \mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}/\mathcal{K}_{r,s} \to 0,$$

we have the exact sequence

$$0 \to \mathbb{C} \to \mathbb{C} \to \mathrm{Ext}(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}, L(h_{r,s})) \to \mathrm{Ext}^1(\mathcal{K}_{r,s}, L(h_{r,s})) \to 0.$$

Thus we have $\mathrm{Ext}(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}, L(h_{r,s})) \simeq \mathrm{Ext}^1(\mathcal{K}_{r,s}, L(h_{r,s}))$. Assume

$$\mathrm{Ext}^1(\mathcal{K}_{r,s}, L(h_{r,s})) \neq 0,$$

and fix a non-trivial extension $[E] \in \mathrm{Ext}^1(\mathcal{K}_{r,s}, L(h_{r,s}))$. Since

$$\mathrm{Ext}^1(L(h_{r,s}), L(h_{r,s})) = 0,$$

E has a submodule which is isomorphic to $\mathcal{K}^*_{r,s}$. Thus, by Theorem 6.1.3, we see that E has L_0-nilpotent rank two. Let $\{u_0, u_1\}$ be a basis of the highest weight space of E such that

$$(L_0 - h_{r,s})u_0 \in \mathbb{C}^\times u_1. \tag{7.3.10}$$

Then, by Theorem 6.1.3 and (7.3.10), we have

$$S^*_{r,s} S_{r,s} u_0 \in \mathbb{C}^\times u_1.$$

In particular we have $S_{r,s}u_0 \neq 0$. Thus E has $\mathcal{X}^+_{r^\vee,s^\vee}$ as a composition factor. But this is a contradiction. Thus we obtain $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}, L(h_{r,s})) = 0$. The other equations can be proved in the same way, so we omit the proofs. □

Proposition 7.3.8. *Let (a,b) be $(r^\vee, s)$ or $(r, s^\vee)$. Then we have*

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{a,b})_{a^\vee,b}, L(h_{a^\vee,b})) = \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{a,b})_{a,b^\vee}, L(h_{a,b^\vee})) = 0.$$

Proof. By Proposition 7.2.8, we have

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}, L(h_{r,s})) \simeq \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}/\mathcal{X}^-_{r^\vee,s}, L(h_{r,s})).$$

Assume $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}/\mathcal{X}^-_{r^\vee,s}, L(h_{r,s})) \neq 0$. Then, by considering the contragredient of any non-trivial extension of $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r,s}/\mathcal{X}^-_{r^\vee,s}, L(h_{r,s}))$, we see that

$$\mathrm{Ext}^1(\mathcal{K}_{r,s}, \mathcal{X}^-_{r^\vee,s}) \neq 0.$$

Since $\mathrm{Ext}^1(L(h_{r,s}), \mathcal{X}^-_{r^\vee,s}) = 0$, any non-trivial extension of $\mathrm{Ext}^1(\mathcal{K}_{r,s}, \mathcal{X}^-_{r^\vee,s})$ has a submodule in $\mathrm{Ext}^1(\mathcal{X}^+_{r,s}, \mathcal{X}^-_{r^\vee,s}) \setminus \{0\}$. In particular, we have

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h_{r,s}, \Delta^+_{r,s;0}), L(\Delta^-_{r^\vee,s;0})) \neq 0.$$

On the other hand, by the structure of Virasoro Verma modules, we see that

$$\mathrm{Ext}^1_{\mathcal{L}}(L(h_{r,s}, \Delta^+_{r,s;0}), L(\Delta^-_{r^\vee,s;0})) = 0.$$

Thus we have a contradiction. The other equations can be proved in the same way, so we omit the proofs. □

The following is a summary of Proposition 7.3.1, 7.3.2, 7.3.4, 7.3.5, 7.3.6, 7.3.7 and 7.3.8.

Proposition 7.3.9. *Let* (ϵ, a, b, c, d) *be any element in*

$$\begin{aligned}\{(\epsilon, a, b, c, d)\} =& \big\{(+, r, s, r^\vee, s), (+, r, s, r, s^\vee), (+, r^\vee, s^\vee, r^\vee, s), (+, r^\vee, s^\vee, r, s^\vee),\\ & (-, r^\vee, s, r, s), (-, r^\vee, s, r^\vee, s^\vee), (-, r, s^\vee, r, s), (-, r, s^\vee, r^\vee, s^\vee)\big\}.\end{aligned}$$

Then we have

$$\begin{aligned}&\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{c,d}, L(h_{r,s})) = 0, \quad \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{c,d}, \mathcal{X}^\epsilon_{a,b}) = 0,\\ &\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{c,d}, \mathcal{X}^\epsilon_{a^\vee,b^\vee}) = 0, \quad \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{c,d}, \mathcal{X}^{-\epsilon}_{c,d}) = 0.\end{aligned}$$

Next we will prove that the four indecomposable modules $\mathcal{P}^\pm_{\bullet,\bullet} \in C^{thick}_{r,s}$ are projective. By Propositions 7.2.8, 7.3.1 and 7.3.4, we have the following lemma.

Lemma 7.3.10. *Fix any* (ϵ, a, b, c, d) *in*

$$\begin{aligned}\{(\epsilon, a, b, c, d)\} =& \big\{(+, r, s, r^\vee, s), (+, r, s, r, s^\vee), (+, r^\vee, s^\vee, r^\vee, s), (+, r^\vee, s^\vee, r, s^\vee),\\ & (-, r^\vee, s, r, s), (-, r^\vee, s, r^\vee, s^\vee), (-, r, s^\vee, r, s), (-, r, s^\vee, r^\vee, s^\vee)\big\}.\end{aligned}$$

Then, any indecomposable module whose composition factors are the same as those of $\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{c,d}$ *is isomorphic to* $\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{c,d}$.

By Lemma 7.3.10 and the structure of $\mathcal{P}^\pm_{\bullet,\bullet}$, we have the following proposition.

Proposition 7.3.11. *Fix any* (a, b, ϵ) *in*

$$\{(r, s, +), (r^\vee, s^\vee, +), (r^\vee, s, -), (r, s^\vee, -)\}.$$

Then the logarithmic module $\mathcal{P}^\epsilon_{a,b}$ *has the following sequences of quotient modules:*

$$\begin{aligned}&0 \le U_1(\mathcal{P}^\epsilon_{a,b}) \le U_2(\mathcal{P}^\epsilon_{a,b}) \le U_3(\mathcal{P}^\epsilon_{a,b}) \le U_4(\mathcal{P}^\epsilon_{a,b}) = \mathcal{P}^\epsilon_{a,b}\\ &0 \le V_1(\mathcal{P}^\epsilon_{a,b}) \le V_2(\mathcal{P}^\epsilon_{a,b}) \le V_3(\mathcal{P}^\epsilon_{a,b}) \le V_4(\mathcal{P}^\epsilon_{a,b}) = \mathcal{P}^\epsilon_{a,b},\end{aligned}$$

with

$$\begin{aligned}&U_1 = \mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{a^\vee,b},\ U_2/U_1 = U_3/U_2 = \mathcal{Q}(\mathcal{X}^{-\epsilon}_{a,b^\vee})_{a^\vee,b^\vee},\ U_4/U_3 = \mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{a^\vee,b}\\ &V_1 = \mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{a,b^\vee},\ V_2/V_1 = V_3/V_2 = \mathcal{Q}(\mathcal{X}^{-\epsilon}_{a^\vee,b})_{a^\vee,b^\vee},\ V_4/V_3 = \mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{a,b^\vee}.\end{aligned}$$

Remark 7.3.12. *Figure 7.4 represents the sequence of the subquotients given in Proposition 7.3.11.*

By Propositions 7.3.9 and 7.3.11, we obtain the following theorems.

Theorem 7.3.13.

$$\begin{aligned}&\mathrm{Ext}^1(\mathcal{P}^+_{r,s}, L(h_{r,s})) = \mathrm{Ext}^1(\mathcal{P}^+_{r^\vee,s^\vee}, L(h_{r,s})) = 0,\\ &\mathrm{Ext}^1(\mathcal{P}^-_{r^\vee,s}, L(h_{r,s})) = \mathrm{Ext}^1(\mathcal{P}^-_{r,s^\vee}, L(h_{r,s})) = 0.\end{aligned}$$

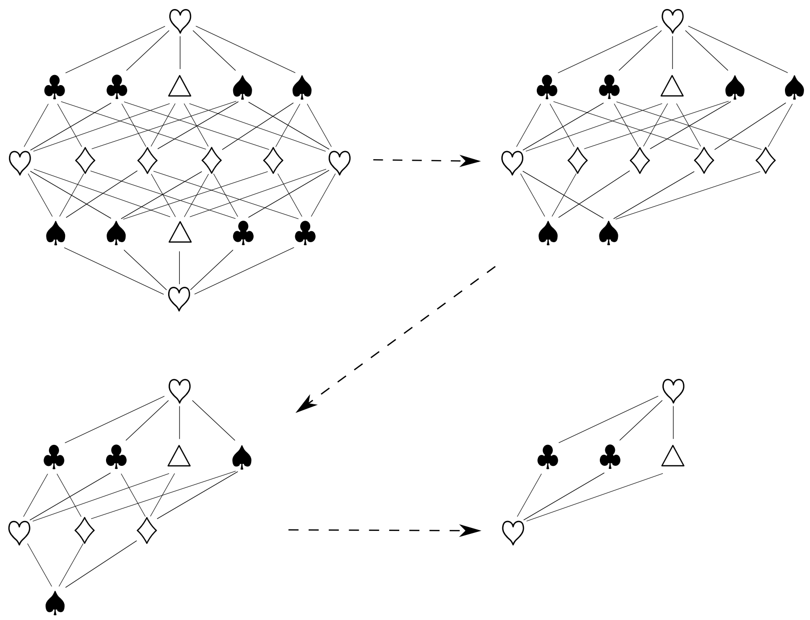


Figure 7.4: The sequence $0 \leq U_1(\mathcal{P}^+_{r,s}) \leq U_2(\mathcal{P}^+_{r,s}) \leq U_3(\mathcal{P}^+_{r,s}) \leq U_4(\mathcal{P}^+_{r,s}) = \mathcal{P}^+_{r,s}$.

Theorem 7.3.14. *Let (a, b, ϵ) be any element in*

$$\{(r, s, +), (r^\vee, s^\vee, +), (r^\vee, s, -), (r, s^\vee, -)\}.$$

Then we have

$$\begin{aligned}
&\mathrm{Ext}^1(\mathcal{P}^\epsilon_{a,b}, \mathcal{X}^\epsilon_{a^\vee,b^\vee}) = \mathrm{Ext}^1(\mathcal{P}^\epsilon_{a^\vee,b^\vee}, \mathcal{X}^\epsilon_{a,b}) = 0,\\
&\mathrm{Ext}^1(\mathcal{P}^\epsilon_{a,b}, \mathcal{X}^{-\epsilon}_{a^\vee,b}) = \mathrm{Ext}^1(\mathcal{P}^\epsilon_{a,b}, \mathcal{X}^{-\epsilon}_{a,b^\vee}) = 0,\\
&\mathrm{Ext}^1(\mathcal{P}^\epsilon_{a^\vee,b^\vee}, \mathcal{X}^{-\epsilon}_{a^\vee,b}) = \mathrm{Ext}^1(\mathcal{P}^\epsilon_{a^\vee,b^\vee}, \mathcal{X}^{-\epsilon}_{a,b^\vee}) = 0.
\end{aligned}$$

By Proposition 7.3.11, we obtain the following proposition.

Proposition 7.3.15. *Each logarithmic module $\mathcal{P}^\pm_{\bullet,\bullet}$ is generated from the top composition factor and has the following socle series as a $\mathcal{W}_{p_+,p_-}$-module:*

1. *For $\mathcal{P}^+_{r,s}$,*

$$\begin{aligned}
&0 \leq S_1(\mathcal{P}^+_{r,s}) \leq S_2(\mathcal{P}^+_{r,s}) \leq S_3(\mathcal{P}^+_{r,s}) \leq S_4(\mathcal{P}^+_{r,s}) \leq S_5(\mathcal{P}^+_{r,s}) = \mathcal{P}^+_{r,s},\\
&S_1 = \mathcal{X}^+_{r,s},\\
&S_2/S_1 = \mathcal{X}^-_{r,s^\vee} \oplus \mathcal{X}^-_{r,s^\vee} \oplus L(h_{r,s}) \oplus \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s},\\
&S_3/S_2 = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r,s},\\
&S_4/S_3 = \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s} \oplus L(h_{r,s}) \oplus \mathcal{X}^-_{r,s^\vee} \oplus \mathcal{X}^-_{r,s^\vee},\\
&S_5/S_4 = \mathcal{X}^+_{r,s}.
\end{aligned}$$

2. *For* $\mathcal{P}^+_{r^\vee,s^\vee}$,

$$\begin{aligned}
&0 \leq S_1(\mathcal{P}^+_{r^\vee,s^\vee}) \leq S_2(\mathcal{P}^+_{r^\vee,s^\vee}) \leq S_3(\mathcal{P}^+_{r^\vee,s^\vee}) \leq S_4(\mathcal{P}^+_{r^\vee,s^\vee}) \leq S_5(\mathcal{P}^+_{r^\vee,s^\vee}) = \mathcal{P}^+_{r^\vee,s^\vee},\\
&S_1 = \mathcal{X}^+_{r^\vee,s^\vee},\\
&S_2/S_1 = \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s} \oplus L(h_{r,s}) \oplus \mathcal{X}^-_{r,s^\vee} \oplus \mathcal{X}^-_{r,s^\vee},\\
&S_3/S_2 = \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee},\\
&S_4/S_3 = \mathcal{X}^-_{r,s^\vee} \oplus \mathcal{X}^-_{r,s^\vee} \oplus L(h_{r,s}) \oplus \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s},\\
&S_5/S_4 = \mathcal{X}^+_{r^\vee,s^\vee}.
\end{aligned}$$

3. *For* $\mathcal{P}^-_{r,s^\vee}$,

$$\begin{aligned}
&0 \leq S_1(\mathcal{P}^-_{r,s^\vee}) \leq S_2(\mathcal{P}^-_{r,s^\vee}) \leq S_3(\mathcal{P}^-_{r,s^\vee}) \leq S_4(\mathcal{P}^-_{r,s^\vee}) \leq S_5(\mathcal{P}^-_{r,s^\vee}) = \mathcal{P}^-_{r,s^\vee},\\
&S_1 = \mathcal{X}^-_{r,s^\vee},\\
&S_2/S_1 = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee},\\
&S_3/S_2 = \mathcal{X}^-_{r,s^\vee} \oplus \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s} \oplus L(h_{r,s}) \oplus L(h_{r,s}) \oplus \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r,s^\vee},\\
&S_4/S_3 = \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s},\\
&S_5/S_4 = \mathcal{X}^-_{r,s^\vee}.
\end{aligned}$$

4. *For* $\mathcal{P}^-_{r^\vee,s}$,

$$\begin{aligned}
&0 \leq S_1(\mathcal{P}^-_{r^\vee,s}) \leq S_2(\mathcal{P}^-_{r^\vee,s}) \leq S_3(\mathcal{P}^-_{r^\vee,s}) \leq S_4(\mathcal{P}^-_{r^\vee,s}) \leq S_5(\mathcal{P}^-_{r^\vee,s}) = \mathcal{P}^-_{r^\vee,s},\\
&S_1 = \mathcal{X}^-_{r^\vee,s},\\
&S_2/S_1 = \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s},\\
&S_3/S_2 = \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r,s^\vee} \oplus L(h_{r,s}) \oplus L(h_{r,s}) \oplus \mathcal{X}^-_{r,s^\vee} \oplus \mathcal{X}^-_{r,s^\vee} \oplus \mathcal{X}^-_{r^\vee,s},\\
&S_4/S_3 = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee},\\
&S_5/S_4 = \mathcal{X}^-_{r^\vee,s}.
\end{aligned}$$

We define the following notation.

Definition 7.3.16. *For any* $\mathcal{W}_{p_+,p_-}$*-module* $M \in C^{thick}_{r,s}$ *with*

$$0 \leq \mathrm{Soc}_1(M) \leq \cdots \leq \mathrm{Soc}_n(M) = M,$$

we say that a simple module of the composition factors of M *is at level* i *if it is contained in* $\mathrm{Soc}_{n-i}(M)/\mathrm{Soc}_{n-i-1}(M)$ $(0 \leq i \leq n-1)$.

Definition 7.3.17. *Let* (a,b) *be* (r,s) *or* $(r^\vee, s^\vee)$*. We define the following indecomposable modules:*

1. *Let* $\mathcal{P}^{+d}_{a,b}$ *be the indecomposable submodule of* $\mathcal{P}^+_{a,b}$ *which is generated from* $2\mathcal{X}^+_{a,b}$ *at level 2.*

2. *Let* $\mathcal{P}^{+u}_{a,b}$ *be the quotient module of* $\mathcal{P}^+_{a,b}$*, which is quotiented by the submodule generated from* $4\mathcal{X}^+_{a^\vee,b^\vee}$ *at level 2.*

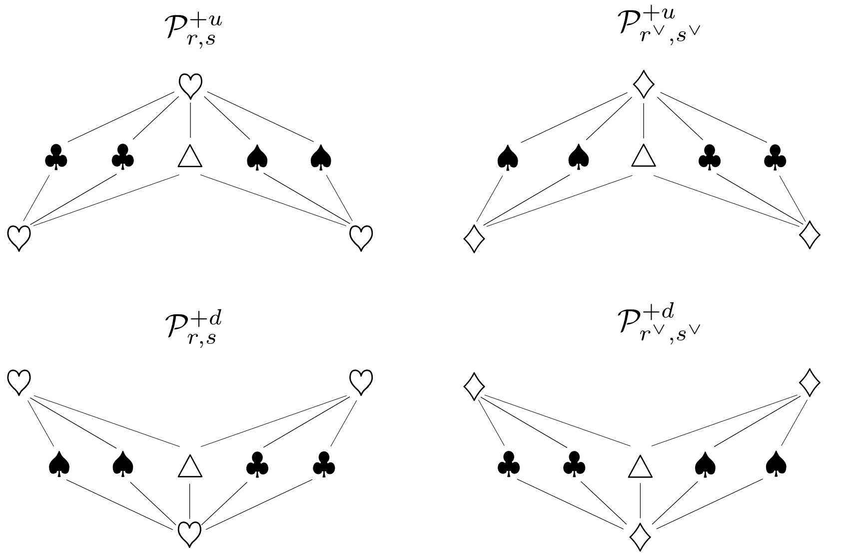


Figure 7.5: The embedding structure of the logarithmic $\mathcal{W}_{p_+,p_-}$-modules $\mathcal{P}^{+u}_{\bullet,\bullet}$ and $\mathcal{P}^{+d}_{\bullet,\bullet}$. The triangle $\triangle$ corresponds to the simple module $L(h_{r,s})$, $\heartsuit$ to $\mathcal{X}^+_{r,s}$, $\diamondsuit$ to $\mathcal{X}^+_{r^\vee,s^\vee}$, $\spadesuit$ to $\mathcal{X}^-_{r,s^\vee}$ and $\clubsuit$ to $\mathcal{X}^-_{r^\vee,s}$.

Remark 7.3.18. *Figure 7.5 represents the embedding structure of the logarithmic $\mathcal{W}_{p_+,p_-}$-modules given in Definition 7.3.17.*

Proposition 7.3.19.

$$\mathrm{Ext}^1(\mathcal{P}^{+d}_{r,s}, \mathcal{X}^+_{r,s}) = \mathrm{Ext}^1(\mathcal{P}^{+d}_{r^\vee,s^\vee}, \mathcal{X}^+_{r^\vee,s^\vee}) = 0.$$

Proof. By Proposition 7.3.11, we see that $\mathcal{P}^{+d}_{r,s}$ has $\mathcal{Q}(\mathcal{X}^+_{r,s})_{r,s^\vee}$ as a submodule. Then by the exact sequence

$$0 \to \mathcal{Q}(\mathcal{X}^+_{r,s})_{r,s^\vee} \to \mathcal{P}^{+d}_{r,s} \to \mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s}/\mathcal{K}_{r,s} \to 0$$

and by the proof of Proposition 7.3.1, we obtain

$$\mathrm{Ext}^1(\mathcal{P}^{+d}_{r,s}, \mathcal{X}^+_{r,s}) = 0.$$

The second equation can be proved in the same way, so we omit the proofs. □

Proposition 7.3.20.

$$\mathrm{Ext}^1(\mathcal{P}^{+u}_{r,s}/(2\mathcal{X}^+_{r,s}), \mathcal{X}^+_{r,s}) = \mathrm{Ext}^1(\mathcal{P}^{+u}_{r^\vee,s^\vee}/(2\mathcal{X}^+_{r^\vee,s^\vee}), \mathcal{X}^+_{r^\vee,s^\vee}) = \mathbb{C}^2.$$

Proof. Let us prove the first equation. The second equation can be proved in the same way, so we omit the proof. Then it is sufficient to show that

$$\mathrm{Ext}^1(\mathcal{P}^{+u}_{r,s}, \mathcal{X}^+_{r,s}) = 0.$$

Assume $\mathrm{Ext}^1(\mathcal{P}^{+u}_{r,s}, \mathcal{X}^+_{r,s}) \neq 0$ and fix any non-trivial extension

$$0 \to \mathcal{X}^+_{r,s} \xrightarrow{\iota} E \xrightarrow{p} \mathcal{P}^{+u}_{r,s} \to 0.$$

For $(a,b) = (r^\vee, s), (r, s^\vee)$, let $\mathcal{E}(\mathcal{X}^-_{a,b})_{r,s}$ be the indecomposable module defined by the following exact sequence

$$0 \to \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s} \to \mathcal{E}(\mathcal{X}^-_{a,b})_{r,s} \to \mathcal{X}^-_{a,b} \to 0.$$

By Propositions 7.2.8 and 7.3.1, we see that at least one of the following sequences of submodules holds

$$\iota(\mathcal{X}^+_{r,s}) \subset \mathcal{E}(\mathcal{X}^-_{r^\vee,s})_{r,s} \subset E, \qquad \iota(\mathcal{X}^+_{r,s}) \subset \mathcal{E}(\mathcal{X}^-_{r,s^\vee})_{r,s} \subset E. \tag{7.3.11}$$

For the Virasoro decomposition $\mathcal{X}^+_{r,s} = \bigoplus_{n\geq 0}(2n+1)L(\Delta^+_{r,s;n})$, let $u \in \mathcal{X}^+_{r,s}$ be the highest weight vector of $L(\Delta^+_{r,s;0})$ and let $w \in \mathcal{X}^+_{r,s}$ be the Virasoro highest weight vector of $L(\Delta^+_{r,s;1})$ such that

$$W^{\pm}[0]w \neq 0 \text{ mod } L(\Delta^+_{r,s;0}).$$

For the surjection $\pi : E \to \mathcal{X}^+_{r,s}$, we fix any L_0-homogeneous vectors $\tilde{u}, \tilde{w} \in E$ such that $\pi(\tilde{u}) = u$ and $\pi(\tilde{w}) = w$. Define the following vectors

$$x_1 = S^*_{r^\vee+p_+,s} S_{r^\vee+p_+,s}\tilde{u} + S_{r^\vee,s^\vee+2p_-} S^*_{r^\vee,s^\vee+2p_-}\tilde{w},$$
$$x_2 = S^*_{r,s^\vee+p_-} S_{r,s^\vee+p_-}\tilde{u} + S_{r^\vee+2p_+,s^\vee} S^*_{r^\vee+2p_+,s^\vee}\tilde{w}.$$

Let us consider the submodule $E_0 \subset E$ defined by $E_0 = \mathcal{W}_{p_+,p_-}.x_1 + \mathcal{W}_{p_+,p_-}.x_2$. By (7.3.11) and by the structure of $\mathcal{E}(\mathcal{X}^-_{r^\vee,s})_{r,s}$, $\mathcal{E}(\mathcal{X}^-_{r,s^\vee})_{r,s}$, $\mathcal{E}(\mathcal{X}^+_{r,s})_{r^\vee,s}$ and $\mathcal{E}(\mathcal{X}^+_{r,s})_{r,s^\vee}$, we see that $E_0 \simeq \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s}$ and

$$p(E_0) \simeq \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s}.$$

Then, by Theorem 6.2.14, we have

$$E_0 = \mathcal{W}_{p_+,p_-}.x_1 + \mathcal{W}_{p_+,p_-}.x_2 + \mathcal{W}_{p_+,p_-}.S_{r,s}S^*_{r,s}\tilde{u}. \tag{7.3.12}$$

Let $E' = E/E_0$. Note that E' is indecomposable. Then, by (7.3.12) and by Propositions 4.2.8 and 4.2.9, we see that L_0 acts semisimply on E' and $L(h_{r,s}) \subset E'$ as the submodule. Let $E'' = E'/L(h_{r,s})$. For the surjection $\pi'' : E'' \to \mathcal{X}^+_{r,s}$, let $u_0 \in E''$ be any highest weight vector such that $\pi''(u_0) = u$ and let u_1 be the highest weight vector of the submodule $\mathcal{X}^+_{r,s} \subset E''$. By (7.3.12), we see that

$$S_{r^\vee,s^\vee+2p_-}S_{r,s^\vee+p_-}u_0 = 0. \tag{7.3.13}$$

By the structure of the indecomposable modules $\mathcal{P}^{+u}_{r,s}/2\mathcal{X}^+_{r,s}$ and $\mathcal{E}^\pm(\mathcal{X}^+_{r,s})_{\bullet,\bullet}$ and by Proposition 7.3.1, we must have

$$\langle u_1^*, Y_{E''}(W^\bullet; z)u_0\rangle \neq 0, \tag{7.3.14}$$

where $W^\bullet$ is one of W^+, W^0 or W^- and u_1^* is any L_0-homogeneous vector of E''^* such that $\langle u_1^*, u_1\rangle \neq 0$. Thus, by (7.3.13) and Proposition 7.2.5, we have

$$\begin{aligned} 0 &= \langle u_1^*, Y_{E''}(W^\bullet; z)S_{r^\vee,s^\vee+2p_-}S_{r,s^\vee+p_-}u_0\rangle \\ &= \prod_{i=1}^{r^\vee}\prod_{j=1}^{s^\vee+2p_-}(h_{4p_+-1,1} - h_{2r^\vee+p_+-2i+1,3p_--2j+1}) \\ &\quad\times \prod_{k=1}^{r}\prod_{l=1}^{s^\vee+p_-}(h_{4p_+-1,1} - h_{2r-2k+1,2s^\vee+2p_--2l+1})\langle u_1^*, Y_{E''}(W^\bullet; z)u_0\rangle. \end{aligned}$$

The coefficient in the above equation is nonzero, so we have $\langle u_1^*, Y_{E''}(W^\bullet; z)u_0\rangle = 0$. But this contradicts (7.3.14).

□

Theorem 7.3.21. *For $\mathcal{P}^+_{r,s}$, $\mathcal{P}^+_{r^\vee,s^\vee} \in C^{thick}_{r,s}$, we have*

$$\mathrm{Ext}^1(\mathcal{P}^+_{r,s}, \mathcal{X}^+_{r,s}) = \mathrm{Ext}^1(\mathcal{P}^+_{r^\vee,s^\vee}, \mathcal{X}^+_{r^\vee,s^\vee}) = 0.$$

Proof. From the exact sequence

$$0 \to \mathcal{P}^{+d}_{r,s} \to \mathcal{P}^+_{r,s} \to \mathcal{P}^+_{r,s}/\mathcal{P}^{+d}_{r,s} \to 0$$

and Proposition 7.3.19, we have the following exact sequence

$$0 \to \mathbb{C} \to \mathbb{C} \to \mathbb{C}^2 \to \mathrm{Ext}^1(\mathcal{P}^+_{r,s}/\mathcal{P}^{+d}_{r,s}, \mathcal{X}^+_{r,s}) \to \mathrm{Ext}^1(\mathcal{P}^+_{r,s}, \mathcal{X}^+_{r,s}) \to 0.$$

By Proposition 7.3.20, we have

$$\mathrm{Ext}^1(\mathcal{P}^+_{r,s}/\mathcal{P}^{+d}_{r,s}, \mathcal{X}^+_{r,s}) = \mathbb{C}^2.$$

Thus we have

$$\mathrm{Ext}^1(\mathcal{P}^+_{r,s}, \mathcal{X}^+_{r,s}) = 0.$$

The second equation can be proved in the same way. □

Definition 7.3.22. *Let (a,b) be $(r^\vee, s)$ or $(r, s^\vee)$. Let $\mathcal{P}^{-u}_{a,b}$ be a quotient module of $\mathcal{P}^-_{a,b}$ quotiented by the submodule generated from $4\mathcal{X}^-_{a^\vee,b^\vee}$ and $2L(h_{r,s})$ at level 2.*

Proposition 7.3.23.

$$\mathrm{Ext}^1(\mathcal{P}^{-u}_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = \mathrm{Ext}^1(\mathcal{P}^{-u}_{r,s^\vee}, \mathcal{X}^-_{r,s^\vee}) = 0.$$

Proof. Assume $\mathrm{Ext}^1(\mathcal{P}^{-u}_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) \neq 0$. Fix a non-trivial extension

$$0 \to \mathcal{X}^-_{r^\vee,s} \xrightarrow{\iota} E \xrightarrow{p} \mathcal{P}^{-u}_{r^\vee,s} \to 0.$$

Since $\mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0$, we see that

$$\mathrm{Socle}(E) = \iota(\mathcal{X}^-_{r^\vee,s}) \oplus 2\mathcal{X}^-_{r^\vee,s}.$$

Let ι_1 and ι_2 be injections from $\mathcal{X}^-_{r^\vee,s}$ to E such that $p \circ \iota_1(\mathcal{X}^-_{r^\vee,s})$ and $p \circ \iota_2(\mathcal{X}^-_{r^\vee,s})$ are $\mathrm{Socle}(\mathcal{P}^{-u}_{r^\vee,s})$. Let $\{v_+, v_-\}$ be the basis of the highest weight space of $\mathcal{X}^-_{r^\vee,s}$ such that

$$W^\pm[0]v_\pm = 0, \qquad W^\pm[0]v_\mp \in \mathbb{C}^\times v_\pm.$$

For the canonical surjection $\pi : E \to \mathcal{X}^-_{r^\vee,s}$, we fix any L_0-homogeneous vectors $\tilde{v}_-, \tilde{v}_+ \in E$ such that $\pi(\tilde{v}_\pm) = v_\pm$. Note that the Virasoro module $U(\mathcal{L}).\tilde{v}_\pm$ has $\widetilde{P}(\Delta^-_{r^\vee,s;0})$ as a subquotient. By Proposition 7.3.4, we see that $\iota(\mathcal{X}^-_{r^\vee,s})$ is contained in both submodules of E generated from each of $2\mathcal{X}^+_{r,s}$ and $2\mathcal{X}^+_{r^\vee,s^\vee}$ at level one. Thus, by the structure of $\widetilde{P}(\Delta^-_{r^\vee,s;0})$ and the structure of the non-trivial extensions in $\mathrm{Ext}^1(\mathcal{E}^\pm(\mathcal{X}^+_{r,s})_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s})$ and $\mathrm{Ext}^1(\mathcal{E}^\pm(\mathcal{X}^+_{r^\vee,s^\vee})_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s})$, we have

$$(L_0 - \Delta^-_{r^\vee,s;0})\tilde{v}_+ \in \mathbb{C}^\times \iota(v_-) + \mathbb{C}\iota(v_+) + \mathbb{C}^\times \iota_1(v_+) + \mathbb{C}^\times \iota_2(v_+),$$
$$(L_0 - \Delta^-_{r^\vee,s;0})\tilde{v}_- \in \mathbb{C}^\times \iota(v_+) + \mathbb{C}\iota(v_-) + \mathbb{C}^\times \iota_1(v_-) + \mathbb{C}^\times \iota_2(v_-).$$

From this, we can see the contradiction as in the proof of Proposition 7.3.4. The second equation can be proved in the same way, so we omit the proof. □

By this proposition, we have the following proposition.

Proposition 7.3.24.

$$\mathrm{Ext}^1(\mathcal{P}^{-u}_{r^\vee,s}/2\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = \mathrm{Ext}^1(\mathcal{P}^{-u}_{r,s^\vee}/2\mathcal{X}^-_{r^\vee,s}, \mathcal{X}^-_{r,s^\vee}) = \mathbb{C}^2.$$

Theorem 7.3.25.

$$\mathrm{Ext}^1(\mathcal{P}^-_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = \mathrm{Ext}^1(\mathcal{P}^-_{r,s^\vee}, \mathcal{X}^-_{r,s^\vee}) = 0.$$

Proof. $\mathcal{P}^-_{r^\vee,s}$ has the indecomposable submodule generated from $\mathcal{X}^-_{r^\vee,s}$ at level 2 and whose components are the same as those of $\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r^\vee,s^\vee}$. By Proposition 7.3.4, this submodule is isomorphic to $\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r^\vee,s^\vee}$. Then by the exact sequence

$$0 \to \mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r^\vee,s^\vee} \to \mathcal{P}^-_{r^\vee,s} \to \mathcal{P}^-_{r^\vee,s}/\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r^\vee,s^\vee} \to 0$$

and Proposition 7.3.4, we have the following exact sequence

$$0 \to \mathbb{C} \to \mathbb{C} \to \mathbb{C} \to \mathrm{Ext}^1(\mathcal{P}^-_{r^\vee,s}/\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r^\vee,s^\vee}, \mathcal{X}^-_{r^\vee,s}) \to \mathrm{Ext}^1(\mathcal{P}^-_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) \to 0. \quad (7.3.15)$$

Let $\mathcal{M}$ be the quotient module of $\mathcal{P}^-_{r^\vee,s}$ defined by the following exact sequences

$$0 \to \mathcal{K}_{r,s} \oplus \mathcal{K}_{r,s} \to \mathcal{P}^-_{r^\vee,s}/\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r^\vee,s^\vee} \to \mathcal{M} \to 0.$$

By this exact sequence and by $\mathrm{Ext}^1(\mathcal{K}_{r,s}, \mathcal{X}^-_{r^\vee,s}) = 0$ (see the proof of Proposition 7.3.8), we have

$$\mathrm{Ext}^1(\mathcal{M}, \mathcal{X}^-_{r^\vee,s}) \simeq \mathrm{Ext}^1(\mathcal{P}^-_{r^\vee,s}/\mathcal{Q}(\mathcal{X}^-_{r^\vee,s})_{r^\vee,s^\vee}, \mathcal{X}^-_{r^\vee,s}). \quad (7.3.16)$$

Note that $\mathcal{M}$ satisfies the exact sequence

$$0 \to \mathcal{X}^-_{r^\vee,s} \oplus 4\mathcal{X}^-_{r,s^\vee} \to \mathcal{M} \to \mathcal{P}^{-u}_{r^\vee,s}/2\mathcal{X}^-_{r^\vee,s} \to 0.$$

By this exact sequence and by Proposition 7.3.24, we have the following exact sequence

$$0 \to \mathbb{C} \to \mathbb{C} \to \mathbb{C} \to \mathbb{C}^2 \to \mathrm{Ext}^1(\mathcal{M}, \mathcal{X}^-_{r^\vee,s}) \to 0.$$

Thus we have $\mathrm{Ext}^1(\mathcal{M}, \mathcal{X}^-_{r^\vee,s}) \simeq \mathbb{C}$. Therefore, by (7.3.15) and (7.3.16), we obtain

$$\mathrm{Ext}^1(\mathcal{P}^-_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) = 0.$$

The second equation can be proved in the same way. □

Since all logarithmic modules $\mathcal{P}^\pm_{\bullet,\bullet}$ in $C^{thick}_{r,s}$ are generated from the top composition factors, by Theorems 7.3.13, 7.3.14, 7.3.21 and 7.3.25, we obtain the following theorem.

