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Generalization of Knuth's Formula for the Number of Skew Tableaux

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We take an elementary approach to derive a generalization of Knuth's formula using Lassalle's explicit formula. In particular, we give a formula for the Kostka numbers of a shape $\mu \vdash n$ and weight $(m, 1^{n-m})$ for $m = 3, 4$.

KEYWORDS: Knuth formula, skew tableau, Kostka number, Lassalle's explicit formula, symmetric group

1. Introduction

Throughout this paper, n will denote a positive integer. We write $\mu \vdash n$ if μ is a partition of n, that is, a non-increasing sequence $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ of positive integers such that $|\mu| = \sum_{i=1}^{k} \mu_i = n$. We say that k is the height of μ and denote it by $h(\mu)$. We denote by D_μ the Young diagram of μ. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h) \vdash m$ and $D_\lambda \subset D_\mu$, then the skew shape μ/λ is obtained by removing from D_μ all the boxes belonging to D_λ.

Let $\mu, \lambda \vdash n$ and $\nu \vdash m \leq n$. A semistandard Young tableau (SSYT) of shape μ and weight λ is a filling of the Young diagram D_μ with the numbers $1, 2, \dots, h(\lambda)$ in such a way that

(i) i occupies λ_i boxes, for $i = 1, 2, \dots, h(\lambda)$,

(ii) the numbers are strictly increasing down the columns and weakly increasing along the rows.

The Kostka number $K(\mu, \lambda)$ is the number of SSYTs of shape μ and weight λ. In particular, if $\lambda = (1^n)$ then such a tableau is called a standard Young tableau (SYT) of shape μ, and for a skew shape μ/ν and weight (1^{n-m}) such a tableau is called a skew SYT of skew shape μ/ν. We denote by $f^{\mu/\nu}$ the number of skew SYTs of skew shape μ/ν. Obviously, if $\lambda = (m, 1^{n-m}) \vdash n$ and $m \leq \mu_1$, then for all SSYTs of shape μ and weight λ, a box $(1, j) \in D_\mu$ is filled by 1 for $1 \leq j \leq m$, so $K(\mu, (m, 1^{n-m})) = f^{\mu/(m)}$. Naturally, if $\nu = \emptyset$ then f^μ is the number of SYTs of shape μ. We can easily compute f^μ using the hook formula (see [4]), but the problem of computing Kostka numbers is in general difficult (see [8]). There is a recurrence formula for Kostka numbers (see [6] and [7]), but we have no explicit formula for Kostka numbers.

For $z \in \mathbb{C}$, the falling factorial is defined by $[z]_n = z(z-1)\cdots(z-n+1) = n!\binom{z}{n}$, and $[z]_0 = 1$. Let $\mu = (\mu_1, \mu_2, \dots, \mu_k) \vdash n$ and μ' be the conjugate of μ. Knuth [5, p. 67, Exercise 19] shows:

$$f^{\mu/(2)} = \frac{f^\mu}{[n]_2}\left(\sum_{i=1}^{k}\binom{\mu_i}{2} - \sum_{j\geq 1}\binom{\mu'_j}{2} + \binom{n}{2}\right). \tag{1}$$

In fact, we can also compute $f^{\mu/\lambda}$ using [1, p. 310], [3, Theorem] and [9, Corollary 7.16.3], but this requires evaluation of determinants and knowledge of Schur functions. If we compute $\lambda = (2)$ using [9, Corollary 7.16.3], then we get the following:

$$f^{\mu/(2)} = \frac{f^\mu}{[n]_2}\left(\sum_{i=1}^{k}\left(\binom{\mu_i}{2} - \mu_i(i-1)\right) + \binom{n}{2}\right). \tag{2}$$

Since the following equation is well known (see [7, (1.6)], also see Proposition 6 for a generalization):

$$\sum_{i=1}^{k}\mu_i(i-1) = \sum_{j\geq 1}\binom{\mu'_j}{2}, \tag{3}$$

we have (1). As previously stated, since $K(\mu, (m, 1^{n-m})) = f^{\mu/(m)}$, we know the value of $K(\mu, (2, 1^{n-2}))$ from (1), so we are interested in the extent to which (1) can be generalized to an arbitrary positive integer m. In fact, if $\lambda = (3)$ then we get the following using [9, Corollary 7.16.3]:

