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An Algorithm for Solving Linear Ordinary Differential Equations of Fuchsian Type with Three Singular Points

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We propose a method for representing the solutions of a certain type of ordinary differential operator L in terms of those of more fundamental differential operators. This method consists of two steps, decomposing L in the ring of differential operators and then describing the projections to the components of this decomposition also in terms of some differential operators. We provide a concrete algorithm for the application of this method and show that this algorithm is successful for a specific example of a 6-th order homogeneous linear ordinary differential operator of Fuchsian type with three singular points.

KEYWORDS: ordinary linear differential equation, Fuchsian type, factorization of differential operators, Riemann's P-equation, hypergeometric function

0. Introduction

Let $u(z)$ be a function defined on some domain in the complex plane $\mathbf{C}$, and $Lu(z) = 0$ be a homogeneous linear differential equation of Fuchsian type with three singular points, $z = 0$, $z = 1$ and $z = \infty$. In this paper, we propose an algorithm for solving the equation $Lu(z) = 0$.

Riemann's P-equation, given in (3) below, is a particular type of homogeneous linear differential equation of Fuchsian type with singular points $z = 0$, $z = 1$ and $z = \infty$. This equation is integrable, and its solutions can be expressed in terms of Riemann's P-function. In the algorithm we propose, more general homogeneous linear differential equations of Fuchsian type are expressed in factorized form entirely in terms of operators of the type contained in Riemann's P-equation.

The algorithm that we propose is divided into two parts. The purpose of the first part is to determine a factorization of the differential operator L. The factors of this factorization of L are operators of the type appearing in Riemann's P-equation. The purpose of the second part is to identify differential operators that provide some relationships between these factors such that each of these operators gives a linearly isomorphic mapping from the solution space of Riemann's P-equation to a subspace of the solution space of the equation $Lu(z) = 0$. In other words, the operators provide projections from the solution space of the equation $Lu(z) = 0$ to its subspaces.

The first part of the algorithm can be perfomed only on homogeneous linear differential equations of Fuchsian type with the three singular points $z = 0$, $z = 1$ and $z = \infty$. The second part, however, can be applied to any differential operator that has a factorization. Most of the argument used for the second part can also be applied to any associative algebra, such as matrix algebras, universal enveloping algebras, etc. We do not consider the matter of this straightforward generalization here.

In terms of $\mathscr{D}$-modules, we find a differential operator that yields a splitting morphism of a given short exact sequence of $\mathscr{D}$-modules (i.e., sub and quotient modules). We remark that in some cases, an abstract argument, such as that of monodromy, can be made to prove the existence of such an operator. However, it is non-trivial to derive such an operator explicitly.

Below, we give an example to which the algorithm can be applied successfully and present an explicit expression representing a fundamental system of its solutions. The ordinary differential equation of this example is that which is obtained using separation of variables from the harmonic equation on vector fields of some hyperbolic 3-manifold (see [1–3] for details).

The algorithm for determining the operators that provide the relationships between the factors that we give in this paper is just one example of this type of algorithm. Certainly, there may be other methods for determining such relationships between factors. We leave the demonstration of this point as a problem (see Problem 3.1.1).

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1. Linear Ordinary Differential Equations of Fuchsian Type

In this section, we review fundamental definitions and well-known properties concerning linear ordinary differential equations of Fuchsian type needed in later sections. (For details regarding the content of this section, see [4].)

Let

$$\frac{d^n u}{dz^n} + p_1(z)\frac{d^{n-1}u}{dz^{n-1}} + \cdots + p_{n-1}(z)\frac{du}{dz} + p_n(z)u = 0 \tag{1}$$

be a homogeneous linear differential equation of n-th order, and let there be a domain $S(\subset \mathbf{C})$ in which $p_1(z), \ldots, p_n(z)$ are all analytic, except at a finite number of points. If a point c in S has the property that each $p_i(z)$ has a pole at c such that $P_i(z) := (z-c)^i p_i(z)$ is analytic at c, then c is called a 'regular singular point' of the differential equation (1). The differential equation (1) is said to be of Fuchsian type if all points of non-analyticity are regular singular points. The point at infinity is said to be a regular singular point if $\zeta = 0$ is a regular singular point for the differential equation obtained by substituting ζ for $1/z$.

Let c be a regular singular point of the differential equation (1). By multiplying both sides of (1) by $(z-c)^n$, we obtain

$$(z-c)^n \frac{d^n u}{dz^n} + (z-c)^{n-1}P_1(z)\frac{d^{n-1}u}{dz^{n-1}} + \cdots + P_n(z)u = 0. \tag{2}$$

Solutions of the equation

$$\rho(\rho-1)(\rho-2)\times\cdots\times(\rho-n+1) + P_1(c)\rho(\rho-1)\times\cdots\times(\rho-n+2)$$
$$+ P_2(c)\rho(\rho-1)\times\cdots\times(\rho-n+3) + \cdots + P_{n-1}(c)\rho + P_n(c) = 0$$

are called 'characteristic exponents at $z = c$' of the differential equation (1).

In this paper, we consider only differential equations of Fuchsian type with three singular points. Without loss of generality, we may study equations for which the three regular singular points are $z = 0$, $z = 1$ and $z = \infty$.

