Application of the Study's Coordinate of a Straight Line to Geometrical Optics. Part I

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Synopsis

In the present paper, the Study's coordinate of a straight line has been introduced into geometrical optics. From the result the law of reflection of light and that of refraction have been expressed respectively as follows :

$$\mathfrak{S}_r=\mathfrak{S}_e-(\mathfrak{N}\mathfrak{S}_e)\mathfrak{N}$$
$$\mathfrak{S}_b=\frac{1}{\mu}[\mathfrak{S}_e+\sqrt{\mu^2-1+(\mathfrak{N}\mathfrak{S}_e)^2}-\mathfrak{N}\mathfrak{S}_e]\mathfrak{N},$$

where $\mathfrak{S}_e$, $\mathfrak{S}_r$, $\mathfrak{S}_b$ and $\mathfrak{N}$ are Study's coordinates respectively of the incident ray, reflected ray, refracted ray and normal to the surface of reflection or refraction at the point of incidence, and μ is the index of refraction.

For example of application of this method a problem with respect to the curvature of the lines in a prismatic spectrum has been treated.

I. Intorduction

By Dr. Kubota's(1) suggestion, the late Dr. Maeda(2) successfully introduced the Study's(3) coordinate of a straight line into the theory of skew bevel gears toothed with skew ruled surfaces. The writer, who mainly studies the theory of skew bevel gears in line with the Maeda's theory, tried to introduce the Study's coordinate of a straight line into geometrical optics.

II. Study's Coordinate of a Straight Line

In this section some properties of the Study's coordinate of a straight line are mentioned.

Take a unit vector $\mathfrak{a}$ on a straight line and let $\mathfrak{r}$ be the radius vector drawn from the origin to any point on the straight line. Study's coordinate of a straight line is defined by $\mathfrak{a}+\varepsilon\mathfrak{r}\times\mathfrak{a}$; where ε denotes an imaginary number which itself is not zero but its square is zero, and $\mathfrak{r}\times\mathfrak{a}$ represents the vector product of $\mathfrak{r}$ and $\mathfrak{a}$. There is a relation that the scalar product

$$(\mathfrak{a}+\varepsilon\mathfrak{r}\times\mathfrak{a})^2=1. \qquad (1)$$

In the following treatise German capital letters represent the Study's coordinates of straight lines, and German small letters represent real vectors.

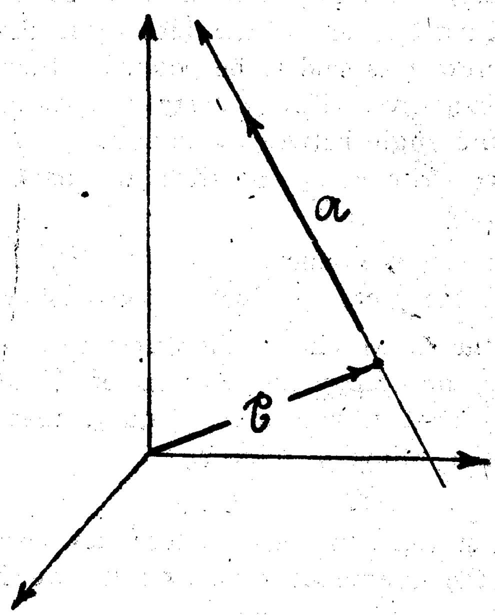

Fig. 1

Let $\mathfrak{C}$ be the Study's coordinate of the straight line which intersects at right angles simultaneously with straight lines $\mathfrak{A}$ and $\mathfrak{B}$, and has the direction determined by any choice.