Theorem 7.3.26. *$\mathcal{P}^+_{r,s}, \mathcal{P}^+_{r^\vee,s^\vee}, \mathcal{P}^-_{r^\vee,s}$ and $\mathcal{P}^-_{r,s^\vee}$ are the projective covers of $\mathcal{X}^+_{r,s}$, $\mathcal{X}^+_{r^\vee,s^\vee}$, $\mathcal{X}^-_{r^\vee,s}$ and $\mathcal{X}^-_{r,s^\vee}$, respectively.*

Remark 7.3.27. *Figure 7.6 represents the embedding structure of the series of quotient modules given in the proof of Theorem 7.3.25.*

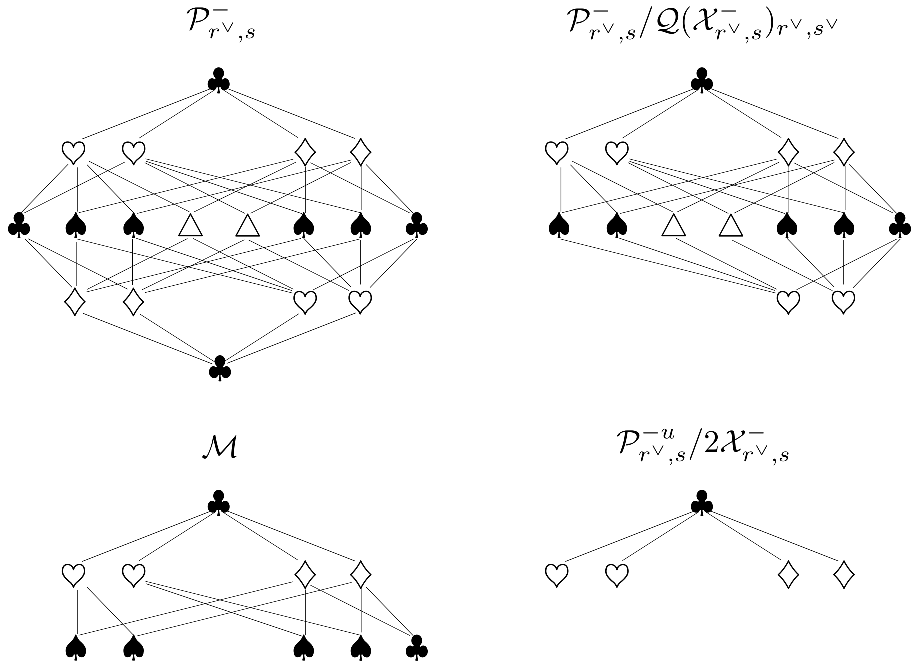


Figure 7.6: The embedding structure of the $\mathcal{W}_{p_+,p_-}$-modules given in the proof of Theorem 7.3.25. The triangle $\triangle$ corresponds to the simple module $L(h_{r,s})$, $\heartsuit$ to $\mathcal{X}^+_{r,s}$, $\diamondsuit$ to $\mathcal{X}^+_{r^\vee,s^\vee}$, $\spadesuit$ to $\mathcal{X}^-_{r,s^\vee}$ and $\clubsuit$ to $\mathcal{X}^-_{r^\vee,s}$.

7.4 The projective covers of the minimal simple modules $L(h_{r,s})$

Fix any thick block $C^{thick}_{r,s}$. Let $\mathcal{P}(h_{r,s})$ be the projective cover of the minimal simple module $L(h_{r,s})$. By Corollary 4.2.11, we can see that $\mathcal{P}(h_{r,s})$ has L_0 nilpotent rank three. In the following, we determine the structure of $\mathcal{P}(h_{r,s})$.

Let $\mathcal{N}_{r,s}$ and $\mathcal{N}_{r^\vee,s^\vee}$ be the submodules of $\mathcal{P}^+_{r,s}$ and $\mathcal{P}^+_{r^\vee,s^\vee}$ generated from $L(h_{r,s})$ at level 1. $\mathcal{N}_{r,s}$ and $\mathcal{N}_{r^\vee,s^\vee}$ are indecomposable and have the following socle series whose socle lengths are four:

1. For $\mathcal{N}_{r,s}$, we have

$$\begin{aligned}
&\mathrm{Soc}_1(\mathcal{N}_{r,s}) = \mathcal{X}^+_{r,s}\\
&\mathrm{Soc}_2(\mathcal{N}_{r,s})/\mathrm{Soc}_1(\mathcal{N}_{r,s}) = 2\mathcal{X}^-_{r^\vee,s} \oplus L(h_{r,s}) \oplus 2\mathcal{X}^-_{r,s^\vee}\\
&\mathrm{Soc}_3(\mathcal{N}_{r,s})/\mathrm{Soc}_2(\mathcal{N}_{r,s}) = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee}\\
&\mathcal{N}_{r,s}/\mathrm{Soc}_3(\mathcal{N}_{r,s}) = L(h_{r,s}).
\end{aligned}$$

2. For $\mathcal{N}_{r^\vee,s^\vee}$, we have

$$\begin{aligned}
&\mathrm{Soc}_1(\mathcal{N}_{r^\vee,s^\vee}) = \mathcal{X}^+_{r^\vee,s^\vee}\\
&\mathrm{Soc}_2(\mathcal{N}_{r^\vee,s^\vee})/\mathrm{Soc}_1(\mathcal{N}_{r^\vee,s^\vee}) = 2\mathcal{X}^-_{r^\vee,s} \oplus L(h_{r,s}) \oplus 2\mathcal{X}^-_{r,s^\vee}\\
&\mathrm{Soc}_3(\mathcal{N}_{r^\vee,s^\vee})/\mathrm{Soc}_2(\mathcal{N}_{r^\vee,s^\vee}) = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee}\\
&\mathcal{N}_{r^\vee,s^\vee}/\mathrm{Soc}_3(\mathcal{N}_{r^\vee,s^\vee}) = L(h_{r,s}).
\end{aligned}$$

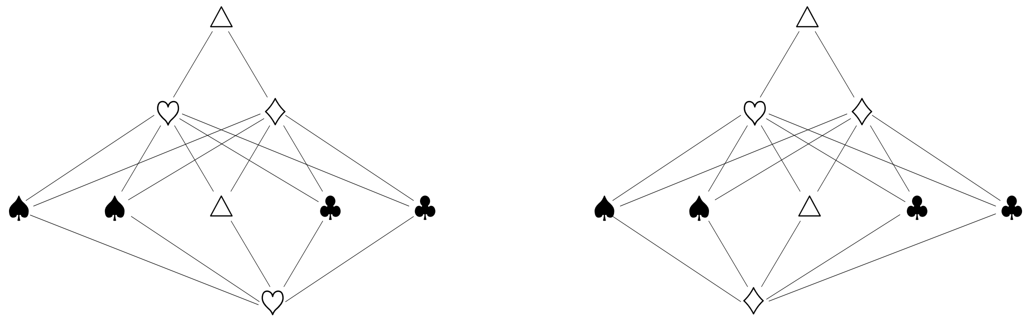


Figure 7.7: The embedding structure of the logarithmic $\mathcal{W}_{p_+,p_-}$-modules $\mathcal{N}_{r,s}$ and $\mathcal{N}_{r^\vee,s^\vee}$. The triangle $\triangle$ corresponds to the simple module $L(h_{r,s})$, $\heartsuit$ to $\mathcal{X}^+_{r,s}$, $\diamondsuit$ to $\mathcal{X}^+_{r^\vee,s^\vee}$, $\spadesuit$ to $\mathcal{X}^-_{r,s^\vee}$ and $\clubsuit$ to $\mathcal{X}^-_{r^\vee,s}$.

Remark 7.4.1. *Figure 7.7 represents the embedding structure of the logarithmic $\mathcal{W}_{p_+,p_-}$-modules $\mathcal{N}_{r,s}$ and $\mathcal{N}_{r^\vee,s^\vee}$.*

As the quotient of $\mathcal{N}_{r,s}$, we define the indecomposable module $\mathcal{Q}(h_{r,s})$ which satisfies

$$[\mathcal{Q}(h_{r,s})] \in \mathrm{Ext}^1(\mathcal{K}_{r,s}, \mathcal{K}^*_{r^\vee,s^\vee}) \setminus \{0\}.$$

By Propositions 7.2.8 and by the proof of Proposition 7.3.7, we have the following lemma.

Lemma 7.4.2.

$$\mathrm{Ext}^1(\mathcal{Q}(h_{r,s}), L(h_{r,s})) = 0.$$

Recall the indecomposable modules $\mathcal{P}^{+u}_{r,s}$ and $\mathcal{P}^{+u}_{r^\vee,s^\vee}$ given in Definition 7.3.17. We define the indecomposable modules

$$\mathcal{R}_{r,s} := \mathcal{P}^{+u}_{r,s}/\mathcal{K}_{r,s}, \qquad \mathcal{R}_{r^\vee,s^\vee} := \mathcal{P}^{+u}_{r^\vee,s^\vee}/\mathcal{K}_{r^\vee,s^\vee}.$$

Note that, by Theorem 6.1.3, $\mathcal{R}_{r,s}$ and $\mathcal{R}_{r^\vee,s^\vee}$ have L_0 nilpotent rank two. By Propositions 7.2.8, 7.3.20, we obtain the following lemma.

Lemma 7.4.3.

$$\mathrm{Ext}^1(\mathcal{R}_{r,s}, \mathcal{X}^+_{r,s}) = \mathrm{Ext}^1(\mathcal{R}_{r^\vee,s^\vee}, \mathcal{X}^+_{r^\vee,s^\vee}) = 0.$$

Lemma 7.4.4.

$$\mathrm{Ext}^1(\mathcal{N}_{r,s}/\mathcal{K}_{r,s}, \mathcal{X}^+_{r,s}) = \mathrm{Ext}^1(\mathcal{N}_{r^\vee,s^\vee}/\mathcal{K}_{r^\vee,s^\vee}, \mathcal{X}^+_{r^\vee,s^\vee}) = 0.$$

Proof. We only prove $\mathrm{Ext}^1(\mathcal{N}_{r,s}/\mathcal{K}_{r,s}, \mathcal{X}^+_{r,s}) = 0$. The second equation can be proved in the same way.

Assume that $\mathrm{Ext}^1(\mathcal{N}_{r,s}/\mathcal{K}_{r,s}, \mathcal{X}^+_{r,s}) \neq 0$ and fix any non-trivial extension E in this Ext^1-group. By Lemma 7.4.3, we see that E has the submodule $\mathcal{R}_{r,s}$. Let $t \in E$ be the highest weight vector in $E[h_{r,s}]$. and let $\{u_0, u_1\}$ be a basis of the highest weight space of the submodule $\mathcal{R}_{r,s} \subset E$ such that

$$(L_0 - \Delta^+_{r,s;0})u_0 \in \mathbb{C}^\times u_1.$$

On the other hand, by

$$(L_0 - h_{r,s})t = 0, \qquad S_{r^\vee,s^\vee}t \in \mathbb{C}^\times u_0 + \mathbb{C}u_1,$$

we have $(L_0 - \Delta^+_{r,s;0})u_0 = 0$. Thus we have a contradiction. □

Note that there exists surjections from $\mathcal{P}(h_{r,s})$ to $\mathcal{N}_{r,s}$ and from $\mathcal{P}(h_{r,s})$ to $\mathcal{N}_{r^\vee,s^\vee}$. Thus, by Corollary 4.2.11 and Lemma 7.4.4, as the quotient of $\mathcal{P}(h_{r,s})$ we have the indecomposable module $\mathcal{P}'(h_{r,s})$ whose socle series is given by

$$\begin{aligned}
&\mathrm{Soc}_1(\mathcal{P}'(h_{r,s})) = L(h_{r,s}),\\
&\mathrm{Soc}_2(\mathcal{P}'(h_{r,s}))/\mathrm{Soc}_1(\mathcal{P}'(h_{r,s})) = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee},\\
&\mathrm{Soc}_3(\mathcal{P}'(h_{r,s}))/\mathrm{Soc}_2(\mathcal{P}'(h_{r,s})) = 2\mathcal{X}^-_{r^\vee,s} \oplus L(h_{r,s}) \oplus 2\mathcal{X}^-_{r,s^\vee},\\
&\mathrm{Soc}_4(\mathcal{P}'(h_{r,s}))/\mathrm{Soc}_3(\mathcal{P}'(h_{r,s})) = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee},\\
&\mathcal{P}'(h_{r,s})/\mathrm{Soc}_4(\mathcal{P}'(h_{r,s})) = L(h_{r,s}).
\end{aligned}$$

Lemma 7.4.5.

$$\mathrm{Ext}^1(\mathcal{P}'(h_{r,s}), L(h_{r,s})) = 0.$$

Proof. Note that $\mathcal{P}'(h_{r,s})$ has the submodule isomorphic to $\mathcal{Q}(h_{r,s})$. Thus by Lemma 7.4.2 we have

$$\mathrm{Ext}^1(\mathcal{P}'(h_{r,s}), L(h_{r,s})) = 0.$$

□

Lemma 7.4.6.

$$\mathrm{Ext}^1(\mathcal{P}'(h_{r,s}), \mathcal{X}^+_{r,s}) = \mathrm{Ext}^1(\mathcal{P}'(h_{r,s}), \mathcal{X}^+_{r^\vee,s^\vee}) = 0.$$

Proof. We only prove the first equation. The second equation can be proved in the same way. By Proposition 7.3.1, we see that

$$\mathrm{Ext}^1(\mathcal{Q}(h_{r,s}), \mathcal{X}^+_{r,s}) = 0. \tag{7.4.1}$$

Note that $\mathcal{P}'(h_{r,s})/\mathcal{Q}(h_{r,s}) \simeq \mathcal{N}_{r,s}/\mathcal{K}_{r,s}$. Thus, by Lemma 7.4.4 and by (7.4.1), we obtain $\mathrm{Ext}^1(\mathcal{P}'(h_{r,s}), \mathcal{X}^+_{r,s}) = 0$. □

Lemma 7.4.7.

$$\mathrm{Ext}^1(\mathcal{P}'(h_{r,s}), \mathcal{X}^-_{r^\vee,s}) = \mathrm{Ext}^1(\mathcal{P}'(h_{r,s}), \mathcal{X}^-_{r,s^\vee}) = 0.$$

Proof. We only prove the first equation. The second equation can be proved in the same way. Assume

$$\mathrm{Ext}^1(\mathcal{P}'(h_{r,s}), \mathcal{X}^-_{r^\vee,s}) \neq 0$$

and fix any non-trivial extension E in this Ext^1-group. Then we see that E has a submodule which has a indecomposable subquotient in

$$\mathrm{Ext}^1(\mathcal{E}^\pm(\mathcal{X}^+_{r,s})^*_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}) \qquad \text{or} \qquad \mathrm{Ext}^1(\mathcal{E}^\pm(\mathcal{X}^+_{r^\vee,s^\vee})^*_{r^\vee,s}, \mathcal{X}^-_{r^\vee,s}).$$

But this contradicts Proposition 7.3.4. □

By Lemmas 7.4.5, 7.4.6 and 7.4.7, we have $\mathcal{P}(h_{r,s}) \simeq \mathcal{P}'(h_{r,s})$. Therefore we obtain the following theorem.

Theorem 7.4.8. *The projective module $\mathcal{P}(h_{r,s})$ has the following socle series:*

$$\begin{aligned}
&\mathrm{Soc}_1(\mathcal{P}(h_{r,s})) = L(h_{r,s}),\\
&\mathrm{Soc}_2(\mathcal{P}(h_{r,s}))/\mathrm{Soc}_1(\mathcal{P}(h_{r,s})) = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee},\\
&\mathrm{Soc}_3(\mathcal{P}(h_{r,s}))/\mathrm{Soc}_2(\mathcal{P}(h_{r,s})) = 2\mathcal{X}^-_{r^\vee,s} \oplus L(h_{r,s}) \oplus 2\mathcal{X}^-_{r,s^\vee},\\
&\mathrm{Soc}_4(\mathcal{P}(h_{r,s}))/\mathrm{Soc}_3(\mathcal{P}(h_{r,s})) = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee},\\
&\mathcal{P}(h_{r,s})/\mathrm{Soc}_4(\mathcal{P}(h_{r,s})) = L(h_{r,s}).
\end{aligned}$$

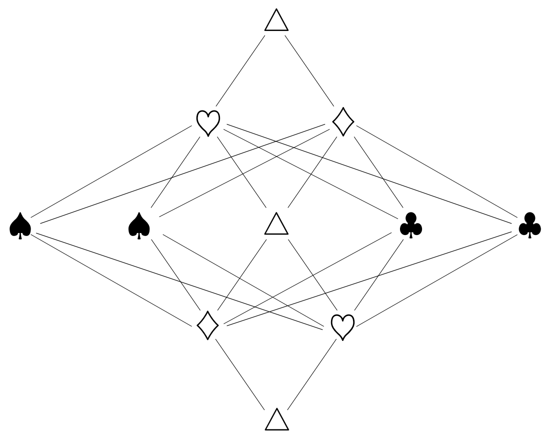


Figure 7.8: The embedding structure of the logarithmic $\mathcal{W}_{p_+,p_-}$-module $\mathcal{P}(h_{r,s})$. The triangle $\triangle$ corresponds to the simple module $L(h_{r,s})$, $\heartsuit$ to $\mathcal{X}^+_{r,s}$, $\diamondsuit$ to $\mathcal{X}^+_{r^\vee,s^\vee}$, $\spadesuit$ to $\mathcal{X}^-_{r,s^\vee}$ and $\clubsuit$ to $\mathcal{X}^-_{r^\vee,s}$.

Remark 7.4.9. *Figure 7.8 represents the embedding structure of the projective module $\mathcal{P}(h_{r,s})$. This embedding structure is conjectured in [33], and it is shown that $P(h_{r,s})$ has no dual. In this thesis, we do not go any further into the properties of the projective modules $\mathcal{P}(h_{r,s})$ on the tensor category of $\mathcal{W}_{p_+,p_-}$-modules.*

7.5 The projective covers of simple modules in the thin blocks

Fix any two thin blocks C^{thin}_{r,p_-}, $C^{thin}_{p_+,s}$ $(1 \le r < p_+,\ 1 \le s < p_-)$. Let us consider the logarithmic modules $\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}, \mathcal{Q}(\mathcal{X}^-_{r^\vee,p_-})_{r,p_-} \in C^{thin}_{r,p_-}$ and $\mathcal{Q}(\mathcal{X}^+_{p_+,s})_{p_+,s^\vee}, \mathcal{Q}(\mathcal{X}^-_{p_+,s^\vee})_{p_+,s} \in C^{thin}_{p_+,s}$.

As in the case of the logarithmic modules $\mathcal{Q}(\mathcal{X}^\pm_{\bullet,\bullet})_{\bullet,\bullet}$ in the thick blocks, we can prove the following propositions.

Proposition 7.5.1. *The logarithmic modules $\mathcal{Q}(\mathcal{X}^\pm_{\bullet,p_-})_{\bullet,p_-}$ and $\mathcal{Q}(\mathcal{X}^\pm_{p_+,\bullet})_{p_+,\bullet}$ are generated from the top composition factors and the socle series are given by :*

1. *The socle series of $\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}$ is given by*

$$\begin{aligned}
\mathrm{Soc}_1 &= \mathcal{X}^+_{r,p_-}, \\
\mathrm{Soc}_2/\mathrm{Soc}_1 &= \mathcal{X}^-_{r^\vee,p_-} \oplus \mathcal{X}^-_{r^\vee,p_-}, \\
\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}/\mathrm{Soc}_2 &= \mathcal{X}^+_{r,p_-}.
\end{aligned}$$

2. *The socle series of $\mathcal{Q}(\mathcal{X}^-_{r^\vee,p_-})_{r,p_-}$ is given by*

$$\begin{aligned}
\mathrm{Soc}_1 &= \mathcal{X}^-_{r^\vee,p_-}, \\
\mathrm{Soc}_2/\mathrm{Soc}_1 &= \mathcal{X}^+_{r,p_-} \oplus \mathcal{X}^+_{r,p_-}, \\
\mathcal{Q}(\mathcal{X}^-_{r^\vee,p_-})_{r,p_-}/\mathrm{Soc}_2 &= \mathcal{X}^-_{r^\vee,p_-}.
\end{aligned}$$

3. *The socle series of $\mathcal{Q}(\mathcal{X}^+_{p_+,s})_{p_+,s^\vee}$ is given by*

$$\begin{aligned}
\mathrm{Soc}_1 &= \mathcal{X}^+_{p_+,s}, \\
\mathrm{Soc}_2/\mathrm{Soc}_1 &= \mathcal{X}^-_{p_+,s^\vee} \oplus \mathcal{X}^-_{p_+,s^\vee}, \\
\mathcal{Q}(\mathcal{X}^+_{p_+,s})_{p_+,s^\vee}/\mathrm{Soc}_2 &= \mathcal{X}^+_{p_+,s}.
\end{aligned}$$

4. *The socle series of $\mathcal{Q}(\mathcal{X}^-_{p_+,s^\vee})_{p_+,s}$ is given by*

$$\begin{aligned}
\mathrm{Soc}_1 &= \mathcal{X}^-_{p_+,s^\vee}, \\
\mathrm{Soc}_2/\mathrm{Soc}_1 &= \mathcal{X}^+_{p_+,s} \oplus \mathcal{X}^+_{p_+,s}, \\
\mathcal{Q}(\mathcal{X}^-_{p_+,s^\vee})_{p_+,s}/\mathrm{Soc}_2 &= \mathcal{X}^-_{p_+,s^\vee}.
\end{aligned}$$

Proposition 7.5.2.

1. *In the thin block C^{thin}_{r,p_-}, we have*

$$\mathrm{Ext}^1(\mathcal{X}^+_{r,p_-}, \mathcal{X}^-_{r^\vee,p_-}) = \mathrm{Ext}^1(\mathcal{X}^-_{r^\vee,p_-}, \mathcal{X}^+_{r,p_-}) = \mathbb{C}^2.$$

The other extensions between the simple modules in this block are trivial.

2. In the thin block $C^{thin}_{p_+,s}$, we have

$$\mathrm{Ext}^1(\mathcal{X}^+_{p_+,s}, \mathcal{X}^-_{p_+,s^\vee}) = \mathrm{Ext}^1(\mathcal{X}^-_{p_+,s^\vee}, \mathcal{X}^+_{p_+,s}) = \mathbb{C}^2.$$

The other extensions between the simple modules in this block are trivial.

The proofs of these propositions are the same as in Theorems 7.1.1, 7.1.3 and Proposition 7.2.8, so we omit them.

Proposition 7.5.3.

$$\begin{aligned}
&\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}, \mathcal{X}^+_{r,p_-}) = \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{p_+,s})_{p_+,s^\vee}, \mathcal{X}^+_{p_+,s}) = 0,\\
&\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,p_-})_{r,p_-}, \mathcal{X}^+_{r,p_-}) = \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{p_+,s^\vee})_{p_+,s}, \mathcal{X}^+_{p_+,s}) = 0,\\
&\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{r^\vee,p_-})_{r,p_-}, \mathcal{X}^-_{r^\vee,p_-}) = \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^-_{p_+,s^\vee})_{p_+,s}, \mathcal{X}^-_{p_+,s^\vee}) = 0,\\
&\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}, \mathcal{X}^-_{r^\vee,p_-}) = \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{p_+,s})_{p_+,s^\vee}, \mathcal{X}^-_{p_+,s^\vee}) = 0.
\end{aligned}$$

Proof. We will only prove $\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}, \mathcal{X}^+_{r,p_-}) = 0$. The other equations can be shown in the same way as Propositions 7.3.2, 7.3.4 and 7.3.5, so we omit the proofs.

Since $\mathrm{Ext}^1(\mathcal{X}^+_{r,p_-}, \mathcal{X}^+_{r,p_-}) = 0$, it is sufficient to show that

$$\mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}/\mathcal{X}^+_{r,p_-}, \mathcal{X}^+_{r,p_-}) = \mathbb{C}.$$

Fix any extension $[E] \in \mathrm{Ext}^1(\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}/\mathcal{X}^+_{r,p_-}, \mathcal{X}^+_{r,p_-})$. Assume that E has L_0-nilpotent rank two. Let $\overline{E}$ be the highest weight space of E. Note that the Virasoro module $U(\mathcal{L}).\overline{E}$ has L_0 nilpotent rank two and

$$[U(\mathcal{L}).\overline{E}] \in \mathrm{Ext}^1_{\mathcal{L}}(K(\tau), L(h_{r,p_-})) \setminus \{0\},$$

where $\tau = (\alpha_{r^\vee,p_-;-1}, \alpha_{r,p_-}, \alpha_{r^\vee,p_-;1})$. Then, by Theorem 6.2.9, as the Baer sum of extensions obtained from E and $\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}$, we have a extension

$$0 \to \mathcal{X}^+_{r,p_-} \xrightarrow{\iota} E' \xrightarrow{p} \mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}/\mathcal{X}^+_{r,p_-} \to 0$$

such that L_0 acts semisimply on the highest weight space of E'. Then, by Propositions 4.2.8 and 4.2.9, we see that L_0 acts semisimply on E'.

Assume $[E'] \neq 0$. Fix any homogeneous vector $u_0 \in E'$ such that $p(u_0)$ is the highest weight vector of $p(E') = \mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}/\mathcal{X}^+_{r,p_-}$, and let u_1 be the highest weight vector of the submodule $\mathcal{X}^+_{r,s} \subset E'$. Since $[E'] \neq 0$, we must have

$$\langle u_1^*, Y_{E'}(W^\bullet; z)u_0\rangle \neq 0, \tag{7.5.1}$$

where $W^\bullet$ is one of W^+, W^0 or W^- and u_1^* is a homogeneous vector of E'^* such that $\langle u_1^*, u_1\rangle \neq 0$. Since L_0 acts semisimply on E', by Theorem 6.2.9, we have $S_{r^\vee,2p_-}S_{r,p_-}u_0 = 0$. Then, by Proposition 7.2.5, we have

$$\begin{aligned}
0 &= \langle u_1^*, Y_{E'}(W^\bullet; z)S_{r^\vee,2p_-}S_{r,p_-}u_0\rangle\\
&= \prod_{i=1}^{r^\vee}\prod_{j=1}^{2p_-}(h_{4p_+-1,1} - h_{r+r^\vee-2i+1,3p_--2j+1})\\
&\quad\times \prod_{k=1}^{r}\prod_{l=1}^{p_-}(h_{4p_+-1,1} - h_{2r-2k+1,2p_--2l+1})\langle u_1^*, Y_{E'}(W^\bullet; z)u_0\rangle.
\end{aligned}$$

The coefficient in the above equation is nonzero, so we have $\langle u_1^*, Y_{E'}(W^\bullet; z)u_0\rangle = 0$. But this contradicts (7.5.1). In the case where L_0 acts semisimply on E, we see that $[E] = 0$ as shown above. $\square$

Since the logarithmic modules $\mathcal{Q}(\mathcal{X}^\pm_{\bullet,p_-})_{\bullet,p_-}$ and $\mathcal{Q}(\mathcal{X}^\pm_{p_+,\bullet})_{p_+,\bullet}$ are generated from the top composition factors, by Proposition 7.5.3, we obtain the following theorem.

Theorem 7.5.4.

1. *$\mathcal{Q}(\mathcal{X}^+_{r,p_-})_{r^\vee,p_-}$ and $\mathcal{Q}(\mathcal{X}^-_{r^\vee,p_-})_{r,p_-}$ are the projective covers of $\mathcal{X}^+_{r,p_-}$ and $\mathcal{X}^-_{r^\vee,p_-}$, respectively.*

2. *$\mathcal{Q}(\mathcal{X}^+_{p_+,s})_{p_+,s^\vee}$ and $\mathcal{Q}(\mathcal{X}^-_{p_+,s^\vee})_{p_+,s}$ are the projective covers of $\mathcal{X}^+_{p_+,s}$ and $\mathcal{X}^-_{p_+,s^\vee}$, respectively.*

Chapter 8

Non-semisimple fusion rules

Since the triplet W algebra $\mathcal{W}_{p_+,p_-}$ is C_2-cofinite, Theorem 4.13 in [36] show that $\mathcal{C}_{p_+,p_-}$ has braided tensor category structure as developed in the series of papers [37, 38, 39, 40, 41, 42, 43, 44]. We denote $(\mathcal{C}_{p_+,p_-}, \boxtimes)$ by the tensor category on $\mathcal{C}_{p_+,p_-}$, where the unit object is given by $\mathcal{K}_{1,1} := \mathcal{W}_{p_+,p_-}.|\alpha_{1,1}\rangle$. Note that the tensor product $\boxtimes$ of $(\mathcal{C}_{p_+,p_-}, \boxtimes)$ is right exact (see Proposition 4.26 in [39]).

The tensor category $(\mathcal{C}_{p_+,p_-}, \boxtimes)$ is not rigid. In fact, if we assume that $(\mathcal{C}_{p_+,p_-}, \boxtimes)$ is rigid, then by the exact sequence

$$0 \to L(h_{1,1}) \to \mathcal{X}^+_{1,1} \boxtimes \mathcal{X}^+_{1,1} \to \mathcal{X}^+_{1,1} \to 0$$

and

$$L(h_{1,1}) \boxtimes \mathcal{X}^+_{1,1} = 0$$

which will be proved in this section (see Proposition 8.4.3 and Proposition 8.2.2), we have the exact sequence

$$0 \to L(h_{1,1}) \boxtimes L(h_{1,1}) \to 0 \to 0 \to 0.$$

But, since $L(h_{1,1}) \boxtimes L(h_{1,1}) = L(h_{1,1})$, this is a contradiction.

This makes it more difficult to study the structure of the tensor category $(\mathcal{C}_{p_+,p_-}, \boxtimes)$ compared to the triplet W-algebra $\mathcal{W}_p$ (cf.[56],[64]). In this chapter, we will show the rigidity of the indecomposable modules $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$ in Theorems 8.3.7 and 8.3.15, using the methods detailed in [15] and [56]. Using the rigidity of $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$, we show that all indecomposable modules $\mathcal{K}_{r,s}$, $\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ and $\mathcal{P}^{\pm}_{r,s}$ can be obtained by repeatedly multiplying $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$. As a result we see that all simple modules in the thin blocks, all indecomposable modules $\mathcal{K}_{r,s}$, $\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ and $\mathcal{P}^{\pm}_{r,s}$ are rigid objects. The rigidity of these indecomposable modules was conjectured in [32].

We also determine the tensor product between all simple modules. These results are stated in Propositions 8.4.3, 8.4.4, 8.4.5, 8.4.7 and 8.4.8.

8.1 Tensor product $\boxtimes$ and $P(w)$-intertwining operators

In this section, we review the definition of the tensor product $\boxtimes$ and $P(w)$-intertwining operators in accordance with [8],[39],[51] and derive some identities known as the Nahm-Gaberdiel-Kausch fusion algorithm(cf. [35]).

Definition 8.1.1. *Let V be a vertex operator (super)algebra and let $\mathcal{C}$ be a category of grading-restricted generalized V-modules. A* tensor product *(or* fusion product*) of M_1 and M_2 in $\mathcal{C}$ is a pair $(M_1 \boxtimes M_2, \mathcal{Y}_\boxtimes)$, with $M_1 \boxtimes M_2$ and $\mathcal{Y}_\boxtimes$ an intertwining operator of type $\binom{M_1 \boxtimes M_2}{M_1\ M_2}$, which satisfies the following universal property: For any $M_3 \in \mathcal{C}$ and intertwining operator $\mathcal{Y}$ of type $\binom{M_3}{M_1\ M_2}$, there is a unique V-module homomorphism $f : M_1 \boxtimes M_2 \to M_3$ such that $\mathcal{Y} = f \circ \mathcal{Y}_\boxtimes$.*

In the paper [39], Huang, Lepowsky and Zhang introduced the notion of $P(w)$-intertwining operators and the $P(w)$-tensor product. The definitions are as follows.

Definition 8.1.2 ([39])**.** *Fix $w \in \mathbb{C}^\times$. Let V be a vertex operator (super)algebra and let $\mathcal{C}$ be a category of grading-restricted generalized V-modules. Given M_1, M_2 and M_3 in $\mathcal{C}$, a $P(w)$-*intertwining operator *I* of type *$\binom{M_3}{M_1\ M_2}$ is a bilinear map $I : M_1 \otimes M_2 \to M_3$ that satisfies the following properties:*

1. *For any $\psi_1 \in M_1$ and $\psi_2 \in M_2$, $\pi_h(I[\psi_1 \otimes \psi_2]) = 0$ for all $h \ll 0$, where π_h denotes the projection onto the generalized eigenspace $M_3[h]$ of L_0-eigenvalue h.*

2. *For any $\psi_1 \in M_1$, $\psi_2 \in M_2$, $\psi_3^* \in M_3^*$ and $A \in V$, the three point functions*

$$\langle \psi_3^*, Y_3(A; z)I[\psi_1 \otimes \psi_2]\rangle,\ \langle \psi_3^*, I[Y_1(A; z-w)\psi_1 \otimes \psi_2]\rangle,\ \langle \psi_3^*, I[\psi_1 \otimes Y_2(A; z)\psi_2]\rangle$$

   are absolutely convergent in the regions $|z| > |w| > 0$, $|w| > |z-w| > 0$, $|w| > |z| > 0$, respectively, where Y_i is the action of V-module.

3. *Given any $f(t) \in R_{P(w)} := \mathbb{C}[t, t^{-1}, (t-w)^{-1}]$, we have the following identity*

$$\begin{aligned}&\oint_{0,w} f(z)\langle \psi_3^*, Y_3(A; z)I[\psi_1 \otimes \psi_2]\rangle \frac{\mathrm{d}z}{2\pi i}\\ &= \oint_w f(z)\langle \psi_3^*, I[Y_1(A; z-w)\psi_1 \otimes \psi_2]\rangle \frac{\mathrm{d}z}{2\pi i} + \mu \oint_0 f(z)\langle \psi_3^*, I[\psi_1 \otimes Y_2(A; z)\psi_2]\rangle \frac{\mathrm{d}z}{2\pi i},\end{aligned}$$

   where μ is the mutual locality index of A with ψ_1.

Definition 8.1.3 ([39])**.** *Let V be a vertex operator (super)algebra and let $\mathcal{C}$ be a category of grading-restricted generalized V-modules. A $P(w)$-*tensor product *of M_1 and M_2 in $\mathcal{C}$ is a pair $(M_1 \boxtimes_{P(w)} M_2, \boxtimes_{P(w)})$, with $M_1 \boxtimes_{P(w)} M_2$ and $\boxtimes_{P(w)}$ a $P(w)$-intertwining operator of type $\binom{M_1 \boxtimes_{P(w)} M_2}{M_1\ M_2}$, which satisfies the following universal property: For any $M_3 \in \mathcal{C}$ and $P(w)$-intertwining operator I of type $\binom{M_3}{M_1\ M_2}$, there is a unique V-module homomorphism $\eta : M_1 \boxtimes_{P(w)} M_2 \to M_3$ such that*

$$\overline{\eta} \circ \boxtimes_{P(w)}[\psi_1 \otimes \psi_2] = I[\psi_1 \otimes \psi_2]$$

for all $\psi_1 \in M_1$ and $\psi_2 \in M_2$, where $\overline{\eta}$ denotes the extension of η to a map between the completions of $M_1 \boxtimes_{P(w)} M_2$ and M_3.

Remark 8.1.4. *It is known that the definition $P(w)$-tensor product $\boxtimes_{P(w)}$ does not depend on the choice of $w \in \mathbb{C}^\times$. See Remark 4.22 in [39].*

The following proposition is due to [39].

Proposition 8.1.5. *Let V be a vertex operator (super)algebra and M_1, M_2 and M_3 be V-modules. Then there exists a linear isomorphism from the space of intertwining operators of type $\binom{M_3}{M_1\ M_2}$ to the space of $P(w)$-intertwining operators of the same type.*

By this proposition, the structure of the space of intertwining operators can be determined from the structure of $P(w)$-intertwining operators of the same type. In the following, we will introduce some formulas derived from the $P(w)$-compatibility conditions.

We define a translation map

$$T_1 : \mathbb{C}(t) \to \mathbb{C}(t), \qquad \text{by} \qquad f(t) \mapsto f(t+1),$$

and a expansion map

$$\iota_+ : \mathbb{C}(t) \hookrightarrow \mathbb{C}((t))$$

that expands a given rational function in t as a power series around $t = 0$. Given V-modules M_1, M_2, M_3 and a $P(1)$-intertwining operator I of type $\binom{M_3}{M_1\ M_2}$, as detailed in [8],[51], we can define the action of $V \otimes \mathbb{C}[t, t^{-1}, (t-1)^{-1}]$ or $V \otimes \mathbb{C}((t))$ on M_3^* as

$$\begin{aligned} &\langle Af(t)\psi_3^*, I[\psi_1 \otimes \psi_1]\rangle = \langle A\iota_+(f(t))\psi_3^*, I[\psi_1 \otimes \psi_1]\rangle \\ &= \langle \psi_3^*, I[\iota_+ \circ T_1\big(A^{\text{opp}} f(t^{-1})\big)\psi_1 \otimes \psi_2]\rangle + \mu\langle \psi_3^*, I[\psi_1 \otimes \iota_+\big(A^{\text{opp}} f(t^{-1})\big)\psi_2]\rangle \end{aligned} \tag{8.1.1}$$

where $\psi_3^* \in M_3^*$, $\psi_i \in M_i$,

$$A^{\text{opp}} := e^{t^{-1}L_1}(-t^2)^{L_0} A t^{-2},$$

and (assuming $A \in V$ has L_0 weight h) $At^n\psi_i = A[n-h+1]\psi_i$.

By (8.1.1), we have the following lemma.

