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A Remark on Löwner's Theorem

Toshiyuki SUGAWA*

Graduate School of Information Sciences, Tohoku University, Aoba-ku, Sendai 980-8579, Japan

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We give a refinement of Löwner's inequality with argument of the equality case. To this end, we establish a complete univalence criterion for meromorphic functions of special type. We also give applications of the refinement.

KEYWORDS: Univalent function, Möbius transformation, univalence criterion, area theorem

1. Introduction

Let Σ be the set of univalent meromorphic functions F on $\Delta = \{\zeta \in \mathbb{C} : |\zeta| > 1\} \cup \{\infty\}$ of the form

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}, \quad |\zeta| > 1.$$

Gronwall's area theorem

$$\sum_{n=1}^{\infty} n|b_n|^2 \le 1 \tag{1}$$

for $F \in \Sigma$ leads to many valuable results in the theory of univalent functions (see [2] for instance). We remark that equality holds in (1) precisely when the image $F(\Delta)$ is of full measure, namely, $\mathbb{C} \setminus F(\Delta)$ is of area zero.

As a simple consequence of the area theorem, K. Löwner's theorem [5, Satz V] can be deduced.

Theorem A (Löwner). *Let $F \in \Sigma$. Then*

$$|F'(\zeta)| \le \frac{|\zeta|^2}{|\zeta|^2 - 1}, \quad |\zeta| > 1,$$

where equality holds at a finite point $\zeta = \zeta_0$ in Δ if and only if

$$F(\zeta) = \zeta + b_0 - \frac{\zeta_0 - \bar{\zeta}_0^{-1}}{\bar{\zeta}_0 \zeta - 1}, \quad |\zeta| > 1$$

for a constant b_0.

We slightly improve this theorem as in the following.

Theorem 1. *For $F \in \Sigma$, the inequality*

$$|F'(\zeta) - 1| \le \frac{1}{|\zeta|^2 - 1} \tag{2}$$

holds for each $\zeta \in \mathbb{C}$ with $|\zeta| > 1$. Moreover, equality holds at $\zeta = \zeta_0$ if and only if

$$F(\zeta) = \zeta + b_0 - \frac{\zeta_0 - \bar{\zeta}_0^{-1}}{\bar{\zeta}_0 \zeta - 1}, \quad |\zeta| > 1,$$

for a constant b_0.

Since $|F'(\zeta)| - 1 \le |F'(\zeta) - 1|$, Löwner's theorem immediately follows from Theorem 1. Indeed, as will be seen later, Löwner's estimation in [5] works for (2) as well. However, discussion for the equality case becomes more subtle in the proof of our result.

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* Corresponding author. E-mail: sugawa@math.is.tohoku.ac.jp

We have to mention a theorem of Goluzin (cf. [3, (12) in p. 133]), which is a special case of well-known Goluzin's inequality.

Theorem B. *Let $F \in \Sigma$ and $-\pi/2 \le \alpha \le \pi/2$. Then*

$$\operatorname{Re}\left\{e^{2i\alpha} \log F'(\zeta)\right\} \le \left|\log F'(\zeta)\right| \le \log \frac{|\zeta|^2}{|\zeta|^2 - 1}, \quad |\zeta| > 1.$$

Here, for each finite point $\zeta = \zeta_0 \in \Delta$ and for each α, equality holds at the same time in the above inequalities when F maps Δ onto the extended plane slit along an arc of the curve $\{w : \operatorname{Re}(e^{i\alpha} \log(w - F(\zeta_0))) = K\}$ for a constant K.

Indeed, this inequality implies (2) (see Remark in Section 3). Since the description of the equality case in Goluzin's theorem is somewhat implicit, our theorem seems to have its own merit.

The present paper is organized as follows. In the second section, we give two elementary lemmas, which may be useful in another situation. The proof of Theorem 1 is provided in Section 3. The final section is devoted to a few applications of Theorem 1.

2. Some Lemmas

Denote by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. We now show the following elementary lemma.