A well-known differential equation of Fuchsian type with the three singular points 0, 1 and ∞ is Riemann's P-equation, mentioned in the previous section:

$$\frac{d^2u}{dz^2} + \left(\frac{1-\alpha-\alpha'}{z} + \frac{1-\gamma-\gamma'}{z-1}\right)\frac{du}{dz} + \left(\frac{-\alpha\alpha'}{z^2(z-1)} + \frac{\beta\beta'}{z(z-1)} + \frac{\gamma\gamma'}{z(z-1)^2}\right)u = 0. \tag{3}$$

The characteristic exponents of this equation are α and α' at $z = 0$, β and β' at $z = \infty$, and γ and γ' at $z = 1$. These characteristic exponents satisfy the relation

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1.$$

The solutions of Riemann's P-equation can be expressed in terms of Riemann's P-function,

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ \alpha & \gamma & \beta & z \\ \alpha' & \gamma' & \beta' & \end{matrix}\right\}.$$

If the difference $\alpha - \alpha'$ between the exponents at $z = 0$ is not an integer, then

$$z^{\alpha}(1-z)^{\gamma}F(\alpha+\beta+\gamma, \alpha+\beta'+\gamma; 1+\alpha-\alpha'; z)$$

and

$$z^{\alpha'}(1-z)^{\gamma}F(\alpha'+\beta+\gamma, \alpha'+\beta'+\gamma; 1+\alpha'-\alpha; z)$$

are linearly independent solutions of (3) in the neighborhood of $z = 0$, where $F(a, b; c; z)$ is the hypergeometric function. Note that on the unit disk, $|z| < 1$, the hypergeometric function $F(a, b; c; z)$ is given by the hypergeometric series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(a)_n := a(a+1)(a+2)\ldots(a+n-1)$, and similarly for $(b)_n$ and $(c)_n$.

2. Factorizations of Differential Operators

In this section, we present an algorithm for factorizing differential operators of Fuchsian type and apply the algorithm to an example.

2.1 An algorithm for factorizing differential operators

Let L denote the differential operator of the differential equation (1):

$$L = \frac{d^n}{dz^n} + p_1(z)\frac{d^{n-1}}{dz^{n-1}} + \cdots + p_{n-1}(z)\frac{d}{dz} + p_n(z).$$

We assume that the differential equation $Lu(z) = 0$ is Fuchsian with three singular points, 0, 1 and ∞. We also assume that the order n of the operator L is even. In this subsection, we obtain an algorithm for factorizing the operator L into a set of operators such that each is the type appearing in Riemann's P-equation.

First, we compute the characteristic exponents of the differential operator L and consider the set

$$\mathscr{C} := \left\{ (\alpha, \alpha', \beta, \beta', \gamma, \gamma') \;\middle|\; \begin{array}{l} \alpha \text{ and } \alpha' \text{ are the exponents at } z = 0, \\ \beta \text{ and } \beta' \text{ are the exponents at } z = \infty, \\ \gamma \text{ and } \gamma' \text{ are the exponents at } z = 1, \\ \text{and } \alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1 \end{array} \right\}.$$

Suppose that the set $\mathscr{C}$ is not empty. (Note that this set could be empty.) Then we choose an element $(\alpha, \alpha', \beta, \beta', \gamma, \gamma')$ of $\mathscr{C}$, and define

$$P_1 := \frac{d^2}{dz^2} + \left(\frac{1-\alpha-\alpha'}{z} + \frac{1-\gamma-\gamma'}{z-1}\right)\frac{d}{dz} + \left(\frac{\alpha\alpha'}{z^2(1-z)} + \frac{\beta\beta'}{z(z-1)} + \frac{\gamma\gamma'}{z(z-1)^2}\right).$$

Then we divide the operator L by the operator P_1 from the right. Suppose there is an element $(\alpha_1, \alpha_1', \beta_1, \beta_1', \gamma_1, \gamma_1')$ of $\mathscr{C}$ such that this division can be done without a remainder, and denote its quotient by Q_1. In this case, the operator L can be factorized into Q_1 and P_1 as

$$L = Q_1 P_1.$$

The operator Q_1 is Fuchsian with the three singular points $z = 0$, $z = 1$ and $z = \infty$, and its order is $n - 2$.

Second, we compute the characteristic exponents of Q_1 and repeat the above process for Q_1. If there is an operator P_2 that divides Q_1 from the right without a remainder, we denote the quotient by Q_2. Then we have

$$Q_1 = Q_2 P_2.$$

This process is carried out repeatedly, and if it can be done until L is factorized entirely into operators of order 2, we denote these operators by P_i $(i = 1, \ldots, n/2)$. In this case, we have

$$L = P_{n/2} P_{(n-2)/2} \cdots P_2 P_1. \tag{4}$$

If L can be factorized as in (4), the differential equation $Lu(z) = 0$ can be solved by applying the method of variations of parameters to non-homogeneous linear equations. More precisely, let f be a fundamental solution of the equation $P_2 f = 0$. Then any solution of the equation $P_2 P_1 g = 0$ can be obtained by solving the non-homogeneous linear equation $P_1 g = f$ by the method of variations of parameters. Performing this process repeatedly, beginning from the right of the operator $L = P_{n/2} \cdots P_1$, any solution g of the equation $Lg = 0$ can be obtained in this manner.

2.2 An example of harmonic vector fields on cone manifolds

Let a and b be non-zero real numbers. We consider the case in which L takes the following form:

$$\begin{aligned}
L := {} & \frac{d^6}{dz^6} + \frac{9(-1+2z)}{z(z-1)}\frac{d^5}{dz^5} + \frac{72 - 3a^2 - 394z + 3a^2z - 3b^2z + 387z^2 + 3b^2z^2}{4z^2(z-1)^2}\frac{d^4}{dz^4} \\
& + \frac{1}{2z^3(z-1)^3}(-12 + 3a^2 + 212z - 12a^2z + 6b^2z - 543z^2 + 9a^2z^2 - 18b^2z^2 + 348z^3 \\
& + 12b^2z^3)\frac{d^3}{dz^3} + \frac{1}{16z^4(z-1)^4}(-12a^2 + 3a^4 - 272z + 92a^2z - 6a^4z - 24b^2z + 6a^2b^2z \\
& + 1792z^2 - 158a^2z^2 + 3a^4z^2 + 212b^2z^2 - 12a^2b^2z^2 + 3b^4z^2 - 2828z^3 + 78a^2z^3 - 362b^2z^3 \\
& + 6a^2b^2z^3 - 6a^4z^3 + 1323z^4 + 174b^2z^4 + 3b^4z^4)\frac{d^2}{dz^2} + \frac{1}{16z^5(z-1)^5}(-12a^2 + 3a^4 + 24a^2z \\
& - 6a^4z - 56z^2 - 6a^2z^2 + 3a^4z^2 - 44b^2z^2 + 6a^2b^2z^2 - 3b^4z^2 + 136z^3 - 12a^2z^3 + 148b^2z^3 \\
& - 12a^2b^2z^3 + 12b^4z^3 - 149z^4 + 6a^2z^4 - 164b^2z^4 + 6a^2b^2z^4 - 15b^4z^4 + 54z^5 + 60b^2z^5
\end{aligned}$$

