博 士 論 文

Analytic studies of Banach function spaces and martingales

（バナッハ関数空間とマルチンゲールに関する解析的研究）

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1. Introduction and Preliminaries.

Since J. L. Doob established the fundamental theory of martingales in [20], many Banach spaces of martingales have been studied. In particular, studies of the Hardy spaces H^p $(1 \leq p \leq \infty)$, BMO space and so on are essential for the development of the martingale theory. To study the martingale spaces, it is indispensable to consider martingale inequalities. Doob's inequality shows that, when $p > 1$, the maximal random variable $\sup_{n \geq 0} |X_n|$ of a martingale $X = (X_n)_{n \geq 0}$ is in L^p if and only if X is written as $X_n = E[X_\infty \,|\, \mathcal{F}_n]$, $n \geq 0$, with some $X_\infty \in L^p$. This means that the Hardy space H^p, defined by means of the maximal random variable, is isomorphic to L^p. On the integrability of the maximal random variable, D. L. Burkholder and R. F. Gundy [12] proved another important inequality: when $p > 1$, $\sup_{n \geq 0} |X_n|$ is in L^p if and only if the square function $\mathcal{S}X$ is in L^p. Thus another Hardy space H^p, defined by means of the square function, is isomorphic to H^p. Immediately B. Davis [15] extended this result to the case where $p = 1$. Furthermore, it was proved in [11] that $\sup_{n \geq 0} |X_n|$ is in L^Φ if and only if $\mathcal{S}X_\infty$ is in L^Φ for every Young function satisfying the Δ_2-condition. These two types of martingale inequalities, Doob's inequality and the Burkholder–Davis–Gundy inequality have been one of the core of the martingale theory.

On the other hand, M. Izumisawa and N. Kazamaki [24] investigated the Doob type inequality in a weighted L^p-space. They introduced the probabilistic A_p-condition for weights and showed that the A_p-condition is necessary and nearly sufficient for the Doob type inequality to hold in the weighted L^p-space. Their work was not only the beginning of the successive studies of A. Uchiyama [55], A. Bonami and D. Lépingle [9], T. Sekiguchi [49, 50], R. L. Long [39] and so on, but also a remarkable work as a new phase of martingale inequalities. In fact, nowadays, it is an efficacious means of estimating a price process, in Mathematical Finance, to consider a change of probability measure and weighted norm inequalities.

Although the necessity of the weight theory of martingales increases recently, there are not many works on the Hardy space associated with a weighted L^p-norm. In this article, we shall consider martingale Hardy spaces in more general setting: our objects are martingale spaces associated with Banach function spaces, such as L^p-spaces, Orlicz spaces, Lorentz spaces or such spaces with weights. Outlines of each chapter are as follows.

In Chapter 1, we will give preliminaries, including some concepts in the abstract theory of Banach function space expounding in [8] or [57].

In Chapter 2, we shall study the distribution of the terminal r.v. A_∞ of increasing processes $A = (A_t)_{t \geq 0}$. We can obtain some inequalities for decreasing rearrangements of the terminal r.v.'s of increasing processes. Using these inequalities, we shall prove the equivalence of BMO_1-norm and BMO_Φ-norm of martingales (cf. [6]). The distribution inequality will be used to establish the ratio inequalities for martingales. The ratio inequalities are extensions of the Burkholder–Davis–Gundy inequality. The corresponding inequalities in the classical analysis are due to Murai and Uchiyama [44].

In Chapter 3, we shall consider the Doob type inequality in Banach function spaces which is not necessarily rearrangement invariant. The Doob type inequality in rearrangement invariant function spaces was also considered in [1] and [46]. We shall prove that if the Doob type inequality holds with respect to an arbitrary filtration, then the function space is rearrangement invariant. Using this result we shall prove that if a weight function satisfies some A_p-condition with respect to any filtration, then the weight is useless. We shall study in Chapter 4 the convergence of martingales with respect to the norm topology in function spaces. It is well-known that a uniformly integrable martingale $(X_n)_{n \geq 0}$ such that $X_\infty \in L^p$ $(1 \leq p < \infty)$ converges in L^p with respect to the norm topology. It is not familiar, however, that if $X_\infty \in L \log L$, then (X_n) converges in $L \log L$. More generally, an analogous consequence is true for a wide class of Banach function spaces. We shall also consider such a problem for the sequence of conditional expectations $E[\, x \mid \mathcal{F}_n \,]$ when $(\mathcal{F}_n)$ is not necessarily increasing in n (cf. [2, 33]): see also Appendix.

Chapter 5 is devoted to establishing a representation theorem of continuous linear functionals on a martingale Hardy space associated with a Banach function space. The celebrated (H^1, BMO)–duality theorem will be extended naturally.

In Appendix, We shall give a simple alternative proof of the result in [2] on the convergence of conditional expectations of a r.v. (Theorem 4.11).