Lemma 8.1.6. *Let V be a vertex operator (super)algebra. Let $A \in V$ be a Virasoro non-zero primary vector with L_0 conformal weight h, that is*

$$L_0 A = hA, \qquad\qquad L_n A = 0, \quad n \geq 1.$$

Given V-modules M_1, M_2, M_3 and a $P(1)$-intertwining operator I of type $\binom{M_3}{M_1\ M_2}$, we have the following identities:

$$\begin{aligned} &\langle A[n]\psi_3^*, I[\psi_1 \otimes \psi_2]\rangle \\ &= \sum_{i=0}^{\infty} \binom{h-n-1}{i} \langle \psi_3^*, I[\big(A[i-h+1]\psi_1\big) \otimes \psi_2]\rangle + \mu\langle \psi_3^*, I[\psi_1 \otimes \big(A[-n]\psi_2\big)]\rangle \end{aligned}$$

and

$$\begin{aligned} &\langle At^{2h-3}(t^{-1}-1)^{n+h-2}\psi_3^*, I[\psi_1 \otimes \psi_2]\rangle \\ &= \langle \psi_3^*, I[\big((A[n-1] + A[n])\psi_1\big) \otimes \psi_2]\rangle \\ &\qquad + \mu \sum_{i=0}^{\infty} \binom{n+h-2}{i} (-1)^{n-i+h-2} \langle \psi_3^*, I[\psi_1 \otimes \big(A[i-h+2]\psi_2\big)]\rangle, \end{aligned}$$

where $\psi_3^* \in M_3^*$, $\psi_i \in M_i$.

In particular, in the case of the conformal vector $A = T$ in Lemma 8.1.6, we have the following lemma.

Lemma 8.1.7. *Let V be a vertex operator (super)algebra. Given V-modules M_1, M_2, M_3 and a $P(1)$-*intertwining operator *I of type $\binom{M_3}{M_1\ M_2}$, we have the following identities:*

$$\langle L_n\psi_3^*, I[\psi_1\otimes\psi_2]\rangle = \sum_{m=0}^{\infty}\binom{-n+1}{m}\langle\psi_3^*, I[(L_{m-1}\psi_1)\otimes\psi_2]\rangle + \langle\psi_3^*, I[\psi_1\otimes(L_{-n}\psi_2)]\rangle,$$

$$\sum_{m=0}^{\infty}(-1)^m\langle L_{m-n}\psi_3^*, I[\psi_1\otimes\psi_2]\rangle$$
$$= \langle\psi_3^*, I[((L_{n-1}+L_n)\psi_1)\otimes\psi_2]\rangle + \sum_{m=0}^{\infty}\binom{n}{m}(-1)^{n-m}\langle\psi_3^*, I[\psi_1\otimes(L_m\psi_2)]\rangle,$$

where $\psi_3^*\in M_3^*$, $\psi_i\in M_i$.

Hereafter we omit $P(1)$-intertwining operators and use the abbreviation as

$$\langle\psi_3^*, \psi_1\otimes\psi_2\rangle = \langle\psi_3^*, I[\psi_1\otimes\psi_2]\rangle$$

unless otherwise noted.

From this chapter, we will use the following notation frequently.

Definition 8.1.8. *Let V be a vertex operator (super)algebra*

1. *For any $M\in V$-*Mod*, we define the following vector space*

$$A_0(M) = \{\psi\in M \mid \psi\neq 0,\ A[n]\psi = 0,\ {}^\forall A\in V,\ n>0\}.$$

2. *For any $M\in V$-*Mod*, we define* the top composition factors *of M as follows*

$$\mathrm{top}(M) = \mathrm{Socle}(M^*),$$

where M^* is the contragredient of M.

3. *Given V-modules M_1, M_2, M_3, we define*

$$I\binom{M_3}{M_2\ \ M_1} = \left\{\textit{intertwining operators of type } \binom{M_3}{M_2\ \ M_1}\right\}.$$

8.2 Tensor product $L(h_{r,s})\boxtimes\bullet$

First we introduce the tensor products between any pair of minimal simple modules $L(h_{r,s})$. Since the maximal ideal $\mathcal{X}_{1,1}^+$ of $\mathcal{W}_{p_+,p_-}$ acts on the minimal simple modules $L(h_{r,s})$ trivially, we have the following minimal model fusion rules.

Proposition 8.2.1 ([10])**.** *For $1\le r,r'\le p_+-1$, $1\le s,s'\le p_--1$, we have*

$$L(h_{r,s})\boxtimes L(h_{r',s'}) = \bigoplus_{\substack{i=1+|r-r'|\\ i+r+r'=1 \bmod 2}}^{\min\{r+r'-1,2p_+-r-r'-1\}}\ \bigoplus_{\substack{j=1+|s-s'|\\ j+s+s'=1 \bmod 2}}^{\min\{s+s'-1,2p_--s-s'-1\}} L(h_{i,j}).$$

Proposition 8.2.2.

$$\mathcal{X}^{\pm}_{r,s} \boxtimes L(h_{1,1}) = 0, \quad 1 \leq r \leq p_+,\ 1 \leq s \leq p_-.$$

Proof. Assume $\mathcal{X}^{+}_{r,s} \boxtimes L(h_{1,1}) \neq 0$. Fix any non-zero vector $\psi_3^* \in A_0((\mathcal{X}^{+}_{r,s} \boxtimes L(h_{1,1}))^*)$ and let ψ_1 and ψ_2 be the highest weight vectors of $\mathcal{X}^{+}_{r,s}$ and $L(h_{1,1})$, respectively. Note that the maximal ideal $\mathcal{X}^{+}_{1,1}$ of $\mathcal{W}_{p_+,p_-}$ acts trivially on any minimal simple modules $L(h_{r,s})$. Then, by Lemmas 8.1.6 and 8.1.7, we can see that $\langle \psi_3^*, \mathcal{X}^{+}_{r,s} \otimes L(h_{1,1})\rangle$ is determined by the numbers

$$\langle \psi_3^*, \psi_1 \otimes L_{-1}^k \psi_2 \rangle,$$

for $k \geq 0$. Since $L_{-1}\psi_2 = 0$, $\langle \psi_3^*, \mathcal{X}^{+}_{r,s} \otimes L(h_{1,1})\rangle$ is determined by $\langle \psi_3^*, \psi_1 \otimes \psi_2\rangle$. In particular, the intertwining operator $\mathcal{Y}_{\boxtimes}$ is non-logarithmic. Using Lemma 8.1.7, we have

$$\begin{aligned}
&\langle L_0\psi_3^*, \psi_1 \otimes \psi_2\rangle \\
&= \langle \psi_3^*, L_{-1}\psi_1 \otimes \psi_2\rangle + \langle \psi_3^*, L_0\psi_1 \otimes \psi_2\rangle + \langle \psi_3^*, \psi_1 \otimes L_0\psi_2\rangle \\
&= \langle \psi_3^*, L_0\psi_1 \otimes \psi_2\rangle.
\end{aligned}$$

Thus, the L_0-eigenvalue of ψ_3^* is the same as that of ψ_1. Then, by restricting the action of $\mathcal{W}_{p_+,p_-}$ to the Virasoro action in the intertwining operator $\mathcal{Y}_{\boxtimes}$, we have a non-trivial non-logarithmic Virasoro intertwining operator of type

$$\begin{pmatrix} & L(\Delta^{+}_{r,s;0}) & \\ L(\Delta^{+}_{r,s;0}) & & L(h_{1,1}) \end{pmatrix}.$$

Note that $S_{p_+-1,p_--1}\psi_2 = 0$. Then by using Proposition 7.2.5, we have

$$\begin{aligned}
0 &= \langle \psi_3^*, \psi_1 \otimes S_{p_+-1,p_--1}\psi_2\rangle \\
&= \prod_{i=1}^{p_+-1}\prod_{j=1}^{p_--1} (h_{r^\vee+p_+,s} - h_{p_+-1+r^\vee+p_+-2i+1,p_--1+s-2i+1})\langle \psi_3^*, \psi_1 \otimes \psi_2\rangle.
\end{aligned}$$

We see that the coefficient of the above equation is non-zero. Thus we obtain $\langle \psi_3^*, \psi_1 \otimes \psi_2\rangle = 0$. But this contradicts the assumption. Similarly, we can prove $L(h_{1,1}) \boxtimes \mathcal{X}^{-}_{r,s} = 0$. □

By Propositions 8.2.1 and 8.2.2, we obtain the following proposition.

Proposition 8.2.3. *For any simple module $\mathcal{X}^{\pm}_{r,s}$, we have*

$$\mathcal{X}_{r,s} \boxtimes L(h_{r',s'}) = 0, \quad 1 \leq r' \leq p_+ - 1,\ 1 \leq s' \leq p_- - 1.$$

Corollary 8.2.4. *For $1 \leq r, r' \leq p_+ - 1$, $1 \leq s, s' \leq p_- - 1$, we have*

$$\mathcal{K}_{r,s} \boxtimes L(h_{r',s'}) \simeq L(h_{r,s}) \boxtimes L(h_{r',s'}).$$

Corollary 8.2.5. *For any $(r,s) \in \mathcal{T}$ and $M \in \mathcal{W}_{p_+,p_-}$, $L(h_{r,s}) \boxtimes M$ becomes a direct sum of minimal simple modules.*

8.3 Self-dual objects $\mathcal{K}_{r,s}$

Let us extend the definition of $\mathcal{K}_{r,s}$ given in Definition 7.2.3 as follows.

Definition 8.3.1. *We define the following $\mathcal{W}_{p_+,p_-}$-modules*

1. *For $1 \le r \le p_+ - 1$, $1 \le s \le p_- - 1$,*
$$\mathcal{K}_{r,s} := \mathcal{W}_{p_+,p_-}.\,|\alpha_{r,s}\rangle\,.$$

2. *For $1 \le r \le p_+$, $1 \le s \le p_-$,*
$$\mathcal{K}_{r,p_-} := \mathcal{X}^+_{r,p_-}, \qquad \mathcal{K}_{p_+,s} := \mathcal{X}^+_{p_+,s}.$$

In this section, we compute some tensor product $\mathcal{K}_{1,2} \boxtimes \bullet$ and $\mathcal{K}_{2,1} \boxtimes \bullet$, and show that the indecomposable modules $\mathcal{K}_{r,s}$ are rigid and self-dual.

Proposition 8.3.2.

1. *For $1 \le r \le p_+$, $2 \le s < p_-$, the dimension of the vector space $A_0((\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s})^*)$ is at most two dimensional. The L_0 eigenvalues are contained in $\{h_{r,s-1}, h_{r,s+1}\}$, which corresponds to the highest weights of $\mathcal{K}_{r,s-1}$ and $\mathcal{K}_{r,s+1}$, respectively.*

2. *For $1 \le r \le p_+$, the dimension of the vector space $A_0((\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,1})^*)$ is at most one dimensional. The L_0 eigenvalues is given by $h_{r,2}$ which corresponds to the highest weight of $\mathcal{K}_{r,2}$.*

3. *For $1 \le r \le p_+$, $2 \le s < p_-$, the dimension of the vector space $A_0((\mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r,s})^*)$ is at most two dimensional. The L_0 eigenvalues are contained in $\{\Delta^+_{r,s-1;0}, \Delta^+_{r,s+1;0}\}$, which corresponds to the highest weights of $\mathcal{X}^+_{r,s-1}$ and $\mathcal{X}^+_{r,s+1}$, respectively.*

4. *For $1 \le r \le p_+$, $2 \le s < p_-$, the dimension of the vector space $A_0((\mathcal{K}_{1,2} \boxtimes \mathcal{X}^-_{r,s})^*)$ is at most four dimensional. The L_0 eigenvalues are contained in $\{\Delta^-_{r,s-1;0}, \Delta^-_{r,s+1;0}\}$, which corresponds to the highest weights of $\mathcal{X}^-_{r,s-1}$ and $\mathcal{X}^-_{r,s+1}$, respectively.*

5. *For $1 \le r \le p_+$, $s = p_-$, the vector space $A_0((\mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r,p_-})^*)$ is at most two dimensional. The L_0 eigenvalues are contained in $\{h_{r,p_--1}, \Delta^+_{r,p_--1;0}\}$, which corresponds to the highest weights of $L(h_{r,p_--1})$ and $\mathcal{X}^+_{r,p_--1}$, respectively.*

6. *For $1 \le r \le p_+$, $s = p_-$, the vector space $A_0((\mathcal{K}_{1,2} \boxtimes \mathcal{X}^-_{r,p_-})^*)$ is at most four dimensional. The L_0 eigenvalues are contained in $\{\Delta^+_{r,1;0}, \Delta^-_{r,p_--1;0}\}$ which corresponds to the highest weights of $\mathcal{X}^+_{r,1}$ and $\mathcal{X}^-_{r,p_--1}$, respectively.*

Proof. Let $p_- > 3$ and fix any $1 \le r \le p_+ - 1$, $2 \le s < p_- - 2$. Assume $\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s} \ne 0$. Let ψ^*, ϕ_1, ϕ_2 be arbitrary elements of $A_0((\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s})^*)$, $\mathcal{K}_{1,2}$ and $\mathcal{K}_{r,s}$, respectively. For $n \ge 1$, let $w^{(n)}_i (i = -n, -n+1 \dots, n)$ be the Virasoro highest weight vectors of the vector subspace $(2n+1)L(\Delta^+_{r,s;n}) \subset \mathcal{K}_{r,s}$. Then, by Lemma 8.1.6, we see that the value $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the values

$$\langle \psi^*, U(\mathcal{L}).\,|\alpha_{1,2}\rangle \otimes |\alpha_{1,2}\rangle\rangle, \qquad \langle \psi^*, U(\mathcal{L}).\,|\alpha_{1,2}\rangle \otimes w^{(n)}_i\rangle,$$

for some finite $n \geq 1$ and i. By Lemma 8.1.7, we can see that the values $\langle \psi^*, U(\mathcal{L}).\, |\alpha_{1,2}\rangle \otimes |\alpha_{r,s}\rangle \rangle$ and $\langle \psi^*, U(\mathcal{L}).\, |\alpha_{1,2}\rangle \otimes w_i \rangle$ are determined by the numbers

$$\langle \psi^*, (L_{-1}^k \, |\alpha_{1,2}\rangle) \otimes |\alpha_{r,s}\rangle \rangle, \qquad \langle \psi^*, (L_{-1}^l \, |\alpha_{1,2}\rangle) \otimes w_i^{(n)} \rangle,$$

for $k, l \geq 0$, respectively. Note that the highest weight vector $|\alpha_{1,2}\rangle$ satisfies

$$\big(L_{-1}^2 - \frac{p_+}{p_-} L_{-2}\big) \, |\alpha_{1,2}\rangle = 0.$$

Thus, the value $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the numbers

$$\begin{aligned} &\langle \psi^*, |\alpha_{1,2}\rangle \otimes |\alpha_{r,s}\rangle \rangle, \qquad \langle \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes |\alpha_{r,s}\rangle \rangle, \\ &\langle \psi^*, |\alpha_{1,2}\rangle \otimes w_i^{(n)} \rangle, \qquad \langle \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes w_i^{(n)} \rangle. \end{aligned}$$

Let us determine the eigenvalues of ψ^*. By Lemma 8.1.7, we have

$$\begin{aligned} \langle L_0 \psi^*, |\alpha_{1,2}\rangle \otimes |\alpha_{r,s}\rangle \rangle &= (h_{1,2} + h_{r,s}) \langle \psi^*, |\alpha_{1,2}\rangle \otimes |\alpha_{r,s}\rangle \rangle + \langle \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes |\alpha_{r,s}\rangle \rangle, \\ \langle L_0 \psi^*, |\alpha_{1,2}\rangle \otimes w_i^{(n)} \rangle &= (h_{1,2} + \Delta^+_{r,s;n}) \langle \psi^*, |\alpha_{1,2}\rangle \otimes w_i^{(n)} \rangle + \langle \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes w_i^{(n)} \rangle, \end{aligned}$$

and

$$\begin{aligned} \langle L_0 \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes |\alpha_{r,s}\rangle \rangle &= (h_{1,2} + h_{r,s} + 1) \langle \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes |\alpha_{r,s}\rangle \rangle \\ &\quad + \frac{p_+}{p_-} \langle \psi^*, (L_{-2} \, |\alpha_{1,2}\rangle) \otimes |\alpha_{r,s}\rangle \rangle \\ &= \frac{p_+}{p_-} h_{r,s} \langle \psi^*, |\alpha_{1,2}\rangle \otimes |\alpha_{r,s}\rangle \rangle \\ &\quad + (h_{1,2} + h_{r,s} + 1 - \frac{p_+}{p_-}) \langle \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes |\alpha_{r,s}\rangle \rangle, \end{aligned}$$

$$\begin{aligned} \langle L_0 \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes w_i^{(n)} \rangle &= (h_{1,2} + \Delta^+_{r,s;n} + 1) \langle \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes w_i^{(n)} \rangle \\ &\quad + \frac{p_+}{p_-} \langle \psi^*, (L_{-2} \, |\alpha_{1,2}\rangle) \otimes w_i^{(n)} \rangle \\ &= \frac{p_+}{p_-} \Delta^+_{r,s;n} \langle \psi^*, |\alpha_{1,2}\rangle \otimes w_i^{(n)} \rangle \\ &\quad + (h_{1,2} + \Delta^+_{r,s;n} + 1 - \frac{p_+}{p_-}) \langle \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes w_i^{(n)} \rangle. \end{aligned}$$

Then we have

$$\begin{pmatrix} \langle L_0 \psi^*, |\alpha_{1,2}\rangle \otimes |\alpha_{r,s}\rangle \rangle \\ \langle L_0 \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes |\alpha_{r,s}\rangle \rangle \end{pmatrix} = A_1 \begin{pmatrix} \langle \psi_i^*, |\alpha_{1,2}\rangle \otimes |\alpha_{r,s}\rangle \rangle \\ \langle \psi_i^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes |\alpha_{r,s}\rangle \rangle \end{pmatrix}$$
$$\begin{pmatrix} \langle L_0 \psi^*, |\alpha_{1,2}\rangle \otimes w_i^{(n)} \rangle \\ \langle L_0 \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes w_i^{(n)} \rangle \end{pmatrix} = A_2 \begin{pmatrix} \langle \psi^*, |\alpha_{1,2}\rangle \otimes w_i^{(n)} \rangle \\ \langle \psi^*, (L_{-1} \, |\alpha_{1,2}\rangle) \otimes w_i^{(n)} \rangle \end{pmatrix}.$$

where

$$A_1 = \begin{pmatrix} h_{1,2} + h_{r,s} & \frac{p_+}{p_-} h_{r,s} \\ 1 & h_{1,2} + h_{r,s} + 1 - \frac{p_+}{p_-} \end{pmatrix},$$
$$A_2 = \begin{pmatrix} h_{1,2} + \Delta^+_{r,s;n} & \frac{p_+}{p_-} \Delta^+_{r,s;n} \\ 1 & h_{1,2} + \Delta^+_{r,s;n} + 1 - \frac{p_+}{p_-} \end{pmatrix}.$$

We see that A_1 and A_2 are diagonalizable and eigenvalues are given by $\{h_{r,s-1}, h_{r,s-1}\}$ and $\{\Delta^+_{r,s+1;n}, \Delta^+_{r,s-1;n}\}$, respectively. Note that the eigenvalues of A_2 do not correspond to any L_0 eigenvalues of the highest weight space of the simple $\mathcal{W}_{p_+,p_-}$-modules. Thus we see that the value $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the numbers

$$\langle \psi^*, |\alpha_{1,2}\rangle \otimes |\alpha_{r,s}\rangle\rangle, \qquad \langle \psi^*, (L_{-1}\, |\alpha_{1,2}\rangle) \otimes |\alpha_{r,s}\rangle\rangle,$$

L_0 acts semisimply on ψ^*, and the L_0 eigenvalue of ψ^* is given by $h_{r,s+1}$ or $h_{r,s-1}$.

The other cases can be proved in the same way, so we omit the proofs.

□

Similar to Proposition 8.3.2, we obtain the following proposition.

Proposition 8.3.3.

- *For $p_+ \geq 3$, we have*

  1. *For $2 \leq r < p_+$, $1 \leq s \leq p_-$, the vector space $A_0((\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{r,s})^*)$ is at most two dimensional. The L_0 eigenvalues are contained in $\{h_{r-1,s}, h_{r+1,s}\}$, which corresponds to the highest weights of $\mathcal{K}_{r-1,s}$ and $\mathcal{K}_{r+1,s}$, respectively.*
  2. *For $1 \leq s \leq p_-$, the dimension of the vector space $A_0((\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{1,s})^*)$ is at most one dimensional. The L_0 eigenvalues is given by $h_{2,s}$ which corresponds to the highest weight of $\mathcal{K}_{2,s}$.*
  3. *For $2 \leq r < p_+$, $1 \leq s \leq p_-$, the vector space $A_0((\mathcal{K}_{2,1} \boxtimes \mathcal{X}^+_{r,s})^*)$ is at most two dimensional. The L_0 eigenvalues are contained in $\{\Delta^+_{r-1,s;0}, \Delta^+_{r+1,s;0}\}$, which corresponds to the highest weights of $\mathcal{X}^+_{r-1,s}$ and $\mathcal{X}^+_{r+1,s}$, respectively.*
  4. *For $2 \leq r < p_+$, $1 \leq s \leq p_-$, the dimension of the vector space $A_0((\mathcal{K}_{2,1} \boxtimes \mathcal{X}^-_{r,s})^*)$ is at most four dimensional. The L_0 eigenvalues are contained in $\{\Delta^-_{r-1,s;0}, \Delta^-_{r+1,s;0}\}$, which corresponds to the highest weights of $\mathcal{X}^-_{r-1,s}$ and $\mathcal{X}^-_{r+1,s}$, respectively.*
  5. *For $r = p_+$, $1 \leq s \leq p_-$, the vector space $A_0((\mathcal{K}_{2,1} \boxtimes \mathcal{X}^+_{p_+,s})^*)$ is at most two dimensional. The L_0 eigenvalues are contained in $\{h_{p_+-1,s}, \Delta^+_{p_+-1,s;0}\}$, which corresponds to the highest weights of $L(h_{p_+-1,s})$ and $\mathcal{X}^+_{p_+-1,s}$, respectively.*
  6. *For $r = p_+$, $1 \leq s \leq p_-$, the vector space $A_0((\mathcal{K}_{2,1} \boxtimes \mathcal{X}^-_{p_+,s})^*)$ is at most four dimensional. The L_0 eigenvalues are contained in $\{\Delta^+_{1,s;0}, \Delta^-_{p_+-1,s;0}\}$ which corresponds to the highest weights of $\mathcal{X}^+_{1,s}$ and $\mathcal{X}^-_{p_+-1,s}$, respectively.*

- *For $p_+ = 2$, we have*

  1. *For $1 \leq s \leq p_-$, the vector space $A_0((\mathcal{K}_{2,1} \boxtimes \mathcal{X}^+_{2,s})^*)$ is at most two dimensional. The L_0 eigenvalues are contained in $\{h_{1,s}, \Delta^+_{1,s;0}\}$, which corresponds to the highest weights of $L(h_{1,s})$ and $\mathcal{X}^+_{1,s}$, respectively.*
  2. *For $1 \leq s \leq p_-$, the vector space $A_0((\mathcal{K}_{2,1} \boxtimes \mathcal{X}^-_{2,s})^*)$ is at most four dimensional. The L_0 eigenvalues are contained in $\{\Delta^+_{1,s;0}, \Delta^-_{1,s;0}\}$ which corresponds to the highest weights of $\mathcal{X}^+_{1,s}$ and $\mathcal{X}^-_{1,s}$, respectively.*

For any $\alpha \in A_{p_+,p_-} = \{\ \alpha_{r,s;n} \mid r,s,n \in \mathbb{Z}\}$, let

$$V_\alpha = \bigoplus_{n\in\mathbb{Z}} F_{\alpha+n\sqrt{2p_+p_-}}$$

be the simple $\mathcal{V}_{[p_+,p_-]}$-module. For any $\alpha, \alpha' \in A_{p_+,p_-}$, it can be proved easily that there are no $\mathcal{V}_{[p_+,p_-]}$-module intertwining operators of type $\begin{pmatrix} V_{\alpha''} \\ V_{\alpha'} \quad V_\alpha \end{pmatrix}$ unless $\alpha'' \equiv \alpha' + \alpha \mod \mathbb{Z}\sqrt{2p_+p_-}$, and $\dim_{\mathbb{C}} I \begin{pmatrix} V_{\alpha'+\alpha} \\ V_{\alpha'} \quad V_\alpha \end{pmatrix} = 1$. Let Y be the $\mathcal{V}_{[p_+,p_-]}$-module intertwining operator of type $\begin{pmatrix} V_{\alpha'+\alpha} \\ V_{\alpha'} \quad V_\alpha \end{pmatrix}$. Then, by restricting the action of $\mathcal{V}_{[p_+,p_-]}$ to $\mathcal{W}_{p_+,p_-}$, Y defines a $\mathcal{W}_{p_+,p_-}$-module intertwining operator of type $\begin{pmatrix} V_{\alpha'+\alpha} \\ V_{\alpha'} \quad V_\alpha \end{pmatrix}$. We denote this $\mathcal{W}_{p_+,p_-}$-module intertwining operator by $Y_{\alpha',\alpha}$.

Lemma 8.3.4. *For $1 \le r \le p_+$, $2 \le s \le p_+ - 1$, we have*

$$I \begin{pmatrix} \mathcal{X}^+_{r,s-1} \\ \mathcal{K}_{1,2} \quad \mathcal{X}^+_{r,s} \end{pmatrix} \neq \emptyset, \qquad I \begin{pmatrix} \mathcal{X}^+_{r,s+1} \\ \mathcal{K}_{1,2} \quad \mathcal{X}^+_{r,s} \end{pmatrix} \neq \emptyset.$$

Proof. Let us consider the $\mathcal{W}_{p_+,p_-}$-module intertwining operator $Y = Y_{\alpha_1,\alpha_2}$, where $\alpha_1 = \alpha_{1,2}$ and $\alpha_2 = \alpha_{r,p_--s;1}$. Then we have

$$\langle \alpha_{r,p_--s+1;1} |\, Y(|\alpha_{1,2}\rangle\, ; z)\, |\alpha_{r,p_--s;1}\rangle \neq 0.$$

Thus, we have a non-zero $\mathcal{W}_{p_+,p_-}$-module intertwining operator of type

$$I \begin{pmatrix} \mathcal{X}^+_{r,s-1} \\ \mathcal{K}_{1,2} \quad \mathcal{W}_{p_+,p_-}.\, |\alpha_2\rangle \end{pmatrix} \neq \emptyset. \tag{8.3.1}$$

Note that following exact sequence

$$0 \to \mathcal{X}^-_{r,s^\vee} \to \mathcal{W}_{p_+,p_-}.\, |\alpha_{r,p_--s;1}\rangle \to \mathcal{X}^+_{r,s} \to 0.$$

Then, by the exact sequence

$$\mathcal{K}_{1,2} \boxtimes \mathcal{X}^-_{r,s^\vee} \to \mathcal{K}_{1,2} \boxtimes \mathcal{W}_{p_+,p_-}.\, |\alpha_{r,p_--s;1}\rangle \to \mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r,s} \to 0,$$

we have the following exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r,s}, \mathcal{X}^+_{r,s-1}) \to \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{1,2} \boxtimes \mathcal{W}_{p_+,p_-}.\, |\alpha_{r,p_--s;1}\rangle\, , \mathcal{X}^+_{r,s-1}) \\ &\to \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{1,2} \boxtimes \mathcal{X}^-_{r,s^\vee}, \mathcal{X}^+_{r,s-1}). \end{aligned} \tag{8.3.2}$$

Thus, by Proposition 8.3.2 and (8.3.1), (8.3.2), we obtain

$$I \begin{pmatrix} \mathcal{X}^+_{r,s-1} \\ \mathcal{K}_{1,2} \quad \mathcal{X}^+_{r,s} \end{pmatrix} \neq \emptyset.$$

The second equation can be proved in the same way, so we omit the proof.

□

Lemma 8.3.5. *For $1 \le r \le p_+$, $1 \le s \le p_+ - 1$, we have*

$$I\begin{pmatrix} & \mathcal{X}^+_{r,s} \\ \mathcal{K}_{r,s} & \mathcal{X}^+_{1,1} \end{pmatrix} \neq \emptyset.$$

Proof. By the exact sequence

$$0 \to \mathcal{X}^+_{1,1} \to \mathcal{K}_{1,1} \to L(h_{1,1}) \to 0$$

and by Corollary 8.2.4, we have the following exact sequence

$$\mathcal{K}_{r,s} \boxtimes \mathcal{X}^+_{1,1} \to \mathcal{K}_{r,s} \to L(h_{r,s}) \to 0.$$

Thus, by this exact sequence, we obtain the claim of the theorem. □

Proposition 8.3.6.

1. *We have*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s} = \mathcal{K}_{r,s-1} \oplus \Gamma_{s,p_- -1}(\mathcal{K}_{r,s+1}),$$

where $\Gamma_{s,p_- -1}(\mathcal{K}_{r,s+1})$ is defined as $\mathcal{K}_{r,s+1}$ in the case of $s \le p_- - 2$ and a certain highest weight module with $\mathrm{top}(\Gamma_{s,p_- -1}(\mathcal{K}_{r,s+1})) = \mathcal{X}^+_{r,p_-}$ in the case of $s = p_- - 1$.

2. *For $p_+ \ge 3$, $2 \le r \le p_+ - 1$, $1 \le s \le p_- - 1$, we have*

$$\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{r,s} = \mathcal{K}_{r-1,s} \oplus \Gamma_{r,p_+ -1}(\mathcal{K}_{r+1,s}),$$

where $\Gamma_{r,p_+ -1}(\mathcal{K}_{r+1,s})$ is defined as $\mathcal{K}_{r+1,s}$ in the case of $r \le p_+ - 2$ and a certain highest weight module with $\mathrm{top}(\Gamma_{r,p_+ -1}(\mathcal{K}_{r+1,s})) = \mathcal{X}^+_{p_+,s}$ in the case of $r = p_+ - 1$.

Proof. Let $p_- \ge 4$, $1 \le r \le p_+ - 1$ and $2 \le s \le p_- - 2$. By the exact sequence

$$0 \to \mathcal{X}^+_{r,s} \to \mathcal{K}_{r,s} \to L(h_{r,s}) \to 0,$$

we have the following exact sequence

$$\mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r,s} \to \mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s} \to L(h_{r,s-1}) \oplus L(h_{r,s+1}) \to 0. \tag{8.3.3}$$

By Lemma 8.3.5, we have the following exact sequence

$$(\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s}) \boxtimes \mathcal{X}^+_{1,1} \to \mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r,s} \to 0. \tag{8.3.4}$$

Then, by two exact sequences (8.3.3),(8.3.4) and by Lemma 8.3.4, we see that $\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s}$ has $\mathcal{X}^+_{r,s-1}$ and $\mathcal{X}^+_{r,s+1}$ as composition factors. Note that by Proposition 8.3.2 we have

$$\mathrm{top}(\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s}) = L(h_{r,s-1}) \oplus L(h_{r,s+1}). \tag{8.3.5}$$

Since

$$\mathrm{Ext}^1(\mathcal{K}_{a,b}, \mathcal{X}^-_{a^\vee,b}) = \mathrm{Ext}^1(\mathcal{K}_{a,b}, \mathcal{X}^-_{a,b^\vee}) = 0, \quad 1 \le a < p_+,\ 1 \le b < p_- \tag{8.3.6}$$

as shown in the proof of Proposition 7.3.8, it is sufficient to show that $\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s}$ does not contain either $\mathcal{X}^{+}_{r^\vee,s^\vee+1}$ or $\mathcal{X}^{+}_{r^\vee,s^\vee-1}$ as a composition factor.

Assume that $\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s}$ contains $\mathcal{X} = \mathcal{X}^{+}_{r^\vee,s^\vee+1}$ or $\mathcal{X}^{+}_{r^\vee,s^\vee-1}$ as a composition factor. Then, we see that the composition factor $\mathcal{X}$ of $\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s}$ comes from that of $\mathcal{K}_{1,2} \boxtimes \mathcal{X}^{+}_{r,s}$ in the exact sequence (8.3.3). Note that by Proposition 7.2.8

$$\mathrm{Ext}^1(\mathcal{X}^{+}_{r,s-1} \oplus \mathcal{X}^{+}_{r,s+1}, \mathcal{X}^{+}_{r^\vee,s^\vee+1} \oplus \mathcal{X}^{+}_{r^\vee,s^\vee-1}) = 0,$$

and by Proposition 8.3.2 $A_0((\mathcal{K}_{1,2} \boxtimes \mathcal{X}^{+}_{r,s})^*) = \mathcal{X}^{+}_{r,s-1} \oplus \mathcal{X}^{+}_{r,s+1}$. Thus, noting (8.3.5), as a quotient module of $\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s}$ we have a non-trivial extension in $\mathrm{Ext}^1(\mathcal{K}_{r,s-1}, \mathcal{X}^{-}_{r^\vee,s-1})$ or $\mathrm{Ext}^1(\mathcal{K}_{r,s-1}, \mathcal{X}^{-}_{r,s^\vee+1})$. But this contradicts (8.3.6). The other cases can be proved in a similar way, so we omit the proofs.

□

Theorem 8.3.7. $\mathcal{K}_{1,2}$ *is rigid and self-dual in* $(\mathcal{C}_{p_+,p_-}, \boxtimes)$.

Proof. We show the rigidity of $\mathcal{K}_{1,2}$ using the methods detailed in [15] and [56] (cf.[64]). By Proposition 8.3.6, we have homomorphisms

$$\begin{aligned}
&i_1 : \mathcal{K}_{1,1} \to \mathcal{K}_{1,2} \boxtimes \mathcal{K}_{1,2},\\
&p_1 : \mathcal{K}_{1,2} \boxtimes \mathcal{K}_{1,2} \to \mathcal{K}_{1,1},\\
&i_3 : \Gamma_{2,p_--1}(\mathcal{K}_{1,3}) \to \mathcal{K}_{1,2} \boxtimes \mathcal{K}_{1,2},\\
&p_3 : \mathcal{K}_{1,2} \boxtimes \mathcal{K}_{1,2} \to \Gamma_{2,p_--1}(\mathcal{K}_{1,3})
\end{aligned}$$

such that

$$p_1 \circ i_1 = \mathrm{id}_{\mathcal{K}_{1,1}}, \qquad p_3 \circ i_3 = \mathrm{id}_{\Gamma_{2,p_--1}(\mathcal{K}_{1,3})}$$

and

$$i_1 \circ p_1 + i_3 \circ p_3 = \mathrm{id}_{\mathcal{K}_{1,2}\boxtimes\mathcal{K}_{1,2}}.$$

To prove that $\mathcal{K}_{1,2}$ is rigid, it is sufficient to prove that the homomorphisms $f, g : \mathcal{K}_{1,2} \to \mathcal{K}_{1,2}$ defined by the commutative diagrams

$$\begin{array}{ccccc}
\mathcal{K}_{1,2} & \xrightarrow{r^{-1}} & \mathcal{K}_{1,2} \boxtimes \mathcal{K}_{1,1} & \xrightarrow{\mathrm{id}\boxtimes i_1} & \mathcal{K}_{1,2} \boxtimes (\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{1,2}) \\
f\big\downarrow & & & & \big\downarrow\mathcal{A} \\
\mathcal{K}_{1,2} & \xleftarrow{l} & \mathcal{K}_{1,1} \boxtimes \mathcal{K}_{1,2} & \xleftarrow{p_1\boxtimes\mathrm{id}} & (\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{1,2}) \boxtimes \mathcal{K}_{1,2}
\end{array}$$

and

$$\begin{array}{ccccc}
\mathcal{K}_{1,2} & \xrightarrow{l^{-1}} & \mathcal{K}_{1,1} \boxtimes \mathcal{K}_{1,2} & \xrightarrow{i_1\boxtimes\mathrm{id}} & (\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{1,2}) \boxtimes \mathcal{K}_{1,2} \\
g\big\downarrow & & & & \big\downarrow\mathcal{A}^{-1} \\
\mathcal{K}_{1,2} & \xleftarrow{r} & \mathcal{K}_{1,2} \boxtimes \mathcal{K}_{1,1} & \xleftarrow{\mathrm{id}\boxtimes p_1} & \mathcal{K}_{1,2} \boxtimes (\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{1,2})
\end{array}$$

are non-zero multiples of the identity, where $\mathcal{A}$ is the associativity isomorphism and l and r are the left and right unit isomorphisms. Since $\mathrm{Hom}(\mathcal{K}_{1,2}, \mathcal{K}_{1,2}) \simeq \mathbb{C}$, it is sufficient to show that f and g are non-zero. We only show $f \neq 0$. The proof of $g \neq 0$ is similar.