Lemma 2. *Let $L(z) = (az + b)/(cz + d)$ be a Möbius transformation, where $a, b, c, d \in \mathbb{C}$ with $ad - bc \ne 0$. Then L maps the unit disk $\mathbb{D}$ into itself if and only if*

$$|a\bar{c} - b\bar{d}| + |ad - bc| \le |d|^2 - |c|^2.$$

Proof. First we assume that $L(\mathbb{D}) \subset \mathbb{D}$. Then $L^{-1}(\infty) = -d/c \in \Delta$, and thus, $|c| < |d|$. Let $\alpha = \bar{c}/\bar{d} \in \mathbb{D}$. We can then express L by

$$L(z) = A\,\frac{z + \alpha}{1 + \bar{\alpha} z} + B,$$

where

$$A = \frac{ad - bc}{d^2(1 - |c/d|^2)}, \quad B = \frac{b\bar{d} - a\bar{c}}{|d|^2 - |c|^2}.$$

Since the Möbius transformation $(z + \alpha)/(1 + \bar{\alpha} z)$ maps $\mathbb{D}$ onto itself, the image $L(\mathbb{D})$ is the disk centered at B with radius $|A|$. Therefore, $L(\mathbb{D}) \subset \mathbb{D}$ implies $|A| + |B| \le 1$, which is equivalent to the inequality in the assertion.

By tracing back the above argument, we prove the other way round, as well. □

Secondly, we give a characterization of univalence for meromorphic functions of a specific form.

Lemma 3. *Let b_0, A and B be complex numbers with $AB \ne 0$. Then the meromorphic function*

$$F(\zeta) = \zeta + b_0 + \frac{A}{B\zeta - 1}$$

is univalent in Δ if and only if

$$|A + \bar{B} - B^{-1}| + |AB| \le |B|^2 - 1. \tag{3}$$

Proof. Suppose that $F(\zeta) = F(\omega)$ for some $\zeta, \omega \in \mathbb{C}$ with $\zeta \ne \omega$. A straightforward computation yields the relation $(B\zeta - 1)(B\omega - 1) = AB$, which is equivalent to $\omega = L(1/\zeta)$, where

$$L(z) = \frac{(A - 1/B)z + 1}{-z + B}.$$

If $L(\mathbb{D}) \subset \mathbb{D}$, then $L(1/\zeta) \in \mathbb{D}$ for each $\zeta \in \Delta$. Therefore, according to the above computation, there is no other point $\omega \in \Delta$ than ζ such that $F(\zeta) = F(\omega)$. Conversely, if F is univalent in Δ, the point $\omega = L(1/\zeta)$ does not belong to $\Delta \setminus \{\zeta\}$ for each $\zeta \in \Delta$, in other words, $|L(z)| \le 1$ for each $z \in \mathbb{D}$ except for the fixed points of the Möbius transformation $1/L(z)$. Since the number of fixed points of a Möbius transformation (except for the identity) is at most two, we conclude that $L(\mathbb{D}) \subset \mathbb{D}$. In this way, we have shown that F is univalent in Δ if and only if $L(\mathbb{D}) \subset \mathbb{D}$. We now apply Lemma 2 to deduce the assertion. □

3. Proof of Theorem 1

We follow the argument in [5] for the first part. Let $F(\zeta) = \zeta + b_0 + b_1/\zeta + b_2/\zeta^2 + \cdots$ be in Σ. Then

$$F'(\zeta) = 1 - \sum_{n=1}^{\infty} n b_n \zeta^{-n-1}.$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |F'(\zeta) - 1| &\le \sum_{n=1}^{\infty} n|b_n||\zeta|^{-n-1} \\ &\le \sqrt{\sum_{n=1}^{\infty} n|b_n|^2 \cdot \sum_{n=1}^{\infty} n|\zeta|^{-2n-2}} \\ &= \sqrt{\sum_{n=1}^{\infty} n|b_n|^2 \cdot \frac{1}{(|\zeta|^2 - 1)^2}}. \end{aligned}$$

We now use (1) to obtain (2).