The straight line $\mathfrak{A}$ may occupy the posi-

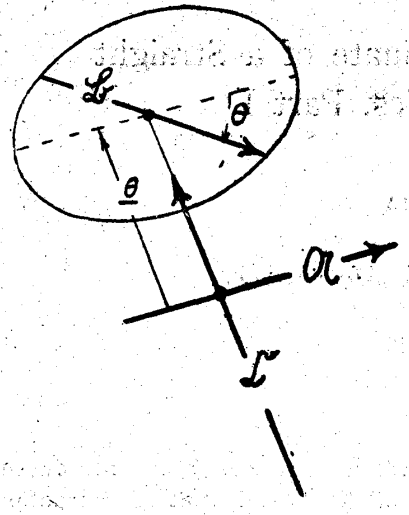

Fig. 2

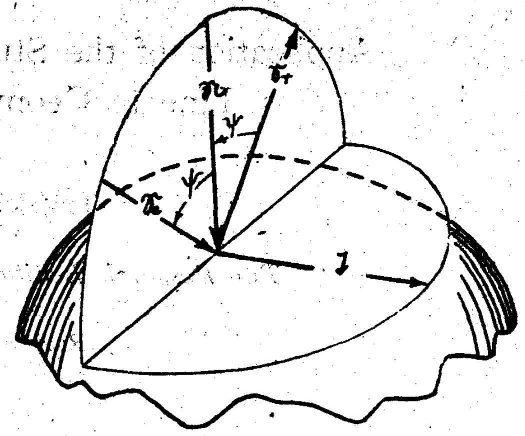

Fig. 3

tion of the straight line $\mathfrak{B}$ by the translation of magnitude $\underline{\theta}$ along $\mathfrak{C}$ and the rotation of angle θ around $\mathfrak{C}$. Consider the right-handed screw motion along $\mathfrak{C}$ by which a screw travels toward the direction of $\mathfrak{C}$. If the direction of the above described translation of $\mathfrak{A}$ coincides with that of the screw, $\underline{\theta}$ is said to be positive, otherwise to be negative. Similarly if the direction of the above described rotation of $\mathfrak{A}$ coincides with that of the screw, θ is said to be positive, otherwise to be negative. The quantity $\theta+\varepsilon\underline{\theta}$ is called the dual angle between $\mathfrak{A}$ and $\mathfrak{B}$.

If we write $\theta+\varepsilon\underline{\theta}=\Theta$, then we have the following relations

the scalar product $\mathfrak{A}\mathfrak{B}=\cos\Theta$ (2)

and the vector product $\mathfrak{A}\times\mathfrak{B}=\mathfrak{C}\sin\Theta$ (3)

Further if we choose the direction of $\mathfrak{C}$ so that $\mathfrak{A}$ can occupy the position of $\mathfrak{B}$ under merely positive values of θ and $\underline{\theta}$, then we have the following relation

$$\mathfrak{C}=\mathfrak{A}\times\mathfrak{B}/\sqrt{(\mathfrak{A}\times\mathfrak{B})^2}. \qquad (4)$$

If a straight line $\mathfrak{A}$ occupies a new position $\mathfrak{B}$ by the screw motion of axis $\mathfrak{G}$, we have

$$\mathfrak{B}=(\mathfrak{A}\mathfrak{G})\mathfrak{G}-(\mathfrak{A}\times\mathfrak{G})\times\mathfrak{G}\cos(\phi+\varepsilon\underline{\phi})$$
$$-\mathfrak{A}\times\mathfrak{G}\sin(\phi+\varepsilon\underline{\phi})$$

where ϕ is the angle of rotation and $\underline{\phi}$ the magnitude of translation along $\mathfrak{G}$.

III. Law of Reflexion

In Fig. 3 $\mathfrak{S}_e$, $\mathfrak{S}_r$ and $\mathfrak{N}$ are Study's coordinates respectively of the incident ray, reflected ray and normal to the reflecting surface at the point of incidence, and ψ is the angle of incidence. Consider a straight line $\mathfrak{T}$ defined by the expression

$$\mathfrak{T}=\mathfrak{N}\times\mathfrak{S}_e/\sqrt{(\mathfrak{N}\times\mathfrak{S}_e)^2}. \qquad (6)$$