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$$f^{\mu/(3)} = \frac{f^\mu}{[n]_3}\left(\sum_{i=1}^{k}\left(\mu_i(i-1)+\binom{\mu_i}{2}\right)+(n-2)\sum_{i=1}^{k}\left(\binom{\mu_i}{2}-\mu_i(i-1)\right)\right)$$
$$+\frac{f^\mu}{[n]_3}\left(2\sum_{i=1}^{k}\left(\mu_i\binom{i-1}{2}+\binom{\mu_i}{3}\right)-2\sum_{i=1}^{k}\binom{\mu_i}{2}(i-1)+\binom{n}{3}-\binom{n}{2}\right). \quad (4)$$

The proof of (4) using Lassalle's explicit formula for characters will be given in Section 4.

Let l be a nonnegative integer. Let $C(\mu) = \{j - i \mid (i,j) \in D_\mu\}$ be the multiset of contents of the partition μ, and

$$p_l[C(\mu)] = \sum_{(i,j)\in D_\mu} (j-i)^l$$

be the lth power sum symmetric function evaluated at the contents of μ. In this paper, we take an elementary approach to derive a formula for $f^{\mu/(m)}$ using [2, Section 5.3] and $p_l[C(\mu)]$.

This paper is organized as follows. After giving preliminaries in Section 2, we prove that $p_l[C(\mu)]$ can be written as a linear combination of $q^{\pm}_{r,t}$ in Section 3. We give an expression for $f^{\mu/(m)}$ in terms of $q^{\pm}_{r,t}$ for $m \le 4$ in Section 4. Finally, we prove a generalization of (3) in Section 5.

2. Preliminaries

Throughout this section, h, l, r and t be nonnegative integers. We denote by $S(n,k)$ the Stirling numbers of the second kind. First of all, we define

$$\mathfrak{C}(r,t) = t!S(r+1,t+1).$$

Then

$$\begin{aligned}\mathfrak{C}(r,t) &= t!S(r+1,t+1)\\ &= t!(S(r,t)+(t+1)S(r,t+1))\\ &= t\mathfrak{C}(r-1,t-1)+(t+1)\mathfrak{C}(r-1,t),\end{aligned} \quad (5)$$

since $S(r+1,t+1) = S(r,t)+(t+1)S(r,t+1)$.

Set

$$\varphi_l(h,r,t) = \binom{l}{h}\mathfrak{C}(h,r)\mathfrak{C}(l-h,t). \quad (6)$$

Clearly,

$$\begin{aligned}\varphi_l(h,r,t) &= \binom{l}{l-h}\mathfrak{C}(l-h,t)\mathfrak{C}(h,r)\\ &= \varphi_l(l-h,t,r).\end{aligned} \quad (7)$$

We define

$$R_l(t) = \sum_{i=1}^{t} i^l.$$

Lemma 1. *We have*

$$R_{l+1}(t) = (t+1)R_l(t) - \sum_{i=1}^{t} R_l(i).$$

Proof. We have

$$\begin{aligned}(t+1)R_l(t) &= (t+1)\sum_{i=1}^{t} i^l\\ &= \sum_{i=1}^{t} i^{l+1} + \sum_{i=1}^{t}\sum_{j=1}^{i} j^l\\ &= R_{l+1}(t) + \sum_{i=1}^{t} R_l(i).\end{aligned}$$ □