Next we consider the equality case. We suppose now that equality holds in (2) for $\zeta = \zeta_0$. Then the sequences $\{b_n\}$ and $\{\bar{\zeta}_0^{-n-1}\}$ must be proportional, in other words, there exists a complex number a such that $b_n = a\bar{\zeta}_0^{-n}$ for $n \ge 1$. Also, since equality must hold in (1), we have $|a| = R - R^{-1}$, where $R = |\zeta_0| > 1$.

Then

$$F(\zeta) = \zeta + b_0 + a\sum_{n=1}^{\infty} \bar{\zeta}_0^{-n}\zeta^{-n} = \zeta + b_0 + \frac{a}{\bar{\zeta}_0\zeta - 1}.$$

For this function, clearly equality holds in (2) at $\zeta = \zeta_0$. It remains to see when this function F is univalent in Δ. We now apply Lemma 3 to see that F is univalent precisely when

$$\left|a + \zeta_0 - \bar{\zeta}_0^{-1}\right| + |a|R = \left|a + \zeta_0 - \bar{\zeta}_0^{-1}\right| + R^2 - 1 \le R^2 - 1,$$

which is equivalent to $|a + \zeta_0 - \bar{\zeta}_0^{-1}| = 0$. Therefore, F is univalent precisely if $a = -(\zeta_0 - \bar{\zeta}_0^{-1})$, which proves the equality part.

Remark. Inequality (2) also follows from Goluzin's inequality in Theorem B. Indeed, for a fixed ζ, we set $w = \log F'(\zeta)$. Then

$$\begin{aligned} |F'(\zeta) - 1| = |e^w - 1| &= \left|\sum_{n=1}^{\infty} \frac{w^n}{n!}\right| \\ &\le \sum_{n=1}^{\infty} \frac{|w|^n}{n!} = e^{|w|} - 1. \end{aligned}$$

Since Goluzin's inequality yields $e^{|w|} \le |\zeta|^2/(|\zeta|^2 - 1)$, we obtain (2). Equality holds precisely when $w = \log(|\zeta|^2/(|\zeta|^2 - 1))$.

4. Remarks and Applications

First we remark on the image of Δ under the extremal map

$$F_{\zeta_0}(\zeta) = \zeta - \frac{\zeta_0 - \bar{\zeta}_0^{-1}}{\bar{\zeta}_0\zeta - 1}$$

for $\zeta_0 = Re^{i\alpha}$, $R > 1$. As is stated in [5], the image $F_{\zeta_0}(\Delta)$ is the slit domain along a proper subarc of a circle of radius R. More precisely, we have the following.

Lemma 4. *Let $\zeta_0 = Re^{i\alpha} \in \Delta$. Then $F_{\zeta_0}(\Delta) = \widehat{\mathbb{C}} \setminus C_{\zeta_0}$, where C_{ζ_0} is the closed circular arc which is contained in the circle $\{\zeta : |\zeta - (\zeta_0 - \bar{\zeta}_0^{-1})| = R\}$, has the endpoints at $(1 \pm i\sqrt{R^2 - 1})/\bar{\zeta}_0$, and contains the point $-\bar{\zeta}_0^{-1}$.*

Proof. First observe that the image of the unit circle is contained in the circle in the assertion. Indeed, since

$$F_{\zeta_0}(\zeta) - \zeta_0 + \frac{1}{\bar{\zeta}_0} = \frac{(\bar{\zeta}_0\zeta - R^2)\zeta}{\bar{\zeta}_0\zeta - 1} = -\bar{\zeta}_0\zeta \cdot \frac{\zeta - \zeta_0}{1 - \bar{\zeta}_0\zeta},$$

we have $|F_{\zeta_0}(\zeta) - \zeta_0 + \bar{\zeta}_0^{-1}| = R$ for $|\zeta| = 1$.