Let $\mathcal{Y}_{2\boxtimes 2}$, $\mathcal{Y}_{(2\boxtimes 2)\boxtimes 2}$ and $\mathcal{Y}_{2\boxtimes(2\boxtimes 2)}$ be the non-zero intertwining operators of type

$$\begin{pmatrix} \mathcal{K}_{1,2}\boxtimes\mathcal{K}_{1,2} \\ \mathcal{K}_{1,2}\ \ \mathcal{K}_{1,2} \end{pmatrix}, \qquad \begin{pmatrix} (\mathcal{K}_{1,2}\boxtimes\mathcal{K}_{1,2})\boxtimes\mathcal{K}_{1,2} \\ \mathcal{K}_{1,2}\boxtimes\mathcal{K}_{1,2}\ \ \mathcal{K}_{1,2} \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{K}_{1,2}\boxtimes(\mathcal{K}_{1,2}\boxtimes\mathcal{K}_{1,2}) \\ \mathcal{K}_{1,2}\ \ \mathcal{K}_{1,2}\boxtimes\mathcal{K}_{1,2} \end{pmatrix},$$

respectively.

To prove $f\neq 0$, it is sufficient to show that the intertwining operator

$$\mathcal{Y}^2_{21} = l_{\mathcal{K}_{1,2}}\circ(p_1\boxtimes \mathrm{id}_{\mathcal{K}_{1,2}})\circ\mathcal{A}_{\mathcal{K}_{1,2},\mathcal{K}_{1,2},\mathcal{K}_{1,2}}\circ\mathcal{Y}_{2\boxtimes(2\boxtimes 2)}\circ(\mathrm{id}_{\mathcal{K}_{1,2}}\otimes i_1)$$

is non-zero.

Define the following intertwining operator

$$\mathcal{Y}^2_{23} = l_{\mathcal{K}_{1,2}}\circ(p_3\boxtimes \mathrm{id}_{\mathcal{K}_{1,2}})\circ\mathcal{A}_{\mathcal{K}_{1,2},\mathcal{K}_{1,2},\mathcal{K}_{1,2}}\circ\mathcal{Y}_{2\boxtimes(2\boxtimes 2)}\circ(\mathrm{id}_{\mathcal{K}_{1,2}}\otimes i_3).$$

Then, for highest weight vectors $v\in\mathcal{K}_{1,2}[h_{1,2}]$, $v^*\in\mathcal{K}^*_{1,2}[h_{1,2}]$, and for some $x\in\mathbb{R}$ such that $1>x>1-x>0$, we have

$$\begin{aligned}
&\langle v^*,\mathcal{Y}^2_{21}(v;1)(p_1\circ\mathcal{Y}_{2\boxtimes 2})(v;x)v\rangle+\langle v^*,\mathcal{Y}^2_{23}(v;1)(p_3\circ\mathcal{Y}_{2\boxtimes 2})(v;x)v\rangle\\
&=\langle v^*,\overline{l_{\mathcal{K}_{1,2}}\circ(p_1\boxtimes\mathrm{id}_{\mathcal{K}_{1,2}})\circ\mathcal{A}_{\mathcal{K}_{1,2},\mathcal{K}_{1,2},\mathcal{K}_{1,2}}}\big(\mathcal{Y}_{2\boxtimes(2\boxtimes 2)}(v;1)\mathcal{Y}_{2\boxtimes 2}(v;x)v\big)\rangle\\
&=\langle v^*,\overline{l_{\mathcal{K}_{1,2}}\circ(p_1\boxtimes\mathrm{id}_{\mathcal{K}_{1,2}})}\big(\mathcal{Y}_{(2\boxtimes 2)\boxtimes 2}(\mathcal{Y}_{2\boxtimes 2}(v;1-x)v;x)v\big)\rangle\\
&=\langle v^*,\overline{l_{\mathcal{K}_{1,2}}}\big(\mathcal{Y}_{1\boxtimes 2}((p_1\circ\mathcal{Y}_{2\boxtimes 2})(v;1-x)v;x)v\big)\rangle,\\
&=\langle v^*,Y_{\mathcal{K}_{1,2}}\big((p_1\circ\mathcal{Y}_{2\boxtimes 2})(v;1-x)v;x\big)v\rangle,
\end{aligned}$$

where $\mathcal{Y}_{1\boxtimes 2}$ is the intertwining operator of type $\binom{\mathcal{K}_{1,2}}{\mathcal{K}_{1,1}\ \mathcal{K}_{1,2}}$. Since $p_1\circ\mathcal{Y}_{2\boxtimes 2}$ is the non-zero intertwining operator of type $\binom{\mathcal{K}_{1,1}}{\mathcal{K}_{1,2}\ \mathcal{K}_{1,2}}$, we have

$$\langle v^*,Y_{\mathcal{K}_{1,2}}\big((p_1\circ\mathcal{Y}_{2\boxtimes 2})(v;1-x)v;x\big)v\rangle\in\mathbb{C}^\times(1-x)^{-2h_{1,2}}\big(1+(1-x)\mathbb{C}[[1-x]]\big). \quad (8.3.7)$$

Set

$$\phi_1(x)=\langle v^*,\mathcal{Y}^2_{21}(v;1)(p_1\circ\mathcal{Y}_{2\boxtimes 2})(v;x)v\rangle,\qquad \phi_3(x)=\langle v^*,\mathcal{Y}^2_{23}(v;1)(p_3\circ\mathcal{Y}_{2\boxtimes 2})(v;x)v\rangle.$$

Then as in [15],[56],[64], we see that $\phi_1(x)$ and $\phi_3(x)$ satisfy the following Fuchsian differential equation (cf. [10])

$$\begin{aligned}
&\phi''(x)+\frac{p_+}{p_-}\Big(\frac{1}{x-1}+\frac{1}{x}\Big)\phi'(x)\\
&\qquad-\frac{p_+h_{1,2}}{p_-}\Big\{\frac{1}{(x-1)^2}+\frac{1}{x^2}-2\Big(\frac{1}{x-1}-\frac{1}{x}\Big)\Big\}=0,
\end{aligned}$$

with the following Riemann scheme

$$\begin{bmatrix} 0 & 1 & \infty\\ \lambda_+ & \mu_+ & \nu_+\\ \lambda_- & \mu_- & \nu_- \end{bmatrix}=\begin{bmatrix} 0 & 1 & \infty\\ \frac{p_+}{2p_-} & \frac{p_+}{2p_-} & 0\\ 1-\frac{3p_+}{2p_-} & 1-\frac{3p_+}{2p_-} & \frac{2p_+}{p_-}-1 \end{bmatrix}.$$

Let $\{u_+, u_-\}$ be the fundamental system of solutions at $x = 0$ whose characteristic exponents are λ_+, λ_-, respectively, and $\{v_+, v_-\}$ the fundamental system of solutions at $x = 1$ whose characteristic exponents are μ_+, μ_-, respectively. Then, the connection matrix between $x = 0$ and $x = 1$ is given by

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \begin{pmatrix} F_{++} & F_{-+} \\ F_{+-} & F_{--} \end{pmatrix} \begin{pmatrix} v_+ \\ v_- \end{pmatrix},$$

where

$$F_{\epsilon\epsilon'} = \frac{\Gamma(-\epsilon(\mu_+ - \mu_-))\Gamma(\epsilon'(\lambda_+ - \lambda_-) + 1)}{\Gamma(\lambda_{\epsilon'} + \mu_{-\epsilon} + \nu_+)\Gamma(\lambda_{\epsilon'} + \mu_{-\epsilon} + \nu_-)}, \qquad \epsilon, \epsilon' = \pm.$$

We can see that this connection matrix is regular. Note that the characteristic exponents of $\phi_1(x)$ and $\phi_3(x)$ at the regular singular point $x = 0$ are given by

$$h_{1,1} - 2h_{1,2} = 1 - \frac{3p_+}{2p_-}, \qquad\qquad h_{1,3} - 2h_{1,2} = \frac{p_+}{2p_-},$$

respectively. Therefore by the non-zero four point function (8.3.7), we see that $\phi_1(x)$ is non-zero. In particular $\mathcal{Y}^2_{21}$ is non-zero. Thus $\mathcal{K}_{1,2}$ is rigid in the tensor category $(\mathcal{C}_{p_+,p_-}, \boxtimes)$.

□

Lemma 8.3.8. *For $1 \le r \le p_+$, $2 \le s \le p_- - 1$, we have*

$$I\begin{pmatrix} & \mathcal{X}^-_{r,s-1} & \\ \mathcal{K}_{1,2} & & \mathcal{X}^-_{r,s} \end{pmatrix} \neq \emptyset, \qquad I\begin{pmatrix} & \mathcal{X}^-_{r,s+1} & \\ \mathcal{K}_{1,2} & & \mathcal{X}^-_{r,s} \end{pmatrix} \neq \emptyset.$$

Proof. We will only prove the first equation. The second equation can be proved in the same way, so we omit the proofs. Let us consider the $\mathcal{W}_{p_+,p_-}$-module intertwining operator $Y_1 = Y_{\alpha_1,\alpha_2}$, where $\alpha_1 = \alpha_{1,2}$, $\alpha_2 = \alpha_{r^\vee,s;-2}$. Note that

$$\langle \alpha_{r^\vee,s+1;-2} |\, Y_1(|\alpha_{1,2}\rangle\,; z)\, |\alpha_{r^\vee,s;-2}\rangle \neq 0. \tag{8.3.8}$$

Thus, we have a non-zero $\mathcal{W}_{p_+,p_-}$-module intertwining operator of type

$$I\begin{pmatrix} & \mathcal{X}^-_{r,s-1} & \\ \mathcal{K}_{1,2} & & \mathcal{W}_{p_+,p_-}.\,|\alpha_2\rangle \end{pmatrix} \neq \emptyset. \tag{8.3.9}$$

Note that $\mathcal{W}_{p_+,p_-}.S_{r,3p_--s}\,|\alpha_{r^\vee,s;-2}\rangle \simeq \mathcal{X}^+_{r^\vee,s}$. Then, by the exact sequence

$$0 \to \mathcal{X}^+_{r^\vee,s} \to \mathcal{W}_{p_+,p_-}.\,|\alpha_{r^\vee,s;-2}\rangle \to \mathcal{X}^-_{r,s} \to 0,$$

we have the following exact sequence

$$\mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r^\vee,s} \to \mathcal{K}_{1,2} \boxtimes \mathcal{W}_{p_+,p_-}.\,|\alpha_{r^\vee,s;-2}\rangle \to \mathcal{K}_{1,2} \boxtimes \mathcal{X}^-_{r,s} \to 0.$$

Thus we obtain the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{1,2} \boxtimes \mathcal{X}^-_{r,s}, \mathcal{X}^-_{r,s-1}) \to \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{W}_{p_+,p_-}.\,|\alpha_{r^\vee,s;-2}\rangle\,, \mathcal{X}^-_{r,s-1}) \\ &\to \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r^\vee,s}, \mathcal{X}^-_{r,s-1}). \end{aligned} \tag{8.3.10}$$

By Proposition 8.3.2, we see that

$$\mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r^\vee,s}, \mathcal{X}^-_{r,s-1}) = 0.$$

Thus, by (8.3.9) and (8.3.10), we obtain

$$I\begin{pmatrix} \mathcal{X}^-_{r,s-1} \\ \mathcal{K}_{1,2} \quad \mathcal{X}^-_{r,s} \end{pmatrix} \neq \emptyset.$$

□

By Proposition 8.3.2 and Lemma 8.3.8, we obtain the following proposition.

Proposition 8.3.9.

1. *For* $1 \leq r \leq p_+$, $2 \leq s \leq p_- - 1$, *we have*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{X}^-_{r,s} = \mathcal{X}^-_{r,s-1} \oplus \mathcal{X}^-_{r,s+1}.$$

2. *For* $1 \leq r \leq p_+$, *we have*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{X}^-_{r,1} = \mathcal{X}^-_{r,2}.$$

3. *For* $p_+ \geq 3$, $2 \leq r \leq p_+ - 1$, $1 \leq s \leq p_-$, *we have*

$$\mathcal{K}_{2,1} \boxtimes \mathcal{X}^-_{r,s} = \mathcal{X}^-_{r-1,s} \oplus \mathcal{X}^-_{r+1,s}.$$

4. *For* $1 \leq s \leq p_-$, *we have*

$$\mathcal{K}_{2,1} \boxtimes \mathcal{X}^-_{1,s} = \mathcal{X}^-_{2,s}.$$

For the following lemma, see, for example, [20].

Lemma 8.3.10. *Let* $(\mathcal{C}, \otimes)$ *be a braided tensor category and let* V *be a rigid object in* $\mathcal{C}$. *Then there is a natural adjunction isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(U \otimes V, W) \simeq \mathrm{Hom}_{\mathcal{C}}(U, W \otimes V^\vee),$$

where U, W *are any objects in* $\mathcal{C}$ *and* $V^\vee$ *is the dual object of* V.

Lemma 8.3.11.

1. *For* $1 \leq r \leq p_+$, $2 \leq s \leq p_- - 1$, *we have*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r,s} = \mathcal{X}^+_{r,s-1} \oplus \mathcal{X}^+_{r,s+1}.$$

2. *For* $1 \leq r \leq p_+$, *we have*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r,1} = \mathcal{X}^+_{r,2}.$$

Proof. We will only prove

$$\mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r,s} = \mathcal{X}^+_{r,s-1} \oplus \mathcal{X}^+_{r,s+1}$$

for $1 \le r \le p_+ - 1$, $1 \le s \le p_+ - 2$. The other cases can be proved in a similar way, so we omit the proofs.

By Proposition 8.3.9, Lemma 8.3.10 and the self-duality of $\mathcal{K}_{1,2}$, we see that

$$\mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{X}, \mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r,s}) = 0,$$

where $\mathcal{X} = \mathcal{X}^-_{r^\vee,s-1}, \mathcal{X}^-_{r,s^\vee+1}, \mathcal{X}^-_{r^\vee,s+1}, \mathcal{X}^-_{r,s^\vee-1}$. Thus, by Lemmas 8.3.2 and 8.3.4, we obtain

$$\mathcal{K}_{1,2} \boxtimes \mathcal{X}^+_{r,s} = \mathcal{X}^+_{r,s-1} \oplus \mathcal{X}^+_{r,s+1}.$$

□

By Proposition 8.3.6 and Lemma 8.3.11, we obtain the following proposition.

Proposition 8.3.12.

1. *For $1 \le r \le p_+$, $2 \le s \le p_- - 1$, we have*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,s} = \mathcal{K}_{r,s-1} \oplus \mathcal{K}_{r,s+1}.$$

2. *For $1 \le r \le p_+$, we have*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{K}_{r,1} = \mathcal{K}_{r,2}.$$

The following Lemma is due to Proposition 3.46 in [38].

Lemma 8.3.13. *For any $U, V \in \mathcal{C}_{p_+,p_-}$, we have a natural isomorphism.*

$$\mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(U, V) \simeq \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(U \boxtimes V^*, \mathcal{K}^*_{1,1}).$$

Lemma 8.3.14. *For $1 \le r \le p_+ - 1$, $1 \le s \le p_- - 1$, we have*

$$I\begin{pmatrix} & \mathcal{K}^*_{p_+-1,s} & \\ \mathcal{K}_{2,1} & & \mathcal{K}_{p_+,s} \end{pmatrix} \neq \emptyset, \qquad I\begin{pmatrix} & \mathcal{K}^*_{r,p_--1} & \\ \mathcal{K}_{1,2} & & \mathcal{K}_{r,p_-} \end{pmatrix} \neq \emptyset.$$

Proof. We only prove the first equation. The second equation can be proved in the same way, so we omit the proof.

Let us consider $\mathcal{W}_{p_+,p_-}$-module intertwining operator $Y = Y_{\alpha_1,\alpha_2}$, where $\alpha_1 = \alpha_{2,1}$ and $\alpha_2 = \alpha_{p_+,s^\vee;1}$. Note that

$$\langle \alpha_{p_+-1,s^\vee} | \, Y(|\alpha_{2,1}\rangle\,; z)\, |\alpha_{p_+,s^\vee;1}\rangle \neq 0.$$

Then, by Proposition 7.2.5, we have

$$\langle \alpha_{p_+-1,s^\vee} | \, S^*_{p_+-1,s} Y(|\alpha_{2,1}\rangle\,; z)\, |\alpha_{p_+,s^\vee;1}\rangle \neq 0.$$

Thus we have

$$I\begin{pmatrix} & \mathcal{K}^*_{p_+-1,s} & \\ \mathcal{K}_{2,1} & & \mathcal{W}_{p_+,p_-}\cdot|\alpha_{p_+,s^\vee;1}\rangle \end{pmatrix} \neq \emptyset, \qquad I\begin{pmatrix} & \mathcal{X}^+_{p_+-1,s} & \\ \mathcal{K}_{2,1} & & \mathcal{W}_{p_+,p_-}\cdot|\alpha_{p_+,s^\vee;1}\rangle \end{pmatrix} \neq \emptyset \tag{8.3.11}$$

Note that $\mathcal{W}_{p_+,p_-}.\,|\alpha_{p_+,s^\vee;1}\rangle$ satisfies the following exact sequence

$$0 \to \mathcal{X}^-_{p_+,s^\vee} \to \mathcal{W}_{p_+,p_-}.\,|\alpha_{p_+,s^\vee;1}\rangle \to \mathcal{X}^+_{p_+,s}(=\mathcal{K}_{p_+,s}) \to 0.$$

Then we have the following exact sequence

$$0 \to \mathcal{K}_{2,1} \boxtimes \mathcal{X}^-_{p_+,s^\vee} \to \mathcal{K}_{2,1} \boxtimes \mathcal{W}_{p_+,p_-}.\,|\alpha_{p_+,s^\vee;1}\rangle \to \mathcal{K}_{2,1} \boxtimes \mathcal{K}_{p_+,s} \to 0.$$

By this exact sequence we obtain the following exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{p_+,s}, \mathcal{K}^*_{1,1}) \to \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{2,1} \boxtimes \mathcal{W}_{p_+,p_-}.\,|\alpha_{p_+,s^\vee;1}\rangle, \mathcal{K}^*_{1,1}) \\ &\to \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{2,1} \boxtimes \mathcal{X}^-_{p_+,s^\vee}, \mathcal{K}^*_{1,1}). \end{aligned} \tag{8.3.12}$$

By Proposition 8.3.2, we have

$$\mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{2,1} \boxtimes \mathcal{X}^-_{p_+,s^\vee}, \mathcal{K}^*_{1,1}) = 0. \tag{8.3.13}$$

Since $L(h_{1,1}) \boxtimes \mathcal{K}_{p_+,s} = 0$, by Lemma 8.3.13, we have

$$\mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{p_+,s}, L(h_{1,1})) = 0.$$

Thus, by (8.3.11), (8.3.12) and (8.3.13), we have a surjective module map from $\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{p_+,s}$ to $\mathcal{K}^*_{p_+-1,s}$.

□

Theorem 8.3.15. *$\mathcal{K}_{2,1}$ is rigid and self-dual in $(\mathcal{C}_{p_+,p_-}, \boxtimes)$.*

Proof. In the case $p_+ \geq 3$, the rigidity can be proved in the same way as in Theorem 8.3.7. Therefore let $p_+ = 2$. Note that in this case it is $\mathcal{K}_{2,1} = \mathcal{X}^+_{2,1}$ from the definition. First we prove

$$\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1} = \mathcal{Q}(\mathcal{X}^+_{1,1})_{1,1}.$$

By Proposition 8.3.3 and by Lemma 8.3.14, we have

$$\mathrm{top}(\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1}) = \mathcal{X}^+_{1,1} \in C^{thick}_{1,1}.$$

By Lemma 8.3.14, we can define the following module map

$$p_1 : \mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1} \to \mathcal{K}_{1,1}.$$

Let $\mathcal{Y}_{2\boxtimes 2}$, $\mathcal{Y}_{(2\boxtimes 2)\boxtimes 2}$ and $\mathcal{Y}_{2\boxtimes(2\boxtimes 2)}$ be the non-zero intertwining operators of type

$$\begin{pmatrix} \mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1} \\ \mathcal{K}_{2,1} \quad \mathcal{K}_{2,1} \end{pmatrix}, \qquad \begin{pmatrix} (\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1}) \boxtimes \mathcal{K}_{2,1} \\ \mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1} \quad \mathcal{K}_{2,1} \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{K}_{2,1} \boxtimes (\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1}) \\ \mathcal{K}_{2,1} \quad \mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1} \end{pmatrix},$$

respectively. We define the intertwining operator

$$\mathcal{Y}^2_{2\boxtimes 2,2} = r_{\mathcal{K}_{2,1}} \circ (\mathrm{id}_{\mathcal{K}_{2,1}} \boxtimes p_1) \circ \mathcal{A}^{-1}_{\mathcal{K}_{2,1},\mathcal{K}_{2,1},\mathcal{K}_{2,1}} \circ \mathcal{Y}_{(2\boxtimes 2)\boxtimes 2}$$

of type $\binom{\mathcal{K}_{2,1}}{\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1}\ \ \mathcal{K}_{2,1}}$.

Let us consider the four point function

$$\langle v^*, \mathcal{Y}^2_{2\boxtimes 2,2}(\mathcal{Y}_{2\boxtimes 2}(v,1-x)v,x)v\rangle, \tag{8.3.14}$$

where v and v^* are the highest weight vectors of $\mathcal{K}_{2,1}$ and $\mathcal{K}^*_{2,1}$, respectively. Then, for some $x\in\mathbb{R}$ such that $1>x>1-x>0$, we have

$$\begin{aligned}
&\langle v^*, \mathcal{Y}^2_{2\boxtimes 2,2}(\mathcal{Y}_{2\boxtimes 2}(v,1-x)v,x)v\rangle\\
&=\langle v^*, \overline{r_{\mathcal{K}_{2,1}}\circ(\mathrm{id}_{\mathcal{K}_{2,1}}\boxtimes p_1)\mathcal{A}^{-1}_{\mathcal{K}_{2,1},\mathcal{K}_{2,1},\mathcal{K}_{2,1}}}\big(\mathcal{Y}_{(2\boxtimes 2)\boxtimes 2}(\mathcal{Y}_{2\boxtimes 2}(v,1-x)v,x)v\big)\rangle\\
&=\langle v^*, \overline{r_{\mathcal{K}_{2,1}}\circ(\mathrm{id}_{\mathcal{K}_{2,1}}\boxtimes p_1)}\big(\mathcal{Y}_{2\boxtimes(2\boxtimes 2)}(v,1)\mathcal{Y}_{2\boxtimes 2}(v,x)v\big)\rangle\\
&=\langle v^*, \overline{r_{\mathcal{K}_{2,1}}}\big(\mathcal{Y}_{2\boxtimes 1}(v,1)(p_1\circ\mathcal{Y}_{2\boxtimes 2})(v,x)v\big)\rangle\\
&=\langle v^*, \Omega(Y_{\mathcal{K}_{2,1}})(v,1)(p_1\circ\mathcal{Y}_{2\boxtimes 2})(v,x)v\rangle,
\end{aligned}$$

where Ω represents the skew-symmetry operation on vertex operators defined by

$$\Omega(Y_{\mathcal{K}_{2,1}})(v,z)w = e^{zL_{-1}}Y_{\mathcal{K}_{2,1}}(w,-z)v$$

for $w\in\mathcal{K}_{1,1}$. Note that

$$\begin{aligned}
&\langle v^*, \Omega(Y_{\mathcal{K}_{2,1}})(v,1)(p_1\circ\mathcal{Y}_{2\boxtimes 2})(v,x)v\rangle\\
&=cx^{\frac{p_-}{4}}(1-x)^{\frac{p_-}{4}}{}_2F_1\big(\frac{p_-}{2},\frac{3p_-}{2}-1,p_-;x\big)\\
&=cx^{\frac{p_-}{4}}(1-x)^{-\frac{3p_-}{4}+1}{}_2F_1\big(-\frac{p_-}{2}+1,\frac{p_-}{2},p_-;x\big)
\end{aligned}$$

where c is a non-zero constant. Then using Equation 15.8.10 in [19], we can see that the constant term of ${}_2F_1\big(-\frac{p_-}{2}+1,\frac{p_-}{2},p_-;x\big)$ is given by $(\Gamma(\frac{p_-}{2})\Gamma(\frac{3p_-}{2}-1))^{-1}$. Therefore, by Lemma 8.3.14, the coefficient of $(1-x)^{-2h_{2,1}}=(1-x)^{-\frac{3p_-}{4}+1}$ in (8.3.14) is non-zero.

Note that $\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1}\in C^{thick}_{1,1}$ does not contain $\mathcal{X}^+_{1,p_--1},\mathcal{X}^-_{1,p_--1}\in C^{thick}_{1,1}$ as composition factors. In fact, assuming that $\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1}$ contains $\mathcal{X}^\pm_{1,p_--1}$, then from the rigidity of $\mathcal{K}_{1,2}$ and from Proposition 8.3.9, $\mathcal{K}_{1,2}\boxtimes(\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1})$ contains $\mathcal{X}^\pm_{1,p_-}\in C^{thin}_{1,p_-}$ as a composition factor, but since

$$\mathcal{K}_{1,2}\boxtimes(\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1})=\mathcal{K}_{2,2}\boxtimes\mathcal{K}_{2,1}\in C^{thick}_{1,2}$$

by Propositions 8.3.3 and 8.3.12, so we have a contradiction. Thus, by Propositions 7.2.8 and 7.3.9, assuming that $\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1}$ is logarithmic, we have $\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1}\simeq\mathcal{Q}(\mathcal{X}^+_{1,1})_{1,1}$, and assuming that $\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1}$ is not logarithmic, we see that $\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1}$ has $L(h_{1,1})$ as a submodule. Let us assume $\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1}$ is not logarithmic. Then $\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1}$ has $L(h_{1,1})$ as a submodule. Since $L(h_{1,1})\boxtimes\mathcal{K}_{2,1}=0$, the coefficient of $(1-x)^{-2h_{2,1}}$ in (8.3.14) must be zero. But this is a contradiction. Therefore we obtain

$$\mathcal{K}_{2,1}\boxtimes\mathcal{K}_{2,1}\simeq\mathcal{Q}(\mathcal{X}^+_{1,1})_{1,1}.$$

We define the following module map i_1

$$i_1:\mathcal{K}_{1,1}\to\mathcal{Q}(\mathcal{X}^+_{1,1})_{1,1}.$$

To prove that $\mathcal{K}_{2,1}$ is rigid, it is sufficient to prove that the homomorphisms $f, g : \mathcal{K}_{2,1} \to \mathcal{K}_{2,1}$ defined by the commutative diagrams

$$\begin{array}{ccccc}
\mathcal{K}_{2,1} & \xrightarrow{r^{-1}} & \mathcal{K}_{2,1} \boxtimes \mathcal{K}_{1,1} & \xrightarrow{\mathrm{id}\boxtimes i_1} & \mathcal{K}_{2,1} \boxtimes (\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1}) \\
{\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle \mathcal{A}} \\
\mathcal{K}_{2,1} & \xleftarrow{l} & \mathcal{K}_{1,1} \boxtimes \mathcal{K}_{2,1} & \xleftarrow{p_1\boxtimes \mathrm{id}} & (\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1}) \boxtimes \mathcal{K}_{2,1}
\end{array}$$

and

$$\begin{array}{ccccc}
\mathcal{K}_{2,1} & \xrightarrow{l^{-1}} & \mathcal{K}_{1,1} \boxtimes \mathcal{K}_{2,1} & \xrightarrow{i_1\boxtimes \mathrm{id}} & (\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1}) \boxtimes \mathcal{K}_{2,1} \\
{\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle \mathcal{A}^{-1}} \\
\mathcal{K}_{2,1} & \xleftarrow{r} & \mathcal{K}_{2,1} \boxtimes \mathcal{K}_{1,1} & \xleftarrow{\mathrm{id}\boxtimes p_1} & \mathcal{K}_{2,1} \boxtimes (\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{2,1})
\end{array}$$

are non-zero multiples of the identity. Since $\mathrm{Hom}(\mathcal{K}_{2,1}, \mathcal{K}_{2,1}) \simeq \mathbb{C}$, it is sufficient to show that f and g are non-zero. We only show $g \neq 0$. The proof of $f \neq 0$ is similar.

Note that

$$\begin{aligned}
&(i_1 \boxtimes \mathrm{id}_{\mathcal{K}_{2,1}}) \circ l^{-1}_{\mathcal{K}_{2,1}}(|\alpha_{2,1}\rangle) \\
&= (i_1 \boxtimes \mathrm{id}_{\mathcal{K}_{2,1}})(\mathcal{Y}_{1\boxtimes 2}(|0\rangle, 1)\,|\alpha_{2,1}\rangle) = \mathcal{Y}_{(2\boxtimes 2)\boxtimes 2}(i_1(|0\rangle), 1)\,|\alpha_{2,1}\rangle,
\end{aligned}$$

where $\mathcal{Y}_{1\boxtimes 2}$ is the intertwining operator of type $\binom{\mathcal{K}_{2,1}}{\mathcal{K}_{1,1}\ \mathcal{K}_{2,1}}$. Since $i_1(|0\rangle)$ is the cofficient of $x^{-2h_{2,1}}$ in $\mathcal{Y}_{2\boxtimes 2}(|\alpha_{2,1}\rangle, x)\,|\alpha_{2,1}\rangle$, we can see that $(i_1 \boxtimes \mathrm{id}_{\mathcal{K}_{2,1}}) \circ l^{-1}_{\mathcal{K}_{2,1}}(|\alpha_{2,1}\rangle)$ is the coefficient of $(1-x)^{-2h_{2,1}}$ in the expansion of

$$\begin{aligned}
&\mathcal{Y}_{(2\boxtimes 2)\boxtimes 2}(\mathcal{Y}_{2\boxtimes 2}(|\alpha_{2,1}\rangle, 1-x)\,|\alpha_{2,1}\rangle, x)\,|\alpha_{2,1}\rangle \\
&= \mathcal{Y}_{(2\boxtimes 2)\boxtimes 2}(\mathcal{Y}_{2\boxtimes 2}(|\alpha_{2,1}\rangle, 1-x)\,|\alpha_{2,1}\rangle, 1-(1-x))\,|\alpha_{2,1}\rangle
\end{aligned}$$

as a series in $1-x$. Therefore, since the coefficient of $(1-x)^{-2h_{2,1}}$ in (8.3.14) is non-zero, we obtain the rigidity of $\mathcal{K}_{2,1}$.

□

Similar to Proposition 8.3.12, we can prove the following proposition.

Proposition 8.3.16.

1. *For $p_+ \geq 3$, $2 \leq r \leq p_+ - 1$, $1 \leq s \leq p_-$, we have*

$$\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{r,s} = \mathcal{K}_{r-1,s} \oplus \mathcal{K}_{r+1,s}.$$

2. *For $1 \leq s \leq p_-$, we have*

$$\mathcal{K}_{2,1} \boxtimes \mathcal{K}_{1,s} = \mathcal{K}_{2,s}.$$

The following standard lemma holds for the product of rigid objects in any tensor category.

Lemma 8.3.17. *Let $(\mathcal{C}, \otimes)$ be a tensor category. Let V_1 and V_2 be rigid object in $\mathcal{C}$. Then $V_1 \otimes V_2$ is also rigid with dual $V_2^\vee \otimes V_1^\vee$, where $V_i^\vee$ be the dual of V_i.*

By Propositions 8.3.12, 8.3.16, Theorems 8.3.7, 8.3.15 and Lemma 8.3.17, we obtain the following theorem.

Theorem 8.3.18. *For $1 \leq r \leq p_+, 1 \leq s \leq p_-$, the indecomposable modules $\mathcal{K}_{r,s}$ are rigid and self-dual.*

8.4 Tensor product between simple modules $\mathcal{X}^{\pm}_{r,s}$

In this section, we compute some tensor product between simple modules $\mathcal{X}^{\pm}_{r,s}$, using the rigidity of $\mathcal{K}_{r',s'}$.

By Cororally 8.2.4 and by Theorem 8.3.18, we obtain the following proposition.

Proposition 8.4.1. *For $1 \leq r \leq p_+$, $1 \leq s \leq p_-$, we have*

$$\mathcal{X}^+_{1,1} \boxtimes \mathcal{K}_{r,s} = \mathcal{X}^+_{r,s}.$$

Note that $\mathcal{X}^+_{1,1}$ is self-contragredient and $L(h_{1,1}) \boxtimes \mathcal{X}^+_{1,1} = 0$. Then, by Lemma 8.3.13, we have

$$\mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{X}^+_{1,1} \boxtimes \mathcal{X}^+_{1,1}, \mathcal{K}^*_{1,1}) = \mathbb{C}, \qquad \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{X}^+_{1,1} \boxtimes \mathcal{X}^+_{1,1}, L(h_{1,1})) = 0.$$

Thus we obtain the following lemma.

Lemma 8.4.2. *There exists a surjective module map from $\mathcal{X}^+_{1,1} \boxtimes \mathcal{X}^+_{1,1}$ to $\mathcal{K}^*_{1,1}$.*

Proposition 8.4.3. *We have*

$$\mathcal{X}^+_{1,1} \boxtimes \mathcal{X}^+_{1,1} = \mathcal{K}^*_{1,1}.$$

Proof. By Lemma 8.4.2, $\mathcal{X}^+_{1,1} \boxtimes \mathcal{X}^+_{1,1} \neq 0$. Let us determine the top composition factors of $\mathcal{X}^+_{1,1} \boxtimes \mathcal{X}^+_{1,1}$. Let π be the surjection from $\mathcal{X}^+_{1,1} \boxtimes \mathcal{X}^+_{1,1}$ to $\mathrm{top}(\mathcal{X}^+_{1,1} \boxtimes \mathcal{X}^+_{1,1})$. Let ψ^* be an arbitrary element of $A_0((\mathrm{top}(\mathcal{X}^+_{1,1} \boxtimes \mathcal{X}^+_{1,1}))^*)$. Let ϕ_1, ϕ_2 be arbitrary elements of $\mathcal{X}^+_{1,1}$, and let u be the highest weight vector of $\mathcal{X}^+_{1,1}$. For $n \geq 1$, let $w^{(n)}_i (i = -n, -n+1 \dots, n)$ be the Virasoro highest weight vectors of the vector subspace $(2n+1)L(\Delta^+_{1,1;n}) \subset \mathcal{X}^+_{1,1}$. Let us consider the value

$$\langle \psi^*, \phi_1 \otimes \phi_2 \rangle = \langle \psi^*, \pi \circ \mathcal{Y}_{\boxtimes}(\phi_1 \otimes \phi_2) \rangle.$$

By Lemma 8.1.6, we see that the value $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the values

$$\langle \psi^*, U(\mathcal{L}).u \otimes u \rangle, \qquad \langle \psi^*, U(\mathcal{L}).u \otimes w^{(n)}_i \rangle$$

for some finite $n \geq 1$ and i. Let h be the L_0 weight of ψ^*.

Let us assume that $\langle \psi^*, u \otimes w^{(n)}_i \rangle \neq 0$ for some $n \geq 1$. Then, by Proposition 7.2.6, we see that h must satisfies the following equations

$$\prod_{j=1}^{2p_--1} (h - h_{1,(2n+4)p_- -2-2j+1}) = 0, \qquad \prod_{i=1}^{2p_+-1} (h - h_{(2n+4)p_+ -2-2i+1,1}) = 0.$$

We see that h satisfying these equations is given by $\Delta^+_{1,1;n} = h_{(2+2n)p_+-1,1}$. We see that $\Delta^+_{1,1;n}$ does not correspond to any highest weight of the simple modules. Thus we have a contradiction. Therefore $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the values

$$\langle \psi^*, U(\mathcal{L}).u \otimes u \rangle.$$

Then, by Proposition 7.2.6, h must satisfies the following equations

$$\prod_{j=1}^{2p_{-}-1}(h-h_{1,4p_{-}-2-2j+1})=0, \qquad \prod_{i=1}^{2p_{+}-1}(h-h_{4p_{+}-2-2i+1,1})=0.$$

We see that h satisfying these equations is given by $\Delta^{+}_{1,1;0}=h_{2p_{+}-1,1}$. Thus we have

$$\mathrm{top}(\mathcal{X}^{+}_{1,1}\boxtimes\mathcal{X}^{+}_{1,1})=(m+1)\mathcal{X}^{+}_{1,1}$$

for some $m\geq 0$. By Theorem 7.2.6, we see that $m=0$. In particular we have $\mathcal{X}^{+}_{1,1}\boxtimes\mathcal{X}^{+}_{1,1}\in C^{thick}_{1,1}$.