Lemma 2. *We have*

$$R_l(t) = \sum_{i=0}^{l} \mathcal{C}(l,i)\binom{t}{i+1}.$$

Proof. Setting $n = q = 0$ in [2, Proposition 5.1.2]. We have

$$\sum_{k=0}^{l}\binom{k}{m} = \binom{l+1}{m+1}. \tag{8}$$

We prove the statement by induction on l. If $l = 0$, then the statement holds since $\mathcal{C}(0,0) = 1$. Assume that the statement holds for $l-1$. Then

$$\begin{aligned}
R_l(t) &= (t+1)R_{l-1}(t) - \sum_{j=1}^{t} R_{l-1}(j) \qquad \text{(by Lemma 1)}\\
&= (t+1)\sum_{i=0}^{l-1}\mathcal{C}(l-1,i)\binom{t}{i+1} - \sum_{j=1}^{t}\sum_{i=0}^{l-1}\mathcal{C}(l-1,i)\binom{j}{i+1}\\
&= \sum_{i=0}^{l-1}(i+2)\mathcal{C}(l-1,i)\binom{t+1}{i+2} - \sum_{i=0}^{l-1}\mathcal{C}(l-1,i)\binom{t+1}{i+2} \qquad \text{(by (8))}\\
&= \sum_{i=0}^{l-1}(i+1)\mathcal{C}(l-1,i)\binom{t+1}{i+2}\\
&= \sum_{i=0}^{l-1}(i+1)\mathcal{C}(l-1,i)\binom{t}{i+2} + \sum_{i=0}^{l-1}(i+1)\mathcal{C}(l-1,i)\binom{t}{i+1}\\
&= \sum_{i=1}^{l} i\mathcal{C}(l-1,i-1)\binom{t}{i+1} + \sum_{i=0}^{l-1}(i+1)\mathcal{C}(l-1,i)\binom{t}{i+1}\\
&= \sum_{i=0}^{l}(i\mathcal{C}(l-1,i-1) + (i+1)\mathcal{C}(l-1,i))\binom{t}{i+1}\\
&= \sum_{i=0}^{l}\mathcal{C}(l,i)\binom{t}{i+1} \qquad \text{(by (5))}.
\end{aligned}$$

□

Lemma 3. *For $z \in \mathbb{C}$, we have*

$$z^l = \sum_{i=0}^{l}\mathcal{C}(l,i)\binom{z-1}{i}.$$

Proof. From [2, p. 211, (4.65)], we have

$$z^l = \sum_{i=0}^{l} S(l,i)[z]_i,$$

so

$$\begin{aligned}
z^l &= \sum_{i=0}^{l} S(l,i)[z]_i\\
&= \sum_{i=0}^{l} S(l,i)z[z-1]_{i-1}\\
&= \sum_{i=0}^{l} S(l,i)[z-1]_{i-1}(z-i+i)\\
&= \sum_{i=0}^{l} S(l,i)[z-1]_i + \sum_{i=1}^{l} iS(l,i)[z-1]_{i-1}\\
&= \sum_{i=0}^{l} S(l,i)[z-1]_i + \sum_{i=0}^{l-1}(i+1)S(l,i+1)[z-1]_i\\
&= \sum_{i=0}^{l}(S(l,i) + (i+1)S(l,i+1))[z-1]_i
\end{aligned}$$

$$= \sum_{i=0}^{l} S(l+1, i+1)[z-1]_i$$
$$= \sum_{i=0}^{l} i!S(l+1, i+1)\binom{z-1}{i}$$
$$= \sum_{i=0}^{l} \mathcal{C}(l, i)\binom{z-1}{i}. \qquad \square$$

Let μ, $\lambda \vdash n$. We denote by $\chi^\mu(\lambda)$ the value of the character of the Specht module S^μ evaluated at a permutation π belonging to the conjugacy class of type λ. From [2, Example 5.3.3], we have

$$\chi^\mu(2, 1^{n-2}) = \frac{f^\mu}{[n]_2} 2p_1[C(\mu)],$$
$$\chi^\mu(3, 1^{n-3}) = \frac{f^\mu}{[n]_3} 3\left(p_2[C(\mu)] - \binom{n}{2}\right),$$
$$\chi^\mu(4, 1^{n-4}) = \frac{f^\mu}{[n]_4} 4(p_3[C(\mu)] - (2n-3)p_1[C(\mu)]),$$
$$\chi^\mu(5, 1^{n-5}) = \frac{f^\mu}{[n]_5} 5\left(p_4[C(\mu)] - (3n-10)p_2[C(\mu)] - 2p_1[C(\mu)]^2 + 5\binom{n}{3} - 3\binom{n}{2}\right),$$
$$\chi^\mu(6, 1^{n-6}) = \frac{f^\mu}{[n]_6} 6(p_5[C(\mu)] + (25-4n)p_3[C(\mu)] + 2(3n-4)(n-5)p_1[C(\mu)]) - \frac{f^\mu}{[n]_6} 36p_1[C(\mu)]p_2[C(\mu)]. \tag{9}$$