§ 1. Processes.

Throughout this article we shall work with a complete probability space $(\Omega, \mathcal{F}, P)$ with a *filtration* $(\mathcal{F}_t)_{t\in\mathbb{T}}$, where $\mathbb{T}$ denotes the set of time parameters; for example $\mathbb{T} = [0, \infty[$, $[0, \infty]$, $\mathbb{N}$ or $\mathbb{N} \cup \{\infty\}$. Instead of $(\mathcal{F}_t)_{t\in\mathbb{T}}$, we write $(\mathcal{F}_t)_{t\geq 0}$ or $(\mathcal{F}_t)_{0\leq t\leq\infty}$ if $\mathbb{T} = [0, \infty[$ or $[0, \infty]$; we write $(\mathcal{F}_n)_{n\geq 0}$ or $(\mathcal{F}_n)_{0\leq n\leq\infty}$ if $\mathbb{T} = \mathbb{N}$ or $\mathbb{N} \cup \{\infty\}$. By filtration we mean a family of sub-σ-fields $(\mathcal{F}_t)_{t\in\mathbb{T}}$ of $\mathcal{F}$ which increases with t. We always assume that the filtration satisfies the *usual hypothesis*, i.e. that $\mathcal{F}_0$ contains all P-negligible sets in $\mathcal{F}$ and, in addition, $(\mathcal{F}_t)$ is right continuous in the sense that $\bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$, when $\mathbb{T} = [0, \infty[$ or $[0, \infty]$ (cf. [17, p. 183]).

In continuous parameter case, since the filtration is right continuous, it is natural to deal with right continuous processes. However this assumption is not enough in martingale theory: we shall deal only with càdlàg processes (except for the left limits processes of càdlàg processes). A process $X = (X_t)_{t\geq 0}$ is said to be *càdlàg* if almost every path $t \mapsto X_t(\omega)$ is right continuous and has left limits. The meaningless word "càdlàg" is abbreviation of the French "continu à droite avec limites à gauche". The left limit of X at $t > 0$ is denoted by X_{t-}: we set $X_{0-} = 0$ for any process X. The jump of X at $t \geq 0$ is denoted by ΔX_t, and more generally, the jump at a stopping time (or random time) T is denoted by ΔX_T. Furthermore, unless otherwise stated, each process $(X_t)_{t\in\mathbb{T}}$ is assumed to be *adapted* to a given filtration $(\mathcal{F}_t)_{t\in\mathbb{T}}$, that is, X_t is $\mathcal{F}_t$-measurable for every $t \in \mathbb{T}$.

Now let $X = (X_t)_{t\in\mathbb{T}}$ be a process. The *maximal process* of X is the process given by $X_t^{\circledast} = \sup\{|X_s|\colon s \in \mathbb{T},\, t \leq s\}$; we denote simply by $X^{\circledast}$ the random variable (r.v.) $X_\infty^{\circledast}$. In most of the references on martingales, the maximal process of X is denoted by (X_t^*). In this paper, however, we reserve the symbol "*" to denote the decreasing rearrangement of a r.v.

If the time parameter set is discrete, the process $(\mathcal{S}X_n)_{n\geq 0}$ given by

$$\mathcal{S}X_n = \Big\{X_0^2 + \sum_{k=1}^{n}(X_k - X_{k-1})^2\Big\}^{1/2}$$

is called the square function of X. The notion of square function is naturally extended to semi-martingales with continuous parameter: for a semi-martingale $X = (X_t)_{t\geq 0}$ we set

$$[X, X]_t := X_0^2 + \lim_{|\Gamma|\to 0} \sum_{t_k\in\Gamma}(X_{t_k} - X_{t_{k-1}})^2, \qquad \text{(in probability)}$$

where Γ denotes a partition of the interval $[0, t]$ and $\lim_{|\Gamma| \to 0}$ means the limit in probability as $|\Gamma| = \max_{t_k \in \Gamma} |t_k - t_{k-1}| \to 0$. The process $([X, X]_t)_{t \ge 0}$ is called the *quadratic variation* (*process*) of X, which is adapted, right-continuous and increasing. We have

$$[X, X]_t = <X^c, X^c>_t + \sum_{0 \le s \le t} (\Delta X_s)^2, \tag{1.1}$$

where X^c denotes the continuous martingale part of X and $\big(<X^c, X^c>_t\big)_{t \ge 0}$ is a unique continuous increasing process such that $\big((X^c)^2 - <X^c, X^c>_t\big)$ is a local martingale.

We assume that the reader has some knowledge on the martingale theory expounded in [18] and besides, in Chapter 4, we use the notion of the general theory of processes, such as optional (or predictable) projection of a process, dual optional (predictable) projection of an increasing process, predictable stopping times and so forth. For details, see [16], [17, 18] or [38].