$$+ 6b^4z^5)\frac{d}{dz} + \frac{1}{64z^6(z-1)^6}(-64a^2 + 20a^4 - a^6 + 232a^2z - 70a^4z + 3a^6z + 12a^2b^2z$$
$$- 3a^4b^2z - 316a^2z^2 + 83a^4z^2 - 3a^6z^2 - 56a^2b^2z^2 + 9a^4b^2z^2 - 3a^2b^4z^2 - 16z^3 + 180a^2z^3$$
$$- 36a^4z^3 + a^6z^3 - 48b^2z^3 + 82a^2b^2z^3 - 9a^4b^2z^3 - 18b^4z^3 + 9a^2b^4z^3 - b^6z^3 + 56z^4$$
$$- 35a^2z^4 + 3a^4z^4 + 100b^2z^4 - 44a^2b^2z^4 + 3a^4b^2z^4 + 47b^4z^4 - 9a^2b^4z^4 + 3b^6z^4 - 34z^5$$
$$+ 3a^2z^5 - 71b^2z^5 + 6a^2b^2z^5 - 40b^4z^5 + 3a^2b^4z^5 - 3b^6z^5 + 9z^6 + 19b^2z^6 + 11b^4z^6 + b^6z^6).$$

Solutions of the differential equation $Lu(z) = 0$ with this form of L are harmonic vector fields in the neighborhood of a singular locus of a hyperbolic 3-cone-manifold. (See [1–3] for the original work on this differential operator L and further discussions.)

We now apply our algorithm to this case. The characteristic exponents of $Lu(z) = 0$ are

$$\pm\frac{a}{2},\ \frac{2\pm a}{2},\ \frac{4\pm a}{2} \quad (z=0);$$
$$\frac{-1}{2},\ \frac{3}{2},\ \frac{1}{2},\ \frac{5}{2},\ \frac{2\pm\sqrt{5}}{2} \quad (z=1);$$
$$\frac{-1\pm b\sqrt{-1}}{2},\ \frac{1\pm b\sqrt{-1}}{2},\ \frac{3\pm b\sqrt{-1}}{2} \quad (z=\infty).$$

Let

$$\alpha_1 := \frac{a}{2},\ \alpha_1' := \frac{-a}{2};$$
$$\beta_1 := \frac{-1+b\sqrt{-1}}{2},\ \beta_1' := \frac{-1-b\sqrt{-1}}{2};$$
$$\gamma_1 := \frac{2+\sqrt{5}}{2},\ \gamma_1' := \frac{2-\sqrt{5}}{2}.$$

Then we have

$$\alpha_1 + \alpha_1' + \beta_1 + \beta_1' + \gamma_1 + \gamma_1' = 1.$$

Next, setting

$$P_1 := \frac{d^2}{dz^2} + \left(\frac{1}{z} - \frac{1}{z-1}\right)\frac{d}{dz} + \left(\frac{a^2}{4z^2(z-1)} + \frac{b^2+1}{4z(z-1)} + \frac{-1}{4z(z-1)^2}\right),$$

then we can factorize L as

$$L = Q_1P_1,$$

where

$$Q_1 := \frac{d^4}{dz^4} + \frac{2(-4+9z)}{z(z-1)}\frac{d^3}{dz^3} + \frac{28 - a^2 - 176z + a^2z - b^2z + 193z^2 + b^2z^2}{2z^2(z-1)^2}\frac{d^2}{dz^2}$$
$$+ \frac{-8 + 2a^2 + 164z - 8a^2z + 3b^2z - 484z^2 + 6a^2z^2 - 10b^2z^2 + 343z^3 + 7b^2z^3}{2z^3(z-1)^3}\frac{d}{dz}$$
$$+ \frac{1}{16z^4(z-1)^4}(-4a^2 + a^4 - 160z + 48a^2z - 2a^4z - 8b^2z + 2a^2b^2z + 1320z^2$$
$$- 94a^2z^2 + a^4z^2 + 80b^2z^2 - 4a^2b^2z^2 + b^4z^2 - 2400z^3 + 50a^2z^3 - 146b^2z^3$$
$$+ 2a^2b^2z^3 - 2b^4z^3 + 1225z^4 + 74b^2z^4 + b^4z^4).$$

Now, returning to the first step, the characteristic exponents of $Q_1u(z) = 0$ are

$$\pm\frac{a}{2},\ \frac{-2\pm a}{2} \quad (z=0);$$
$$\frac{-5}{2},\ \frac{-3}{2},\ \frac{-1}{2},\ \frac{1}{2} \quad (z=1);$$

$$\frac{5 \pm b\sqrt{-1}}{2}, \; \frac{7 \pm b\sqrt{-1}}{2} \quad (z = \infty).$$

Let

$$\alpha_2 := \frac{a}{2}, \; \alpha_2' := \frac{-2-a}{2};$$
$$\beta_2 := \frac{5 + b\sqrt{-1}}{2}, \; \beta_2' := \frac{5 - b\sqrt{-1}}{2};$$
$$\gamma_2 := \frac{-5}{2}, \; \gamma_2' := \frac{-1}{2}.$$

Then we obtain

$$\alpha_2 + \alpha_2' + \beta_2 + \beta_2' + \gamma_2 + \gamma_2' = 1.$$

Now, we set

$$P_2 := \frac{d^2}{dz^2} + \left(\frac{2}{z} + \frac{4}{z-1}\right)\frac{d}{dz} + \left(\frac{a(a+2)}{4z^2(z-1)} + \frac{b^2+25}{4z(z-1)} + \frac{5}{4z(z-1)^2}\right),$$

and then Q_1 can be factorized as

$$Q_1 = Q_2 P_2,$$

where

$$Q_2 := \frac{d^2}{dz^2} + \left(\frac{6}{z} + \frac{6}{z-1}\right)\frac{d}{dz} + \left(\frac{(a-6)(a+4)}{4z^2(z-1)} + \frac{b^2+121}{4z(z-1)} + \frac{21}{4z(z-1)^2}\right).$$