Remark 4. In [2, Example 5.3.3], the coefficient of $d_3(\lambda)$ (in this paper, we denote it by $p_3[C(\mu)]$) in the character value $\hat{\chi}^\lambda_{6,1^{n-6}}$ is $24(7-n)$. Since c_6^λ and c_7^λ are incorrect in [2, p. 251], the value of the character $\hat{\chi}^\lambda_{6,1^{n-6}}$ is also incorrect. In fact, the coefficient of $d_3(\lambda)$ in the character value $\hat{\chi}^\lambda_{6,1^{n-6}}$ is $6(25-4n)$, as given in (9).

We obtain [2, Example 5.3.8]:

$$\chi^\mu(2, 2, 1^{n-4}) = \frac{f^\mu}{[n]_4} 4\left(p_1[C(\mu)]^2 - 3p_2[C(\mu)] + 2\binom{n}{2}\right). \tag{10}$$

In general, for $\mu \vdash n$ and $\lambda \vdash m \le n$, the character $\chi^\mu(\lambda, 1^{n-m})$ can be expressed as a polynomial of $c_r^\mu(t)$ using Lassalle's explicit formula [2, Theorem 5.3.11].

3. $p_l[C(\mu)]$ and $q^\pm_{r,t}$

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$, and let r, t be nonnegative integers. We define

$$q^\pm_{r,t} = \sum_{i=1}^{k}\left(\binom{\mu_i}{r+1}\binom{i-1}{t} \pm \binom{\mu_i}{t+1}\binom{i-1}{r}\right). \tag{11}$$

Observe that if $r = t$ then

$$q^-_{r,r} = 0, \tag{12}$$

and

$$q^+_{r,t} = q^+_{t,r}, \tag{13}$$
$$q^-_{r,t} = -q^-_{t,r}. \tag{14}$$

Proposition 5. *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$ and l be a nonnegative integer. Then*

$$p_{2l+1}[C(\mu)] = \sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^h \varphi_{2l+1}(h, r, t) q^-_{t,r},$$
$$p_{2l}[C(\mu)] = \sum_{h=0}^{l-1}\sum_{r=0}^{h}\sum_{t=0}^{2l-h}(-1)^h \varphi_{2l}(h, r, t) q^+_{r,t} + \frac{1}{2}(-1)^l \sum_{r=0}^{l}\sum_{t=0}^{l}\varphi_{2l}(l, r, t) q^+_{r,t}.$$

Proof. By the definition of $p_l[C(\mu)]$, we get the following:

$$
\begin{aligned}
p_l[C(\mu)] &= \sum_{i=1}^{k}\sum_{j=1}^{\mu_i}(j-i)^l \\
&= \sum_{i=1}^{k}\sum_{j=1}^{\mu_i}\sum_{h=0}^{l}(-1)^{l-h}\binom{l}{h}j^h i^{l-h} \\
&= \sum_{i=1}^{k}\sum_{h=0}^{l}(-1)^{l-h}\binom{l}{h}i^{l-h}R_h(\mu_i) \\
&= \sum_{i=1}^{k}\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{l-h}(-1)^{l-h}\binom{l}{h}\mathcal{C}(h,r)\mathcal{C}(l-h,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \\
&= \sum_{i=1}^{k}\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{l-h}(-1)^{l-h}\varphi_l(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \qquad \text{(by (6))},
\end{aligned}
$$

