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Test of finite temperature random-phase approximation on a Lipkin model

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We investigate the applicability of the finite temperature random phase approximation (RPA) using a solvable Lipkin model. We show that the finite temperature RPA reproduces reasonably well the temperature dependence of total strength, both for the positive energy (i.e., the excitation) and the negative energy (i.e., the de-excitation) parts. This is the case even at very low temperatures, which may be relevant to astrophysical purposes.

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The random phase approximation (RPA) [1] and its extension, quasiparticle RPA (QRPA), have successfully described nuclear giant resonances [2,3]. Usually, those giant resonances are built on the ground state, but the collective states can be excited also from excited states [4]. In the early 1980s, heavy-ion fusion experiments revealed the existence of giant resonances in hot nuclei [5–9], and properties of collective excitation at finite temperatures have attracted much interest. To discuss giant resonances in hot nuclei, RPA has been extended by including thermal effects [10–17]. Such extension is referred to as finite temperature RPA or thermal RPA.

Recently, there has been renewed interest in atomic nuclei at finite temperatures, in connection to nuclear astrophysics [18–22] (see also Ref. [23] for collective excitations in hot exotic nuclei). The finite temperature RPA has often been used to estimate, e.g., β decay rates in a stellar environment [19,20,24]. In order to make quantitative calculations for nuclear astrophysical purposes, especially for r-process nucleosynthesis, it is necessary to know the accuracy of finite temperature RPA also at relatively low temperatures.

The aim of this Brief Report is to assess the applicability of finite temperature RPA using the Lipkin-Meshkov-Glick model [25]. This is a schematically solvable model and has been employed extensively to test many-body methods [1]. A similar study for finite temperature RPA has already been done by Rossignoli and Ring [26] (see also Ref. [27]), but they explicitly investigated only the positive energy part of a strength function. Here we investigate both the positive and negative energy parts separately. Since the sensitivity to the thermal occupation probability is large for de-excitation processes at low temperatures, such investigation provides an interesting test of finite temperature RPA. We also consider both the canonical and the grand-canonical ensembles for the exact solutions of the Lipkin model, while Ref. [26] considered only the grand-canonical ensemble. This is important because the finite temperature RPA is based on the grand canonical ensemble despite the number of particle being well-defined in actual nuclei. For compound nuclei formed in heavy-ion fusion reactions, the grand-canonical ensemble may be justified because of neutron evaporation processes [14]. However, it is not obvious whether the same argument holds for nuclei in a stellar condition at low temperatures. By comparing the results of the canonical ensemble to those of the grand-canonical ensemble, one can get some insight about the applicability of many-body theories based on the grand-canonical ensemble, such as finite temperature RPA.

In the Lipkin model, one considers two single-particle levels, at energies of $-\epsilon/2$ and $\epsilon/2$, respectively, each of which has 2Ω-fold degeneracy. The Hamiltonian for this model reads [25]

$$H = \epsilon \hat{K}_0 - \frac{V}{2}(\hat{K}_+\hat{K}_+ +, \hat{K}_-\hat{K}_-), \tag{1}$$

where V is the strength of a two-body interaction. The operators $\hat{K}_0$, $\hat{K}_+$, and $\hat{K}_-$ are defined as

$$\hat{K}_0 = \frac{1}{2}\sum_{i=1}^{2\Omega}(c^\dagger_{1i}c_{1i} - c^\dagger_{0i}c_{0i}), \tag{2}$$

$$\hat{K}_+ = \sum_{i=1}^{2\Omega} c^\dagger_{1i}c_{0i},\ \hat{K}_- = (\hat{K}_+)^\dagger. \tag{3}$$

Here, $c^\dagger_{0i}$ and $c^\dagger_{1i}$ are the creation operators for the lower and upper levels, respectively.

The exact solutions of the Lipkin model can be obtained with the quasispin formalism [1,25]. The eigenstates are then classified in terms of the eigenvalue of the operator $\hat{K}^2 = \hat{K}_0{}^2 + (\hat{K}_+\hat{K}_+ + \hat{K}_-\hat{K}_-)/2$. Denoting those states and their energy as $|J\alpha\rangle$ and $E_{J\alpha}$, respectively, the strength function for the canonical ensemble is given by [26]

$$S_C(E) = \frac{1}{Z_C}\sum_{J,\alpha,\alpha'} Y_C(J)\, e^{-\beta E_{J\alpha}} |\langle J\alpha'|\hat{F}|J\alpha\rangle|^2 \times \delta(E - E_{J\alpha'} + E_{J\alpha}), \tag{4}$$