From the expressions (4) and (6) it is clear that $\mathfrak{T}$ intersects simultaneously with $\mathfrak{N}$ and $\mathfrak{S}_e$ at right angles, then $\mathfrak{T}$ is one of the tangents of the reflecting surface at the point of incidence. Accordingly we may get $\mathfrak{S}_r$ by the rotation of $\mathfrak{S}_e$ about $\mathfrak{T}$ through the angle $\pi-2\psi$. Then applying the formulas (1), (2) and (4) we get

$$\cos\psi=\mathfrak{N}\mathfrak{S}_e \qquad (7)$$
$$\mathfrak{T}\sin\psi=\mathfrak{N}\times\mathfrak{S}_e \qquad (8)$$

and

$$\mathfrak{S}_r=(\mathfrak{S}_e\mathfrak{T})\mathfrak{T}-(\mathfrak{S}_e\times\mathfrak{T})\times\mathfrak{T}\cos(\pi-2\psi)$$
$$-\mathfrak{S}_e\times\mathfrak{T}\sin(\pi-2\psi). \qquad (9)$$

We have from (6)

$$\mathfrak{S}_e\mathfrak{T}=0,$$

from (7) and (8)

$$\cos(\pi-2\psi)=1-2(\mathfrak{N}\mathfrak{S}_e)^2$$
$$\mathfrak{T}\sin(\pi-2\psi)=2(\mathfrak{N}\mathfrak{S}_e)\mathfrak{N}\times\mathfrak{S}_e.$$

Then from (9)

$$\mathfrak{S}_r=-(\mathfrak{S}_e\times\mathfrak{T})\times\mathfrak{T}\{1-2(\mathfrak{N}\mathfrak{S}_e)^2\}$$
$$-2\mathfrak{S}_e\times(\mathfrak{N}\times\mathfrak{S}_e)(\mathfrak{N}\mathfrak{S}_e)$$
$$=\{(\mathfrak{S}_e\mathfrak{T})\mathfrak{T}-\mathfrak{S}_e\mathfrak{T}^2\}\{2(\mathfrak{N}\mathfrak{S}_e)^2-1\}$$
$$-2\{\mathfrak{N}\mathfrak{S}_e^{\,2}-(\mathfrak{N}\mathfrak{S}_e)\mathfrak{S}_e\}(\mathfrak{N}\mathfrak{S}_e)$$
$$=-\mathfrak{S}_e\{2(\mathfrak{N}\mathfrak{S}_e)^2-1\}-2\{\mathfrak{N}-(\mathfrak{N}\mathfrak{S}_e)\mathfrak{S}_e\}(\mathfrak{N}\mathfrak{S}_e)$$
$$=\mathfrak{S}_e-2(\mathfrak{N}\mathfrak{S}_e)\mathfrak{N}.$$

Thus we obtain an expression for the law of reflection in the form

$$\mathfrak{S}_r=\mathfrak{S}_e-2(\mathfrak{N}\mathfrak{S}_e)\mathfrak{N}. \qquad (10)$$

IV. Law of Refraction

In Fig. 4 $\mathfrak{S}_e$, $\mathfrak{S}_b$ and $\mathfrak{N}$ are respectively the incident ray, refracted ray and normal to the surface of refraction at the point of incidence, and ψ_e is the angle of incidence and ψ_b is that of refraction. In similar manner as

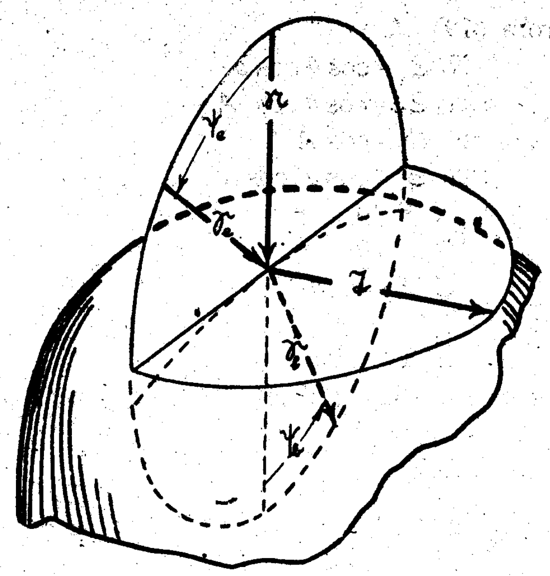

Fig. 4

before, the tangent $\mathfrak{T}$ is defined by the expression (6). Also in like manner as before we may get $\mathfrak{S}_b$ by the rotation $\mathfrak{S}_e$ about $\mathfrak{T}$ through the angle $\psi_b-\psi_e$. Then apylying the formulas (2), (3) and (5) we get