where the fourth equality follows from Lemma 2 and Lemma 3. Thus

$$
\begin{aligned}
p_{2l+1}[C(\mu)] &= \sum_{i=1}^{k}\sum_{h=0}^{2l+1}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{2l+1-h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \\
&= \sum_{i=1}^{k}\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{2l+1-h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \\
&\quad + \sum_{i=1}^{k}\sum_{h=l+1}^{2l+1}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{2l+1-h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \\
&= \sum_{i=1}^{k}\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{h-1}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \\
&\quad + \sum_{i=1}^{k}\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{t+1}\binom{i-1}{r} \\
&= \sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{h}\varphi_{2l+1}(h,r,t) \\
&\quad \cdot\left\{\sum_{i=1}^{k}\binom{\mu_i}{t+1}\binom{i-1}{r} - \sum_{i=1}^{k}\binom{\mu_i}{r+1}\binom{i-1}{t}\right\} \\
&= \sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{h}\varphi_{2l+1}(h,r,t)q^{-}_{t,r},
\end{aligned}
$$

where the third equality can be shown as follows:

$$
\begin{aligned}
&\sum_{h=l+1}^{2l+1}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{2l+1-h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \\
&= \sum_{h=0}^{l}\sum_{r=0}^{2l+1-h}\sum_{t=0}^{h}(-1)^{h}\varphi_{2l+1}(2l+1-h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \\
&= \sum_{h=0}^{l}\sum_{r=0}^{2l+1-h}\sum_{t=0}^{h}(-1)^{h}\varphi_{2l+1}(h,t,r)\binom{\mu_i}{r+1}\binom{i-1}{t} \qquad \text{(by (7))} \\
&= \sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{t+1}\binom{i-1}{r}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
p_{2l}[C(\mu)] &= \sum_{h=0}^{l-1}\sum_{r=0}^{h}\sum_{t=0}^{2l-h}(-1)^{h}\varphi_{2l}(h,r,t)q^{+}_{r,t} \\
&\quad + \sum_{i=1}^{k}\sum_{r=0}^{l}\sum_{t=0}^{l}(-1)^{l}\varphi_{2l}(l,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t}
\end{aligned}
$$

$$= \sum_{h=0}^{l-1}\sum_{r=0}^{h}\sum_{t=0}^{2l-h}(-1)^h\varphi_{2l}(h,r,t)q^+_{r,t} + \frac{1}{2}(-1)^l\sum_{r=0}^{l}\sum_{t=0}^{l}\varphi_{2l}(l,r,t)q^+_{r,t},$$

where the second equality can be shown as follows:

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{r=0}^{l}\sum_{t=0}^{l}(-1)^l\varphi_{2l}(l,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t}\\
&\quad= \frac{1}{2}(-1)^l\sum_{i=1}^{k}\sum_{r=0}^{l}\sum_{t=0}^{l}\varphi_{2l}(l,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t}\\
&\qquad+ \frac{1}{2}(-1)^l\sum_{i=1}^{k}\sum_{r=0}^{l}\sum_{t=0}^{l}\varphi_{2l}(l,r,t)\binom{\mu_i}{t+1}\binom{i-1}{r}\\
&\quad= \frac{1}{2}(-1)^l\sum_{r=0}^{l}\sum_{t=0}^{l}\varphi_{2l}(l,r,t)q^+_{r,t}.
\end{aligned}$$

□

By Proposition 5, we have

$$\begin{aligned}
p_0[C(\mu)] &= \frac{1}{2}q^+_{0,0} = n,\\
p_1[C(\mu)] &= q^-_{0,0} + q^-_{1,0}\\
&= q^-_{1,0}, \qquad \text{(by (12))}\\
p_2[C(\mu)] &= 2q^+_{0,1} + 2q^+_{0,2} - q^+_{1,0} - q^+_{1,1}\\
&= q^+_{0,1} + 2q^+_{0,2} - q^+_{1,1}, \qquad \text{(by (13))}\\
p_3[C(\mu)] &= -2q^-_{1,0} + 6q^-_{2,0} + 6q^-_{3,0} - 3q^-_{0,1} - 9q^-_{1,1} - 6q^-_{2,1}\\
&= q^-_{1,0} + 6q^-_{2,0} + 6q^-_{3,0} - 6q^-_{2,1} \qquad \text{(by (12) and (14)).}
\end{aligned} \tag{15}$$