where $\beta = 1/kT$ is the inverse temperature and $\hat{F}$ is the transition operator. We have assumed that $\hat{F}$ does not change the value of J. $Y_C(J)$ is the degeneracy of the J state given by [28]

$$Y_C(J) = W_C(J) - W_C(J+1)(1 - \delta_{J,J_{\max}}), \tag{5}$$

$$W_C(J) = \binom{2\Omega}{N/2 - J}\binom{2\Omega}{N/2 + J}, \tag{6}$$

where N is the number of particles in the system, and $J_{\max} = \min[N/2, 2\Omega - N/2]$ is the maximum J for a given N. Z_C in

Eq. (4) is the partition function given by

$$Z_C = \sum_{J,\alpha} Y_C(J) e^{-\beta E_{J\alpha}}. \tag{7}$$

In this Brief Report, we consider only a system with $N = 2\Omega$ (that is, half-filling). In this case, the chemical potential μ in the grand-canonical ensemble is zero, and the exact strength function for the grand-canonical ensemble is given by a similar formula as in Eq. (4) but with a different value of degeneracy [28],

$$Y_{GC}(J) = W_{GC}(J) - W_{GC}(J+1)(1 - \delta_{J,J_{\max}}), \tag{8}$$

$$W_{GC}(J) = \binom{4\Omega}{2\Omega - 2J}. \tag{9}$$

In addition to the exact solution, we also seek an approximate solution for the strength function using the finite temperature RPA. To this end, we first solve the thermal Hartree-Fock equation [1,10,27]

$$h_{\text{HF}} \begin{pmatrix} D_{0k} \\ D_{1k} \end{pmatrix} = e_k \begin{pmatrix} D_{0k} \\ D_{1k} \end{pmatrix}, \tag{10}$$

with

$$(h_{\text{HF}})_{00} = -\epsilon/2, (h_{\text{HF}})_{11} = \epsilon/2, \tag{11}$$

$$(h_{\text{HF}})_{01} = (h_{\text{HF}})_{01} = -V(N-1) \sum_{k=0,1} f_k D_{0k}^* D_{1k}, \tag{12}$$

where

$$f_k = \frac{1}{1 + e^{(e_k - \mu)/kT}}, \tag{13}$$

is the thermal occupation probability of the Hartree-Fock state k. With the Hartree-Fock basis, we assume that the excitation operator of the system is given by

$$\hat{Q}^\dagger = x_{10} \sum_{i=1}^{N} a_{1i}^\dagger a_{0i} + x_{01} \sum_{i=1}^{N} a_{0i}^\dagger a_{1i}, \tag{14}$$

where $a_{1i}^\dagger$ and $a_{0i}^\dagger$ are the creation operators for the Hartree-Fock states. The finite temperature RPA equation then reads [10]

$$\begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{01} \end{pmatrix} = \omega \begin{pmatrix} x_{10} \\ x_{01} \end{pmatrix}, \tag{15}$$

with

$$a = e_1 - e_0 + (f_0 - f_1) \times \frac{1}{2}(N-1)V \sin^2 2\alpha, \tag{16}$$

$$b = -(f_0 - f_1) \times (N-1)V \left(1 - \frac{1}{2}\sin^2 2\alpha\right), \tag{17}$$

where $D_{00} = \cos\alpha$. With the solutions of the RPA equations, the RPA strength function is obtained as [10,12,13,29]

$$S_{\text{RPA}}(E) = \sum_n \frac{\omega_n}{|\omega_n|} \times \frac{1}{1 - e^{-\beta E}} \left| \sum_{kl} \langle k|\hat{F}|l\rangle (f_k - f_l) x_{kl}^{(n)} \right|^2, \times \delta(\omega_n - E), \tag{18}$$

where the matrix elements for $\hat{F}$ are taken with the Hartree-Fock basis.