$$\cos\psi_e=\mathfrak{N}\mathfrak{S}_e \tag{11}$$

$$\mathfrak{T}\sin\psi_e=\mathfrak{N}\times\mathfrak{S}_e \tag{12}$$

$$\cos\psi_b=\mathfrak{N}\mathfrak{S}_b \tag{13}$$

$$\mathfrak{T}\sin\psi_b=\mathfrak{N}\times\mathfrak{S}_b \tag{14}$$

and

$$\mathfrak{S}_b=(\mathfrak{S}_e\mathfrak{T})\mathfrak{T}-(\mathfrak{S}_e\times\mathfrak{T})\times\mathfrak{T}\cos(\psi_b-\psi_e)$$
$$-\mathfrak{S}_e\times\mathfrak{T}\sin(\psi_b-\psi_e). \tag{15}$$

Moreover Snell's law claims

$$\sin\psi_e\Big/\sin\psi_b=\mu \tag{16}$$

where μ is the index of refraction.

We have from (6)

$$\mathfrak{S}_e\mathfrak{T}=0,$$

from (11), (12), (13), (14) and (16)

$$(\mathfrak{S}_e\times\mathfrak{T})\times\mathfrak{T}\cos(\psi_b-\psi_e)=(\mathfrak{S}_e\times\mathfrak{T})$$
$$\times\mathfrak{T}(\cos\psi_b\cos\psi_e+\sin\psi_b\sin\psi_e)$$
$$=\{(\mathfrak{S}_e\mathfrak{T})\mathfrak{T}-\mathfrak{S}_e\mathfrak{T}^2\}\Big[\cos\psi_e\sqrt{1-\frac{1}{\mu^2}\{1-(\mathfrak{N}\mathfrak{S}_e)^2\}}$$
$$+\frac{1}{\mu}(1-\cos^2\psi_e)\Big]$$
$$=-\mathfrak{S}_e\Big[(\mathfrak{N}\mathfrak{S}_e)\sqrt{1-\frac{1}{\mu^2}\{1-(\mathfrak{N}\mathfrak{S}_e)^2\}}$$
$$+\frac{1}{\mu}\{1-(\mathfrak{N}\mathfrak{S}_e)^2\}\Big]$$

and

$$\mathfrak{S}_e\times\mathfrak{T}\sin(\psi_b-\psi_e)=\mathfrak{S}_e\times\mathfrak{T}(\sin\psi_b\cos\psi_e$$
$$-\sin\psi_e\cos\psi_b)$$
$$=\mathfrak{S}_e\times\mathfrak{T}\sin\psi_e\Big[\frac{1}{\mu}\cos\psi_e-\sqrt{1-\frac{1}{\mu^2}(1-\cos^2\psi_e)}\Big]$$
$$=\mathfrak{S}_e\times(\mathfrak{N}\times\mathfrak{S}_e)\Big[\frac{1}{\mu}\mathfrak{N}\mathfrak{S}_e-\sqrt{1-\frac{1}{\mu^2}(1-\cos^2\psi_e)}\Big]$$
$$=\{\mathfrak{N}-\mathfrak{S}_e(\mathfrak{N}\mathfrak{S}_e)\}\Big[\frac{1}{\mu}\mathfrak{N}\mathfrak{S}_e-\sqrt{1-\frac{1}{\mu^2}\{1-(\mathfrak{N}\mathfrak{S}_e)^2\}}\Big].$$

Then from (15) we have the law of refraction in the form

$$\mathfrak{S}_b=\frac{1}{\mu}[\mathfrak{S}_e+\{\sqrt{\mu^2-1+(\mathfrak{N}\mathfrak{S}_e)^2}-\mathfrak{N}\mathfrak{S}_e\}\mathfrak{N}]. \tag{17}$$

Let μ_e be the absolute index of refraction of the medium through which the incident ray $\mathfrak{S}_e$ passes, and let μ_b be that of the medium through which the refracted ray $\mathfrak{S}_b$ passes.