4. Main results

For any $i \geq 1$, $m_i(\mu) = |\{j \mid \mu_j = i\}|$ is the multiplicity of i in μ. Set

$$z_\mu = \prod_{i\geq 1} i^{m_i(\mu)} m_i(\mu)!.$$

Let $\mu \vdash n$ and $\lambda \vdash m \leq n$. From [10, Theorem 3.1], we have

$$f^{\mu/\lambda} = \sum_{\nu\vdash m} z_\nu^{-1}\chi^\mu(\nu, 1^{n-m})\chi^\lambda(\nu).$$

If $\lambda = (m)$, then

$$\begin{aligned}
f^{\mu/(m)} &= \sum_{\nu\vdash m} z_\nu^{-1}\chi^\mu(\nu, 1^{n-m})\chi^{(m)}(\nu)\\
&= \sum_{\nu\vdash m} z_\nu^{-1}\chi^\mu(\nu, 1^{n-m}).
\end{aligned} \tag{16}$$

We already proved that $p_l[C(\mu)]$ can be expressed as a linear combination of $q^\pm_{r,t}$ (Proposition 5), so the character value $\chi^\mu(\lambda, 1^{n-m})$ can be written as a polynomial in $q^\pm_{r,t}$ using Lassalle's explicit formula [2, Theorem 5.3.11]. We compute $\chi^\mu(m, 1^{n-m})$ for $2 \leq m \leq 4$ and $\chi^\mu(2, 2, 1^{n-4})$ using (9), (10) and (15).

$$\begin{aligned}
\chi^\mu(2, 1^{n-2}) &= \frac{f^\mu}{[n]_2}2p_1[C(\mu)]\\
&= \frac{f^\mu}{[n]_2}2q^-_{1,0},\\
\chi^\mu(3, 1^{n-3}) &= \frac{f^\mu}{[n]_3}3\left(p_2[C(\mu)] - \binom{n}{2}\right)\\
&= \frac{f^\mu}{[n]_3}3\left(q^+_{0,1} + 2q^+_{0,2} - q^+_{1,1} - \binom{n}{2}\right),\\
\chi^\mu(4, 1^{n-4}) &= \frac{f^\mu}{[n]_4}4(p_3[C(\mu)] - (2n-3)p_1[C(\mu)])
\end{aligned}$$

$$= \frac{f^\mu}{[n]_4} 4((4-2n)q^-_{1,0} + 6q^-_{2,0} + 6q^-_{3,0} - 6q^-_{2,1}),$$

$$\chi^\mu(2,2,1^{n-4}) = \frac{f^\mu}{[n]_4} 4\left(p_1[C(\mu)]^2 - 3p_2[C(\mu)] + 2\binom{n}{2}\right)$$

$$= \frac{f^\mu}{[n]_4} 4\left((q^-_{1,0})^2 - 3q^+_{0,1} - 6q^+_{0,2} + 3q^+_{1,1} + 2\binom{n}{2}\right). \tag{17}$$

Substituting (17) into (16), we find

$$f^{\mu/(2)} = \frac{1}{z_{(2)}}\chi^\mu(2,1^{n-2}) + \frac{1}{z_{(1,1)}}\chi^\mu(1^n)$$

$$= \frac{1}{2}\frac{f^\mu}{[n]_2}\cdot 2q^-_{1,0} + \frac{1}{2}f^\mu$$

$$= \frac{f^\mu}{[n]_2}\left(q^-_{1,0} + \binom{n}{2}\right), \tag{18}$$

$$f^{\mu/(3)} = \frac{1}{z_{(3)}}\chi^\mu(3,1^{n-3}) + \frac{1}{z_{(2,1)}}\chi^\mu(2,1^{n-2}) + \frac{1}{z_{(1,1,1)}}\chi^\mu(1^n)$$