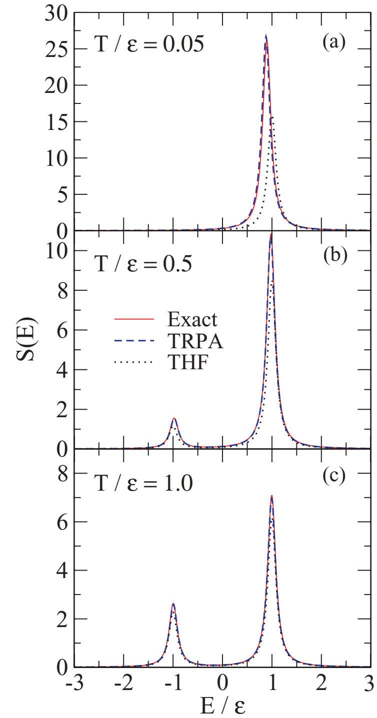

FIG. 1. (Color online) The strength function for the operator $\hat{F} = (\hat{K}_+ + \hat{K}_-)/2$ obtained with several methods for $VN/\epsilon = 0.5$. (a), (b), and (c) correspond to the temperature of $T/\epsilon = 0.05, 0.5$, and 1.0, respectively. The solid line is the exact result for the grand-canonical ensemble, while the dashed and the dotted lines denote the solutions of finite temperature RPA and thermal Hartree-Fock, respectively. The exact result for the canonical ensemble is almost the same as the solid line, and is not shown in the figure.

Let us now solve the model Hamiltonian numerically and compute the strength function. For this purpose, we take the particle number to be $N = 20$, and set $VN/\epsilon = 0.5$. As a transition operator, we consider $\hat{F} = (\hat{K}_+ + \hat{K}_-)/2$. Following Ref. [26], we smear the strength function with a width of $\eta/\epsilon = 0.1$.

Figure 1(a) shows the strength function at a temperature of $T/\epsilon = 0.05$. For our choice of parameters, the difference in the strength function between the canonical and the grand-canonical ensembles is small, and we only plot the result of the grand-canonical ensemble as the exact solution (see the solid line). The dashed and the dotted lines are the result of finite temperature RPA and thermal Hartree-Fock, respectively. At this low temperature, the thermal effect is almost negligible, and the strength function is actually almost the same as that at zero temperature. Notice that the RPA well reproduces the exact strength function. The Hartree-Fock result is not satisfactory, and the RPA correlation plays an important role.

At finite temperatures, the excited levels are thermally occupied with a finite probability. The probability for de-excitation of the excited states with the operator $\hat{F}$ appears in the negative energy part of the strength function. Figure 2 shows the negative energy part of Fig. 1(a). Since the temperature is low, the thermal occupation probability of the

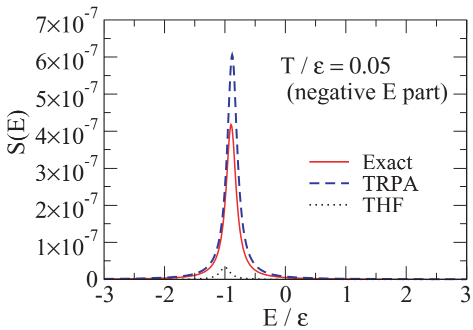

FIG. 2. (Color online) Same as Fig. 1(a), but only with the negative energy part.

excited states is negligibly small. Nevertheless, we find that the RPA works reasonably well. The energy of the first excited state is 0.897ϵ for the exact solution, while it is 0.88ϵ in RPA. As the energy is slightly underestimated in RPA, the peak of the strength function is somewhat overestimated. Despite this, we will show later that the temperature dependence of the total strength is well reproduced with finite temperature RPA (see Figs. 3 and 4).

The strength functions at higher temperatures, $T/\epsilon = 0.5$ and 1.0, are shown in Figs. 1(b) and 1(c). At these temperatures also, one sees that the finite temperature RPA works well both for the positive and the negative energy parts. Especially, the shift of the peak position in the strength function due to the finite temperature effects is well reproduced with RPA. As the temperature increases, even the thermal Hartree-Fock method reproduces the exact strength function.

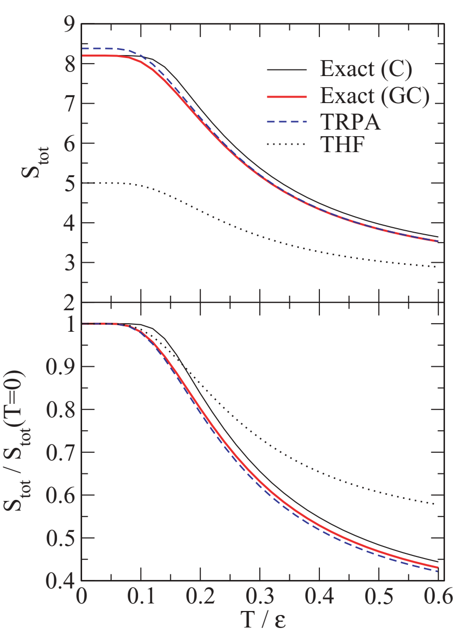

FIG. 3. (Color online) The total strength obtained with the several methods as a function of temperature (the upper panel). The meaning of each line is the same as in Fig. 1, except for the thin solid line which denotes the exact result of the canonical ensemble. The lower panel shows the ratio of the total strength to that at zero temperature.