Note that, by Lemma 8.4.2, $\mathcal{X}^{+}_{1,1}\boxtimes\mathcal{X}^{+}_{1,1}$ has $\mathcal{K}^{*}_{1,1}$ as a subquotient. Assume that $\mathcal{X}^{+}_{1,1}\boxtimes\mathcal{X}^{+}_{1,1}\not\cong\mathcal{K}^{*}_{1,1}$. Then, by Proposition 7.3.9, $\mathcal{X}^{+}_{1,1}\boxtimes\mathcal{X}^{+}_{1,1}$ has a submodule whose top composition factors are given by some direct sum of $\mathcal{X}^{-}_{p_{+}-1,1}$ and $\mathcal{X}^{-}_{1,p_{-}-1}$. Then, by the rigidity of $\mathcal{K}_{2,1}$ and $\mathcal{K}_{1,2}$ and by Proposition 8.3.9, either $\mathcal{K}_{1,2}\boxtimes(\mathcal{X}^{-}_{1,1}\boxtimes\mathcal{X}^{-}_{1,1})$ or $\mathcal{K}_{2,1}\boxtimes(\mathcal{X}^{-}_{1,1}\boxtimes\mathcal{X}^{-}_{1,1})$ contains $\mathcal{X}^{-}_{1,p_{-}}$ or $\mathcal{X}^{-}_{p_{+},1}$ as composition factors, respectively. On the other hand, by the associativity and Proposition 8.4.1, we have

$$\begin{aligned}\mathcal{K}_{1,2}\boxtimes(\mathcal{X}^{+}_{1,1}\boxtimes\mathcal{X}^{+}_{1,1})&=(\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{1,1})\boxtimes\mathcal{X}^{+}_{1,1}=\mathcal{X}^{+}_{1,2}\boxtimes\mathcal{X}^{+}_{1,1},\\ \mathcal{K}_{2,1}\boxtimes(\mathcal{X}^{+}_{1,1}\boxtimes\mathcal{X}^{+}_{1,1})&=(\mathcal{K}_{2,1}\boxtimes\mathcal{X}^{+}_{1,1})\boxtimes\mathcal{X}^{+}_{1,1}=\mathcal{X}^{+}_{2,1}\boxtimes\mathcal{X}^{+}_{1,1}.\end{aligned}$$

Similar to the case of $\mathcal{X}^{+}_{1,1}\boxtimes\mathcal{X}^{+}_{1,1}$, by using Proposition 7.2.6, we can show that the top composition factors of $\mathcal{X}^{+}_{1,2}\boxtimes\mathcal{X}^{+}_{1,1}$ and $\mathcal{X}^{+}_{2,1}\boxtimes\mathcal{X}^{+}_{1,1}$ are $\mathcal{X}^{+}_{1,2}$ and $\mathcal{X}^{+}_{2,1}$, respectively. In particular, we have $\mathcal{X}^{+}_{1,2}\boxtimes\mathcal{X}^{+}_{1,1}\in C^{thick}_{1,2}$ and $\mathcal{X}^{+}_{2,1}\boxtimes\mathcal{X}^{+}_{1,1}\in C^{a}_{2,1}$, where $a=thick$ for $p_{+}\geq 3$ and $a=thin$ for $p_{+}=2$. But since $\mathcal{X}^{-}_{1,p_{-}}\in C^{thin}_{p_{+}-1,p_{-}}$ and $\mathcal{X}^{-}_{p_{+},1}\in C^{thin}_{p_{+},p_{-}-1}$, we have a contradiction.

□

By Propositions 8.4.1 and 8.4.3, we obtain the following proposition.

Proposition 8.4.4. *For $1\leq r,r'\leq p_{+}$, $1\leq s,s'\leq p_{-}$, we have*

$$\mathcal{X}^{+}_{r,s}\boxtimes\mathcal{X}^{+}_{r',s'}=(\mathcal{K}_{r,s}\boxtimes\mathcal{K}_{r',s'})\boxtimes\mathcal{K}^{*}_{1,1}.$$

From Proposition 8.4.4 and the following proposition, we can compute the tensor product between the simple modules $\mathcal{X}^{+}_{r,s}$ by using the tensor product between the indecomposable modules $\mathcal{K}^{+}_{r,s}$.

Proposition 8.4.5. *For $1\leq r\leq p_{+}$, $1\leq s\leq p_{-}$, we have*

$$\mathcal{K}^{*}_{1,1}\boxtimes\mathcal{K}_{r,s}=\mathcal{K}^{*}_{r,s}.$$

Proof. By Proposition 8.4.3, we have

$$\mathcal{X}^{+}_{1,1}\boxtimes\mathcal{X}^{+}_{1,1}=\mathcal{K}^{*}_{1,1}. \tag{8.4.1}$$

Multiplying both sides by $\mathcal{K}_{r,s}$ and using Proposition 8.4.1, we have

$$\mathcal{X}^{+}_{1,1}\boxtimes\mathcal{X}^{+}_{r,s}=\mathcal{K}^{*}_{1,1}\boxtimes\mathcal{K}_{r,s}.$$

for $1 \leq r \leq p_+ - 1, 1 \leq s \leq p_- - 1$. By the exact sequence

$$0 \to L(h_{1,1}) \to \mathcal{K}^*_{1,1} \to \mathcal{X}^+_{1,1} \to 0$$

and the rigidity of $\mathcal{K}_{r,s}$, we have the following exact sequence

$$0 \to L(h_{r,s}) \to \mathcal{K}^*_{1,1} \boxtimes \mathcal{K}_{r,s} \to \mathcal{X}^+_{r,s} \to 0. \tag{8.4.2}$$

By Theorem 8.3.18 and Lemma 8.3.10, we have

$$\mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}^*_{1,1} \boxtimes \mathcal{K}_{r,s}, L(h_{r,s})) \simeq \mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}^*_{1,1}, L(h_{r,s}) \boxtimes \mathcal{K}_{r,s}) = 0.$$

Thus the exact sequence (8.4.2) does not split. Therefore, since $\mathrm{Ext}^1(\mathcal{X}^+_{r,s}, L(h_{r,s})) = \mathbb{C}$ by Proposition 7.2.8, we obtain

$$\mathcal{K}^*_{1,1} \boxtimes \mathcal{K}_{r,s} \simeq \mathcal{K}^*_{r,s}.$$

□

Similar to Lemma 8.4.2, by using Lemma 8.3.13, we obatin the following lemma.

Lemma 8.4.6. *There exists a surjective module map from $\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1}$ to $\mathcal{K}^*_{1,1}$.*

Proposition 8.4.7. *We have*

$$\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1} = \mathcal{K}^*_{1,1}, \qquad \mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^+_{1,1} = \mathcal{X}^-_{1,1}.$$

Proof. First we prove $\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1} = \mathcal{K}^*_{1,1}$. By Lemma 8.4.6, we see that $\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1} \neq 0$. Let us determine the top composition factors of $\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1}$. Let π be the surjection from $\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1}$ to $\mathrm{top}(\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1})$. Let ψ^* be an arbitrary element of $A_0((\mathrm{top}(\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1}))^*)$. Let ϕ_1 and ϕ_2 be arbitrary elements of $\mathcal{X}^-_{1,1}$. Let $\{v_+, v_-\}$ be a baisis of the highest weight space of $\mathcal{X}^-_{1,1}$. For $n \geq 1$, let $\{v^{(n)}_{\frac{i}{2}}, v^{(n)}_{\frac{-i}{2}}\}_{i=1}^{n+1}$ be the basis of the Virasoro highest weight space of the vector subspace $(2n+2)L(\Delta^-_{1,1;n}) \subset \mathcal{X}^-_{1,1}$. Let us consider the value

$$\langle \psi^*, \phi_1 \otimes \phi_2 \rangle = \langle \psi^*, \pi \circ \mathcal{Y}_{\boxtimes}(\phi_1 \otimes \phi_2) \rangle.$$

By using Lemma 8.1.6 we see that $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the values

$$\langle \psi^*, U(\mathcal{L}).v_\epsilon \otimes v_{\epsilon'} \rangle, \qquad \langle \psi^*, U(\mathcal{L}).v_\epsilon \otimes v^{(n)}_{\pm\frac{i}{2}} \rangle$$

for some $n \geq 1$ and i, where $\epsilon = +$ or $-$ and $\epsilon' = +$ or $-$. By Proposition 7.2.6, we can see that there is no highest weight of simple modules that gives the L_0 weight of ψ^* such that $\langle \psi^*, v_\epsilon \otimes v^{(n)}_{\pm\frac{i}{2}} \rangle$ is non-zero. Thus by using Lemma 8.1.6 again, we see that $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the values

$$\langle \psi^*, U(\mathcal{L}).v_\epsilon \otimes v_{\epsilon'} \rangle.$$

By Proposition 7.2.6, we see that the L_0-eigenvalue of ψ^* is the highest weight of $\mathcal{X}^+_{1,1}$. Thus, noting $W^\pm[0]\psi^* = 0$, $\langle \psi^*, v_\epsilon \otimes v^{(n)}_{\pm\frac{i}{2}} \rangle = 0$ and

$$W^\pm[-h]v_\pm = 0 \quad \text{for } h < \Delta^-_{1,1;1} - \Delta^-_{1,1;0},$$

we have $\langle \psi^*, v_\pm \otimes v_\pm \rangle = 0$. Therefore $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the values

$$\langle \psi^*, U(\mathcal{L}).v_+ \otimes v_- \rangle.$$

Then, by Proposition 7.2.6, we have

$$\mathrm{top}(\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1}) = \mathcal{X}^+_{1,1}. \tag{8.4.3}$$

Note that by Lemma 8.4.6 $\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1}$ has $\mathcal{K}^*_{1,1}$ as a subquotient. Assume $\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1} \not\cong \mathcal{K}^*_{1,1}$. Then noting (8.4.3) we see that $\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1}$ has a submodule whose top composition factors are given by some direct sum of $\mathcal{X}^-_{p_+-1,1}$ and $\mathcal{X}^-_{1,p_--1}$. Hence, by the rigidity of $\mathcal{K}_{2,1}$ and $\mathcal{K}_{1,2}$ and by Proposition 8.3.9, we see that at least one of $\mathcal{K}_{1,2} \boxtimes (\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1})$ and $\mathcal{K}_{2,1} \boxtimes (\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1})$ contains $\mathcal{X}^-_{1,p_-}$ or $\mathcal{X}^-_{p_+,1}$ as composition factors, respectively. On the other hand, by the associativity and Proposition 8.3.9, we have

$$\mathcal{K}_{1,2} \boxtimes (\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1}) = (\mathcal{K}_{1,2} \boxtimes \mathcal{X}^-_{1,1}) \boxtimes \mathcal{X}^-_{1,1} = \mathcal{X}^-_{1,2} \boxtimes \mathcal{X}^-_{1,1},$$
$$\mathcal{K}_{2,1} \boxtimes (\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1}) = (\mathcal{K}_{2,1} \boxtimes \mathcal{X}^-_{1,1}) \boxtimes \mathcal{X}^-_{1,1} = \mathcal{X}^-_{2,1} \boxtimes \mathcal{X}^-_{1,1}.$$

Similar to the case of $\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1}$, we can see that the top composition factors of $\mathcal{X}^-_{1,2} \boxtimes \mathcal{X}^-_{1,1}$ and $\mathcal{X}^-_{2,1} \boxtimes \mathcal{X}^-_{1,1}$ are $\mathcal{X}^+_{1,2}$ and $\mathcal{X}^+_{2,1}$, respectively. In particular, we have $\mathcal{X}^-_{1,2} \boxtimes \mathcal{X}^-_{1,1} \in C^{thick}_{1,2}$ and $\mathcal{X}^-_{2,1} \boxtimes \mathcal{X}^-_{1,1} \in C^a_{2,1}$, where $a = thick$ for $p_+ \geq 3$ and $a = thin$ for $p_+ = 2$. But since $\mathcal{X}^-_{1,p_-} \in C^{thin}_{p_+-1,p_-}$ and $\mathcal{X}^-_{p_+,1} \in C^{thin}_{p_+,p_--1}$, we have a contradiction.

Next we prove the second equation. By the exact sequence

$$0 \to \mathcal{X}^+_{1,1} \to \mathcal{K}_{1,1} \to L(h_{1,1}) \to 0,$$

we have a surjection

$$\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^+_{1,1} \xrightarrow{\pi} \mathcal{X}^-_{1,1} \to 0.$$

As in the case of the proof of the first equation, by using the rigidities of $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$, we can show $\mathrm{Ker}\pi = 0$. Thus we obtain $\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^+_{1,1} \simeq \mathcal{X}^-_{1,1}$ □

By Propositions 8.3.9, 8.4.5 and 8.4.7, we obtain the following proposition.

Proposition 8.4.8.

1. *For* $1 \leq r \leq p_+ - 1$, $1 \leq s \leq p_- - 1$, *we have*

$$\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{r,s} = \mathcal{K}^*_{r,s},$$
$$\mathcal{X}^-_{1,1} \boxtimes \mathcal{K}_{r,s} = \mathcal{X}^-_{r,s},$$
$$\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^+_{r,s} = \mathcal{X}^-_{r,s}.$$

2. *For* $1 \leq s \leq p_-$, *we have*

$$\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^\pm_{p_+,s} = \mathcal{X}^\mp_{p_+,s}.$$

3. *For* $1 \leq r \leq p_+$, *we have*

$$\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^\pm_{r,p_-} = \mathcal{X}^\mp_{r,p_-}.$$

8.5 The self-duality of $\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ and $\mathcal{P}^{\pm}_{r,s}$

In this section, we will compute the tensor product $\mathcal{K}\boxtimes\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ and $\mathcal{K}\boxtimes\mathcal{P}^{\pm}_{r,s}$, where $\mathcal{K}=\mathcal{K}_{1,2}$ or $\mathcal{K}_{2,1}$. Then we will show that all simple modules in the thin blocks and all indecomposable modules $\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ and $\mathcal{P}^{\pm}_{r,s}$ are self-dual. In the following Propositions 8.5.1 and 8.5.2, we will determine the tensor product $\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{\pm}_{\bullet,p_-}$ and $\mathcal{K}_{2,1}\boxtimes\mathcal{X}^{\pm}_{p_+,\bullet}$.

Proposition 8.5.1.

$$\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{\pm}_{p_+,p_-}=\mathcal{Q}(\mathcal{X}^{\pm}_{p_+,p_--1})_{p_+,1},$$
$$\mathcal{K}_{2,1}\boxtimes\mathcal{X}^{\pm}_{p_+,p_-}=\mathcal{Q}(\mathcal{X}^{\pm}_{p_+-1,p_-})_{1,p_-}.$$

Proof. Similar to Lemma 8.3.14, we can show that the tensor product $\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{\pm}_{\bullet,p_-}$ and $\mathcal{K}_{2,1}\boxtimes\mathcal{X}^{\pm}_{p_+,\bullet}$ are non-zero. Since $\mathcal{X}^{\pm}_{p_+,p_-}$ is projective, by the rigidity of $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$, $\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{\pm}_{p_+,p_-}$ and $\mathcal{K}_{2,1}\boxtimes\mathcal{X}^{\pm}_{p_+,p_-}$ become projective modules. Thus, by Propositions 8.3.2 and 8.3.3, we obtain the proposition. □

Proposition 8.5.2.

1. *For $1\le r\le p_+-1$, we have*

$$\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{\pm}_{r,p_-}=\mathcal{Q}(\mathcal{X}^{\pm}_{r,p_--1})_{r,1}.$$

2. *For $1\le s\le p_--1$, we have*

$$\mathcal{K}_{2,1}\boxtimes\mathcal{X}^{\pm}_{p_+,s}=\mathcal{Q}(\mathcal{X}^{\pm}_{p_+-1,s})_{1,s}.$$

Proof. We only prove $\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{r,p_-}=\mathcal{Q}(\mathcal{X}^{+}_{r,p_--1})_{r,1}$. The other equations can be proved in the same way, so we omit the proofs.

By Proposition 8.3.2 and Lemma 8.3.14, we have

$$\mathrm{top}(\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{r,p_-})=\mathcal{X}^{+}_{r,p_--1} \tag{8.5.1}$$

and a surjection from $\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{r,p_-}$ to $\mathcal{K}^{*}_{r,p_--1}$.

Let $r=1$. Assume that $\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{1,p_-}$ has $\mathcal{X}^{-}_{p_+-1,p_--1}$ as a composition factor. Then

$$\mathcal{K}_{2,1}\boxtimes(\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{1,p_-})\simeq\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{2,p_-}$$

has $\mathcal{X}^{-}_{p_+,p_--1}\in C^{thin}_{p_+,1}$ as a composition factor. But, by (8.5.1) and Proposition 8.5.1, we have a contradiction because $\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{2,p_-}\in C^{thick}_{2,p_--1}$ for $p_+\ge 3$ and $\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{2,p_-}\in C^{thin}_{2,p_--1}$ for $p_+=2$. Inductively, we see that $\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{r,p_-}$ does not have $\mathcal{X}^{-}_{r^\vee,p_--1}$ as a composition factor.

By Lemma 8.3.10 and the self-duality of $\mathcal{K}_{1,2}$, we see that

$$\mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{X}^{+}_{r,p_--1},\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{r,p_-})=\mathbb{C}.$$

Thus by Proposition 7.3.9 we obtain $\mathcal{K}_{1,2}\boxtimes\mathcal{X}^{+}_{r,p_-}\simeq\mathcal{Q}(\mathcal{X}^{+}_{r,p_--1})_{r,1}$.

□

Proposition 8.5.3. *The tensor products of $\mathcal{K}_{1,2} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ and $\mathcal{K}_{2,1} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ are given by :*

1. *For $1 \leq r \leq p_+ - 1$, $2 \leq s \leq p_- - 1$,*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{r^\vee,s} = \mathcal{Q}(\mathcal{X}^{\pm}_{r,s-1})_{r^\vee,s-1} \oplus \mathcal{Q}(\mathcal{X}^{\pm}_{r,s+1})_{r^\vee,s+1}.$$

2. *For $1 \leq r \leq p_+$, $2 \leq s \leq p_- - 2$,*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{r,s^\vee} = \mathcal{Q}(\mathcal{X}^{\pm}_{r,s-1})_{r,s^\vee+1} \oplus \mathcal{Q}(\mathcal{X}^{\pm}_{r,s+1})_{r,s^\vee-1}.$$

3. *For $p_+ \geq 3$, $2 \leq r \leq p_+ - 1$, $1 \leq s \leq p_- - 1$,*

$$\mathcal{K}_{2,1} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{r,s^\vee} = \mathcal{Q}(\mathcal{X}^{\pm}_{r-1,s})_{r-1,s^\vee} \oplus \mathcal{Q}(\mathcal{X}^{\pm}_{r+1,s})_{r+1,s^\vee}.$$

4. *For $p_+ \geq 3$, $2 \leq r \leq p_+ - 2$, $1 \leq s \leq p_-$,*

$$\mathcal{K}_{2,1} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{r^\vee,s} = \mathcal{Q}(\mathcal{X}^{\pm}_{r-1,s})_{r^\vee+1,s} \oplus \mathcal{Q}(\mathcal{X}^{\pm}_{r+1,s})_{r^\vee-1,s}.$$

5. *For $1 \leq r \leq p_+ - 1$,*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{r,1})_{r^\vee,1} = \mathcal{Q}(\mathcal{X}^{\pm}_{r,2})_{r^\vee,2}.$$

6. *For $1 \leq r \leq p_+$,*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{r,p_- -1})_{r,1} = 2\mathcal{X}^{\pm}_{r,p_-} \oplus \mathcal{Q}(\mathcal{X}^{\pm}_{r,p_- -2})_{r,2}.$$

7. *For $1 \leq s \leq p_- - 1$,*

$$\mathcal{K}_{2,1} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{1,s})_{1,s^\vee} = \mathcal{Q}(\mathcal{X}^{\pm}_{2,s})_{2,s^\vee}.$$

8. *For $1 \leq s \leq p_-$,*

$$\mathcal{K}_{2,1} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{p_+-1,s})_{1,s} = 2\mathcal{X}^{\pm}_{p_+,s} \oplus \mathcal{Q}(\mathcal{X}^{\pm}_{p_+-2,s})_{2,s}.$$

9. *For $1 \leq r \leq p_+$, $1 \leq s \leq p_-$,*

$$\mathcal{K}_{1,2} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{r,1})_{r,p_- -1} = 2\mathcal{X}^{\mp}_{r,p_-} \oplus \mathcal{Q}(\mathcal{X}^{\pm}_{r,2})_{r,p_- -2},$$
$$\mathcal{K}_{2,1} \boxtimes \mathcal{Q}(\mathcal{X}^{\pm}_{1,s})_{p_+-1,s} = 2\mathcal{X}^{\mp}_{p_+,s} \oplus \mathcal{Q}(\mathcal{X}^{\pm}_{2,s})_{p_+-2,s}.$$

Proof. We will only prove

$$\mathcal{K}_{1,2} \boxtimes \mathcal{Q}(\mathcal{X}^{+}_{r,s})_{r^\vee,s} = \mathcal{Q}(\mathcal{X}^{+}_{r,s-1})_{r^\vee,s-1} \oplus \mathcal{Q}(\mathcal{X}^{+}_{r,s+1})_{r^\vee,s+1} \tag{8.5.2}$$

for $1 \leq r \leq p_+ - 1$, $2 \leq s \leq p_- - 1$. The other equations can be proved in a similar way, so we omit the proofs. By the rigidity of $\mathcal{K}_{1,2}$ the composition factors of $\mathcal{K}_{1,2} \boxtimes \mathcal{Q}(\mathcal{X}^{+}_{r,s})_{r^\vee,s}$ is given by

$$\mathcal{X}^{+}_{r,s-1}, \quad \mathcal{K}_{r,s-1}, \quad 2\mathcal{X}^{-}_{r^\vee,s-1}, \quad \mathcal{X}^{+}_{r,s+1}, \quad \mathcal{K}_{r,s+1}, \quad 2\mathcal{X}^{-}_{r^\vee,s+1}.$$

By using Lemma 8.3.10 and the self-duality of $\mathcal{K}_{1,2}$, we can see that

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{1,2}\boxtimes\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s},\mathcal{X}^+_{r,s-1})=\mathbb{C},\\
&\mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{1,2}\boxtimes\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s},\mathcal{X}^+_{r,s+1})=\mathbb{C},\\
&\mathrm{Hom}_{\mathcal{C}_{p_+,p_-}}(\mathcal{K}_{1,2}\boxtimes\mathcal{Q}(\mathcal{X}^+_{r,s})_{r^\vee,s},2\mathcal{X}^-_{r^\vee,s-1}\oplus 2\mathcal{X}^-_{r^\vee,s+1})=0.
\end{aligned}$$

Thus by Proposition 7.3.9 we obtain (8.5.2).

□

By Proposition 8.5.3, we obtain the following theorems.

Theorem 8.5.4. *All indecomposable modules of type $\mathcal{Q}(\mathcal{X}_{\bullet,\bullet})_{\bullet,\bullet}$ are rigid and self-dual.*

Theorem 8.5.5. *All simple modules in the thin blocks and the semi-simple blocks are rigid and self-dual.*

Since $\mathcal{X}^+_{1,1}\boxtimes\mathcal{X}^\pm_{a,b}=\mathcal{X}^\pm_{a,b}$ for any simple module $\mathcal{X}^\pm_{a,b}$ in the thin blocks and the semi-simple blocks, by Propositions 8.5.1, 8.5.2 and 8.5.3, we obtain the following proposition.

Proposition 8.5.6. *For any indecomposable module $\mathcal{Q}(\mathcal{X}^\pm_{a,b})_{c,d}$ of type $\mathcal{Q}(\mathcal{X}_{\bullet,\bullet})_{\bullet,\bullet}$, we have*

$$\mathcal{X}^+_{1,1}\boxtimes\mathcal{Q}(\mathcal{X}^\pm_{a,b})_{c,d}=\mathcal{Q}(\mathcal{X}^\pm_{a,b})_{c,d}.$$

By Propositions 8.4.8, 8.5.1, 8.5.2 and 8.5.3, we obtain the following proposition.

Proposition 8.5.7. *For any indecomposable module $\mathcal{Q}(\mathcal{X}^\pm_{a,b})_{c,d}$ of type $\mathcal{Q}(\mathcal{X}_{\bullet,\bullet})_{\bullet,\bullet}$, we have*

$$\mathcal{X}^-_{1,1}\boxtimes\mathcal{Q}(\mathcal{X}^\pm_{a,b})_{c,d}=\mathcal{Q}(\mathcal{X}^\mp_{a,b})_{c,d}.$$

Since all indecomposable modules of types $\mathcal{Q}(\mathcal{X}^\pm_{\bullet,p_-})_{\bullet,p_-}$, $\mathcal{Q}(\mathcal{X}^\pm_{p_+,\bullet})_{p_+,\bullet}$ and $\mathcal{P}^\pm_{\bullet,\bullet}$ are projective and generated from the top composition factor, by using Lemma 8.3.10, we obtain the following propositions.

Proposition 8.5.8.

1. *For $1\le r\le p_+-1$, we have*

$$\mathcal{K}_{1,2}\boxtimes\mathcal{Q}(\mathcal{X}^\pm_{r,p_-})_{r^\vee,p_-}=\mathcal{P}^\pm_{r,p_--1}.$$

2. *For $1\le s\le p_--1$, we have*

$$\mathcal{K}_{2,1}\boxtimes\mathcal{Q}(\mathcal{X}^\pm_{p_+,s})_{p_+,s^\vee}=\mathcal{P}^\pm_{p_+-1,s}.$$

Proposition 8.5.9. *The fusion products of $\mathcal{K}_{1,2}\boxtimes\mathcal{P}^\pm_{r,s}$ and $\mathcal{K}_{2,1}\boxtimes\mathcal{P}^\pm_{r,s}$ are given by :*

1. *For $1\le r\le p_+-1$, $2\le s\le p_--2$,*

$$\mathcal{K}_{1,2}\boxtimes\mathcal{P}^\pm_{r,s}=\mathcal{P}^\pm_{r,s-1}\oplus\mathcal{P}^\pm_{r,s+1}.$$

2. *For $p_+\ge 3$, $2\le r\le p_+-2$, $1\le s\le p_--1$,*

$$\mathcal{K}_{2,1}\boxtimes\mathcal{P}^\pm_{r,s}=\mathcal{P}^\pm_{r-1,s}\oplus\mathcal{P}^\pm_{r+1,s}.$$

3. *For* $1 \le r \le p_+ - 1$, $s = 1$,

$$\mathcal{K}_{1,2} \boxtimes \mathcal{P}^\pm_{r,p_- - 1} = 2\mathcal{Q}(\mathcal{X}^\pm_{r,p_-})_{r^\vee,p_-} \oplus \mathcal{P}^\pm_{r,p_- - 2}.$$

4. *For* $r = 1$, $1 \le s \le p_- - 1$,

$$\mathcal{K}_{2,1} \boxtimes \mathcal{P}^\pm_{p_+ - 1,s} = 2\mathcal{Q}(\mathcal{X}^\pm_{p_+,s})_{p_+,s^\vee} \oplus \mathcal{P}^\pm_{p_+ - 2,s}.$$

5. *For* $1 \le r \le p_+ - 1$, $1 \le s \le p_- - 1$,

$$\mathcal{K}_{1,2} \boxtimes \mathcal{P}^\pm_{r,1} = 2\mathcal{Q}(\mathcal{X}^\mp_{r,p_-})_{p_+ - r,p_-} \oplus \mathcal{P}^\pm_{r,2},$$
$$\mathcal{K}_{2,1} \boxtimes \mathcal{P}^\pm_{1,s} = 2\mathcal{Q}(\mathcal{X}^\mp_{p_+,s})_{p_+,p_- - s} \oplus \mathcal{P}^\pm_{2,s}.$$

From the above propositions, we obtain the following theorem.

Theorem 8.5.10. *All indecomposable modules* $\mathcal{P}^\pm_{r,s}$ *are rigid and self-dual.*

By Propositions 8.5.7 and 8.5.9, we obtain the following propositions.

Proposition 8.5.11. *For* $1 \le r \le p_+ - 1$, $1 \le s \le p_- - 1$, *we have*

$$\mathcal{X}^+_{1,1} \boxtimes \mathcal{P}^\pm_{r,s} = \mathcal{P}^\pm_{r,s}.$$

Proposition 8.5.12. *For* $1 \le r \le p_+ - 1$, $1 \le s \le p_- - 1$, *we have*

$$\mathcal{X}^-_{1,1} \boxtimes \mathcal{P}^\pm_{r,s} = \mathcal{P}^\mp_{r,s}.$$

Chapter 9

Non-semisimple fusion rings

Let $\mathbb{I}_{p_+,p_-}$ be the set consisting of all simple modules of type $\mathcal{X}^{\pm}_{r,s}$ and all indecomposable modules $\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$ and $\mathcal{P}^{\pm}_{r,s}$, and let $\mathbb{S}_{p_+,p_-}$ be the set consisting of all simple modules of type $\mathcal{X}^{\pm}_{r,s}$. In this chapter, we consider the structure of commutative rings $P^0(\mathcal{C}_{p_+,p_-})$ and $K^0(\mathcal{C}_{p_+,p_-})$ defined on $\mathbb{I}_{p_+,p_-}$ and $\mathbb{S}_{p_+,p_-}$, respectively. We will also discuss the relationship between these commutative rings and the non-semisimple fusion ring $P(\mathcal{C}_p)$ and the Grothendieck ring $K(\mathcal{C}_p)$ of the triplet W-algebra $\mathcal{W}_p$.

In Section 9.3, we introduce a braided tensor category $\mathcal{C}^0_{p_+,p_-}$, which is defined by the quotient of the abelian category $\mathcal{C}_{p_+,p_-}$ by the Serre subcategory Min_{p_+,p_-} consisting of all minimal simple modules $L(h_{r,s})$. This category $\mathcal{C}^0_{p_+,p_-}$ is expected to be equivalent to $\mathfrak{g}_{p_+,p_-}$-mod, as the abelian category, and even ribbon tensor equivalent. See [9],[22],[23] for the quantum group $\mathfrak{g}_{p_+,p_-}$.

9.1 The ring structure of $P^0(\mathcal{C}_{p_+,p_-})$

As in the subsection 5.3 in [64], we introduce the free abelian group $P^0(\mathcal{C}_{p_+,p_-})$ of rank $8p_+p_- - 4p_+ - 4p_- + 2$ generated by all simple modules, all projective modules and all indecomposable modules $\mathcal{Q}(\mathcal{X}^{\pm}_{r,s})_{\bullet,\bullet}$

$$\begin{aligned}
P^0(\mathcal{C}_{p_+,p_-}) = &\bigoplus_{r=1}^{p_+}\bigoplus_{s=1}^{p_-}\bigoplus_{\epsilon=\pm}\mathbb{Z}[\mathcal{X}^{\epsilon}_{r,s}]_P \oplus \bigoplus_{r=1}^{p_+-1}\bigoplus_{s=1}^{p_--1}\bigoplus_{\epsilon=\pm}\mathbb{Z}[\mathcal{P}^{\epsilon}_{r,s}]_P\\
&\oplus \bigoplus_{r=1}^{p_+-1}\bigoplus_{s=1}^{p_--1}\bigoplus_{\epsilon=\pm}\mathbb{Z}[\mathcal{Q}(\mathcal{X}^{\epsilon}_{r,s})_{r^\vee,s}]_P \oplus \bigoplus_{r=1}^{p_+-1}\bigoplus_{s=1}^{p_--1}\bigoplus_{\epsilon=\pm}\mathbb{Z}[\mathcal{Q}(\mathcal{X}^{\epsilon}_{r,s})_{r,s^\vee}]_P\\
&\oplus \bigoplus_{r=1}^{p_+-1}\bigoplus_{\epsilon=\pm}\mathbb{Z}[\mathcal{Q}(\mathcal{X}^{\epsilon}_{r,p_-})_{r^\vee,p_-}]_P \oplus \bigoplus_{s=1}^{p_--1}\bigoplus_{\epsilon=\pm}\mathbb{Z}[\mathcal{Q}(\mathcal{X}^{\epsilon}_{p_+,s})_{p_+,s^\vee}]_P
\end{aligned}$$

For any $M \in \mathcal{C}_{p_+,p_-}$ which have minimal simple modules in the Socle, let $\pi_0(M)$ be the quotient module of M quotiented by all the minimal simple modules in the Socle. We define the endfunctor π of $\mathcal{C}_{p_+,p_-}$ by the condition that for any M in $\mathcal{C}_{p_+,p_-}$

$$\pi(M) = \begin{cases}\pi_0(M) & M \textit{ contains minimal simple modules in } \mathrm{Soc}(M)\\ M & \textit{otherwise}\end{cases}$$

Then, from the results presented in the previous chapter, we can define the structure of a commutative ring on $P^0(\mathcal{C}_{p_+,p_-})$ such that the product as a ring is given by

$$[M_1]_P \cdot [M_2]_P = [\pi(M_1 \boxtimes M_2)]_P,$$

where $M_1, M_2 \in \mathbb{I}_{p_+,p_-}$ and we extend the symbol $[\bullet]_P$ as follows

$$\Big[\bigoplus_{i\geq 1}^{n} N_i\Big]_P = \bigoplus_{i\geq 1}^{n}[N_i]_P$$

for any $N_i \in \mathbb{I}_{p_+,p_-}$ and any $n \in \mathbb{Z}_{\geq 1}$.

Remark 9.1.1. *By Propositions 8.4.3, 8.4.4, 8.4.5, 8.4.7 and 8.4.8, we see that the tensor product does not close on the set $\mathbb{I}_{p_+,p_-}$. Therefore, to define the structure of a commutative ring on $P^0(\mathcal{C}_{p_+,p_-})$, we need to quotient by the minimal simple modules. Note that*

$$\pi(\mathcal{X}^+_{r,s} \boxtimes M) = \mathcal{X}^+_{r,s} \boxtimes M = \mathcal{K}_{r,s} \boxtimes M, \qquad 1 \leq r \leq p_+ - 1,\ 1 \leq s \leq p_- - 1,$$

for any rigid indecomposable module M in $\mathbb{I}_{p_+,p_-}$.

The three operators

$$X = \pi(\mathcal{X}^+_{1,2} \boxtimes -), \qquad Y = \pi(\mathcal{X}^+_{2,1} \boxtimes -), \qquad Z = \pi(\mathcal{X}^-_{1,1} \boxtimes -)$$

define $\mathbb{Z}$-linear endomorphism of $P^0(\mathcal{C}_{p_+,p_-})$. Thus $P^0(\mathcal{C}_{p_+,p_-})$ is a module over $\mathbb{Z}[X,Y,Z]$. We define the following $\mathbb{Z}[X,Y,Z]$-module map

$$\begin{aligned} \psi : \mathbb{Z}[X,Y,Z] &\to P^0(\mathcal{C}_{p_+,p_-}), \\ f(X,Y,Z) &\mapsto f(X,Y,Z) \cdot [\mathcal{X}^+_{1,1}]_P. \end{aligned}$$

Before examining the action of $\mathbb{Z}[X,Y,Z]$ on $P^0(\mathcal{C}_{p_+,p_-})$, we introduce the following Chebyshev polynomials.

Definition 9.1.2. *We define* Chebyshev polynomials $U_n(A)$*, $n = 0, 1, \cdots \in \mathbb{Z}[A]$ recursively*

$$\begin{aligned} &U_0(A) = 1, \qquad U_1(A) = A, \\ &U_{n+1}(A) = AU_n(A) - U_{n-1}(A). \end{aligned}$$

Note that the coefficient of the leading term of $U_n(A)$ is 1.

The goal of this section is to prove the following theorem.

Theorem 9.1.3. *The $\mathbb{Z}[X,Y,Z]$-module map ψ is surjective and the kernel of ψ is given by the following ideal*

$$\mathrm{ker}\psi = \langle Z^2 - 1, U_{2p_- - 1}(X) - 2ZU_{p_- - 1}(X), U_{2p_+ - 1}(Y) - 2ZU_{p_+ - 1}(Y) \rangle.$$

We will show some propositions to prove this theorem.

By Propositions 8.4.4 and 8.4.8, all simple modules can be expressed using the Chebyshev polynomials as follows.

Proposition 9.1.4. *For $1 \leq r \leq p_+$, $1 \leq s \leq p_-$, we have*

$$[\mathcal{X}^+_{r,s}]_P = U_{s-1}(X)U_{r-1}(Y)[\mathcal{X}^+_{1,1}]_P, \qquad [\mathcal{X}^-_{r,s}]_P = ZU_{s-1}(X)U_{r-1}(Y)[\mathcal{X}^+_{1,1}]_P.$$

By Proposition 8.5.2, we have the following proposition.

Proposition 9.1.5. *For $1 \leq r \leq p_+ - 1$, $1 \leq s \leq p_- - 1$, we have*

$$\begin{aligned} &[Q(\mathcal{X}^+_{r,p_- - 1})_{r,1}]_P = XU_{r-1}(Y)U_{p_- - 1}(X)[\mathcal{X}^+_{1,1}]_P, \\ &[Q(\mathcal{X}^+_{p_+ - 1,s})_{1,s}]_P = YU_{p_+ - 1}(Y)U_{s-1}(X)[\mathcal{X}^+_{1,1}]_P. \end{aligned}$$

By Propositions 8.5.3 and 9.1.5, we obtain the following proposition.

Proposition 9.1.6.