Note that the critical points $\zeta^{\pm} = (1 \pm \mathrm{i}\sqrt{R^2 - 1})/\bar{\zeta}_0$ of the rational function F_{ζ_0} are contained in the unit circle. Therefore, the image of the unit circle under F_{ζ_0} is folded at the critical values $F(\zeta^{\pm}) = (1 \pm 2\mathrm{i}\sqrt{R^2 - 1})/\bar{\zeta}_0$. Since $F(\pm \mathrm{e}^{\mathrm{i}\alpha}) = -\bar{\zeta}_0^{-1}$, we now have the required assertion. □

We denote by $\mathscr{M}$ the set of meromorphic functions F in Δ with $F(\zeta) = \zeta + b_0 + O(\zeta^{-1})$ as $\zeta \to \infty$. Also, we denote by $\mathscr{A}$ the set of analytic functions f in $\mathbb{D}$ with $f(z) = z + O(z^2)$ as $z \to 0$. In the theory of univalent functions, the main object is the set $\mathscr{S}$ of univalent functions in $\mathscr{A}$. For $f \in \mathscr{S}$, the function $F(\zeta) = 1/f(1/\zeta)$ belongs to Σ and satisfies $F(\zeta) \neq 0$ for $\zeta \in \Delta$. Note then that $F'(\zeta) = f'(1/\zeta)/(\zeta^2 f(1/\zeta)^2) = z^2 f'(z)/f(z)^2$, where $z = 1/\zeta$. Therefore, Theorem 1 yields the following.

Corollary 5. *For $f \in \mathscr{S}$, the inequality*

$$|z^2 f'(z) - f(z)^2| \leq \frac{|zf(z)|^2}{1 - |z|^2}$$

holds for $z \in \mathbb{D}$. Equality holds at $z_0 = \rho \mathrm{e}^{\mathrm{i}\alpha} \in \mathbb{D} \setminus \{0\}$ precisely when

$$f(z) = \frac{z}{1 - cz - \mathrm{e}^{2\mathrm{i}\alpha} z^2 \frac{1-\rho^2}{1-\bar{z}_0 z}}$$

for some $c \in C_{1/z_0}$. Here, C_{ζ_0} is the circular arc given in Lemma 4.

Of course, inequality (2) is a necessary condition for F to be univalent. As a sufficient condition, we have the following criterion due to Aksent'ev.

Theorem C (Aksent'ev [1])**.** *If $F \in \mathscr{M}$ satisfies the inequality*

$$|F'(\zeta) - 1| \leq 1, \quad |\zeta| > 1, \tag{4}$$

then F is univalent in Δ.

Note that if F is meromorphically convex, namely, if $\mathrm{Re}(1 + \zeta F''(\zeta)/F'(\zeta)) > 0$ in $|\zeta| > 1$, then F satisfies (4) (see [8, Cor. 2.5]). Krzyż [4] further showed that if $|F'(\zeta) - 1| \leq k < 1$, then F extends to a k-quasiconformal mapping of $\widehat{\mathbb{C}}$.

In relation with these results, for $k > 0$ Ponnusamy and others studied the class $\mathscr{U}(k)$ consisting of functions $f \in \mathscr{A}$ satisfying

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| \leq k, \quad |z| < 1$$

(see, for instance, [6]). Aksent'ev's theorem implies that $\mathscr{U}(1) \subset \mathscr{S}$.

For $F \in \mathscr{M}$ and $R > 1$, we define $F_R \in \mathscr{M}$ by $F_R(\zeta) = F(R\zeta)/R$. Since $F_R'(\zeta) = F'(R\zeta)$, Theorem 1 yields

$$|F_R'(\zeta) - 1| \leq \frac{1}{R^2|\zeta|^2 - 1} < \frac{1}{R^2 - 1}, \quad |\zeta| > 1.$$

In particular, for $R \geq \sqrt{1 + 1/k}$, we have $|F_R'(\zeta) - 1| \leq k$, $|\zeta| > 1$. From this observation, we can deduce the following result of Obradović and Ponnusamy [7, Theorem 2.4].

Corollary 6. *Let $f \in \mathscr{S}$ and $k > 0$. Then for $0 < r \leq 1/\sqrt{1 + 1/k}$, the function $f_r(z) = rf(z/r)$ belongs to the class $\mathscr{U}(k)$.*

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