In this way, L has been factorized into the three operators P_1, P_2 and Q_2. Rewriting Q_2 as P_3, the factorzation of L is written as

$$L = P_3 P_2 P_1.$$

In fact, it is not difficult to show that Q_1 can also be factorized as

$$Q_1 = P_3^- P_2^-,$$

where

$$P_2^- := \frac{d^2}{dz^2} + \left(\frac{2}{z} + \frac{4}{z-1}\right)\frac{d}{dz} + \left(\frac{-a(-a+2)}{4z^2(z-1)} + \frac{b^2+25}{4z(z-1)} + \frac{5}{4z(z-1)^2}\right),$$
$$P_3^- := \frac{d^2}{dz^2} + \left(\frac{6}{z} + \frac{6}{z-1}\right)\frac{d}{dz} + \left(\frac{(-a-6)(-a+4)}{4z^2(z-1)} + \frac{b^2+121}{4z(z-1)} + \frac{21}{4z(z-1)^2}\right).$$

It is thus seen that L can be factorized in two ways:

$$L = P_3 P_2 P_1 = P_3^- P_2^- P_1.$$

3. Finding a Splitting Operator

In this section we investigate the problem of finding an operator for splitting a short exact sequence of $\mathscr{D}$-modules corresponding to a differential operator that is factorized into two differential operators. We obtain an algorithm for determining such an operator and apply it to the example given in §2.2. In the last two subsections, we present an explicit expression of a fundamental system of solutions for the example by employing this splitting operator.

3.1 A splitting operator

We consider the following problem:

Problem 3.1.1. Let A and B be linear differential operators. The problem that we consider here is to construct an algorithm for determining an operator R such that $AR - 1$ can be divided by B from the right without a remainder. Let C be the quotient of this division. Then, the problem under study becomes that of constructing an algorithm to determine operators R and C so that

$$AR - 1 = CB.$$

Remark. Depending on the operators A and B, it may be the case that operators R and C as described above do not exist.

The operator R provides a splitting of a short exact sequence of the $\mathscr{D}$-modules as follows:

$$0 \longrightarrow \mathscr{D}/\mathscr{D}B \longrightarrow \mathscr{D}/\mathscr{D}BA \longrightarrow \mathscr{D}/\mathscr{D}A \longrightarrow 0.$$

We obtain the following decomposition of the solution space of the differential equation $BAu(z) = 0$ corresponding to this splitting for $\mathscr{D}$-modules.

Proposition 3.1.2. *Suppose that there exist operators R, C as described in Problem 3.1.1. Then each solution of the homogeneous linear equation*

$$BAu(z) = 0$$

can be written as

$$u(z) = v(z) + Rw(z),$$

where $v(z)$ and $w(z)$ are solutions of the equations $Av(z) = 0$ and $Bw(z) = 0$, respectively. Conversely, if $v(z)$ and $w(z)$ are solutions of the equations $Av(z) = 0$ and $Bw(z) = 0$, respectively, then

$$u(z) := v(z) + Rw(z)$$

satisfies the equation $BAu(z) = 0$.

Proof. Since

$$AR - CB = 1,$$

we have

$$BARw(z) = B(1 + CB)w(z) = (B + BCB)w(z) = (1 + BC)Bw(z).$$

Then, if $Bw(z) = 0$ holds, so too does $BARw(z) = 0$. Thus we can define the mapping

$$\Phi : \{v(z); Av(z) = 0\} \oplus \{w(z); Bw(z) = 0\} \to \{u(z); BAu(z) = 0\}$$

by

$$\Phi(v(z), w(z)) := v(z) + Rw(z).$$

Now, assume that $v(z) + Rw(z) = 0$. Then

$$0 = -Av(z) = ARw(z) = (1 + CB)w(z) = w(z),$$

which hence implies that $v(z) = 0$ also. This shows that Φ is injective. Therefore Φ is a linear isomorphism, because the dimensions of the two spaces $\{v(z); Av(z) = 0\} \oplus \{w(z); Bw(z) = 0\}$ and $\{u(z); BAu(z) = 0\}$ are the same. □

3.2 An algorithm for determining the splitting operator

We now propose an algorithm for determining the splitting operator R. What we present here is just one example of such an algorithm. We remark that there are cases in which either Step 1 or Step 2 cannot be carried out.

Step 1. Find an operator X_1 such that

$$X_1 = R_1 A + s,$$

where $s \in \mathbf{C}$ is nonzero and R_1 is an operator satisfying $AR_1 \neq R_1 A$.

Remark. An intertwining operator of the homogeneous differential equation $BAu(z) = 0$ is one operator of this type that we could use. Here, we call X_1 an intertwining operator if the operator BAX_1 can be factorized by BA from the right. Non-scalar intertwining operators do not necessarily exist. Even if the existence of intertwining operators are shown in some example, it is difficult to determine their explicit forms.

Step 2. Define

$$X_2 := AR_1 + s.$$

Then, find the set $\{x_0, \ldots, x_k\}$ with the fewest number of elements k for which the relations

$$x_0 S_0 + x_1 S_1 + x_2 S_2 + \cdots + x_k S_k = 0$$

and

$$x_0 + x_1 s + x_2 s^2 + \cdots + x_k s^k \neq 0$$

hold, where S_i ($i \in \{1, \ldots, k\}$) denotes the remainder of the division of X_2^i by B from the right. If it is not possible to carry out this procedure, return to Step 1 and try another operator X_1.