1. *For* $1 \le r \le p_+$, $1 \le s \le p_- - 1$, *we have*
$$[Q(\mathcal{X}^+_{r,s})_{r,s^\vee}]_P = \big(U_{2p_- - s-1}(X) + U_{s-1}(X)\big)U_{r-1}(Y)[\mathcal{X}^+_{1,1}]_P.$$

2. *For* $1 \le r \le p_+ - 1$, $1 \le s \le p_-$, *we have*
$$[Q(\mathcal{X}^+_{r,s})_{r^\vee,s}]_P = \big(U_{2p_+ - r-1}(Y) + U_{r-1}(Y)\big)U_{s-1}(X)[\mathcal{X}^+_{1,1}]_P.$$

3. *We have the following relations*
$$\begin{aligned} \big(U_{2p_- -1}(X) - 2ZU_{p_- -1}(X)\big)[\mathcal{X}^+_{1,1}]_P = 0,\\ \big(U_{2p_+ -1}(Y) - 2ZU_{p_+ -1}(Y)\big)[\mathcal{X}^+_{1,1}]_P = 0. \end{aligned} \tag{9.1.1}$$

By Proposition 8.5.8, we have the following proposition.

Proposition 9.1.7. *For* $1 \le r \le p_+ - 1$, *we have*
$$\begin{aligned} [\mathcal{P}^+_{r,p_- -1}]_P &= \big(U_{2p_+ - r-1}(Y) + U_{r-1}(Y)\big)\big(U_{p_-}(X) + U_{p_- -2}(X)\big)[\mathcal{X}^+_{1,1}]_P,\\ [\mathcal{P}^+_{p_+ -1,s}]_P &= \big(U_{2p_- - s-1}(X) + U_{s-1}(X)\big)\big(U_{p_+}(Y) + U_{p_+ -2}(Y)\big)[\mathcal{X}^+_{1,1}]_P. \end{aligned}$$

By Propositions 8.5.9 and 9.1.7, we obtain the following proposition.

Proposition 9.1.8. *For* $1 \le r \le p_+ - 1$, $1 \le s \le p_- - 1$, *we have*
$$[\mathcal{P}^+_{r,s}]_P = \big(U_{2p_+ - r-1}(Y) + U_{r-1}(Y)\big)\big(U_{2p_- - s-1}(X) + U_{s-1}(X)\big)[\mathcal{X}^+_{1,1}]_P.$$

From the above propositions, we can prove Theorem 9.1.3.

Proof of Theorem 9.1.3. By the above propositions and Propositions 8.4.8, 8.5.7, 8.5.12, we can see that ψ is surjective. We define the following ideal of $\mathbb{Z}[X,Y,Z]$
$$I = \langle Z^2 - 1, U_{2p_- -1}(X) - 2ZU_{p_- -1}(X), U_{2p_+ -1}(Y) - 2ZU_{p_+ -1}(Y)\rangle.$$
Then, by the relations (9.1.1) and by Propositions 8.4.8, 8.5.7, 8.5.12, we see that I is contained in $\ker\psi$. It is easy to see that the dimension of the quotient ring $\mathbb{Z}[X,Y,Z]/I$ is $8p_+p_- - 4p_+ - 4p_- + 2$. Therefore we have $\ker\psi = I$. □

For $p \ge 2$ let $\mathcal{C}_p$ be the abelian category of modules of the triplet W-algebra $\mathcal{W}_p$. As in shown [1],[58], $\mathcal{C}_p$ has $2p$ simple modules $\mathcal{X}^\pm_s$ $(s = 1, \dots, p)$ and $2(p-1)$ projective coves $\mathcal{P}^\pm_s$ of simple modules $\mathcal{X}^\pm_s$, where we use the notation in [58]. Since $\mathcal{W}_p$ is C_2-cofinite, $\mathcal{C}_p$ has braided tensor category structure as developed in the series of papers

[37, 38, 39, 40, 41, 42, 43, 44]. The structure of this braided tensor category is completely determined by [56] and [64]. Let

$$P(\mathcal{C}_p) := \bigoplus_{s=1}^{p}\bigoplus_{\epsilon=\pm}\mathbb{Z}[\mathcal{X}_s^{\epsilon}]_P \oplus \bigoplus_{s=1}^{p-1}\bigoplus_{\epsilon=\pm}\mathbb{Z}[\mathcal{P}_s^{\epsilon}]_P$$

be the free abelian group of rank $4p-2$. $P(\mathcal{C}_p)$ has the structure of a commutative ring where the product as a ring is given by $[\bullet]_P \cdot [\bullet]_P = [\bullet \boxtimes \bullet]_P$ and the unit is $[\mathcal{X}_1^+]_P$. As shown in [64], $P(\mathcal{C}_p)$ is generated by $[\mathcal{X}_1^+]_P$, $[\mathcal{X}_2^+]_P$ and $[\mathcal{X}_1^-]_P$, and is isomorphic to

$$\frac{\mathbb{Z}[X,Z]}{\langle Z^2-1, U_{2p-1}(X)-2ZU_{p-1}(X)\rangle},$$

where

$$[\mathcal{X}_1^+]_P \mapsto 1, \qquad [\mathcal{X}_2^+]_P \mapsto X, \qquad [\mathcal{X}_1^-]_P \mapsto Z.$$

By Theorem 9.1.3, we obtain the following proposition.

Proposition 9.1.9. *By setting $X=0$ or $Y=0$ in $P^0(\mathcal{C}_{p_+,p_-})$, we obtain two fusion rings $P(\mathcal{C}_{p_+})$ and $P(\mathcal{C}_{p_-})$:*

$$P(\mathcal{C}_{p_+}) \xleftarrow{X=0} P^0(\mathcal{C}_{p_+,p_-}) \xrightarrow{Y=0} P(\mathcal{C}_{p_-}).$$

9.2 The ring structure of $K^0(\mathcal{C}_{p_+,p_-})$

In this section, we introduce a certain Grothendieck ring $K^0(\mathcal{C}_{p_+,p_-})$ and review the structure of this ring in our setting. The structure of the Grothendieck ring $K^0(\mathcal{C}_{p_+,p_-})$ is determined by Ridout and Wood [61] (cf. [22],[60],[66]). They determine the structure of $K^0(\mathcal{C}_{p_+,p_-})$ using the Verlinde ring of the singlet W-algebra consisting of the characters of the singlet W-algebra.

Let us introduce the rank $2p_+p_- + \frac{(p_+-1)(p_--1)}{2}$ Grothendieck group of $\mathcal{C}_{p_+,p_-}$

$$K(\mathcal{C}_{p_+,p_-}) = \bigoplus_{r=1}^{p_+}\bigoplus_{s=1}^{p_-}\bigoplus_{\epsilon=\pm}\mathbb{Z}[\mathcal{X}_{r,s}^{\epsilon}]_K \oplus \bigoplus_{(r,s)\in\mathcal{T}}\mathbb{Z}[L(h_{r,s})]_K.$$

$K(\mathcal{C}_{p_+,p_-})$ has the structure of a commutative ring where the product as a ring is given by

$$[\bullet]_K \cdot [\bullet]_K = [\bullet \boxtimes \bullet]_K.$$

Note that $K(\mathcal{C}_{p_+,p_-})$ has the ideal generated by all minimal simple modules $L(h_{r,s})$. Let $K^0(\mathcal{C}_{p_+,p_-})$ be the quotient ring of $K(\mathcal{C}_{p_+,p_-})$ quotiented by this ideal. The three operators

$$X = \pi(\mathcal{X}_{1,2}^+ \boxtimes -), \qquad Y = \pi(\mathcal{X}_{2,1}^+ \boxtimes -), \qquad Z = \pi(\mathcal{X}_{1,1}^- \boxtimes -)$$

define $\mathbb{Z}$-linear endomorphism of $K^0(\mathcal{C}_{p_+,p_-})$. Thus $K^0(\mathcal{C}_{p_+,p_-})$ is a module over $\mathbb{Z}[X,Y,Z]$.

We define the following $\mathbb{Z}[X,Y,Z]$-module map

$$\begin{aligned}\phi : \mathbb{Z}[X,Y,Z] &\to K^0(\mathcal{C}_{p_+,p_-}),\\ f(X,Y,Z) &\mapsto f(X,Y,Z)\cdot[\mathcal{X}_{1,1}^+]_K.\end{aligned}$$

As in the case of $P^0(\mathcal{C}_{p_+,p_-})$, we can prove the following theorem.

Theorem 9.2.1. *The* $\mathbb{Z}[X,Y,Z]$*-module map* ϕ *is surjective and the kernel of* ϕ *is given by the following ideal*

$$\mathrm{ker}\phi = \langle Z^2-1, U_{p_-}(X)-U_{p_--2}(X)-2Z, U_{p_+}(Y)-U_{p_+-2}(Y)-2Z\rangle.$$

Let us consider the relationship between $K^0(\mathcal{C}_{p_+,p_-})$ and the rank $2p$ Grothendieck group

$$K(\mathcal{C}_p) := \bigoplus_{s=1}^{p}\bigoplus_{\epsilon=\pm}\mathbb{Z}[\mathcal{X}_s^{\epsilon}]_K.$$

of the triplet W-algebra. $K(\mathcal{C}_p)$ has the structure of a commutative rings and, as shown in [64], is isomorphic to the ring

$$\frac{\mathbb{Z}[X,Z]}{\langle Z^2-1, U_p(X)-U_{p-2}(X)-2Z\rangle},$$

where

$$[\mathcal{X}_1^+]_K \mapsto 1, \qquad [\mathcal{X}_2^+]_K \mapsto X, \qquad [\mathcal{X}_1^-]_K \mapsto Z.$$

By Theorem 9.2.1, we obtain the following proposition.

Proposition 9.2.2. *By setting* $[\mathcal{X}_{1,2}^+]_K=0$ *or* $[\mathcal{X}_{2,1}^+]_K=0$ *in* $K^0(\mathcal{C}_{p_+,p_-})$*, we obtain two Grothendieck rings* $K(\mathcal{C}_{p_+})$ *and* $K(\mathcal{C}_{p_-})$*:*

$$K(\mathcal{C}_{p_+}) \xleftarrow{[\mathcal{X}_{1,2}^+]_K=0} K^0(\mathcal{C}_{p_+,p_-}) \xrightarrow{[\mathcal{X}_{2,1}^+]_K=0} K(\mathcal{C}_{p_-}).$$

Remark 9.2.3. *By setting* $U_{p_+-1}(X)=U_{p_--1}(Y)=1$ *in* $P^0(\mathcal{C}_{p_+,p_-})$*, we obtain* $K^0(\mathcal{C}_{p_+,p_-})$*.*

9.3 The braided tensor category $\mathcal{C}^0_{p_+,p_-}$

Let Min_{p_+,p_-} be the full subcategory of $\mathcal{C}_{p_+,p_-}$ consisting of the whole minimal simple modules $L(h_{r,s})$. Since

$$\mathrm{Ext}^1(L(h_{r,s}),L(h_{r',s'}))=0, \quad 1\le r,r'<p_+,\ 1\le s,s'<p_-,$$

Min_{p_+,p_-} is a Serre subcategory. Let

$$\mathcal{C}^0_{p_+,p_-} := \mathcal{C}_{p_+,p_-}/\mathrm{Min}_{p_+,p_-}$$

be the quotient of the abelian category $\mathcal{C}_{p_+,p_-}$ by the Serre subcategory Min_{p_+,p_-}. Since by Corollary 8.2.5 $L(h_{r,s})\boxtimes M\in\mathrm{Min}_{p_+,p_-}$ for any $M\in\mathcal{C}_{p_+,p_-}$, the abelian category $\mathcal{C}^0_{p_+,p_-}$ has the structure of a braided tensor category.

Before we examine the properties of $\mathcal{C}^0_{p_+,p_-}$, let us review some properties of weakly rigid tensor category according to [53],[64].

Definition 9.3.1. *Let* $(\mathcal{C}, \otimes, \mathbf{1})$ *be a tensor category. We say that an object* M *is* weakly rigid *if the contravariant functor*

$$F_M(-) = \mathrm{Hom}_{\mathcal{C}}(- \otimes M, \mathbf{1})$$

is representable.

For the proof of the following proposition, see Appendix A in [53].

Proposition 9.3.2. *Let* $\mathcal{C}$ *be a weakly rigid tensor category. Assume that* $\mathcal{C}$ *satisfies the following properties*

1. $\mathcal{C}$ *has enough projective and injective objects.*
2. *All projective objects are injective and all injective objects are projective.*
3. *All projective objects are rigid.*

Then if

$$0 \to L \to M \to N \to 0$$

is an exact sequence in $\mathcal{C}$ *such that two of* L*,* M*,* N *are rigid, then the third object is also rigid.*

Note that, in the braided tensor category $\mathcal{C}^0_{p_+,p_-}$, the unit object is given by

$$\mathcal{K}_{1,1} \simeq \mathcal{K}^*_{1,1} \simeq \mathcal{X}^+_{1,1}. \tag{9.3.1}$$

Thus, by (9.3.1) and by Lemma 8.3.13, we obtain the following proposition.

Proposition 9.3.3. *The braided tensor category* $\mathcal{C}^0_{p_+,p_-}$ *is weakly rigid.*

From the results in Chapters 7 and 8, we obtain the following propositions.

Proposition 9.3.4. *In the abelian category* $\mathcal{C}^0_{p_+,p_-}$*, the indecomposable modules* $\mathcal{Q}(\mathcal{X}^\pm_{\bullet,p_-})_{\bullet,p_-}$*,* $\mathcal{Q}(\mathcal{X}^\pm_{p_+,\bullet})_{p_+,\bullet}$ *and* $\mathcal{P}^\pm_{r,s}$ *are projective.*

Proposition 9.3.5. *In the braided tensor category* $\mathcal{C}^0_{p_+,p_-}$*, all simple modules* $\mathcal{X}^\pm_{r,s}$ *and all indecomposable modules* $\mathcal{Q}(\mathcal{X}^\pm_{r,s})_{\bullet,\bullet}$ *and* $\mathcal{P}^\pm_{r,s}$ *are rigid.*

Proof. Since $\mathcal{K}_{r,s} \simeq \mathcal{X}^+_{r,s}$ in $\mathcal{C}^0_{p_+,p_-}$, it is sufficient to show that all simple module $\mathcal{X}^-_{r,s}$ are rigid. Note that

$$\mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^+_{r,s} = \mathcal{X}^-_{r,s}$$

in $\mathcal{C}^0_{p_+,p_-}$. Thus it is sufficient to show that $\mathcal{X}^-_{1,1}$ is rigid. Then we can see that the rigidity of $\mathcal{X}^-_{1,1}$ follows by choosing the evaluation map and coevaluation map as

$$\mathrm{ev}_{\mathcal{X}^-_{1,1}} = \mathrm{Id}_{\mathcal{X}^-_{1,1}} : \mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1} \xrightarrow{\sim} \mathcal{X}^+_{1,1}, \qquad \mathrm{coev}_{\mathcal{X}^-_{1,1}} = \mathrm{Id}_{\mathcal{X}^-_{1,1}} : \mathcal{X}^+_{1,1} \xrightarrow{\sim} \mathcal{X}^-_{1,1} \boxtimes \mathcal{X}^-_{1,1}.$$

□

Since the projective covers of all simple modules are rigid in $\mathcal{C}^0_{p_+,p_-}$, by Proposition 9.3.2, we obtain the following theorem.

Theorem 9.3.6. *The braided tensor category* $\mathcal{C}^0_{p_+,p_-}$ *is rigid.*

Let us define the following full subcategories of $\mathcal{C}^0_{p_+,p_-}$

Definition 9.3.7. *Let* $1 \le r \le p_+ - 1$, $1 \le s \le p_- - 1$.

1. *Let* $C_{1,s}$ *be the full abelian subcategory of* $\mathcal{C}^0_{p_+,p_-}$ *defined by*

$$M \in C_{1,s}$$
$$\Leftrightarrow \text{ all composition factors of } M \text{ are given by } \mathcal{X}^+_{1,s} \text{ and } \mathcal{X}^-_{1,s^\vee}.$$

2. *Let* $C_{r,1}$ *be the full abelian subcategory of* $\mathcal{C}^0_{p_+,p_-}$ *defined by*

$$M \in C_{r,1}$$
$$\Leftrightarrow \text{ all composition factors of } M \text{ are given by } \mathcal{X}^+_{r,1} \text{ and } \mathcal{X}^-_{r^\vee,1}.$$

3. *Let* $C^\pm_{1,p_-}$ *be the full abelian subcategory of* $\mathcal{C}^0_{p_+,p_-}$ *defined by*

$$M \in C^\pm_{1,p_-}$$
$$\Leftrightarrow \text{ all composition factors of } M \text{ are given by } \mathcal{X}^\pm_{1,p_-}.$$

4. *Let* $C^\pm_{1,p_-}$ *be the full abelian subcategory of* $\mathcal{C}^0_{p_+,p_-}$ *defined by*

$$M \in C^\pm_{p_+,1}$$
$$\Leftrightarrow \text{ all composition factors of } M \text{ are given by } \mathcal{X}^\pm_{p_+,1}.$$

Definition 9.3.8. *Let* $\mathcal{C}_{1,p_-}$ *and* $\mathcal{C}_{p_+,1}$ *be the following full subcategories of* $\mathcal{C}^0_{p_+,p_-}$*:*

$$\mathcal{C}_{1,p_-} := \bigoplus_{s=1}^{p_--1} C_{1,s} \oplus C^+_{1,p_-} \oplus C^-_{1,p_-}, \qquad \mathcal{C}_{p_+,1} := \bigoplus_{r=1}^{p_+-1} C_{r,1} \oplus C^+_{p_+,1} \oplus C^-_{p_+,1}.$$

By the results in Chapters 7 we obtain the following proposition.

Proposition 9.3.9. *Let* $1 \le r \le p_+ - 1$, $1 \le s \le p_- - 1$.

1. *In* $\mathcal{C}_{1,p_-}$, $\mathcal{Q}(\mathcal{X}^+_{1,s})_{1,s^\vee}$ *and* $\mathcal{Q}(\mathcal{X}^-_{1,s^\vee})_{1,s}$ *are the projective covers of* $\mathcal{X}^+_{1,s}$ *and* $\mathcal{X}^-_{1,s^\vee}$*, respectively.*

2. *In* $\mathcal{C}_{p_+,1}$, $\mathcal{Q}(\mathcal{X}^+_{r,1})_{r^\vee,1}$ *and* $\mathcal{Q}(\mathcal{X}^-_{r^\vee,1})_{r,1}$ *are the projective covers of* $\mathcal{X}^+_{r,1}$ *and* $\mathcal{X}^-_{r^\vee,1}$*, respectively.*

3. *In* $\mathcal{C}_{1,p_-}$*, the simple module* $\mathcal{X}^\pm_{1,p_-} \in C^\pm_{1,p_-}$ *is projective.*

4. *In* $\mathcal{C}_{p_+,1}$*, the simple module* $\mathcal{X}^\pm_{p_+,1} \in C^\pm_{p_+,1}$ *is projective.*

Then, similar to the arguments in Section 6 of [58], we obtain the following proposition.

Proposition 9.3.10. *We have the following equivalences as the abelian categories*

$$\mathcal{C}_{1,p_-} \simeq \mathcal{C}_{p_-}, \qquad\qquad \mathcal{C}_{p_+,1} \simeq \mathcal{C}_{p_+}.$$

Theorem 9.3.11. *$\mathcal{C}_{1,p_-}$ and $\mathcal{C}_{p_+,1}$ are braided tensor subcategories of $\mathcal{C}^0_{p_+,p_-}$. Furthermore, we have the following equivalences as the braided tensor categories*

$$\mathcal{C}_{1,p_-} \simeq \mathcal{C}_{p_-}, \qquad\qquad \mathcal{C}_{p_+,1} \simeq \mathcal{C}_{p_+}.$$

Proof. The structure of the tensor product between all indecomposable modules of $\mathcal{C}_p$ is determined by [64]. Then, by Proposition 9.3.10 and by the results in Section 9.1, we see that the tensor structures of $\mathcal{C}_{1,p_-}$ and $\mathcal{C}_{p_+,1}$ coincide with those of $\mathcal{C}_{p_-}$ and $\mathcal{C}_{p_+}$, respectively. □

Let $\mathfrak{g}_{p_+,p_-}$-mod be the category of finite dimensional $\mathfrak{g}_{p_+,p_-}$-modules, where $\mathfrak{g}_{p_+,p_-}$ is the quantum group constructed by Feigin et al.[22].

Conjecture 9.3.12. *The braided tensor category $\mathcal{C}^0_{p_+,p_-}$ is ribbon tensor equivalent to* $\mathfrak{g}_{p_+,p_-}$-mod.

Remark 9.3.13. *The equivalence of the ribbon tensor categories $\mathcal{C}_p$ and $\overline{U}_q(sl_2)$*-mod *is proved in [34], where $\overline{U}_q(sl_2)$ ($q = e^{\frac{\pi i}{p}}$) is the restricted quantum group.*

Remark 9.3.14. *$\mathfrak{g}_{p_+,p_-}$ is a Hopf algebra over $\mathbb{C}$ generated by $e_\pm$, $f_\pm$ and $K^{\pm 1}$ with relations*

$$\begin{aligned}
&KK^{-1} = K^{-1}K = 1, \qquad e_\pm^{p_\pm} f_\pm^{p_\pm} = 0, \qquad K^{2p_+p_-} = 1,\\
&Ke_\pm K^{-1} = q_\pm^2 e_\pm, \qquad Kf_\pm K^{-1} = q_\pm^{-2} f_\pm,\\
&e_+e_- = e_-e_+, \qquad f_+f_- = f_-f_+, \qquad e_\pm f_\mp = f_\mp e_\pm,\\
&[e_\pm, f_\pm] = \frac{K^{p_\mp} - K^{-p_\mp}}{q_\pm^{p_\mp} - q_\pm^{-p_\mp}},
\end{aligned}$$

where $q_\pm = q^{2p_\mp} = \exp(\frac{\pi\sqrt{-1}}{p_\pm})$.

In [23], it is shown that the restricted quantum groups $\overline{U}_{q_\pm^{p_\mp}}(sl_2) = \langle e_\pm, f_\pm, K_\pm\rangle$ are embedded in $\mathfrak{g}_{p_+,p_-}$ by

$$e_\pm \mapsto e_\pm, \qquad f_\pm \mapsto f_\pm, \qquad K_\pm \mapsto K^{p_\mp}$$

and that

$$\mathfrak{g}_{p_+,p_-} \simeq \overline{U}_{q_+^{p_-}}(sl_2) \otimes \overline{U}_{q_-^{p_+}}(sl_2)/(K_+^{p_-} \otimes 1 - 1 \otimes K_-^{p_+}),$$

where $(K_+^{p_-} \otimes 1 - 1 \otimes K_-^{p_+})$ is the Hopf ideal generated by $(K_+^{p_-} \otimes 1 - 1 \otimes K_-^{p_+})$. For more detailed structure of $\mathfrak{g}_{p_+,p_-}$, see [9].

Chapter 10

Representation theory of the Neveu-Schwarz algebra

There are two minimal extensions of the Virasoro algebra with $N = 1$ supersymmetry: the Neveu Schwarz algebra and the Ramond algebra. In this thesis, we only consider the Neveu-chwarz algebra. The $N = 1$ Neveu-Schwarz algebra is the Lie superalgebra

$$\mathfrak{ns} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{r\in\frac{1}{2}+\mathbb{Z}} \mathbb{C}G_r \oplus \bigoplus \mathbb{C}C$$

with commutation relations ($k, l \in \mathbb{Z},\ r, s \in \mathbb{Z} + \frac{1}{2}$):

$$\begin{aligned}
&[L_k, L_l] = (k-l)L_{k+l} + \delta_{k+l,0}\frac{k^3-k}{12}C,\\
&[L_k, G_r] = (\frac{1}{2}k - r)G_{k+r},\\
&\{G_r, G_s\} = 2L_{r+s} + \frac{1}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}C,\\
&[L_k, C] = 0, \quad [G_r, C] = 0.
\end{aligned}$$

We identify C with a scalar multiple of the identity, $C = c \cdot \mathrm{id}$, when acting on modules and refer to the number $c \in \mathbb{C}$ as the central charge. In this chapter we fix $m \in \mathbb{Z}_{\geq 1}$ and review basic facts of representation theory of the Neveu-Schwarz algebra whose central charge is

$$c = c^{N=1}_{1,2m+1} = \frac{15}{2} - 3(2m + 1 + \frac{1}{2m+1}),$$

in accordance with the papers [11],[46],[47].

10.1 Free field realisation of the Neveu-Schwarz algebra

The fermion algebra $\mathfrak{f}$ is the Lie superalgebra

$$\mathfrak{f} = \bigoplus_{r\in\mathbb{Z}+\frac{1}{2}} \mathbb{C}b_r \oplus \mathbb{C}\mathbf{1}$$

with anti-commutation relations:

$$\{b_r, b_s\} = \delta_{r+s,0}, \quad \{b_r, \mathbf{1}\} = 0.$$

The fermion algebra $\mathfrak{f}$ has the triangular decomposition

$$\mathfrak{f}^{\pm} = \bigoplus_{r>0} \mathbb{C}b_r, \quad \mathfrak{f}^0 = \mathbb{C}\mathbf{1}.$$

Let $\mathbb{C}\left|0\right\rangle_{\mathrm{NS}}$ be the one dimensional representation of $\mathfrak{f}^{\geq} = \mathfrak{f}^{+} \oplus \mathfrak{f}^{0}$, which satisfy

$$\mathbf{1}\left|0\right\rangle_{\mathrm{NS}} = \left|0\right\rangle_{\mathrm{NS}}, \quad \mathfrak{f}^{+}\left|0\right\rangle_{\mathrm{NS}} = 0.$$

Definition 10.1.1. *The* Neveu-Schwarz fermionic Fock module $F^{\mathfrak{f}}$ *is defined by*

$$F^{\mathfrak{f}} = \mathrm{Ind}_{\mathfrak{f}^{\geq}}^{\mathfrak{f}} \mathbb{C}\left|0\right\rangle_{\mathrm{NS}}.$$

Let $b(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n z^{-n-\frac{1}{2}}$ and we define the following energy-momentum tensor

$$T^{(\mathfrak{f})}(z) = \frac{1}{2} :\partial b(z) b(z): = \sum_{n\in\mathbb{Z}} L_n^{(\mathfrak{f})} z^{-n-2}$$

whose central charge is $c = \frac{1}{2}$.

We set

$$\beta_0 := \sqrt{2m+1} - \frac{1}{\sqrt{2m+1}}.$$

We define the following bosonic energy-momentum tensor

$$T^{(\mathrm{B})}(z) = \frac{1}{2}(: a(z)^2 : + \beta_0 \partial a(z)) = \sum_{n\in\mathbb{Z}} L_n^{(\mathrm{B})} z^{-n-2}$$

whose central charge is

$$c = 1 - 3\beta_0^2 = c_{1,2m+1}^{N=1} - \frac{1}{2}.$$

We introduce an even field and an odd field:

$$T(z) = T^{(B)}(z) + T^{(\mathfrak{f})}(z),$$
$$G(z) = a(z)b(z) + \beta_0 \partial b(z).$$

Proposition 10.1.2. *$T(z)$ and $G(z)$ have the following operator product expansions*

$$T(z)T(w) = \frac{c_{\beta_0}/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^3} + \frac{\partial T(w)}{z-w} + \cdots,$$
$$T(z)G(w) = \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \cdots,$$
$$G(z)G(w) = \frac{2c_{\beta_0}/3}{(z-w)^3} + \frac{2T(w)}{z-w} + \cdots,$$

where $c_{\beta_0} := \frac{3}{2} - 3\beta_0^2 = c_{1,2m+1}^{N=1}$. For the Fourier mode expansions of fields

$$T(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad G(w) := \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r z^{-r-\frac{3}{2}},$$

each mode satisfies the following commutation or anti-commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c_{\beta_0},$$
$$[L_m, G_r] = (\frac{1}{2}m - r)G_{m+r},$$
$$\{G_r, G_s\} = 2L_{r+s} + \frac{1}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}c_{\beta_0}.$$

Namely the each mode of the fields $T(z)$ and $G(z)$ generates the Neveu-Schwarz algebra whose central charge is $c_{\beta_0} = \frac{3}{2} - 3\beta_0^2 = c^{N=1}_{1,2m+1}$.

Definition 10.1.3. *For $\beta \in \mathbb{C}$, we set*

$$F^{\mathfrak{ns}}_\beta := F_\beta \otimes F^{\mathfrak{f}}$$

and call this tensor product Fock module simply.

From this section, we omit $|\beta\rangle \otimes |0\rangle_{\mathrm{NS}}$ as $|\beta\rangle$ for any $\beta \in \mathbb{C}$ and omit the tensor product symbols for brevity, identifying a_n with $a_n \otimes \mathbf{1}$ and b_r with $\mathbf{1} \otimes b_r$.

We define the following two vectors in $F^{\mathfrak{ns}}_0$

$$T = \frac{1}{2}(a_{-1}^2 + \beta_0 a_{-2} + b_{-\frac{1}{2}}b_{-\frac{3}{2}})\,|0\rangle\,,$$
$$G = (a_{-1}b_{-\frac{1}{2}} + \beta_0 b_{-\frac{3}{2}})\,|0\rangle\,.$$

Definition 10.1.4. *The Fock module $F^{\mathfrak{ns}}_0$ carries the structure of a $\frac{1}{2}\mathbb{Z}$-graded vertex operator superalgebra, with*

$$Y(|0\rangle\,; z) = \mathrm{id}, \quad Y(a_{-1}\,|0\rangle\,; z) = a(z), \quad Y(b_{-\frac{1}{2}}\,|0\rangle\,; z) = b(z),$$
$$Y(G; z) = G(z), \quad Y(T; z) = T(z).$$

We denote this vertex operator superalgebra by $\mathcal{F}^{\mathfrak{ns}}_{\beta_0}$.

10.2 Structure of Fock modules

We set

$$\beta_+ = \sqrt{2m+1}, \qquad \beta_- = -\sqrt{\frac{1}{2m+1}}.$$

For $r, s, n \in \mathbb{Z}$, we set

$$\beta_{r,s;n} := \frac{1-r}{2}\beta_+ + \frac{1-s}{2}\beta_- + \frac{n}{2}\beta_+, \qquad \beta_{r,s} = \beta_{r,s;0},$$

and we use the abbreviation as $F^{\mathfrak{ns}}_{r,s;n} = F^{\mathfrak{ns}}_{\beta_{r,s;n}}$, $F^{\mathfrak{ns}}_{r,s} = F^{\mathfrak{ns}}_{\beta_{r,s}}$. For $r, s, n \in \mathbb{Z}$, let us set

$$h^{\mathfrak{ns}}_{r,s} := \frac{1}{8}(r^2-1)(2m+1) - \frac{1}{4}(rs-1) + \frac{1}{8}(s^2-1)\frac{1}{2m+1},$$
$$h^{\mathfrak{ns}}_{r,s;n} := h^{\mathfrak{ns}}_{r-n,s} = h^{\mathfrak{ns}}_{r,s+(2m+1)n}$$

and let $L^{\mathfrak{ns}}(h)$ be the simple $\mathfrak{ns}$-module whose highest weight and central charge are h and $c^{N=1}_{1,2m+1}$.

Theorem 10.2.1 ([47]). *For each $r, s \in \mathbb{Z}$, $r - s \in 2\mathbb{Z}$, the Fock module $F^{\mathfrak{ns}}_{r,s} \in \mathcal{F}^{\mathfrak{ns}}_{\beta_0}$-Mod has the following socle series as an $\mathfrak{ns}$-module:*

1. *For each $F^{\mathfrak{ns}}_{1,s;n}$ $(1 \leq s < 2m+1, n \in \mathbb{Z}, s - n \in 2\mathbb{Z}+1)$, we have*

$$\mathrm{Soc}(F^{\mathfrak{ns}}_{1,s;n}) = \bigoplus_{k \geq 0} L^{\mathfrak{ns}}(h^{\mathfrak{ns}}_{1,2m+1-s;|n|+2k+1}),$$
$$F^{\mathfrak{ns}}_{1,s;n}/\mathrm{Soc}(F^{\mathfrak{ns}}_{1,s;n}) = \bigoplus_{k \geq a} L^{\mathfrak{ns}}(h^{\mathfrak{ns}}_{1,s;|n|+2k}),$$

   where $a = 0$ if $n \geq 0$, $a = 1$ if $n < 0$.

2. *For each $F^{\mathfrak{ns}}_{1,2m+1;2n}(n \in \mathbb{Z})$, we have*

$$\mathrm{Soc}(F^{\mathfrak{ns}}_{1,2m+1;2n}) = F^{\mathfrak{ns}}_{1,2m+1;2n} = \bigoplus_{k \geq 0} L^{\mathfrak{ns}}(h^{\mathfrak{ns}}_{1,2m+1;|2n|+2k}).$$

We introduce the following two fields

$$Q^{\mathfrak{ns}}_{+}(z) := b(z)V_{\beta_+}(z), \qquad Q^{\mathfrak{ns}}_{-}(z) := b(z)V_{\beta_-}(z).$$

The operator product expansions between these fields, $T(z)$ and $G(z)$ are given by:

$$T(z)Q^{\mathfrak{ns}}_{\pm}(w) \sim \partial_w \frac{Q^{\mathfrak{ns}}_{\pm}(w)}{z-w}, \qquad G(z)Q^{\mathfrak{ns}}_{\pm}(w) \sim \frac{1}{\beta_\pm}\partial_w \frac{V_{\beta_\pm}(w)}{z-w}. \tag{10.2.1}$$

By (10.2.1), the following operators become commutative with $U(\mathfrak{ns})$-action:

$$Q^{\mathfrak{ns}}_{+} := \oint_{z=0} Q^{\mathfrak{ns}}_{+}(z)\mathrm{d}z : F^{\mathfrak{ns}}_{1,2k+1} \to F^{\mathfrak{ns}}_{-1,2k+1}, \quad k \in \mathbb{Z},$$
$$Q^{\mathfrak{ns}}_{-} := \oint_{z=0} Q^{\mathfrak{ns}}_{-}(z)\mathrm{d}z : F^{\mathfrak{ns}}_{2k+1,1} \to F^{\mathfrak{ns}}_{2k+1,-1}, \quad k \in \mathbb{Z}.$$

Similar to (3.3.2), we introduce the non-trivial field

$$Q^{\mathfrak{ns}[s]}_{-}(z) : F^{\mathfrak{ns}}_{s+2k,s} \to F^{\mathfrak{ns}}_{s+2k,-s}[[z, z^{-1}]], \quad k \in \mathbb{Z},$$

as follows

$$Q^{\mathfrak{ns}[s]}_{-}(z) = \int_{\overline{\Gamma}_s(\kappa_-)} Q^{\mathfrak{ns}}_{-}(z)Q^{\mathfrak{ns}}_{-}(zx_1)Q^{\mathfrak{ns}}_{-}(zx_2)\cdots Q^{\mathfrak{ns}}_{-}(zx_{s-1})z^{s-1}\mathrm{d}x_1\cdots \mathrm{d}x_{s-1}$$

for $s \geq 2$, where $\overline{\Gamma}_s(\kappa_-)$ is a certain regularized cycle constructed from the simplex

$$\Delta_{s-1} = \{\ (x_1, \ldots, x_{s-1}) \in \mathbb{R}^{s-1} \mid 1 > x_1 > \cdots > x_{s-1} > 0\ \}.$$

The following proposition is due to [47].

Proposition 10.2.2. *For each $s \geq 2$, the zero-mode*

$$Q_{-}^{\mathfrak{ns}[s]} := \oint_{z=0} Q_{-}^{\mathfrak{ns}[s]}(z)\mathrm{d}z : F^{\mathfrak{ns}}_{s+2k,s} \to F^{\mathfrak{ns}}_{s+2k,-s}, \quad k \in \mathbb{Z}$$

is non trivial and commutative with $\mathfrak{ns}$-action of $\mathcal{F}^{\mathfrak{ns}}_{\beta_0}$-modules.

These fields $Q_{\pm}^{\mathfrak{ns}[\bullet]}(z)$ are called screening currents and the zero-modes $Q_{\pm}^{\mathfrak{ns}[\bullet]}$ are called screening operators. The following theorem is due to [47].

Theorem 10.2.3 ([47])**.** *For any $1 \leq s \leq 2m$ and $n \in \mathbb{Z}$ such that $s - n$ is odd, let us define the $\mathfrak{ns}$-modules*

$$K^{\mathfrak{ns}}_{s;n} = \ker Q_{-}^{\mathfrak{ns}[s]} : F^{\mathfrak{ns}}_{1,s;n} \to F^{\mathfrak{ns}}_{1,-s;n}.$$

Then $K^{\mathfrak{ns}}_{s;n} = \mathrm{Soc}(F^{\mathfrak{ns}}_{1,s;n})$.