§ 2. Banach function spaces.

In this section, we shall give some definitions and notation on the general theory of Banach function spaces. We always denote by I the unit interval $[0, 1]$ with the Lebesgue (probability) measure m. Let f be a random variable on Ω. The *decreasing rearrangement* of f is the right continuous decreasing† function f^* on I given by

$$f^*(t) = \inf\{\lambda > 0 : P(|f| > \lambda) \le t\}, \qquad t \in I,$$

where, and in what follows, we employ the convention that $\inf \varnothing = +\infty$. As (I, m) is a probability space, we may define the decreasing rearrangement of a function φ on I in the same way: the decreasing rearrangement is still denoted by φ^*, since there will be no confusion. The decreasing rearrangement of f is characterized as the unique right continuous decreasing function satisfying

$$m\{s \in I : f^*(s) > \lambda\} = P\{\,|f| > \lambda\} \qquad \text{for every} \quad \lambda > 0.$$

As mentioned above, in many references X^* denotes the maximal process; in our notation, however, X_T^* denotes not the r.v. $\sup_{t \le T} |X_t|$ but the decreasing rearrangement of the r.v. X_T.

† The word 'decreasing' (increasing) means 'nonincreasing' (nondecreasing), while 'positive' means 'strictly positive'.

The following inequality, which is called *Hardy's inequality*, will be used frequently: if f and g are r.v.'s, then

$$\int_\Omega fg\,dP \le \int_0^1 f^*(s)\,g^*(s)\,ds. \tag{1.2}$$

In particular, setting $g = 1_A$ for $A \in \mathcal{F}$, we have

$$\int_A f\,dP \le \int_0^{P(A)} f^*(s)\,ds, \qquad A \in \mathcal{F}. \tag{1.3}$$

Now let f be an integrable random variable. Then f^{**} denotes the maximal function of f^* defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \qquad t \in I.$$

If Ω is non-atomic, that is, contains no atom, then

$$f^{**}(t) = \frac{1}{t}\sup\left\{\int_E f\,dP \,:\, P(E) = t\right\}, \tag{1.4}$$

(cf. [8, p. 53]). It then follows immediately that $(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t)$ for all integrable r.v.'s f and g, provided that Ω is non-atomic. In fact, this result remains true for any Ω (cf. [8, p. 55]). Notice that the operation $f \mapsto f^*$ is not subadditive in general. Taking account of these facts, we often use f^{**}, instead of f^*, to define a normed space of r.v.'s.

In connection with the order structure of the rearrangement invariant spaces, we define an order relation between integrable random variables: we write $f \prec g$ if $f^{**} \le g^{**}$ on I. Thus:

$$f \prec g \iff \int_0^t f^*(s)\,ds \le \int_0^t g^*(s)\,ds \quad \text{for every } t \in I.$$

Definition 1.1. Let $(B\,, \|\cdot\|_B)$ be a Banach space of (equivalence classes of) r.v.'s on Ω. B is called a *Banach function space* if B satisfies the following conditions:

(1.5) $L^\infty \hookrightarrow B \hookrightarrow L^1$;

(1.6) $|f| \le |g|$, $g \in B$ implies $f \in B$ and $\|f\|_B \le \|g\|_B$;

(1.7) $0 \le f_n \uparrow f$, $\sup_n \|f_n\|_B < \infty$ implies $f \in B$ and $\|f\|_B = \sup_n \|f_n\|_B$, where $B_1 \hookrightarrow B_2$ means that B_1 is continuously embedded in B_2.

A Banach function space B is said to be *rearrangement invariant* (or *r.i.* briefly), if

(1.8) $f^* = g^*,\ g \in B$ implies $f \in B$ and $\|f\|_B = \|g\|_B$.

A Banach function space B is said to be *universally rearrangement invariant* (*u.r.i.*) if

(1.9) $f \prec g,\ g \in B$ implies $f \in B$ and $\|f\|_B \le \|g\|_B$.

Property (1.7) is called the *Fatou property.*

It is easy to see that L^p spaces, Orlicz spaces and Lorentz spaces are all r.i., and in fact, they are u.r.i. Observe that if B is u.r.i., then it is r.i. and the converse is not true in general. If Ω is non-atomic, then B is r.i. if and only if B is u.r.i. For details, see Section 16 of [13].

Since (I, m) is a probability space, we may consider r.i. spaces over I. The Luxemburg representation theorem makes clear the connection between u.r.i. spaces over Ω and I.

Theorem 1.2 (W. A. J. Luxemburg [41]). *A Banach function space B over Ω is u.r.i. if and only if there exists a r.i. space $\hat{B}$ over I such that*

(1.10) $$\|f\|_B = \|f^*\|_{\hat{B}} \quad \textit{for every} \ f \in B.$$

For the proof, see e.g. [41, p. 121] or [13, p. 113]. Note that, for such a $\hat{B}$, $f \in B$ if and only if $f^* \in \hat{B}$ by the Fatou property (1.7).