Step 3. Define

$$F(t) := x_0 + x_1 t + x_2 t^2 + \cdots + x_k t^k.$$

Then, it is obvious from the relations satisfied by $x_0, \ldots, x_k$ as given in Step 2, that both of the operators $F(X_2)$ and $F(X_2)/F(s)$ can be divided by B from the right without remainders.

Step 4. Define

$$X_3 := F(X_1)/F(s).$$

Then, as seen from the form of X_1 given in Step 1, $1 - X_3$ can be divided by A from the right without a reminder. Let R_2 be the quotient of this division.

Then, combining the above, we have the following:

Proposition 3.2.1. *$AR_2 - 1$ can be divided by B from the right without a remainder.*

Proof. Since $X_1 = R_1 A + s$ and $X_2 = AR_1 + s$, we have

$$AX_1 = A(R_1 A + s) = AR_1 A + sA = (AR_1 + s)A = X_2 A.$$

In the same way, we can show that

$$AF(X_1) = F(X_2)A. \tag{i}$$

Next, from the statement concerning $1 - X_3$ in Step 4,

$$X_3 = 1 - R_2 A.$$

Hence

$$AX_3 = A(1 - R_2 A) = A - AR_2 A = (1 - AR_2)A. \tag{ii}$$

The form of X_3 defined in Step 4 is

$$X_3 = F(X_1)/F(s).$$

This yields, using (i),

$$AX_3 = AF(X_1)/F(s) = F(X_2)A/F(s). \tag{iii}$$

From (ii) and (iii), we find

$$(1 - AR_2)A = F(X_2)A/F(s).$$

Hence

$$1 - AR_2 = F(X_2)/F(s).$$

As stated in Step 3, $F(X_2)/F(s)$ can be divided by B from the right without a remainder. Thus $1 - AR_2$ can also be divided by B from the right without a remainder. □

Step 5. Divide R_2 by B from the right. Let R be the remainder of this division. Then, it is easy to see that $AR - 1$ can be divided by B from the right without a remainder. Let C be the quotient of this division. Then we have

$$AR - 1 = CB.$$

3.3 An example of harmonic vector fields on cone manifolds

Let

$$A := P_1, \qquad B := P_3 P_2,$$

where P_1, P_2 and P_3 are the operators given in §2.2. Below we carry out the algorithm described in §3.2 for this choice of A and B. (Recall here that we assume $a \neq 0$ and $b \neq 0$.)

Step 1. Define

$$X_1 := 4z^2(z-1)^3\left\{\frac{d^4}{dz^4}+\frac{-4+7z}{z(z-1)}\frac{d^3}{dz^3}+\frac{4-a^2-24z+a^2z-b^2z+18z^2+b^2z^2}{2z^2(z-1)^2}\frac{d^2}{dz^2}\right.$$
$$+\frac{8-a^2+2b^2-8z+a^2z-5b^2z+3z^2+3b^2z^2}{4z^2(z-1)^3}\frac{d}{dz}+\frac{1}{16z^4(z-1)^4}(-4a^2+a^4$$
$$+12a^2z-2a^4z+2a^2b^2z+8z^2-8a^2z^2+a^4z^2+12b^2z^2-4a^2b^2z^2+b^4z^2-8z^3$$
$$\left.-16b^2z^3+2a^2b^2z^3-2b^4z^3+3z^4+4b^2z^4+b^4z^4)\right\},$$

$$R_1 := 4z^2(z-1)^3\left\{\frac{d^2}{dz^2}+\left(\frac{3}{z}+\frac{4}{z-1}\right)\frac{d}{dz}+\left(\frac{a^2-4}{4z^2(z-1)}+\frac{b^2+35}{4z(z-1)}+\frac{5}{4z(z-1)^2}\right)\right\}.$$

Then it is easily confirmed by direct computation that

$$X_1 = R_1A - b^2.$$

Remark. In this example, the operator X_1 is an intertwining operator on the solution space of the homogeneous differential equation $BAu(z)=0$. It is confirmed by computation that the operator BAX_1 can be factorized by BA from the right.

Step 2. Define

$$X_2 := AR_1 - b^2.$$

Then, by direct computation, we obtain

$$X_2 = 4z^2(z-1)^3\left\{\frac{d^4}{dz^4}+\frac{-8+17z}{z(z-1)}\frac{d^3}{dz^3}+\frac{28-a^2-166z+a^2z-b^2z+170z^2+b^2z^2}{2z^2(z-1)^2}\frac{d^2}{dz^2}\right.$$
$$+\frac{-16+4a^2+312z-15a^2z+6b^2z-848z^2+11a^2z^2-19b^2z^2+553z^3+13b^2z^3}{4z^3(z-1)^3}\frac{d}{dz}$$
$$+\frac{1}{16z^4(z-1)^4}(-4a^2+a^4-160z+48a^2z-2a^4z-8b^2z+2a^2b^2z+1176z^2-88a^2z^2$$
$$+a^4z^2+76b^2z^2-4a^2b^2z^2+b^4z^2-1916z^3+44a^2z^3-128b^2z^3+2a^2b^2z^3-2b^4z^3$$
$$\left.+875z^4+60b^2z^4+b^4z^4)\right\},$$

and, moreover, we have

$$S_0 = 1,$$

$$S_1 = -4(z-1)^2z^2\frac{d^3}{dz^3}-2z(z-1)(-10+23z)\frac{d^2}{dz^3}-(16-a^2-120z+a^2z-b^2z+133z^2$$
$$+b^2z^2)\frac{d}{dz}-\frac{1}{2(z-1)}(72-3a^2+2b^2-242z+3a^2z-9b^2z+175z^2+7b^2z^2),$$

$$S_2 = 4(4-a^2+b^2)z^2(z-1)^2\frac{d^3}{dz^3}+2(4-a^2+b^2)z(z-1)(-10+23z)\frac{d^2}{dz^2}+(4-a^2+b^2)$$
$$\times(16-a^2-120z+a^2z-b^2z+133z^2+b^2z^2)\frac{d}{dz}-\frac{1}{2(z-1)}(-288+84a^2-3a^4$$
$$-80b^2+7a^2b^2-2b^4+968z-254a^2z+3a^4z+278b^2z-14a^2b^2z+9b^4z-700z^2$$
$$+175a^2z^2-203b^2z^2+7a^2b^2z^2-7b^4z^2),$$

where S_0, S_1 and S_2 are the remainders of the division of 1, X_2 and X_2^2 by B from the right, respectively. Then, defining

$$x_0 := -a^2b^2, \quad x_1 := -a^2+b^2+4, \quad x_2 := 1,$$

it is easily seen that x_0, x_1 and x_2 satisfy

$$x_0S_0 + x_1S_1 + x_2S_2 = 0, \qquad x_0 + x_1(-b^2) + x_2(-b^2)^2 = -4b^2 \neq 0.$$