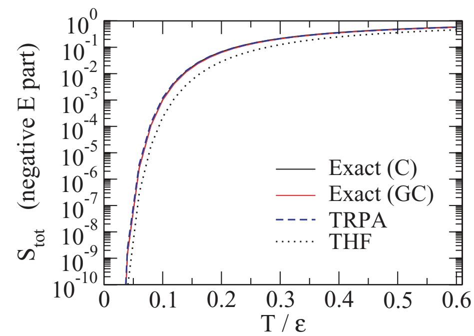

FIG. 4. (Color online) Contribution of the negative energy part to the total strength shown in Fig. 3. The thin and the thick solid lines are indistinguishable on this scale.

The total strength defined as

$$S_{\rm tot} = \int_{-\infty}^{\infty} S(E)\, dE \tag{19}$$

is plotted in the upper panel of Fig. 3 as a function of temperature. The lower panel shows the ratio of the total strength to that at zero temperature. The result of finite temperature RPA closely follows the exact result of the grand-canonical ensemble, as has been noted in Ref. [26]. We have confirmed that this conclusion remains qualitatively the same even for a smaller particle number, e.g., $N = 10$. The result of the canonical ensemble, shown by the thin solid line, is close to the result of the grand-canonical ensemble, although the difference is not negligible. The thermal Hartree-Fock, on the other hand, leads to an inconsistent temperature dependence of total strength, as can be seen in the lower panel.

The contribution of the negative energy part to the total strength is shown separately in Fig. 4. As one can see, the finite temperature RPA yields the correct temperature dependence of total strength even at very low temperatures.

Let us next discuss briefly a case with a stronger coupling. Figure 5 shows the strength function for $VN/\epsilon = 2.0$ at temperature $T/\epsilon = 0.675$. This corresponds to Fig. 2(b) in Ref. [26]. In this case, the canonical and the grand-canonical ensembles yield slightly different strength distributions from each other. That is, the grand-canonical ensemble leads to a smoother strength function because of number fluctuation, although the overall behavior is similar to each other. One

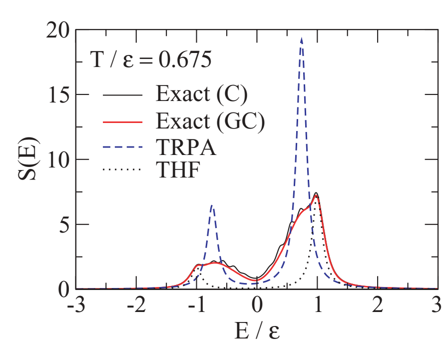

FIG. 5. (Color online) Same as Fig. 1, but for a stronger coupling strength, $VN/\epsilon = 2.0$ at temperature $T/\epsilon = 0.675$.

TABLE I. The total strength for $(VN/\epsilon) = 2.0$ at temperature $(T/\epsilon) = 0.675$ obtained with several methods, that is, the exact result with canonical (C) and grand canonical (GC) ensembles, thermal RPA (TRPA), and thermal Hartree-Fock (THF). The contribution from the negative energy part is also listed in the parentheses.

C	GC	TRPA	THF
7.92	7.40	8.04	2.81
(2.02)	(1.84)	(2.01)	(0.52)

also notices that the finite temperature RPA significantly underestimates the thermal broadening of the strength function and/or the ground state correlation. However, as has been argued in Ref. [26], the finite temperature RPA provides a reasonable estimate for the total strength. Table I summarizes the total strength for this particular choice for the parameters. The agreement between the exact results and the finite temperature RPA is satisfactory. We have checked that this is the case even for a stronger coupling, $VN/\epsilon = 4.0$.

In summary, we have investigated the applicability of finite temperature RPA using a schematic solvable model. We have shown that the finite temperature RPA provides a reasonable estimate for the total strength, both for the excitation and the decay processes. This is the case even at low temperatures. For a small coupling case, the finite temperature RPA also yields a reasonable strength function itself. We have also shown that the canonical and the grand-canonical ensembles lead to similar strength functions, as well as the total strengths, to each other. We thus conclude that the finite temperature RPA, being based on the grand-canonical ensemble, provides a reasonable tool to discuss properties of hot and warm nuclei, including those in a stellar environment.

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