Chapter 11

The $N = 1$ triplet vertex operator superalgebra $\mathcal{SW}(m)$

In Section 11.1, we review some results on the super triplet W-algebra $\mathcal{SW}(m)$ by Admović and Milas in [2] briefly. In Section 11.3, we will construct logarithmic $\mathcal{SW}(m)$-modules $\mathcal{SP}^{\pm}_{\bullet}$ by using the logarithmic deformation by J. Fjeistad et al.[27]. By using the structure of the structure of the logarithmic modules $\mathcal{SP}^{+}_{\bullet}$, in Section 11.4, we review the structure of the Zhu-algebra $A(\mathcal{SW}(m))$ determined in [6].

11.1 Vertex operator superalgebras $\mathcal{V}^{\mathfrak{ns}}_L$ and $\mathcal{SW}(m)$

Fix any $m \in \mathbb{Z}_{\geq 1}$. Let $L = \mathbb{Z}\beta_+ = \mathbb{Z}\sqrt{2m+1}$ be an integer lattice.

Definition 11.1.1. *The* lattice vertex operator superalgebra $\mathcal{V}^{\mathfrak{ns}}_L$ *associated with L is the quadruple*

$$\Big(\bigoplus_{\beta \in L} F^{\mathfrak{ns}}_{\beta}, |0\rangle, T, G, Y \Big)$$

where the fields corresponding to $|0\rangle$, $a_{-1}|0\rangle$, $b_{-\frac{1}{2}}|0\rangle$, T and G are those of $\mathcal{F}^{\mathfrak{ns}}_{\beta_0}$ and

$$Y(|\beta\rangle; z) = V_{\beta}(z), \ \beta \in L.$$

For each $i \in \mathbb{Z}$, we introduce the following symbol

$$\gamma_i = \frac{i}{2m+1}\beta_+ = -i\beta_-.$$

It is a known fact that simple $\mathcal{V}^{\mathfrak{ns}}_L$-modules are given by

$$\mathcal{V}^{\mathfrak{ns}}_{L+\gamma_i} := \bigoplus_{n \in \mathbb{Z}} F^{\mathfrak{ns}}_{\beta_{1,1;2n}+\gamma_i} = \bigoplus_{n \in \mathbb{Z}} F^{\mathfrak{ns}}_{1,1+2i;2n}, \quad i = 0, \dots, 2m.$$

For each $i \in \{0, \dots, m-1\}$, we define

$$\mathcal{SX}^{+}_{i+1} := \ker\, Q^{\mathfrak{ns}[2i+1]}_{-}|_{\mathcal{V}^{\mathfrak{ns}}_{L+\gamma_i}}$$
$$\mathcal{SX}^{-}_{m-i} := \ker\, Q^{\mathfrak{ns}[2(m-i)]}_{-}|_{\mathcal{V}^{\mathfrak{ns}}_{L+\gamma_{2m-i}}}.$$

By Proposition 10.2.3, we have the following decomposition as $\mathfrak{ns}$-modules

$$\mathcal{SX}^{+}_{i+1} \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (2n+1) L^{\mathfrak{ns}}(h^{\mathfrak{ns}}_{1,2i+1;-2n})$$
$$\mathcal{SX}^{-}_{m-i} \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 1}} (2n) L^{\mathfrak{ns}}(h^{\mathfrak{ns}}_{1,2(m-i);-2n+1}),$$

and

$$\mathcal{SX}^+_{i+1} \simeq \mathcal{V}^{\mathfrak{ns}}_{L+\gamma_{2m-i}}/\mathcal{SX}^-_{m-i}$$
$$\mathcal{SX}^-_{m-i} \simeq \mathcal{V}^{\mathfrak{ns}}_{L+\gamma_i}/\mathcal{SX}^+_{i+1}.$$

For $i = m$, we define

$$\mathcal{SX}^+_{m+1} := \mathcal{V}^{\mathfrak{ns}}_{L+\gamma_m} \simeq \bigoplus_{n\in\mathbb{Z}_{\geq 0}} (2n+1)L^{\mathfrak{ns}}(h^{\mathfrak{ns}}_{1,2m+1;-2n}).$$

Define the following vertex superalgebra

$$\mathcal{SW}(m) := \ker\, Q^{\mathfrak{ns}}_-|_{\mathcal{V}^{\mathfrak{ns}}_L}.$$

Proposition 11.1.2 ([2])**.** *$\mathcal{SW}(m)$ has the structure of a vertex operator superalgebra.*

We call this vertex operator superalgebra the $N = 1$ triplet vertex operator superalgebra.

We define the following three elements in $\mathcal{V}^{\mathfrak{ns}}_L$

$$W^- := |\beta_{1,1;-2}\rangle\,, \quad W^0 := Q^{\mathfrak{ns}}_+ W^-, \quad W^+ := Q^{\mathfrak{ns}}_+ \circ Q^{\mathfrak{ns}}_+ W^-.$$

These elements have the same L_0-weight $h_{1,3} = 2m + \frac{1}{2}$. We define the following three elements

$$\widehat{W}^- := b_{-\frac{1}{2}}\,|\beta_{1,1;-2}\rangle\,, \quad \widehat{W}^0 := Q^{\mathfrak{ns}}_+ \widehat{W}^-, \quad \widehat{W}^+ := Q^{\mathfrak{ns}}_+ \circ Q^{\mathfrak{ns}}_+ \widehat{W}^-.$$

These elements have the same L_0-weight $2m + 1$.

Theorem 11.1.3 ([2])**.** *The $N = 1$ triplet vertex operator superalgebra $\mathcal{SW}(m)$ is generated by $Y(W^\pm; z), Y(W^0; z), G(z)$. Furthermore $\mathcal{SW}(m)$ is strongly generated by*

$$G(z),\ T(z),\ Y(W^\pm; z),\ Y(W^0; z),\ Y(\widehat{W}^\pm; z),\ Y(\widehat{W}^0; z).$$

Theorem 11.1.4 ([2])**.** *The $N = 1$ triplet vertex operator superalgebra $\mathcal{SW}(m)$ is C_2-cofinite.*

Let $A(\mathcal{SW}(m))$ be the Zhu-algebra [68] of $\mathcal{SW}(m)$.

Proposition 11.1.5 ([2])**.** *Zhu-algebra $A(\mathcal{SW}(m))$ is generated by $[\widehat{W}^+], [\widehat{W}^0], [\widehat{W}^-]$ and $[T]$. The generators satisfy the following relations:*

$$\begin{aligned}
[\widehat{W}^0] * [\widehat{W}^-] - [\widehat{W}^-] * [\widehat{W}^0] &= -2f([T])[\widehat{W}^-],\\
[\widehat{W}^0] * [\widehat{W}^+] - [\widehat{W}^+] * [\widehat{W}^0] &= 2f([T])[\widehat{W}^+],\\
[\widehat{W}^+] * [\widehat{W}^-] - [\widehat{W}^-] * [\widehat{W}^+] &= -2f([T])[\widehat{W}^0],\\
[\widehat{W}^0] * [\widehat{W}^0] &= g([T]),\\
[\widehat{W}^+] * [\widehat{W}^+] &= 0,\\
[\widehat{W}^-] * [\widehat{W}^-] &= 0,
\end{aligned}$$

where $f([T])$ and $g([T])$ are non-trivial polynomials of $[T]$.

Proposition 11.1.6 ([2]).

1. *For each $0 \le i \le m$, $\mathcal{SX}^+_{i+1}$ becomes an simple $\mathcal{SW}(m)$-module. The highest weight space of $\mathcal{SX}^+_{i+1}$ becomes a one dimensional simple $A(\mathcal{SW}(m))$-module.*

2. *For each $0 \le j \le m-1$, $\mathcal{SX}^-_{m-j}$ becomes an simple $\mathcal{SW}(m)$-module. The highest weight space of $\mathcal{SX}^-_{m-i}$ becomes a two dimensional simple $A(\mathcal{SW}(m))$-module.*

Proposition 11.1.7 ([2]). *For each $0 \le i \le m-1$, the simple $\mathcal{V}^{\mathfrak{ns}}_L$-modules $\mathcal{V}^{\mathfrak{ns}}_{L+\gamma_i}$ and $\mathcal{V}^{\mathfrak{ns}}_{L+\gamma_{2m-i}}$ become $\mathcal{SW}(m)$-modules and satisfy the following exact sequences:*

$$0 \to \mathcal{SX}^+_{i+1} \to \mathcal{V}^{\mathfrak{ns}}_{L+\gamma_i} \to \mathcal{SX}^-_{m-i} \to 0,$$
$$0 \to \mathcal{SX}^-_{m-i} \to \mathcal{V}^{\mathfrak{ns}}_{L+\gamma_{2m-i}} \to \mathcal{SX}^+_{i+1} \to 0.$$

Theorem 11.1.8 ([2]). *All simple $\mathcal{SW}(m)$-modules are completed by $2m+1$ simple $\mathcal{SW}(m)$-modules*

$$\{\mathcal{SX}^-_i \ : \ 1 \le i \le m\} \cup \{\mathcal{SX}^+_i \ : \ 1 \le i \le m+1\}.$$

For the dimension of Zhu-algebra $A(\mathcal{SW}(m))$, the following theorem holds.

Theorem 11.1.9 ([2]).

$$\dim_{\mathbb{C}} A(\mathcal{SW}(m)) = 6m+1.$$

Remark 11.1.10. *In* [6], *the structure of $A(\mathcal{SW}(m))$ was determined. We will review this theorem in Subsection 11.4.*

11.2 The block decomposition of $\mathcal{SC}_m$

Definition 11.2.1. *Let $\mathcal{SC}_m$ be the abelian category of weak $\mathcal{SW}(m)$-modules.*

For any M in $\mathcal{SC}_m$, let M^* be the contragredient of M. Note that $\mathcal{SC}_m$ is closed under contragredient.

We denote $\mathrm{Ext}^n_{\mathcal{SC}_m}(\bullet, \bullet)$ by the n-th Ext groups in the abelian category $\mathcal{SC}_m$. The following theorem can be proved in the same way as Theorem 4.4. in [1].

Theorem 11.2.2. *For all $i \ne j$*

$$\mathrm{Ext}^1_{\mathcal{SC}_m}(\mathcal{SX}^+_{i+1}, \mathcal{SX}^+_{j+1}) = \mathrm{Ext}^1_{\mathcal{SC}_m}(\mathcal{SX}^-_{m-i}, \mathcal{SX}^-_{m-j}) = 0,$$
$$\mathrm{Ext}^1_{\mathcal{SC}_m}(\mathcal{SX}^+_{i+1}, \mathcal{SX}^-_{m-j}) = 0.$$

For each $0 \le i \le m-1$ we denote by C_{i+1} the full abelian subcategory of $\mathcal{SC}_m$ such that

$$M \in C_{i+1} \ \Leftrightarrow \ \text{every composition factors of } M \text{ are given by } \mathcal{SX}^+_{i+1}, \mathcal{SX}^-_{m-i}.$$

We denote by C_{m+1} the full abelian subcategory of $\mathcal{SC}_m$ such that

$$M \in C_{m+1} \ \Leftrightarrow \ \text{every composition factors of } M \text{ are given by } \mathcal{SX}^+_{m+1}.$$

By Theorem 11.2.2, we have the following theorem.

Theorem 11.2.3. *The abelian category $\mathcal{SC}_m$ has the following block decomposition*

$$\mathcal{SC}_m = \bigoplus_{i=0}^{m} C_{i+1}.$$

11.3 Construction of logarithmic modules $\mathcal{SP}^{\pm}_{\bullet}$

Similar to the arguments in Section 5.1, we have the following proposition.

Proposition 11.3.1. *For $0 \le i \le m-1$, the screening operators $Q_-^{[2i+1]}, Q_-^{[2(m-i)]}$ are $\mathcal{SW}(m)$ homomorphisms, that is, for $A \in \mathcal{SW}(m)$*

$$[Q_-^{\mathfrak{ns}[2i+1]}, Y(A;z)] = 0, \qquad [Q_-^{\mathfrak{ns}[2(m-i)]}, Y(A;z)] = 0.$$

For each $0 \le i \le m-1$, we set

$$\mathcal{SP}_{\gamma_i} = \mathcal{V}^{\mathfrak{ns}}_{L+\gamma_i} \oplus \mathcal{V}^{\mathfrak{ns}}_{L+\gamma_{2m-i}} \in C_{i+1}$$

and let $(\mathcal{SP}_{\gamma_i}, Y_{\mathcal{SP}_{\gamma_i}})$ be the ordinary $\mathcal{SW}(m)$-module. Let us consider the logarithmic deformation by the screening currents $Q_-^{\mathfrak{ns}[2i+1]}(z)$ and $Q_-^{\mathfrak{ns}[2(m-i)]}$. Note that, by Proposition 11.3.1, for $A \in \mathcal{SW}(m)$ two fields $\Delta_{Q_-^{\mathfrak{ns}[2i+1]}}(Y(A;z))$ and $\Delta_{Q_-^{\mathfrak{ns}[2(m-i)]}}(Y(A;z))$ does not contain a $\log z$ term. By using Theorem 5.1.4, we can define the structure of a logarithmic module on $\mathcal{SP}_{\gamma_i}$.

Definition 11.3.2. *For $1 \le i \le m-1$, we define $\mathcal{SW}(m)$-modules $(\mathcal{SP}^+_{i+1}, J^+_{i+1})$ and $(\mathcal{SP}^-_{m-i}, J^-_{m-i})$ as follows. As the vector spaces*

$$\mathcal{SP}^+_{i+1} = \mathcal{SP}^-_{m-i} = \mathcal{SP}_{\gamma_i}$$

and the module actions are defined by

$$J^+_{i+1}(A;z) = \begin{cases} Y_{\mathcal{SP}_{\gamma_i}}(A;z) + \Delta_{Q_-^{\mathfrak{ns}[2i+1]}}(Y_{\mathcal{SP}_{\gamma_i}}(A;z)) & \text{on } \mathcal{V}^{\mathfrak{ns}}_{L+\gamma_i} \\ Y_{\mathcal{SP}_{\gamma_i}}(A;z) & \text{on } \mathcal{V}^{\mathfrak{ns}}_{L+\gamma_{2m-i}}, \end{cases}$$

$$J^-_{m-i}(A;z) = \begin{cases} Y_{\mathcal{SP}_{\gamma_i}}(A;z) + \Delta_{Q_-^{\mathfrak{ns}[2(m-i)]}}(Y_{\mathcal{SP}_{\gamma_i}}(A;z)) & \text{on } \mathcal{V}^{\mathfrak{ns}}_{L+\gamma_{2m-i}} \\ Y_{\mathcal{SP}_{\gamma_i}}(A;z) & \text{on } \mathcal{V}^{\mathfrak{ns}}_{L+\gamma_i}. \end{cases}$$

for any $A \in \mathcal{SW}(m)$.

For any $A \in \mathcal{SW}(m)$, we use the following notation

$$J^+_{i+1}(A_n) := \oint_{z=0} J^+_{i+1}(A;z) z^{n+\Delta_A-1} \mathrm{d}z,$$
$$J^-_{m-i}(A_n) := \oint_{z=0} J^-_{m-i}(A;z) z^{n+\Delta_A-1} \mathrm{d}z.$$

For the energy momentum tensor $T(z) = Y(T;z)$, we have the following proposition.

Proposition 11.3.3.

1. *On the $\mathcal{V}^{\mathfrak{ns}}_{L+\gamma_{2m-i}}$, we have*

$$J^+_{i+1}(T;z) = T(z) + \frac{Q_-^{\mathfrak{ns}[2(m-i)]}(z)}{z},$$
$$J^+_{i+1}(L_0) = L_0 + Q_-^{\mathfrak{ns}[2(m-i)]}.$$

2. On the $\mathcal{V}^{\mathfrak{ns}}_{L+\gamma_i}$, we have

$$J^-_{m-i}(T;z) = T(z) + \frac{Q^{\mathfrak{ns}[2i+1]}_-(z)}{z},$$
$$J^-_{m-i}(L_0) = L_0 + Q^{\mathfrak{ns}[2i+1]}_-.$$

By Propositions 10.2.3 and 11.3.3, we see that $\mathcal{SW}(m)$-modules $\mathcal{SP}^\pm_\bullet$ are indecomposable.

11.4 The structure of the Zhu-algebra $A(\mathcal{SW}(m))$

In the following, we review the structure theorem of Zhu-algebra $A(\mathcal{SW}(m))$ which was proved in [6], using the logarithmic modules $\mathcal{SP}^+_{\gamma_i}$, $0 \le i \le m-1$.

We set $R = A(\mathcal{SW}(m))$. For $0 \le i \le m-1$, let $V_{i+1} = H(\mathcal{SP}^+_{i+1})$ be the highest weight space of $\mathcal{SW}(m)$-module $\mathcal{SP}^+_{i+1}$:

$$V_{i+1} = H(\mathcal{SP}^+_{i+1}) = \mathbb{C}\left|\beta_{1,2(m-i);1}\right\rangle + \mathbb{C}\left|\beta_{1,2i+1}\right\rangle.$$

V_{i+1} becomes a left R-module. Then, by Proposition 11.3.3, the action of L_0 on V_{i+1} is given by

$$(J^+_{i+1}(L_0) - h^{\mathfrak{ns}}_{1,2i+1})\left|\beta_{1,2(m-i);1}\right\rangle = \left|\beta_{1,2i+1}\right\rangle.$$

Therefore, the image of the representation of R, $\Phi_{V_{i+1}} : R \to M_2(\mathbb{C})$ contains

$$I^+_{i+1} := \left\{\begin{pmatrix} a & 0 \\ b & a\end{pmatrix};\quad a, b \in \mathbb{C}\right\}.$$

Let $W_{i+1} = H(\mathcal{SX}^-_{m-i})$ be the highest weight space of $\mathcal{SX}^-_{m-i}$. Since W_{i+1} is two dimensional irreducible R-module, the image of the representation of R, $\Phi_{W_{i+1}} : R \to M_2(\mathbb{C})$, becomes

$$I^-_{m-i} := \mathrm{Im}\Phi_{W_{i+1}} = M_2(\mathbb{C}).$$

Let $V_{m+1} = H(\mathcal{SX}^+_{m+1})$ be the highest weight space of $\mathcal{SX}^+_{m+1}$. Since V_{i+1} is the one dimensional R-module, the image of the representation of R, $\Phi_{V_{m+1}} : R \to \mathbb{C}$, becomes

$$I^+_{m+1} := \mathrm{Im}\Phi_{V_{m+1}} = \mathbb{C}.$$

Therefore, by Theorem 11.1.9, we have the following theorem [6].

Theorem 11.4.1 ([6])**.** *The Zhu algebra $A(\mathcal{SW}(m))$ is isomorphic to*

$$I = \bigoplus_{i=0}^{m} I^+_{i+1} \oplus \bigoplus_{i=0}^{m-1} I^-_{m-i}.$$

From Theorem 11.4.1, in particular we obtain the following proposition (see also Proposition 7.7 in [56]).

Proposition 11.4.2. *For* $0 \leq i \leq m-1$,

$$\mathrm{Ext}^1_{\mathcal{SC}_m}(\mathcal{SX}^+_{i+1}, \mathcal{SX}^+_{i+1}) = 0, \qquad \mathrm{Ext}^1_{\mathcal{SC}_m}(\mathcal{SX}^-_{m-i}, \mathcal{SX}^-_{m-i}) = 0.$$

Let us consider the structure of the logarithmic modules $\mathcal{SP}^\pm_\bullet$. By Proposition 11.4.2, we obtain the following proposition.

Proposition 11.4.3. *Let* $0 \leq i \leq m-1$.

1. *At least one of* $\mathcal{SP}^+_{i+1}/\mathcal{SX}^+_{i+1}$ *or* $\mathcal{SP}^{+*}_{i+1}/\mathcal{SX}^+_{i+1}$ *is indecomposable.*
2. *At least one of* $\mathcal{SP}^-_{m-i}/\mathcal{SX}^-_{m-i}$ *or* $\mathcal{SP}^{-*}_{m-i}/\mathcal{SX}^-_{m-i}$ *is indecomposable.*

Chapter 12

The structure of braided tensor category on $\mathcal{SC}_m$

Since the super triplet W algebra $\mathcal{SW}(m)$ is C_2-cofinite, Theorem 4.13 in [36] show that $\mathcal{SW}(m)$ has braided tensor category structure as developed in the series of papers [37, 38, 39, 40, 41, 42, 43, 44] (see also [14]). We denote $(\mathcal{SC}_m,\boxtimes)$ by the tensor category on $\mathcal{SC}_m$, where the unit object is given by $\mathcal{SX}^+_1$. In this chapter, we compute certain non-semisimple fusion rules and determine the structure of the projective covers of all simple modules.

12.1 The tensor product $\mathcal{SX}^-_1 \boxtimes \mathcal{SX}^-_1$

In the following, we abbreviate $h^{\mathfrak{ns}}_{r,s;n}$ as $h_{r,s;n}$.

Lemma 12.1.1. *For $i = 1,\dots,m$, the vector space $A_0((\mathcal{SX}^-_1 \boxtimes \mathcal{SX}^-_i)^*)$ is at most two dimensional. L_0 acts semisimply on $A_0((\mathcal{SX}^-_1 \boxtimes \mathcal{SX}^-_i)^*)$ and any L_0 eigenvalue of this space is contained in $\{h_{1,2i-1}, h_{1,2i+1}\}$, where $h_{1,2i-1}$ and $h_{1,2i+1}$ are the L_0 weights of the highest weight space of $\mathcal{SX}^+_{i-1}$ and $\mathcal{SX}^+_{i+1}$, respectively.*

Proof. We will only prove the first case. The other cases can proved in the same way, so we omit the proofs.

Let ψ^*, ϕ_1 and ϕ_2 be arbitrary elements of $A_0((\mathcal{SX}^-_1 \boxtimes \mathcal{SX}^-_i)^*)$, $\mathcal{SX}^-_1$ and $\mathcal{SX}^-_i$, respectively. For $1 \le j \le m$, let $\{v^+_j, v^-_j\}$ be a basis of the highest weight space of $\mathcal{SX}^-_j$ such that

$$\widehat{W}^{\pm}[0]v^{\pm}_j = 0, \qquad \widehat{W}^{\pm}[0]v^{\mp}_j \in \mathbb{C}^{\times}v^{\pm}_j.$$

For $n \geq 1$, let $w^{(n)}_k (k = 1\dots, 2n+2)$ be the $U(\mathfrak{ns})$-highest weight vectors of the vector subspace $(2n+2)L^{\mathfrak{ns}}(h_{2n+2,2i}) \subset \mathcal{SX}^-_i$. By Lemma 8.1.6 and the relation

$$\{\frac{4t}{t^2-1}G^4_{-\frac{1}{2}} + \frac{t+1}{t-1}G_{-\frac{1}{2}}G_{-\frac{3}{2}} + \frac{t-1}{t+1}G_{-\frac{3}{2}}G_{-\frac{1}{2}}\}v^{\pm}_1 = 0, \quad t = -2m-1, \tag{12.1.1}$$

we see that, depending on whether ψ^* is even or odd, the value $\langle \psi^*, \phi_1 \otimes \phi_2\rangle$ is determined by the values

$$\begin{aligned}
&\langle \psi^*, v^{\epsilon}_1 \otimes v^{\epsilon'}_i\rangle, && \langle \psi^*, L_{-1}v^{\epsilon}_1 \otimes v^{\epsilon'}_i\rangle,\\
&\langle \psi^*, v^{\epsilon}_1 \otimes w^{(2n)}_k\rangle, && \langle \psi^*, L_{-1}v^{\epsilon}_1 \otimes w^{(2n)}_k\rangle,\\
&\langle \psi^*, G_{-\frac{1}{2}}v^{\epsilon}_1 \otimes w^{(2n-1)}_k\rangle, && \langle \psi^*, G^3_{-\frac{1}{2}}v^{\epsilon}_1 \otimes w^{(2n-1)}_k\rangle.
\end{aligned}$$

or

$$\begin{aligned}
&\langle \psi^*, G_{-\frac{1}{2}}v^{\epsilon}_1 \otimes v^{\epsilon'}_i\rangle, && \langle \psi^*, G^3_{-\frac{1}{2}}v^{\epsilon}_1 \otimes v^{\epsilon'}_i\rangle,\\
&\langle \psi^*, v^{\epsilon}_1 \otimes w^{(2n-1)}_k\rangle, && \langle \psi^*, L_{-1}v^{\epsilon}_1 \otimes w^{(2n-1)}_k\rangle,\\
&\langle \psi^*, G_{-\frac{1}{2}}v^{\epsilon}_1 \otimes w^{(2n)}_k\rangle, && \langle \psi^*, G^3_{-\frac{1}{2}}v^{\epsilon}_1 \otimes w^{(2n)}_k\rangle.
\end{aligned}$$

for some finite $n \geq 1$ and k, where $\epsilon = \pm$ and $\epsilon' = \pm$.

By using Lemma 8.1.6 and (12.1.1), we have

$$\begin{pmatrix} \langle L_0\psi^*, v_1^\epsilon \otimes w_k^{(n)} \rangle \\ \langle L_0\psi^*, (L_{-1}v_1^\epsilon) \otimes w_k^{(n)} \rangle \end{pmatrix} = \begin{pmatrix} h_{2,2} + h_{2n+2,2i} & 1 \\ \frac{2m^2}{2m+1} h_{2n+2,2i} & h_{2,2} + h_{2n+2,2i} + 1 - \frac{(2m+1)^2+1}{2(2m+1)} \end{pmatrix} \begin{pmatrix} \langle \psi^*, v_1^\epsilon \otimes w_k^{(n)} \rangle \\ \langle \psi^*, (L_{-1}v_1^\epsilon) \otimes w_k^{(n)} \rangle \end{pmatrix}.$$

We see that the eigenvalues of this matrix do not correspond to L_0 weight of the highest weight of all simple $\mathcal{SW}(m)$-modules. Thus we have

$$\langle \psi^*, v_1^\epsilon \otimes w_k^{(n)} \rangle = 0, \qquad \langle \psi^*, (L_{-1}v_1^\epsilon) \otimes w_k^{(n)} \rangle = 0$$

for any $n \geq 1$. Similary we can show that

$$\langle \psi^*, G_{-\frac{1}{2}} v_1^\epsilon \otimes w_k^{(n)} \rangle = 0, \qquad \langle \psi^*, G^3_{-\frac{1}{2}} v_1^\epsilon \otimes w_k^{(n)} \rangle = 0$$

for any $n \geq 1$.

Therefore the value $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the values

$$\langle \psi^*, v_1^\epsilon \otimes v_i^{\epsilon'} \rangle, \qquad \langle \psi^*, L_{-1} v_1^\epsilon \otimes v_i^{\epsilon'} \rangle$$

or

$$\langle \psi^*, G_{-\frac{1}{2}} v_1^\epsilon \otimes v_i^{\epsilon'} \rangle, \qquad \langle \psi^*, G^3_{-\frac{1}{2}} v_1^\epsilon \otimes v_i^{\epsilon'} \rangle.$$

Let us assume that ψ^* is odd. Then $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the values

$$\langle \psi^*, G_{-\frac{1}{2}} v_1^\epsilon \otimes v_i^{\epsilon'} \rangle, \qquad \langle \psi^*, G^3_{-\frac{1}{2}} v_1^\epsilon \otimes v_i^{\epsilon'} \rangle,$$

By using Lemma 8.1.6 and (12.1.1), we have

$$\begin{pmatrix} \langle L_0\psi^*, G_{-\frac{1}{2}} v_1^\epsilon \otimes v_i^{\epsilon'} \rangle \\ \langle L_0\psi^*, G^3_{-\frac{1}{2}} v_1^\epsilon \otimes v_i^{\epsilon'} \rangle \end{pmatrix} = \begin{pmatrix} h_{2,2} + h_{2,2i} + \frac{1}{2} & 1 \\ \frac{m^2}{2m+1} + \frac{2m^2}{2m+1} h_{2,2i} & h_{2,2} + h_{2,2i} + \frac{3}{2} - \frac{(2m+1)^2+1}{2(2m+1)} \end{pmatrix} \begin{pmatrix} \langle \psi^*, G_{-\frac{1}{2}} v_1^\epsilon \otimes v_i^{\epsilon'} \rangle \\ \langle \psi^*, G^3_{-\frac{1}{2}} v_1^\epsilon \otimes v_i^{\epsilon'} \rangle \end{pmatrix}.$$

We see that this matrix is diagonalizable and eigenvalues are given by $\{h_{1,2i+1}, h_{3,2i-1}\}$. Note that the eigenvalue $h_{3,2i-1}$ does not correspond to any L_0 eigenvalues of the highest weight space of the simple $\mathcal{SW}(m)$-modules. Thus the L_0-weight of ψ^* is $h_{1,2i+1}$.

Next let us assume that ψ^* is even. Then $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the values

$$\langle \psi^*, v_1^\epsilon \otimes v_i^{\epsilon'} \rangle, \qquad \langle \psi^*, L_{-1} v_1^\epsilon \otimes v_i^{\epsilon'} \rangle,$$

By using Lemma 8.1.6 and (12.1.1), we have

$$\begin{pmatrix} \langle L_0\psi^*, v_1^\epsilon \otimes v_i^{\epsilon'} \rangle \\ \langle L_0\psi^*, (L_{-1}v_1^\epsilon) \otimes v_i^{\epsilon'} \rangle \end{pmatrix} = \begin{pmatrix} h_{2,2} + h_{2,2i} & 1 \\ \frac{2m^2}{2m+1} h_{2,2i} & h_{2,2} + h_{2,2i} + 1 - \frac{(2m+1)^2+1}{2(2m+1)} \end{pmatrix} \begin{pmatrix} \langle \psi^*, v_1^\epsilon \otimes v_i^{\epsilon'} \rangle \\ \langle \psi^*, (L_{-1}v_1^\epsilon) \otimes v_i^{\epsilon'} \rangle \end{pmatrix}.$$

We see that this matrix is diagonalizable and eigenvalues are given by $\{h_{1,2i-1}, h_{3,2i+1}\}$. Note that the eigenvalue $h_{3,2i+1}$ does not correspond to any L_0 eigenvalues of the highest weight space of the simple $\mathcal{SW}(m)$-modules. Thus the L_0-weight of ψ^* is $h_{1,2i-1}$.

Note that $\widehat{W}^{\pm}[0]$ acts trivially on the highest weight spaces of $\mathcal{SX}^+_{i-1}$ and $\mathcal{SX}^+_{i+1}$. Then we have

$$\widehat{W}^{\pm}[0]\psi^* = 0.$$

Thus, by Lemma 8.1.6, we see that $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the values

$$\langle \psi^*, v_1^+ \otimes v_i^- \rangle, \qquad \langle \psi^*, L_{-1}v_1^+ \otimes v_i^- \rangle$$

or

$$\langle \psi^*, G_{-\frac{1}{2}}v_1^+ \otimes v_i^- \rangle, \qquad \langle \psi^*, G^3_{-\frac{1}{2}}v_1^+ \otimes v_i^- \rangle.$$

Therefore, from the above results, we can see that the vector space $A_0((\mathcal{SX}^-_1 \boxtimes \mathcal{SX}^-_i)^*)$ is at most two dimensional. □

Remark 12.1.2. *In the proof of Lemma 12.1.1 we see in particular that if $i = 1$ and $L_0\psi^* = h_{1,1}\psi^* = 0$, then $G_{-\frac{1}{2}}\psi^* = 0$.*

For any $\beta \in A_m := \{\ \beta_{r,s;n} \mid r, s, n \in \mathbb{Z}\}$, let

$$\mathcal{V}^{\mathfrak{ns}}_{\beta+L} := \bigoplus_{n\in\mathbb{Z}} F^{\mathfrak{ns}}_{\beta+n\sqrt{2m+1}}$$

be a simple $\mathcal{V}^{\mathfrak{ns}}_L$-module. For any $\beta, \beta' \in A_m$, it can be proved easily that there are no $\mathcal{V}^{\mathfrak{ns}}_L$-module intertwining operators of type $\begin{pmatrix} \mathcal{V}^{\mathfrak{ns}}_{\beta''+L} \\ \mathcal{V}^{\mathfrak{ns}}_{\beta'+L}\ \mathcal{V}^{\mathfrak{ns}}_{\beta+L} \end{pmatrix}$ unless $\beta'' \equiv \beta' + \beta \bmod L$, and $\dim_{\mathbb{C}} I\begin{pmatrix} \mathcal{V}^{\mathfrak{ns}}_{\beta'+\beta+L} \\ \mathcal{V}^{\mathfrak{ns}}_{\beta'+L}\ \mathcal{V}^{\mathfrak{ns}}_{\beta+L} \end{pmatrix} = 1$. Let Y be the $\mathcal{V}^{\mathfrak{ns}}_L$-module intertwining operator of type $\begin{pmatrix} \mathcal{V}^{\mathfrak{ns}}_{\beta'+\beta+L} \\ \mathcal{V}^{\mathfrak{ns}}_{\beta'+L}\ \mathcal{V}^{\mathfrak{ns}}_{\beta+L} \end{pmatrix}$. Then, by restricting the action of $\mathcal{V}^{\mathfrak{ns}}_L$ to $\mathcal{SW}(m)$, Y defines a $\mathcal{SW}(m)$-module intertwining operator of type $\begin{pmatrix} \mathcal{V}^{\mathfrak{ns}}_{\beta'+\beta+L} \\ \mathcal{V}^{\mathfrak{ns}}_{\beta'+L}\ \mathcal{V}^{\mathfrak{ns}}_{\beta+L} \end{pmatrix}$. We denote this $\mathcal{SW}(m)$-module intertwining operator by $Y_{\beta',\beta}$.

Lemma 12.1.3.

$$I\begin{pmatrix} \mathcal{SX}^+_1 \\ \mathcal{SX}^-_1\ \mathcal{SX}^-_1 \end{pmatrix} \neq \emptyset, \qquad I\begin{pmatrix} \mathcal{SX}^+_2 \\ \mathcal{SX}^-_1\ \mathcal{SX}^-_1 \end{pmatrix} \neq \emptyset.$$

Proof. Let us consider $\mathcal{SW}(m)$-module intertwining operator $Y = Y_{\beta_1,\beta_2}$, where $\beta_1 = \beta_{2,2}$ and $\beta_2 = \beta_{1,2m-1;2}$. Then we have

$$\langle \beta_{1,2m;1} |\, Y(|\beta_{2,2}\rangle\,; z)\, |\beta_{1,2m-1;2}\rangle \neq 0.$$

Thus we have

$$I\begin{pmatrix} \mathcal{SX}^+_1 \\ \mathcal{SX}^-_1\ \mathcal{V}^{\mathfrak{ns}}_{\beta_2} \end{pmatrix} \neq \emptyset. \tag{12.1.2}$$

Note the following exact sequence

$$0 \to \mathcal{SX}_m^+ \to \mathcal{V}_{\beta_2}^{\mathfrak{ns}} \to \mathcal{SX}_1^- \to 0.$$

Then we have the following exact sequence

$$\mathcal{SX}_1^- \boxtimes \mathcal{SX}_m^+ \to \mathcal{SX}_1^- \boxtimes \mathcal{V}_{\beta_2}^{\mathfrak{ns}} \to \mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^- \to 0.$$

From this exact sequence, we have the following exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathcal{SW}(m)}(\mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^-, \mathcal{SX}_1^+) \to \mathrm{Hom}_{\mathcal{SW}(m)}(\mathcal{SX}_1^- \boxtimes \mathcal{V}_{\beta_2}^{\mathfrak{ns}}, \mathcal{SX}_1^+) \\ &\to \mathrm{Hom}_{\mathcal{SW}(m)}(\mathcal{SX}_1^- \boxtimes \mathcal{SX}_m^+, \mathcal{SX}_1^+). \end{aligned} \tag{12.1.3}$$

By Lemma 12.1.1, we have

$$\mathrm{Hom}_{\mathcal{SW}(m)}(\mathcal{SX}_1^- \boxtimes \mathcal{SX}_m^+, \mathcal{SX}_1^+) = 0.$$

Therefore by (12.1.2) and (12.1.3), we obatin

$$\mathrm{Hom}_{\mathcal{SW}(m)}(\mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^-, \mathcal{SX}_1^+) \neq 0.$$

The second equation can be shown similarly by considering the intertwining operator $Y_{\beta_1',\beta_2'}$, where $(\beta_1', \beta_2') = (\beta_{2,2}, \beta_{2,2})$. □

By Lemmas 12.1.1 and 12.1.3, we have the following proposition (see Remark 12.1.2).

Proposition 12.1.4.

$$\mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^- = \mathcal{SX}_1^+ \oplus \Gamma(\mathcal{SX}_2^+), \tag{12.1.4}$$

where $\Gamma(\mathcal{SX}_2^+)$ *is a highest weight module such that* $\mathrm{top}(\Gamma(\mathcal{SX}_2^+)) = \mathcal{SX}_2^+$.

12.2 Four point functions

In the next section, we will show the self-duality of the simple module $\mathcal{SX}_1^-$. Before that, we introduce four point functions which satisfy a fourth order Fuchsian differential equation, and examine monodromy property of these four points function by using Dotsenko-Fateev integrals [17],[18],[28],[62].