Step 3. Define

$$F(t) := -a^2b^2 + (-a^2 + b^2 + 4)t + t^2.$$

Then, it is obvious from the relations satisfied by x_0, x_1 and x_2 as given in Step 2, that both of the operators $F(X_2)$ and $F(X_2)/F(-b^2)$ can be divided by B from the right without remainders.

Step 4. Define

$$X_3 := F(X_1)/F(-b^2).$$

It follows that $1 - X_3$ can be divided by A from the right without a remainder. Let R_2 be the quotient of this division. Then, by direct computation, we have

$$\begin{aligned}
16b^2z^2R_2 &= 64z^6(z-1)^6\frac{d^6}{dz^6} + 64z^5(z-1)^5(-15+34z)\frac{d^5}{dz^5} + 16z^4(z-1)^4(252 - 3a^2\\
&\quad - 1438z + 3a^2z - 3b^2z + 1615z^2 + 3b^2z^2)\frac{d^4}{dz^4} + 32z^3(z-1)^3(-156 + 9a^2 + 2000z\\
&\quad - 34a^2z + 12b^2z - 5593z^2 + 25a^2z^2 - 40b^2z^2 + 4144z^3 + 28b^2z^3)\frac{d^3}{dz^3} + 4z^2(z-1)^2\\
&\quad \times (288 - 84a^2 + 3a^4 - 12112z + 820a^2z - 6a^4z - 168b^2z + 6a^2b^2z + 73832z^2\\
&\quad - 1710a^2z^2 + 3a^4z^2 + 1272b^2z^2 - 12a^2b^2z^2 + 3b^4z^2 - 134500z^3 + 974a^2z^3\\
&\quad - 2362b^2z^3 + 6a^2b^2z^3 - 6b^4z^3 + 73827z^4 + 1258b^2z^4 + 3b^4z^4)\frac{d^2}{dz^2} + 4z(z-1)\\
&\quad \times (12a^2 - 3a^4 + 1568z - 480a^2z + 22a^4z + 48b^2z - 12a^2b^2z - 28376z^2 + 2302a^2z^2\\
&\quad - 35a^4z^2 - 1088b^2z^2 + 62a^2b^2z^2 - 9b^4z^2 + 108880z^3 - 3360a^2z^3 + 16a^4z^3\\
&\quad + 4264b^2z^3 - 88a^2b^2z^3 + 40b^4z^3 - 144739z^4 + 1526a^2z^4 - 5604b^2z^4 + 38a^2b^2z^4\\
&\quad - 53b^4z^4 + 62622z^5 + 2380b^2z^5 + 22b^4z^5)\frac{d}{dz} + (-16a^2 + 8a^4 - a^6 + 184a^2z\\
&\quad - 58a^4z + 3a^6z + 12a^2b^2z - 3a^4b^2z + 5600z^2 - 1940a^2z^2 + 147a^4z^2 - 3a^6z^2\\
&\quad + 512b^2z^2 - 164a^2b^2z^2 + 9a^4b^2z^2 + 12b^4z^2 - 3a^2b^4z^2 - 56400z^3 + 5692a^2z^3\\
&\quad - 152a^4z^3 + a^6z^3 - 5064b^2z^3 + 462a^2b^2z^3 - 9a^4b^2z^3 - 138b^4z^3 - 138b^4z^3 + 9a^2b^4z^3\\
&\quad - b^6z^3 + 151500z^4 - 6195a^2z^4 + 55a^4z^4 + 13272b^2z^4 - 480a^2b^2z^4 + 3a^4b^2z^4\\
&\quad + 363b^4z^4 - 9a^2b^4z^4 + 3b^6z^4 - 155750z^5 + 2275a^2z^5 - 13375b^2z^5 + 170a^2b^2z^5\\
&\quad - 360b^4z^5 + 3a^2b^4z^5 - 3b^6z^5 + 55125z^6 + 4655b^2z^6 + 123b^4z^6 + b^6z^6).
\end{aligned}$$

Step 5. Divide R_2 by B from the right, and let R denote the remainder of this division. Then, by direct computation, we have

$$R = z^2(z-1)^4\left\{\frac{d^2}{dz^2} + \left(\frac{3}{z} + \frac{3}{z-1}\right)\frac{d}{dz} + \left(\frac{a^2-4}{4z^2(z-1)} + \frac{b^2+25}{4z(z-1)} + \frac{-1}{4z(z-1)^2}\right)\right\}.$$

Next, let C be the quotient of the division of $AR - 1$ by B from the right. Then, by direct computation, we have

$$C = z^2(z-1)^4.$$

It is easy to confirm that R and C satisfy

$$AR - CB = 1.$$

We thus find that the algorithm has been successful in determining the splitting operator R.

3.4 An explicit expression of solutions

In this subsection, we obtain an explicit expression of solutions of the differential equation $Lu(z) = 0$ given in §2.2 by using the splitting operator R constructed in §3.3. (Refer to [2] for details.)