In (17) if we put μ_b/μ_e in place of μ, and then put $\mu_e\mathbf{S}_e$ and $\mu_b\mathbf{S}_b$ in places of $\mathfrak{S}_e$ and $\mathfrak{S}_b$ respectively, (17) becomes

$$\mathbf{S}_b=\mathbf{S}_e+\{\sqrt{\mu_b^2-\mu_e^2+(\mathfrak{N}\mathbf{S}_e)^2}-\mathfrak{N}\mathbf{S}_e\}\mathfrak{N}. \tag{18}$$

V. An Example of Application

For example of the application of this method a problem connected with the curvature of the lines in the prismatic spectrum is treated as follows:

On this problem, several researchers(4)(5)(6) have already studied mathematically. For the convenience of comparing the result with that of H. C. Plummer's(5) calculation, the same notations A, v, θ, α, α' and μ are adopted as those which Plummer adopted in his treatise.

Here the position of the point of incidence is out of the question, therefore only the directions of rays and that of normals to the faces of the prism are taken seriously. Then a spherical representation of their directions sufficient for discussion. Accordingly we may treat the subject utilizing the laws of refraction (17) in which let the imaginary parts (containing the imaginary number ε) of Study's coordinates be zero in this case.

Fig. 5 represents the elevation and plan of the prism, a penetrating ray ($\mathfrak{S}_1$ $\mathfrak{S}_2$ and $\mathfrak{S}_3$) and normals ($\mathfrak{N}_1$, and $\mathfrak{N}_2$) to the faces of the prism.

Take a rectangular coordinate axes $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ as follows. Let axis $\mathfrak{X}$ coincide with the edge of the prism, and let axis $\mathfrak{Y}$ lie on one of the faces of the prism as shown in Fig. 5.

In Fig. 5 (a) the angle of the prism, the angle between the orthogonal projection of $\mathfrak{N}_1$ and that of $\mathfrak{S}_1$ on the principal plane of the prism and the angle between the orthogonal projection of $\mathfrak{N}_2$ and that of $\mathfrak{S}_3$, on the same plane are respectively denoted by A, v and α'. The angle between the ray