$$= \frac{1}{3}\frac{f^\mu}{[n]_3}\cdot 3\left(q^+_{0,1} + 2q^+_{0,2} - q^+_{1,1} - \binom{n}{2}\right) + \frac{1}{2}\frac{f^\mu}{[n]_2}\cdot 2q^-_{1,0} + \frac{1}{6}f^\mu$$

$$= \frac{f^\mu}{[n]_3}\left(q^+_{0,1} + 2q^+_{0,2} - q^+_{1,1} + (n-2)q^-_{1,0} + \binom{n}{3} - \binom{n}{2}\right), \tag{19}$$

and

$$f^{\mu/(4)} = \frac{1}{z_{(4)}}\chi^\mu(4,1^{n-4}) + \frac{1}{z_{(3,1)}}\chi^\mu(3,1^{n-3}) + \frac{1}{z_{(2,2)}}\chi^\mu(2,2,1^{n-4})$$

$$+ \frac{1}{z_{(2,1,1)}}\chi^\mu(2,1^{n-2}) + \frac{1}{z_{(1,1,1,1)}}\chi^\mu(1^n)$$

$$= \frac{1}{4}\frac{f^\mu}{[n]_4}\cdot 4((4-2n)q^-_{1,0} + 6q^-_{2,0} + 6q^-_{3,0} - 6q^-_{2,1})$$

$$+ \frac{1}{3}\frac{f^\mu}{[n]_3}\cdot 3\left(q^+_{0,1} + 2q^+_{0,2} - q^+_{1,1} - \binom{n}{2}\right)$$

$$+ \frac{1}{8}\frac{f^\mu}{[n]_4}\cdot 4\left((q^-_{1,0})^2 - 3q^+_{0,1} - 6q^+_{0,2} + 3q^+_{1,1} + 2\binom{n}{2}\right)$$

$$+ \frac{1}{4}\frac{f^\mu}{[n]_2}\cdot 2q^-_{1,0} + \frac{1}{24}f^\mu$$

$$= \frac{f^\mu}{[n]_4}\left(\frac{1}{2}(n-2)(n-7)q^-_{1,0} + 6q^-_{2,0} + 6q^-_{3,0} - 6q^-_{2,1} + \frac{1}{2}(q^-_{1,0})^2\right)$$

$$+ \frac{f^\mu}{[n]_4}\left(\left(n-\frac{9}{2}\right)q^+_{0,1} + (2n-9)q^+_{0,2} - \left(n-\frac{9}{2}\right)q^+_{1,1}\right)$$

$$+ \frac{f^\mu}{[n]_4}\left(\binom{n}{4} - 3\binom{n}{3} + 2\binom{n}{2}\right).$$

We get (2) and (4) by substituting (11) into (18) and (19), respectively.

5. A generalization of a polynomial identity for a partition and its conjugate

Proposition 6. *Let μ be a partition of an integer. Then μ' is the conjugate of μ if and only if*

$$\sum_{i=1}^{k}\binom{\mu_i}{t+1}\binom{i-1}{r} = \sum_{j\ge 1}\binom{\mu'_j}{r+1}\binom{j-1}{t},$$

for all nonnegative integers r and t.

Proof. First, we show the "only if" part. Then

$$\sum_{j\geq 1}\binom{\mu'_j}{r+1}\binom{j-1}{t} = \sum_{j\geq t+1}\sum_{\substack{J\subseteq\{1,2,\ldots,\mu_1\},\\ |J|=t+1,\\ \max J=j}} |\{I \mid I\times J\subseteq D_\mu,\ |I|=r+1\}|$$

$$= \sum_{i=r+1}^{k}\sum_{\substack{I\subseteq\{1,2,\ldots,k\},\\ |I|=r+1,\\ \max I=i}} |\{J \mid I\times J\subseteq D_\mu,\ |J|=t+1\}|$$

$$= \sum_{i=r+1}^{k}\sum_{\substack{I\subseteq\{1,2,\ldots,k\},\\ |I|=r+1,\\ \max I=i}} |\{J \mid \max J\leq \mu_i,\ |J|=t+1\}|$$