Recall that Q_1 can be factorized as

$$Q_1 = P_3P_2 = P_3^- P_2^-,$$

and thus that L can be factorized as

$$L = P_3P_2P_1 = P_3^- P_2^- P_1.$$

Next, define the mapping

$$\Psi : \{w^+(z); P_2 w^+(z) = 0\} \oplus \{w^-(z); P_2^- w^-(z) = 0\} \to \{w(z); P_3P_2 w(z) = 0\}$$

by

$$\Psi(w^+(z), w^-(z)) := w^+(z) + w^-(z).$$

If $w^+(z) + w^-(z) = 0$, then from the relation $P_2^- - P_2 = \frac{a}{z^2(1-z)}$, we obtain

$$0 = P_2 w^+(z) = -P_2 w^-(z) = \left(\frac{a}{z^2(1-z)} - P_2^-\right) w^-(z) = \frac{a}{z^2(1-z)} w^-(z).$$

Hence, since $a \neq 0$, we find $w^-(z) = 0$. This implies, by the above assumption, that $w^+(z) = 0$. We thus find that Ψ is injective. Thus, because the dimensions of the two spaces $\{w^+(z); P_2 w^+(z) = 0\} \oplus \{w^-(z); P_2^- w^-(z) = 0\}$ and $\{w(z); P_3P_2 w(z) = 0\}$ are the same, Ψ is a linear isomorphism.

As in the proof of Proposition 3.1.2, define the mapping

$$\Phi : \{v(z); P_1 v(z) = 0\} \oplus \{w(z); P_3P_2 w(z) = 0\} \to \{u(z); Lu(z) = 0\}$$

by

$$\Phi(v(z), w(z)) := v(z) + Rw(z).$$

This is a linear isomorphism. Therefore

$$\begin{aligned} \Phi \circ (1 \oplus \Psi) : \{v(z); P_1 v(z) = 0\} &\oplus \{w^+(z); P_2 w^+(z) = 0\} \oplus \{w^-(z); P_2^- w^-(z) = 0\} \\ &\to \{u(z); Lu(z) = 0\} \end{aligned}$$

is also a linear isomorphism.

Let $\{v_1(z), v_2(z)\}$, $\{w_1^+(z), w_2^+(z)\}$ and $\{w_1^-(z), w_2^-(z)\}$ be fundamental systems of solutions of the equations $P_1 v(z) = 0$, $P_2 w^+(z) = 0$ and $P_2^- w^-(z) = 0$, respectively. For each $i \in \{1, 2\}$, define

$$u_i(z) := v_i(z), \quad u_{i+2}(z) := Rw_i^+(z), \quad u_{i+4}(z) := Rw_i^-(z).$$

Then by the argument given in the previous paragraphs, $\{u_j(z); j = 1, \dots, 6\}$ is a fundamental system of solutions of the differential equation $Lu(z) = 0$.

By dividing R by P_2 and P_2^- from the right, it is confirmed that the second equality in each of the following expressions holds:

$$u_{i+2}(z) = Rw_i^+(z) = z(1-z)^3 \left(\frac{d}{dz} + \frac{a+2}{2z} + \frac{3}{2(z-1)}\right) w_i^+(z),$$

$$u_{i+4}(z) = Rw_i^-(z) = z(1-z)^3 \left(\frac{d}{dz} + \frac{-a+2}{2z} + \frac{3}{2(z-1)}\right) w_i^-(z).$$

3.5 An explicit expression of a fundamental system of solutions

In this subsection, we obtain an explicit expression of the fundamental system $\{u_j(z); j = 1, \dots, 6\}$ of solutions given in §3.4 in terms of hypergeometric functions by imposing a to be generic.

The characteristic exponents of the equations $P_1 v(z) = 0$, $P_2 w^+(z) = 0$ and $P_2^- w^-(z) = 0$ are

- $\frac{a}{2}$, $\frac{-a}{2}$ $(z = 0)$; $\frac{2+\sqrt{5}}{2}$, $\frac{2-\sqrt{5}}{2}$ $(z = 1)$; $\frac{-1+b\sqrt{-1}}{2}$, $\frac{-1-b\sqrt{-1}}{2}$ $(z = \infty)$,
- $\frac{a}{2}$, $\frac{-a-2}{2}$ $(z = 0)$; $\frac{-1}{2}$, $\frac{-5}{2}$ $(z = 1)$; $\frac{5+b\sqrt{-1}}{2}$, $\frac{5-b\sqrt{-1}}{2}$ $(z = \infty)$,

and

- $\frac{a-2}{2}$, $\frac{-a}{2}$ $(z = 0)$; $\frac{-1}{2}$, $\frac{-5}{2}$ $(z = 1)$; $\frac{5+b\sqrt{-1}}{2}$, $\frac{5-b\sqrt{-1}}{2}$ $(z = \infty)$,

respectively. The differences between the exponents at $z = 0$ are a, $a + 1$, and $a - 1$, respectively.

Assumption 3.5.1. The parameter a is not an integer.

This assumption is equivalent to the assumption that none of the values a, $-a$, $a + 1$, $-a - 1$, $a - 1$ and $-a + 1$ is a negative integer. Under this condition, we can choose fundamental systems of solutions near $z = 0$ explicitly as follows:

$$v_1(z) := z^{\frac{a}{2}}(1-z)^{\frac{2+\sqrt{5}}{2}} F\left(\frac{a+1+b\sqrt{-1}+\sqrt{5}}{2}, \frac{a+1-b\sqrt{-1}+\sqrt{5}}{2}; a+1; z\right),$$

$$v_2(z) := z^{\frac{-a}{2}}(1-z)^{\frac{2+\sqrt{5}}{2}} F\left(\frac{-a+1+b\sqrt{-1}+\sqrt{5}}{2}, \frac{-a+1-b\sqrt{-1}+\sqrt{5}}{2}; -a+1; z\right),$$

$$w_1^+(z) := z^{\frac{a}{2}}(1-z)^{\frac{-1}{2}} F\left(\frac{a+4+b\sqrt{-1}}{2}, \frac{a+4-b\sqrt{-1}}{2}; a+2; z\right),$$

$$w_2^+(z) := z^{\frac{-a-2}{2}}(1-z)^{\frac{-1}{2}} F\left(\frac{-a+2+b\sqrt{-1}}{2}, \frac{-a+2-b\sqrt{-1}}{2}; -a; z\right),$$