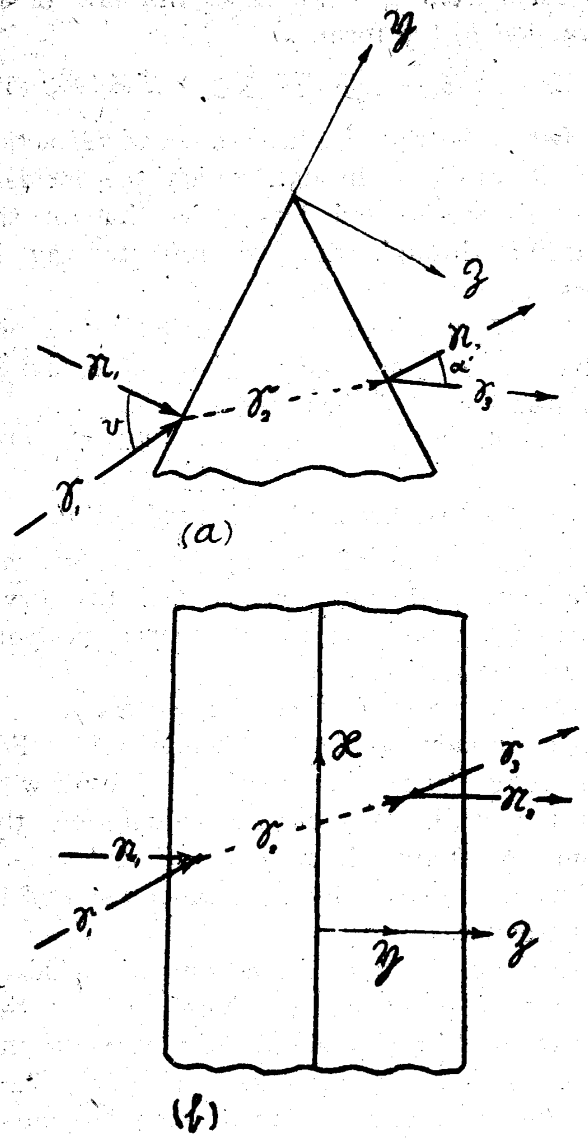

Fig. 5

$\mathfrak{S}_1$ and the principal plane and the angle between the ray $\mathfrak{S}_3$ and the same plane are represented by θ and θ' respectively. Direction-cosines of $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{S}_1$ and $\mathfrak{S}_3$ are given in the following table.

	$\mathfrak{X}$	$\mathfrak{Y}$	$\mathfrak{Z}$
$\mathfrak{N}_1$	0	0	1
$\mathfrak{N}_2$	0	$\sin A$	$\cos A$
$\mathfrak{S}_1$	$\sin\theta$	$\cos\theta\sin v$	$\cos\theta\cos v$
$\mathfrak{S}_2$	$\sin\theta'$	$\cos\theta'\sin(A-\alpha')$	$\cos\theta'\cos(A-\alpha')$

(19)

Applying the formula (17) to this case, we have in connection with the first refraction

$$\mathfrak{S}_2=\frac{1}{\mu}\mathfrak{S}_1+\frac{1}{\mu}\{\sqrt{\mu^2-1+(\mathfrak{N}_1\mathfrak{S}_1)^2}-\mathfrak{N}_1\mathfrak{S}_1\}\mathfrak{N}_1 \tag{20}$$

and in connection with the second refraction

$$\mathfrak{S}_3=\mu\mathfrak{S}_2+\mu\left\{\sqrt{\frac{1}{\mu^2}-1+(\mathfrak{N}_2\mathfrak{S}_2)^2}-\mathfrak{N}_2\mathfrak{S}_2\right\}\mathfrak{N}_2 \tag{21}$$

From (19)

$$\mathfrak{N}_1\mathfrak{S}_1=\cos\theta\cos v \tag{22}$$
$$\mathfrak{N}_2\mathfrak{S}_1=\cos\theta\cos(A-v) \tag{23}$$
$$\mathfrak{N}_1\mathfrak{N}_2=\cos A \tag{24}$$
$$\mathfrak{N}_2\mathfrak{S}_3=\cos\theta'\cos\alpha' \tag{25}$$
$$\mathfrak{N}_1\mathfrak{X}=0 \tag{26}$$
$$\mathfrak{N}_2\mathfrak{X}=0 \tag{27}$$
$$\mathfrak{S}_1\mathfrak{X}=\sin\theta \tag{28}$$

and

$$\mathfrak{S}_3\mathfrak{X}=\sin\theta'. \tag{29}$$

From (20), (21), (26), (27), (28) and (29) we have

$$\begin{aligned}\sin\theta'&=\mathfrak{S}_3\mathfrak{X}\\&=\left[\mu\mathfrak{S}_2+\mu\left\{\sqrt{\frac{1}{\mu^2}-1+(\mathfrak{N}_2\mathfrak{S}_2)^2}-\mathfrak{N}_2\mathfrak{S}_2\right\}\mathfrak{N}_2\right]\mathfrak{X}\\&=\mu\mathfrak{S}_2\mathfrak{X}=\left[\mathfrak{S}_1+\{\sqrt{\mu^2-1+(\mathfrak{N}_1\mathfrak{S}_1)^2}-\mathfrak{N}_1\mathfrak{S}_1\}\mathfrak{N}_1\right]\mathfrak{X}\\&=\mathfrak{S}_1\mathfrak{X}=\sin\theta.\end{aligned} \tag{30}$$