Let a, b, c, a', b', c' be generic complex numbers and let

$$U(u, v; z) := u^a (u-1)^b (u-z)^c v^{a'} (v-1)^{b'} (v-z)^{c'} (u-v)^{-2},$$

where $z \in \mathbb{R}$ such that $0 < z < 1$. Let us consider the following Dotsenko-Fateev integrals

$$I_i(z) := \int_{[\Delta_i]} U(u, v; z) \mathrm{d}u \mathrm{d}v, \qquad (i = 1, 2, 3, 4),$$

where $[\Delta_i]$ are regularized cycles associated to the following regions

$$\begin{aligned} &\Delta_1 := \{1 < u < v\}, &&\Delta_2 := \{0 < u < 1, 1 < v\}, \\ &\Delta_3 := \{1 < u, 0 < v < z\}, &&\Delta_4 := \{0 < u < v, v < z\}. \end{aligned}$$

Remark 12.2.1. *In [62], the meromorphic continuation of the Dotsenko-Fateev integral*

$$\int_{[\Delta_i]} u^a(u-1)^b(u-z)^c v^{a'}(v-1)^{b'}(v-z)^{c'}(u-v)^{2\gamma}F(u,v)\mathrm{d}u\mathrm{d}v$$

is constructed, where $F(u,v)$ is any symmetric polynomial of u and v. It is proved that $\gamma=-1$ is an apparent singularity and is removable.

Using similar methods in [17], we obtain the following relations:

$$\begin{aligned}I_1(z) = & \frac{s(a)s(a')}{s(b+c)s(b'+c')}I_1(1-z) - \frac{s(c)s(a')}{s(b+c)s(b'+c')}I_2(1-z)\\ & - \frac{s(a)s(c')}{s(b+c)s(b'+c')}I_3(1-z) + \frac{s(c)s(c')}{s(b+c)s(b'+c')}I_4(1-z),\end{aligned}$$

$$\begin{aligned}I_2(z) = & -\frac{s(a+b+c)s(a')}{s(b+c)s(b'+c')}I_1(1-z) - \frac{s(b)s(a')}{s(b+c)s(b'+c')}I_2(1-z)\\ & + \frac{s(a+b+c)s(c')}{s(b+c)s(b'+c')}I_3(1-z) + \frac{s(b)s(c')}{s(b+c)s(b'+c')}I_4(1-z),\end{aligned}$$

$$\begin{aligned}I_3(z) = & -\frac{s(a)s(a'+b'+c')}{s(b+c)s(b'+c')}I_1(1-z) + \frac{s(c)s(a'+b'+c')}{s(b+c)s(b'+c')}I_2(1-z)\\ & - \frac{s(a)s(b')}{s(b+c)s(b'+c')}I_3(1-z) + \frac{s(c)s(b')}{s(b+c)s(b'+c')}I_4(1-z),\end{aligned}$$

$$\begin{aligned}I_4(z) = & \frac{s(a+b+c)s(a'+b'+c')}{s(b+c)s(b'+c')}I_1(1-z) + \frac{s(b)s(a'+b'+c')}{s(b+c)s(b'+c')}I_2(1-z)\\ & + \frac{s(a+b+c)s(b')}{s(b+c)s(b'+c')}I_3(1-z) + \frac{s(b)s(c')}{s(b+c)s(b'+c')}I_4(1-z),\end{aligned}$$

where $s(x)=\sin(\pi x)$.

For $r,s\in\mathbb{Z}$ and $t\in\mathbb{C}^\times$, we set

$$\widetilde{\beta}_{r,s} := \frac{(1-r)}{2}\beta_+ + \frac{(1-s)}{2}\beta_- + t.$$

For $i=1,2,3,4$, let us consider the following 4-point function

$$\begin{aligned}\widetilde{\Psi}_i(z,t) :&= \int_{[\Delta_i]} \langle \beta_{2,2}+2t|\, Q_-^{\mathrm{ns}}(u)Q_+^{\mathrm{ns}}(v)V_{\widetilde{\beta}_{2,2}}(1)V_{\widetilde{\beta}_{2,2}}(z)\,|\widetilde{\beta}_{2,2}\rangle\,\mathrm{d}u\mathrm{d}v\\ &= \int_{[\Delta_i]} u^{\beta_- \widetilde{\beta}_{2,2}}(u-1)^{\beta_- \widetilde{\beta}_{2,2}}(u-z)^{\beta_- \widetilde{\beta}_{2,2}}\\ &\qquad \times v^{\beta_+ \widetilde{\beta}_{2,2}}(v-1)^{\beta_+ \widetilde{\beta}_{2,2}}(v-z)^{\beta_+ \widetilde{\beta}_{2,2}}(u-v)^{-2}\mathrm{d}u\mathrm{d}v.\end{aligned}$$

By using the formulas of Dotsenko-Fateev integral [18],[28], we see that $\widetilde{\Psi}_i(z)$ is finite and non-trivial in the limit of $t=0$. Let

$$\Psi_i(z) := \lim_{t\to 0}\widetilde{\Psi}_i(z,t)$$

for $i = 1, 2, 3, 4$. Then, from the above four relations for $I_i(z)$, we obtain

$$\Psi_1(z) = \frac{1}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_1(1-z) - \frac{1}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_2(1-z) - \frac{1}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_3(1-z) + \frac{1}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_4(1-z),$$

$$\Psi_2(z) = -\frac{1}{2}\frac{s(3\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_1(1-z) - \frac{1}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_2(1-z) + \frac{1}{2}\frac{s(3\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_3(1-z) + \frac{1}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_4(1-z),$$

$$\Psi_3(z) = -\frac{3}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_1(1-z) + \frac{3}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_2(1-z) - \frac{1}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_3(1-z) + \frac{1}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_4(1-z),$$

$$\Psi_4(z) = \frac{3}{2}\frac{s(3\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_1(1-z) + \frac{3}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_2(1-z) + \frac{1}{2}\frac{s(3\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_3(1-z) + \frac{1}{2}\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}\Psi_4(1-z).$$

From these relations, we obtain

$$\Psi_1(z) + \Psi_3(z) = -\frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}(\Psi_1(1-z) + \Psi_3(1-z)) + \frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}(\Psi_2(1-z) + \Psi_4(1-z)), \quad (12.2.1)$$

$$\Psi_2(z) + \Psi_4(z) = \frac{s(3\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}(\Psi_1(1-z) + \Psi_3(1-z)) + \frac{s(\beta_-\beta_{2,2})}{s(2\beta_-\beta_{2,2})}(\Psi_2(1-z) + \Psi_4(1-z)). \quad (12.2.2)$$

Similar to the arguments in [45], we can see that each Ψ_i satisfies the following Fuchsian differential equation

$$\Big(\frac{d^4}{dz^4} + \frac{p_3(z)}{z(z-1)}\frac{d^3}{dz^3} + \frac{p_2(z)}{z^2(z-1)^2}\frac{d^2}{dz^2} + \frac{p_1(z)}{z^3(z-1)^3}\frac{d}{dz} + \frac{p_0(z)}{z^4(z-1)^4}\Big)\Psi_i(z) = 0 \quad (12.2.3)$$

where

$$
\begin{aligned}
p_0(z) &= \frac{3m^4}{(2m+1)^4}\big((16m^3-8m^2-16m-4)z^2 \\
&\qquad + (-16m^3+8m^2+16m+4)z + (3m^4+12m^3+2m^2-4m-1)\big) \\
p_1(z) &= \frac{2}{(2m+1)^3}\big((16m^5+48m^4+56m^3+34m^2+12m+2)z^3 \\
&\qquad + (-24m^5-72m^4-84m^3-51m^2-18m-3)z^2 \\
&\qquad + (-12m^6-8m^5+12m^3+13m^2+6m+1)z \\
&\qquad + (6m^6+8m^5+12m^4+8m^3+2m^2)\big) \\
p_2(z) &= \frac{2}{(2m+1)^2}\big((8m^4+32m^3+44m^2+28m+7)z^2 \\
&\quad + (-8m^4-32m^3-44m^2-28m-7)z + (-m^4+2m^3+5m^2+4m+1)\big) \\
p_3(z) &= \frac{4(m+1)^2(2z-1)}{(2m+1)}
\end{aligned}
$$

The Riemann scheme of the Fuchsian differential equation (12.2.3) is given by

$$
\begin{bmatrix}
0 & 1 & \infty \\
-\frac{3m^2}{2m+1} & -\frac{3m^2}{2m+1} & 0 \\
\frac{m^2}{2m+1} & \frac{m^2}{2m+1} & \frac{1}{2m+1} \\
\frac{1-3m^2}{2m+1} & \frac{1-3m^2}{2m+1} & \frac{4m^2}{2m+1} \\
\frac{m^2+4m+1}{2m+1} & \frac{m^2+4m+1}{2m+1} & 2m+1
\end{bmatrix}.
$$

Note that

$$
\begin{aligned}
h_{1,1} - 2h_{2,2} &= -\frac{3m^2}{2m+1}, \\
h_{3,3} - 2h_{2,2} &= \frac{m^2}{2m+1}, \\
h_{1,3} + \frac{1}{2} - 2h_{2,2} &= \frac{1-3m^2}{2m+1}, \\
h_{3,1} + \frac{1}{2} - 2h_{2,2} &= \frac{m^2+4m+1}{2m+1}.
\end{aligned}
$$

Let ρ_i be the characteristic exponent of $\Psi_i(z)$ around $z=0$. Then we have

$$
\begin{aligned}
\rho_1 &= h_{3,3} - 2h_{2,2}, \\
\rho_2 &= h_{3,1} + \frac{1}{2} - 2h_{2,2} \\
\rho_3 &= h_{1,3} + \frac{1}{2} - 2h_{2,2} \\
\rho_4 &= h_{1,1} - 2h_{2,2}.
\end{aligned}
$$

12.3 Self duality of the simple module $\mathcal{S}\mathcal{X}_1^-$

Theorem 12.3.1. *$\mathcal{S}\mathcal{X}_1$ is rigid and self-dual.*

Proof. We show the rigidity of $\mathcal{SX}_1^-$ using the methods detailed in [15] and [56] (cf.[64]). By Proposition 12.1.4, we have homomorphisms

$$\begin{aligned} &i_1 : \mathcal{SX}_1^+ \to \mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^-, \\ &p_1 : \mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^- \to \mathcal{SX}_1^+, \\ &i_3 : \Gamma(\mathcal{SX}_2^+) \to \mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^-, \\ &p_3 : \mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^- \to \Gamma(\mathcal{SX}_2^+) \end{aligned}$$

such that

$$p_1 \circ i_1 = \mathrm{id}_{\mathcal{SX}_1^+}, \qquad p_3 \circ i_3 = \mathrm{id}_{\Gamma(\mathcal{SX}_2^+)}$$

and

$$i_1 \circ p_1 + i_3 \circ p_3 = \mathrm{id}_{\mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^-}.$$

To prove that $\mathcal{SX}_1^-$ is rigid, it is sufficient to prove that the homomorphisms $f, g : \mathcal{SX}_1^- \to \mathcal{SX}_1^-$ defined by the commutative diagrams

$$\begin{CD} \mathcal{SX}_1^- @>{r^{-1}}>> \mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^+ @>{\mathrm{id}\boxtimes i_1}>> \mathcal{SX}_1^- \boxtimes (\mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^-) \\ @V{f}VV @. @VV{\mathcal{A}}V \\ \mathcal{SX}_1^- @<{l}<< \mathcal{SX}_1^+ \boxtimes \mathcal{SX}_1^- @<{p_1\boxtimes\mathrm{id}}<< (\mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^-) \boxtimes \mathcal{SX}_1^- \end{CD}$$

and

$$\begin{CD} \mathcal{SX}_1^- @>{l^{-1}}>> \mathcal{SX}_1^+ \boxtimes \mathcal{SX}_1^- @>{i_1\boxtimes\mathrm{id}}>> (\mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^-) \boxtimes \mathcal{SX}_1^- \\ @V{g}VV @. @VV{\mathcal{A}^{-1}}V \\ \mathcal{SX}_1^- @<{r}<< \mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^+ @<{\mathrm{id}\boxtimes p_1}<< \mathcal{SX}_1 \boxtimes (\mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^-) \end{CD}$$

are non-zero. We only show $f \neq 0$. The proof of $g \neq 0$ is similar.

Let $\mathcal{Y}_{2\boxtimes 2}$, $\mathcal{Y}_{(2\boxtimes 2)\boxtimes 2}$ and $\mathcal{Y}_{2\boxtimes(2\boxtimes 2)}$ be the non-zero intertwining operators of type

$$\begin{pmatrix} \mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^- \\ \mathcal{SX}_1^- \;\; \mathcal{SX}_1^- \end{pmatrix}, \qquad \begin{pmatrix} (\mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^-) \boxtimes \mathcal{SX}_1^- \\ \mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^- \;\; \mathcal{SX}_1^- \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{SX}_1^- \boxtimes (\mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^-) \\ \mathcal{SX}_1^- \;\; \mathcal{SX}_1^- \boxtimes \mathcal{SX}_1^- \end{pmatrix},$$

respectively.

To prove $f \neq 0$, it is sufficient to show that the intertwining operator

$$\mathcal{Y}_{21}^2 = l_{\mathcal{SX}_1^-} \circ (p_1 \boxtimes \mathrm{id}_{\mathcal{SX}_1^-}) \circ \mathcal{A}_{\mathcal{SX}_1^-, \mathcal{SX}_1^-, \mathcal{SX}_1^-} \circ \mathcal{Y}_{2\boxtimes(2\boxtimes 2)} \circ (\mathrm{id}_{\mathcal{SX}_1^-} \otimes i_1)$$

is non-zero.

Define the following intertwining operator

$$\mathcal{Y}_{23}^2 = l_{\mathcal{SX}_1^-} \circ (p_3 \boxtimes \mathrm{id}_{\mathcal{SX}_1^-}) \circ \mathcal{A}_{\mathcal{SX}_1^-, \mathcal{SX}_1^-, \mathcal{SX}_1^-} \circ \mathcal{Y}_{2\boxtimes(2\boxtimes 2)} \circ (\mathrm{id}_{\mathcal{SX}_1^-} \otimes i_3).$$

Fix any highest weight vectors $v \in \mathcal{SX}_1^-[h_{2,2}]$, $v^* \in \mathcal{SX}_1^{-*}[h_{2,2}]$. Then, for some $x \in \mathbb{R}$ such that $1 > x > 1-x > 0$, we have

$$\begin{aligned}
&\langle v^*, \mathcal{Y}^2_{21}(v;1)(p_1 \circ \mathcal{Y}_{2\boxtimes 2})(v;x)v\rangle + \langle v^*, \mathcal{Y}^2_{23}(v;1)(p_3 \circ \mathcal{Y}_{2\boxtimes 2})(v;x)v\rangle \\
&= \langle v^*, \overline{l_{\mathcal{SX}_1^-} \circ (p_1 \boxtimes \mathrm{id}_{\mathcal{SX}_1^-}) \circ \mathcal{A}_{\mathcal{SX}_1^-,\mathcal{SX}_1^-,\mathcal{SX}_1^-}}\big(\mathcal{Y}_{2\boxtimes(2\boxtimes 2)}(v;1)\mathcal{Y}_{2\boxtimes 2}(v;x)v\big)\rangle \\
&= \langle v^*, \overline{l_{\mathcal{SX}_1^-} \circ (p_1 \boxtimes \mathrm{id}_{\mathcal{SX}_1^-})}\big(\mathcal{Y}_{(2\boxtimes 2)\boxtimes 2}(\mathcal{Y}_{2\boxtimes 2}(v;1-x)v;x)v\big)\rangle \\
&= \langle v^*, \overline{l_{\mathcal{SX}_1^-}}\big(\mathcal{Y}_{1\boxtimes 2}((p_1 \circ \mathcal{Y}_{2\boxtimes 2})(v;1-x)v;x)v\big)\rangle, \\
&= \langle v^*, Y_{\mathcal{SX}_1^-}\big((p_1 \circ \mathcal{Y}_{2\boxtimes 2})(v;1-x)v;x\big)v\rangle,
\end{aligned}$$

where $\mathcal{Y}_{1\boxtimes 2}$ is the intertwining operator of type $\binom{\mathcal{SX}_1^-}{\mathcal{SX}_1^+\ \mathcal{SX}_1^-}$. Since $p_1 \circ \mathcal{Y}_{2\boxtimes 2}$ is the non-zero intertwining operator of type $\binom{\mathcal{SX}_1^+}{\mathcal{SX}_1^-\ \mathcal{SX}_1^-}$, we have

$$\langle v^*, Y_{\mathcal{SX}_1^-}\big((p_1 \circ \mathcal{Y}_{2\boxtimes 2})(v;1-x)v;x\big)v\rangle \in \mathbb{C}^\times(1-x)^{-2h_{2,2}}\big(1+(1-x)\mathbb{C}[[1-x]]\big). \tag{12.3.1}$$

We define the following 4-point functions

$$\begin{aligned}
\phi_1(x) &= \langle v^*, \mathcal{Y}^2_{21}(v;1)(p_1 \circ \mathcal{Y}_{2\boxtimes 2})(v;x)v\rangle \in \mathbb{C}x^{h_{1,1}-2h_{2,2}}\big(1+x\mathbb{C}[[x]]\big), \\
\phi_3(x) &= \langle v^*, \mathcal{Y}^2_{23}(v;1)(p_3 \circ \mathcal{Y}_{2\boxtimes 2})(v;x)v\rangle \in \mathbb{C}x^{h_{3,3}-2h_{2,2}}\big(1+x\mathbb{C}[[x]]\big).
\end{aligned}$$

Similar to the arguments in [45], we can show that $\phi_1(z)$ and $\phi_3(z)$ satisfy the Fuchsian differential equation (12.2.3) in Subsection 12.2. Therefore, by the relations (12.2.1) and (12.2.2) given in Subsection 12.2 and by the non-zero four point function (12.3.1), we see that $\phi_1(x)$ is non-zero. In particular $\mathcal{Y}^2_{21}$ is non-zero. Thus $\mathcal{SX}_1^-$ is rigid and self-dual. □

12.4 Non-semisimple fusion rules

From this section we introduce the following symbols:

1. For $0 \le i \le m$
$$X_{2i+1} := \mathcal{SX}^+_{i+1}.$$
2. For $1 \le i \le m$
$$X_{2i} := \mathcal{SX}^-_{i}.$$
3. For $1 \le i \le m$
$$P_{2i} := \mathcal{SP}^+_{m-i+1}.$$
4. For $0 \le i \le m-1$
$$P_{2i+1} := \mathcal{SP}^-_{m-i}.$$

Lemma 12.4.1.

1. *For $i = 1, \ldots, m-1$, the vector space $A_0((X_2 \boxtimes X_{2i+1})^*)$ is at most four dimensional. L_0 acts semisimply on $A_0((X_2 \boxtimes X_{2i+1})^*)$ and any L_0 eigenvalue of this space is contained in $\{h_{2,2i}, h_{2,2i+2}\}$, where $h_{2,2i}$ and $h_{2,2i+2}$ are the L_0 weights of the highest weight spaces of X_{2i} and X_{2i+2}, respectively.*

2. *The vector space $A_0((X_2 \boxtimes X_{2m+1})^*)$ is at most four dimensional. Any L_0 eigenvalue of this space is contained in $\{h_{1,1}, h_{2,2m}\} = \{0, \frac{1}{2}\}$, where $h_{1,1}$ and $h_{2,2m}$ are the L_0 weights of the highest weight spaces of X_1 and X_{2m}, respectively.*

Proof. We only prove the first claim. The second claim can be proved in the same way, so we omit the proof. Let ψ^*, ϕ_1 and ϕ_2 be arbitrary elements of $A_0((X_2 \boxtimes X_{2i+1})^*)$, X_2 and X_{2i+1}, respectively. For $1 \leq j \leq m$, let $\{v^+, v^-\}$ be a basis of the highest weight space of X_2 such that

$$\widehat{W}^{\pm}[0]v^{\pm} = 0, \qquad \widehat{W}^{\pm}[0]v^{\mp} \in \mathbb{C}^{\times}v^{\pm}.$$

For $n \geq 1$, let $w_k^{(n)} (k = 1 \ldots, 2n+1)$ be the $\mathfrak{ns}$-highest weight vectors of the vector subspace $(2n+1)L(h_{2n+1,2i+1}) \subset X_{2i+1}$. Similar to the arguments in Lemma 12.1.1, we see that

$$\langle \psi^*, U(\mathfrak{ns}).v^{\pm} \otimes w_k^{(n)} \rangle = 0$$

for any $n \geq 2$ and k, where $U(\mathfrak{ns})$ is the universal enveloping algebra of the Neveu-Schwarz algebra. Note that

$$\widehat{W}^{\pm}[-h]v^{\epsilon} \in U(\mathfrak{ns}).v^{\epsilon} + U(\mathfrak{ns}).v^{-\epsilon}, \qquad W^{\pm}[-h]v^{\epsilon} \in U(\mathfrak{ns}).v^{\epsilon} + U(\mathfrak{ns}).v^{-\epsilon}$$

for $h \leq h_{3,1} = \frac{1}{2} + 2m$, where $\epsilon = \pm$. Thus, by using Lemma 8.1.6, we see that the value $\langle \psi^*, \phi_1 \otimes \phi_2 \rangle$ is determined by the values

$$\langle \psi^*, U(\mathfrak{ns}).v^{\pm} \otimes u \rangle,$$

where u is the highest weight vector of X_{2i+1}. Then, similar to the arguments in Lemma 12.1.1, we see that L_0 acts semisimply on ψ^* and the L_0 eigenvalue of ψ^* is contained in

$$\{h_{1,2(m-i)+1}, h_{1,2(m-i)-1}, h_{2,2i}, h_{2,2i+2}\}.$$

Let us assume that the L_0 eigenvalue of ψ^* is $h_{1,2(m-i)+1}$ or $h_{1,2(m-i)-1}$. Then, similar to the arguments in Lemma 12.1.1, we see that

$$\langle \psi^*, U(\mathfrak{ns}).v^{\pm} \otimes w_k^{(1)} \rangle = 0 \tag{12.4.1}$$

for any k. Note that $\widehat{W}^{\pm}[0]\psi^* = 0$. Then, by Lemma 8.1.6 and by (12.4.1), we have

$$\langle \psi^*, U(\mathfrak{ns}).v^{\pm} \otimes u \rangle = 0.$$

Thus we have a contradiction.

Next, let us assume that the L_0 eigenvalue of ψ^* is $h_{2,2i}$ or $h_{2,2i+2}$. Note that

$$W^{\pm}[-h]v^{\pm} = 0, \qquad \widehat{W}^{\pm}[-h]v^{\pm} = 0 \tag{12.4.2}$$

for $h \leq h_{3,1}$. Then, by Lemma 8.1.6 and by (12.4.2), we see that

$$\langle \widehat{W}^{\pm}[0]\psi^*, v^{\pm} \otimes U(\mathfrak{ns}).u\rangle = 0.$$

Thus $\langle \psi^*, \phi_1 \otimes \phi_2\rangle$ is determined by the values

$$\langle \widehat{W}^{+}[0]\psi^* + \psi^*, U(\mathfrak{ns}).v^{-} \otimes u\rangle.$$

Therefore, by using the relation 12.1.1 and Lemma 8.1.6, we see that $A((X_2 \boxtimes X_{2i+1})^*)$ is at most four dimensional. □

Proposition 12.4.2. *For $s = 2, \dots, 2m$, we have*

$$X_2 \boxtimes X_s = X_{s-1} \oplus X_{s+1}.$$

Proof. Let us show that

$$X_2 \boxtimes X_{2i} = X_{2i-1} \oplus X_{2i+1}.$$

Similar to Lemma 12.1.3, we can show that

$$I\begin{pmatrix} X_{2i-1} \\ X_2 \quad X_{2i}\end{pmatrix} \neq \emptyset, \qquad I\begin{pmatrix} X_{2i+1} \\ X_2 \quad X_{2i}\end{pmatrix} \neq \emptyset. \tag{12.4.3}$$

By Lemmas 12.1.1, 8.3.10 and by the self-duality of X_2, we see that

$$\mathrm{Hom}_{\mathcal{SW}(m)}(X_{2(m-i+1)} \oplus X_{2(m-i)}, X_2 \boxtimes X_{2i}) = 0.$$

Thus, by (12.4.3), we obtain

$$X_2 \boxtimes X_{2i} = X_{2i-1} \oplus X_{2i+1}.$$

Next let us show the formula

$$X_2 \boxtimes X_{2i+1} = X_{2i} \oplus X_{2i+2}, \qquad i = 0, \dots, m-1.$$

Similar to Lemma 12.1.3, we can show that

$$I\begin{pmatrix} X_{2i} \\ X_2 \quad X_{2i+1}\end{pmatrix} \neq \emptyset, \qquad I\begin{pmatrix} X_{2i+2} \\ X_2 \quad X_{2i+1}\end{pmatrix} \neq \emptyset. \tag{12.4.4}$$

By Lemmas 12.4.1, 8.3.10 and by the self-duality of X_2, we see that

$$\mathrm{Hom}_{\mathcal{SW}(m)}(X_{2(m-i)-1} \oplus X_{2(m-i)+1}, X_2 \boxtimes X_{2i+1}) = 0.$$

Thus, by (12.4.4), we obtain

$$X_2 \boxtimes X_{2i+1} = X_{2i} \oplus X_{2i+2}.$$

□

Since $\mathcal{SW}(m)$ is C_2-cofinite, every simple module has projective cover [36]. For $0 \leq i \leq m-1$, let $\widetilde{P}_{2i+2}$ and $\widetilde{P}_{2i+1}$ be the projective covers of the simple modules $X_{2(m-i)+1}$ and $X_{2(m-i)}$, respectively.

Proposition 12.4.3.

$$X_2 \boxtimes X_{2m+1} = \widetilde{P}_1 = P_1.$$

Proof. Since X_{2m+1} is projective, by the self-duality of X_2, $X_2 \boxtimes X_{2m+1}$ must be projective. By using Lemma 8.3.10 and Proposition 12.4.2, we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{SW}(m)}(X_{2m}, X_2 \boxtimes X_{2m+1}) = \mathbb{C},\\
&\mathrm{Hom}_{\mathcal{SW}(m)}(X_2 \boxtimes X_{2m+1}, X_{2m}) = \mathbb{C},\\
&\mathrm{Hom}_{\mathcal{SW}(m)}(X_1, X_2 \boxtimes X_{2m+1}) = 0,\\
&\mathrm{Hom}_{\mathcal{SW}(m)}(X_2 \boxtimes X_{2m+1}, X_1) = 0.
\end{aligned} \tag{12.4.5}$$

Thus we obtain

$$X_2 \boxtimes X_{2m+1} = \widetilde{P}_1.$$

By Proposition 11.4.3, we can see that $\widetilde{P}_1$ has $2X_1$ as composition factors. Note that L_0 weight of the highest weight spaces of X_{2m} and X_1 are given by $\frac{1}{2}$ and 0, respectively. Thus, by Lemma 12.4.1 and (12.4.5), we can see that $\widetilde{P}_1$ has the socle series

$$\mathrm{Soc}(\widetilde{P}_1) = \mathrm{Soc}_1(\widetilde{P}_1) \leq \mathrm{Soc}_2(\widetilde{P}_1) \leq \mathrm{Soc}_3(\widetilde{P}_1) = \widetilde{P}_1$$

such that

$$\begin{aligned}
&\mathrm{Soc}_1(\widetilde{P}_1) \simeq X_{2m},\ \mathrm{Soc}_2(\widetilde{P}_1)/\mathrm{Soc}_1(\widetilde{P}_1) \simeq X_1 \oplus X_1,\\
&\mathrm{Soc}_3(\widetilde{P}_1)/\mathrm{Soc}_2(\widetilde{P}_1) \simeq X_{2m}.
\end{aligned}$$

Since P_1 and $\widetilde{P}_1$ have the same com composition factors, we see that

$$\widetilde{P}_1 \simeq P_1 \simeq P_1^*.$$

□

By Lemma 8.3.10 and by Propositions 12.4.2, 12.4.3, we obtain the following propositions.

Proposition 12.4.4. *For every $s = 1, \dots, 2m$,*

$$\widetilde{P}_s = P_s.$$

The socle series of the projective covers of the simple modules are given by:

1. *For $1 \leq i \leq m$,*

$$\mathrm{Soc}(P_{2i}) = \mathrm{Soc}_1(P_{2i}) \leq \mathrm{Soc}_2(P_{2i}) \leq \mathrm{Soc}_3(P_{2i}) = P_{2i}$$

such that

$$\begin{aligned}
&\mathrm{Soc}_1(P_{2i}) \simeq X_{2(m-i)+1},\ \mathrm{Soc}_2(P_{2i})/\mathrm{Soc}_1(P_{2i}) \simeq X_{2i} \oplus X_{2i},\\
&\mathrm{Soc}_3(P_{2i})/\mathrm{Soc}_2(P_{2i}) \simeq X_{2(m-i)+1}.
\end{aligned}$$

2. *For* $0 \leq i \leq m-1$,

$$\mathrm{Soc}(P_{2i+1}) = \mathrm{Soc}_1(P_{2i+1}) \leq \mathrm{Soc}_2(P_{2i+1}) \leq \mathrm{Soc}_3(P_{2i+1}) = P_{2i+1}$$

such that

$$\begin{aligned}&\mathrm{Soc}_1(P_{2i+1}) \simeq X_{2(m-i)},\ \mathrm{Soc}_2(P_{2i+1})/\mathrm{Soc}_1(P_{2i+1}) \simeq X_{2i+1} \oplus X_{2i+1},\\&\mathrm{Soc}_3(P_{2i+1})/\mathrm{Soc}_2(P_{2i+1}) \simeq X_{2(m-i)}.\end{aligned}$$

Proposition 12.4.5.

$$X_2 \boxtimes P_1 = 2X_{2m+1} \oplus P_2.$$

Proposition 12.4.6. *For* $2 \leq s \leq 2m-1$,

$$X_2 \boxtimes P_s = P_{s-1} \oplus P_{s+1}.$$

Proposition 12.4.7.

$$X_2 \boxtimes P_{2m} = 2X_{2m+1} \oplus P_{2m-1}.$$

From these proposition, we obtain the following theorem.

Theorem 12.4.8. *For* $1 \leq s \leq 2m+1$ *and* $1 \leq t \leq 2m$, *the simple modules* X_s *and the projective modules* P_t *are rigid and self-dual.*

Similar to the arguments in [64], we obtain the following theorem.

Theorem 12.4.9. *The braided tensor category* $(\mathcal{SC}_m, \boxtimes)$ *is rigid. For any* $M \in \mathcal{SC}_m$, *we have* $M^\vee = M^*$, *where* $M^\vee$ *is the dual of* M.

Let $U_q^{small}(sl_2)$-mod be the abelian category of finite dimensional modules over the small quantum group $U_q^{small}(sl_2)$, where $q = e^{\frac{2\pi i}{2m+1}}$. Similar to the arguments in Section 6 of [58], by Proposition 12.4.4, we obtain the following theorem.

Theorem 12.4.10. *Two abelian categories* $\mathcal{SC}_m$ *and* $U_q^{small}(sl_2)$-mod *are equivalent as abelian categories.*

Remark 12.4.11. *Let* $q = e^{\frac{2\pi i}{2m+1}}$. *The small quantum group* $U_q^{small}(sl_2)$ *is an associative* $\mathbb{C}$*-algebra which is generated by* E, F, K, K^{-1} *satisfying the following fundamental relations*

$$\begin{aligned}&KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\\&EF - FE = \frac{K - K^{-1}}{q - q^{-1}},\\&E^{2m+1} = F^{2m+1} = 0, \quad K^{2m+1} = 1.\end{aligned}$$

See [54] for details.

12.5 Non-semisimple fusion rings

We introduce the free abelian group $P(\mathcal{SC}_m)$ of rank $4m+1$ generated by all simple modules and all projective modules

$$P(\mathcal{SC}_m) = \bigoplus_{s=1}^{2m+1} \mathbb{Z}[X_s]_P \oplus \bigoplus_{s=1}^{2m} \mathbb{Z}[P_s]_P.$$

From the results presented in the previous subsection, we see that $P(\mathcal{SW}(m))$ has the structure of a commutative ring where the product as a ring is given by

$$[\bullet]_P \cdot [\bullet]_P = [\bullet \boxtimes \bullet]_P.$$

The operators

$$X = X_2 \boxtimes -$$

define $\mathbb{Z}$-linear endomorphism of $P(\mathcal{SC}_m)$. Thus $P(\mathcal{SC}_m)$ is a module over $\mathbb{Z}[X]$. We define the following $\mathbb{Z}[X]$-module map

$$\begin{aligned} \psi : \mathbb{Z}[X] &\to P(\mathcal{SC}_m), \\ f(X) &\mapsto f(X) \cdot [X_1]_P. \end{aligned}$$

From the results of previous section, we obtain the following propositions.

Proposition 12.5.1. *For $s = 1, \dots, 2m+1$,*

$$[X_s]_P = U_{s-1}(X)[X_1]_P.$$

Proposition 12.5.2. *For $s = 1, \dots, 2m-1$,*

$$[P_s]_P = (U_{2m+s}(X) + U_{2m-s}(X))[X_1]_P$$

Proposition 12.5.3.

$$U_{4m+1}(X)[X]_P = 2U_{2m}(X)[X_1]_P.$$

From these proposition, we obtain the following theorem.

Theorem 12.5.4. *The $\mathbb{Z}[X]$-module map ψ is surjective and the kernel of ψ is given by the following ideal*

$$\ker\psi = \langle U_{4m+1}(X) - 2U_{2m}(X) \rangle.$$

Proof. By Proposition 12.5.1 and 12.5.2, we see that ψ is surjective. We define the following ideal of $\mathbb{Z}[X]$

$$I = \langle U_{4m+1}(X) - 2U_{2m}(X) \rangle.$$

By Proposition 12.5.3, we see that I is contained in $\ker\psi$. It is easy to see that the dimension of $\mathbb{Z}[X]/I$ is $4m+1$. Therefore we obtain $\ker\psi = I$.

□

Let us show below that, from the non-semisimple fusion ring $P(\mathcal{C}_p)$ $(p = 2m+1)$ determined in [64], we obtain the non-semisimple fusion ring $P(\mathcal{SC}_m)$. Recall that the rank $4p-2$ free abelian group

$$P(\mathcal{C}_p) = \bigoplus_{s=1}^{p} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{X}_s^\epsilon]_P \oplus \bigoplus_{s=1}^{p-1} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{P}_s^\epsilon]_P$$

has the structure of a commutative ring and is isomorphic to

$$\frac{\mathbb{Z}[X,Y]}{\langle Y^2-1, U_{2p-1}(X) - 2YU_{p-1}(X)\rangle},$$

where

$$[\mathcal{X}_1^+]_P \mapsto 1, \qquad [\mathcal{X}_2^+]_P \mapsto X, \qquad [\mathcal{X}_1^-]_P \mapsto Y.$$

Let $p = 2m+1$ and set $Y = 1$ in $P(\mathcal{C}_p)$, and then we obtain the non-semisimple fusion ring $P(\mathcal{SC}_m)$:

$$P(\mathcal{C}_{2m+1}) \xrightarrow{[\mathcal{X}_1^-]_P = 1} P(\mathcal{SC}_m).$$

In the paper [2], Adamović and Milas showed that the characters of the simple $\mathcal{SW}(m)$-modules are intimately related to the characters of the simple $\mathcal{W}_{2m+1}$-modules. From their results and ours, it is expected that there is a deep connection between $\mathcal{SC}_m$ and $\mathcal{C}_{2m+1}$.

Remark 12.5.5. *In the paper [3], Adamović and Milas introduced a certain non-rational vertex operator superalgebra $\mathcal{SW}(p,q)$, where p and q are positive integers such that $q > p$ and $(q, \frac{q-p}{2}) = 1$. This vertex operator superalgebra $\mathcal{SW}(p,q)$ is a natural generalization of $\mathcal{SW}(m)$ and is a extension of the super Virasoro minimal models:*

$$L^{\mathfrak{ns}}(c_{p,q}^{N=1}, 0) \subset \mathcal{SW}(p,q),$$

where $L^{\mathfrak{ns}}(c_{p,q}^{N=1}, 0)$ is the Neveu-Schwarz vertex operator superalgebra of central charge

$$c_{p,q}^{N=1} = \frac{3}{2}\big(1 - 2\frac{(p-q)^2}{pq}\big).$$

Just as $\mathcal{SW}(m)$ and $\mathcal{W}_{2m+1}$ are related, $\mathcal{SW}(p,q)$ is considered to be related to $\mathcal{W}_{2p+1,2q+1}$. We conjecture that the commutative ring $P^0(\mathcal{C}_{2m_1+1,2m_2+1})|_{[\mathcal{X}_{1,1}^-]_P=1}$ $((m_1,m_2)=1)$ corresponds to a non-semisimple fusion ring of $\mathcal{SW}(2m_1+1, 2m_2+1)$.

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