$$w_1^-(z) := z^{\frac{-a}{2}}(1-z)^{\frac{-1}{2}} F\left(\frac{-a+4+b\sqrt{-1}}{2}, \frac{-a+4-b\sqrt{-1}}{2}; -a+2; z\right),$$

$$w_2^-(z) := z^{\frac{a-2}{2}}(1-z)^{\frac{-1}{2}} F\left(\frac{a+2+b\sqrt{-1}}{2}, \frac{a+2-b\sqrt{-1}}{2}; a; z\right),$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function, mentioned in §1. In these expressions, we regard that all of these functions are defined on the simply connected domain $\mathbf{C} - \Lambda$, where Λ is the subset of $\mathbf{C}$ defined by

$$\Lambda := \{z \in \mathbf{R}\ ;\ z \leq 0 \text{ or } 1 \leq z\ \}.$$

Moreover, we choose a branch of the logarithm such that $-\pi < \mathrm{Im}(\log z) < \pi$ and $-\pi < \mathrm{Im}(\log(1-z)) < \pi$. Then the functions $z^\kappa := \exp(\kappa \log z)$ and $(1-z)^\kappa := \exp(\kappa \log(1-z))$ are single-valued functions on the domain $\mathbf{C} - \Lambda$ for any $\kappa \in \mathbf{C}$, and are real-valued if $\kappa \in \mathbf{R}$ and $0 < z < 1$.

Remark. If we do not enforce Assumption 3.5.1, we may have to resort to a standard procedure in the theory of hypergeometric functions, in which logarithmic terms are taken to form the fundamental systems of solutions (see [4]).

Finally, using the formula

$$\frac{d}{dz} F(\alpha, \beta; \gamma; z) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1; \gamma+1; z),$$

and imposing Assumption 3.5.1, we obtain explicitly the fundamental system of solutions near $z = 0$ of the differential equation $Lu(z) = 0$:

$$u_1(z) = z^{\frac{a}{2}}(1-z)^{\frac{2+\sqrt{5}}{2}} F\left(\frac{a+1+b\sqrt{-1}+\sqrt{5}}{2}, \frac{a+1-b\sqrt{-1}+\sqrt{5}}{2}; a+1; z\right),$$

$$u_2(z) = z^{\frac{-a}{2}}(1-z)^{\frac{2+\sqrt{5}}{2}} F\left(\frac{-a+1+b\sqrt{-1}+\sqrt{5}}{2}, \frac{-a+1-b\sqrt{-1}+\sqrt{5}}{2}; -a+1; z\right),$$

$$\begin{aligned} u_3(z) &= Rw_1^+(z) \\ &= z(1-z)^3\left(\frac{d}{dz} + \frac{a+2}{2z} + \frac{3}{2(z-1)}\right) w_1^+(z) \\ &= z^{\frac{a}{2}}(1-z)^{\frac{3}{2}}(-2z - az + 1 + a) F\left(\frac{a+4+b\sqrt{-1}}{2}, \frac{a+4-b\sqrt{-1}}{2}; a+2; z\right) \\ &\quad + \frac{(a+4+b\sqrt{-1})(a+4-b\sqrt{-1})}{4(a+2)} z^{\frac{a+2}{2}}(1-z)^{\frac{5}{2}} F\left(\frac{a+6+b\sqrt{-1}}{2}, \frac{a+6-b\sqrt{-1}}{2}; a+3; z\right), \end{aligned}$$

$$\begin{aligned} u_4(z) &= Rw_2^+(z) \\ &= z(1-z)^3\left(\frac{d}{dz} + \frac{a+2}{2z} + \frac{3}{2(z-1)}\right) w_2^+(z) \\ &= -z^{\frac{-a}{2}}(1-z)^{\frac{3}{2}} F\left(\frac{-a+2+b\sqrt{-1}}{2}, \frac{-a+2-b\sqrt{-1}}{2}; -a; z\right) \end{aligned}$$

$$- \frac{(-a+2+b\sqrt{-1})(-a+2-b\sqrt{-1})}{4a} z^{\frac{-a}{2}}(1-z)^{\frac{5}{2}} F\left(\frac{-a+4+b\sqrt{-1}}{2}, \frac{-a+4-b\sqrt{-1}}{2}; -a+1; z\right),$$

$$\begin{aligned}
u_5(z) &= Rw_1^-(z) \\
&= z(1-z)^3\left(\frac{d}{dz} + \frac{-a+2}{2z} + \frac{3}{2(z-1)}\right) w_1^-(z) \\
&= z^{\frac{-a}{2}}(1-z)^{\frac{3}{2}}(-2z+az+1-a) F\left(\frac{-a+4+b\sqrt{-1}}{2}, \frac{-a+4-b\sqrt{-1}}{2}; -a+2; z\right) \\
&\quad + \frac{(-a+4+b\sqrt{-1})(-a+4-b\sqrt{-1})}{4(-a+2)} z^{\frac{-a+2}{2}}(1-z)^{\frac{5}{2}} F\left(\frac{-a+6+b\sqrt{-1}}{2}, \frac{-a+6-b\sqrt{-1}}{2}; -a+3; z\right),
\end{aligned}$$

$$\begin{aligned}
u_6(z) &= Rw_2^-(z) \\
&= z(1-z)^3\left(\frac{d}{dz} + \frac{-a+2}{2z} + \frac{3}{2(z-1)}\right) w_2^-(z) \\
&= -z^{\frac{a}{2}}(1-z)^{\frac{3}{2}} F\left(\frac{a+2+b\sqrt{-1}}{2}, \frac{a+2-b\sqrt{-1}}{2}; a; z\right) \\
&\quad + \frac{(a+2+b\sqrt{-1})(a+2-b\sqrt{-1})}{4a} z^{\frac{a}{2}}(1-z)^{\frac{5}{2}} F\left(\frac{a+4+b\sqrt{-1}}{2}, \frac{a+4-b\sqrt{-1}}{2}; a+1; z\right).
\end{aligned}$$

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