Substituting (22) in (20)

$$\mathfrak{S}_2=\frac{1}{\mu}\Big[\mathfrak{S}_1+\{\sqrt{\mu^2-1+\cos^2\theta\cos^2v}-\cos\theta\cos v\}\mathfrak{N}_1\Big] \tag{31}$$

Construct the scalar product of $\mathfrak{N}_2$ and $\mathfrak{S}_2$, taking (23) and (24) into consideration, then we have

$$\begin{aligned}\mathfrak{N}_2\mathfrak{S}_2&=\frac{1}{\mu}\Big[\cos\theta\cos(A-v)\\&\quad+(\sqrt{\mu^2-1+\cos^2\theta\cos^2v}-\cos\theta\cos v)\cos A\Big]\\&=\frac{1}{\mu}\Big[\sin A\cos\theta\sin v\\&\quad+(\sqrt{\mu^2-1+\cos^2\theta\cos^2v})\cos A\Big].\end{aligned} \tag{32}$$

Substituting (21) in (25)

$$\begin{aligned}\cos\theta'\cos\alpha'&=\mu\mathfrak{N}_2\mathfrak{S}_2\\&\quad+\mu\left\{\sqrt{\frac{1}{\mu^2}-1+(\mathfrak{N}_2\mathfrak{S}_2)^2}-\mathfrak{N}_2\mathfrak{S}_2\right\}\mathfrak{N}_2^2\\&=\mu\sqrt{\frac{1}{\mu^2}-1+(\mathfrak{N}_2\mathfrak{S}_2)^2}.\end{aligned}$$

Taking (30) and (32) into consideration

$$\begin{aligned}\sin^2\alpha'&=1-\frac{1-\mu^2+\mu^2(\mathfrak{N}_2\mathfrak{S}_2)^2}{\cos^2\theta}=\frac{1}{\cos^2\theta}\Big[\cos^2\theta-1\\&+\mu^2-\sin^2A\cos^2\theta\sin^2v-(\mu^2-1+\cos^2\theta\cos^2v)\cos^2A\\&-2(\sqrt{\mu^2-1+\cos^2\theta\cos^2v})\cos A\sin A\cos\theta\sin v\Big]\\&=\frac{1}{\cos^2\theta}\Big[(\mu^2-1+\cos^2\theta\cos^2v)\sin^2A\\&-2(\sqrt{\mu^2-1+\cos^2\theta\cos^2v})\sin A\cos A\cos\theta\sin v\\&+\cos^2A\cos^2\theta\sin^2v\Big]\end{aligned}$$

$$\therefore \sin\alpha' = \pm\frac{1}{\cos\theta}\Big[(\sqrt{\mu^2-1+\cos^2\theta\cos^2 v})\sin A$$
$$-\cos A\cos\theta\sin v\Big]$$
$$=\pm\Big[\{\sqrt{(\mu^2-1)\tan^2\theta+\mu^2-\sin^2 v}\}\sin A$$
$$-\cos A\sin v\Big]. \qquad (33)$$

To remove the ambiguity on the sign in (33), let $\theta=0$ and $v=0$, then on the first refraction occurs normal incidence, accordingly in this case (33) must be transformed into

$$\mu=\sin\alpha'/\sin A.$$

Then we have

$$\sin\alpha'=\{\sqrt{(\mu^2-1)\tan^2\theta+\mu^2-\sin^2 v}\}\sin A$$
$$-\cos A\sin v. \qquad (34)$$

This expression satisfactly coincides with that of Plummer's.

Some trials on the application of this method to the calculations refering to lenses are in progress.

Summary

If we represent the position and the direction of a straight line employing the Study's coordinate, the laws of reflection and refraction are expressed respectively by the formulas (10) and (11). The treatment on the example given above seems to show that these formulas save the brain work in comparison with the classical method.

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