$$= \sum_{i=r+1}^{k}\sum_{\substack{I\subseteq\{1,2,\ldots,k\},\\ |I|=r+1,\\ \max I=i}} |\{J \mid J\subseteq \{1,2,\ldots,\mu_i\},\ |J|=t+1\}|$$

$$= \sum_{i=r+1}^{k}\sum_{\substack{I\subseteq\{1,2,\ldots,k\},\\ |I|=r+1,\\ \max I=i}} \binom{\mu_i}{t+1}$$

$$= \sum_{i=r+1}^{k}\binom{\mu_i}{t+1}\binom{i-1}{r}.$$

Next, let λ be the conjugate of μ. Set $h(\lambda)=h$. Then

$$\sum_{j=1}^{h}\binom{\lambda_j}{r+1}\binom{j-1}{t} = \sum_{i=1}^{k}\binom{\mu_i}{t+1}\binom{i-1}{r} = \sum_{j\geq 1}\binom{\mu'_j}{r+1}\binom{j-1}{t}. \tag{20}$$

Setting $h(\mu')=l$ and $r=0$ in (20), we have

$$\sum_{j=1}^{h}\lambda_j\binom{j-1}{t} = \sum_{i=1}^{l}\mu'_i\binom{i-1}{t}. \tag{21}$$

Suppose $h>l$ and set $t=h-1$ in (21), then $\lambda_h=0$. Similarly, suppose $h<l$ and set $t=l-1$ in (21). Then $\mu'_l=0$, and both cases are contradictions. Thus $h=l$.

We show that $\lambda_{h-i}=\mu'_{h-i}$ for all i with $0\leq i\leq h-1$ by induction on i. If $i=0$, setting $t=h-1$ in (21), then $\lambda_h=\mu'_h$.

Assume that the assertion holds for some $i\in\{0,1,\ldots,h-2\}$. Let $t=h-(i+2)$ in (21). By the inductive hypothesis, we have

$$\sum_{j=h-i}^{h}\lambda_j\binom{j-1}{h-i-2} = \sum_{j=h-i}^{h}\mu'_j\binom{j-1}{h-i-2}.$$

Therefore, $\lambda_{h-i-1}=\mu'_{h-i-1}$ since $\binom{j-1}{h-i-2}=0$ for all j with $1\leq j\leq h-j-2$. Thus $\lambda=\mu'$ and μ' is the conjugate of μ. □

From Proposition 6, we have

$$q^{\pm}_{r,t} = \sum_{i=1}^{k}\binom{\mu_i}{r+1}\binom{i-1}{t} \pm \sum_{j\geq 1}\binom{\mu'_j}{r+1}\binom{j-1}{t}. \tag{22}$$

By substituting (22) into (18) and (19), we get (1) and

$$f^{\mu/(3)} = \frac{f^\mu}{[n]_3}\left(q^+_{0,1}+2q^+_{0,2}-q^+_{1,1}+(n-2)q^-_{1,0}+\binom{n}{3}-\binom{n}{2}\right)$$

$$= \frac{f^\mu}{[n]_3}\left(q^+_{1,0}+2q^+_{2,0}-q^+_{1,1}+(n-2)q^-_{1,0}+\binom{n}{3}-\binom{n}{2}\right) \qquad \text{(by (13))}$$

$$= \frac{f^{\mu}}{[n]_3}\left(\left(\sum_{i=1}^{k}\binom{\mu_i}{2}+\sum_{j\geq 1}\binom{\mu_j'}{2}\right)+2\left(\sum_{i=1}^{k}\binom{\mu_i}{3}+\sum_{j\geq 1}\binom{\mu_j'}{3}\right)\right)$$
$$-\frac{f^{\mu}}{[n]_3}\left(\sum_{i=1}^{k}\binom{\mu_i}{2}(i-1)+\sum_{j\geq 1}\binom{\mu_j'}{2}(j-1)\right)$$
$$+\frac{f^{\mu}}{[n]_3}\left((n-2)\left(\sum_{i=1}^{k}\binom{\mu_i}{2}-\sum_{j\geq 1}\binom{\mu_j'}{2}\right)+\binom{n}{3}-\binom{n}{2}\right),$$

respectively.

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