Let Ω_1 be the union of all atoms in Ω and $\Omega_0 = \Omega \setminus \Omega_1$; thus Ω_0 contains no atom. Ω_0 is called the *non-atomic part* of Ω. When we consider a r.i. or u.r.i. space, *we always assume that* $P(\Omega_0) > 0$.

Lemma 1.3. *For each u.r.i space B, a r.i. space $\hat{B}$ satisfying* (1.10) *is unique.*

Proof. Suppose that both $\hat{B}_1$ and $\hat{B}_2$ satisfy (1.10) and let $t_0 = P(\Omega_0) > 0$. Since Ω_0 contains no atom, for each $x \in \hat{B}_2$, there exists a r.v. f such that $f^*(t) = x^*(t \wedge t_0)$ for every $t \in I$. As $x^* \le f^* \le x^*(t_0) + x^* \in \hat{B}_2$ on I, we have $f^* \in \hat{B}_2$ or equivalently $f \in B$. Since $f \in B$ is again equivalent to

$f^* \in \hat{B}_1$, we get $x^* \in \hat{B}_1$ and hence $x \in \hat{B}_1$. Thus we obtain $\hat{B}_2 \subset \hat{B}_1$. The closed graph theorem applying to the identity operator $\hat{B}_2 \ni x \mapsto x \in \hat{B}_1$ shows that $\hat{B}_2 \hookrightarrow \hat{B}_1$. In the same way, we obtain $\hat{B}_1 \hookrightarrow \hat{B}_2$ and these spaces have the equivalent norms. $\square$

Now let φ be a real-valued function on I. For each $s > 0$, the *dilation operator* D_s is defined by

$$D_s\varphi(t) = \begin{cases} \varphi(st), & \text{if } \ 0 \le t \le s^{-1} \wedge 1, \\ 0, & \text{if } \ s^{-1} \wedge 1 \le t \le 1. \end{cases} \tag{1.11}$$

For a Banach space $\mathfrak{X}$, we denote by $(\mathcal{L}[\mathfrak{X}]\,,\, \|\cdot\|_{\mathcal{L}[\mathfrak{X}]})$ the Banach space consisting of all the bounded linear operators from $\mathfrak{X}$ into itself. If $\mathfrak{X}$ is a r.i. space over I, then $D_s \in \mathcal{L}[\mathfrak{X}]$ and $\|D_s\|_{\mathcal{L}[\mathfrak{X}]} \le 1 \vee s^{-1}$ for every $s > 0$.

Definition 1.4. Let B be an arbitrary u.r.i. space over Ω. The *lower* and *upper Boyd indices* are defined respectively by

$$\underline{\alpha}_B = \underline{\alpha}_{\hat{B}} = \sup_{0<s<1} \frac{\log \|D_{1/s}\|_{\mathcal{L}[\hat{B}]}}{\log s} = \lim_{s\to 0} \frac{\log \|D_{1/s}\|_{\mathcal{L}[\hat{B}]}}{\log s}\,;$$

$$\overline{\alpha}_B = \overline{\alpha}_{\hat{B}} = \inf_{s>1} \frac{\log \|D_{1/s}\|_{\mathcal{L}[\hat{B}]}}{\log s} = \lim_{s\to \infty} \frac{\log \|D_{1/s}\|_{\mathcal{L}[\hat{B}]}}{\log s}\,,$$

where $\hat{B}$ is a r.i. space over I satisfying (1.10).

Note that the indices are determined without depending on the choice of $\hat{B}$ by Lemma 1.3. These indices were introduced by Boyd [10]. By definition, we have $0 \le \underline{\alpha}_B \le \overline{\alpha}_B \le 1$ for any B. It is easily checked that $\|D_{1/s}\|_{\mathcal{L}[L^p]} = s^{1/p}$ $(1 \le p < \infty)$ and $\|D_{1/s}\|_{\mathcal{L}[L^\infty]} = 1$ for all $s > 0$. Thus we have $\underline{\alpha}_{L^p} = \overline{\alpha}_{L^p} = 1/p$ for every $1 \le p < \infty$ and $\underline{\alpha}_{L^\infty} = \overline{\alpha}_{L^\infty} = 0$. Note that in some other references, for example [1], [25] or [37], the indices of B are taken to be the reciprocals of the ones defined above; $1/\overline{\alpha}_B$ (resp. $1/\underline{\alpha}_B$) is called the lower (resp. upper) Boyd index there.

Since a probability measure is a finite measure, we have $L^p \hookrightarrow L^q$ for all p, q with $1 \le q \le p \le \infty$. The following theorem extends this result naturally.

Theorem 1.5. *If* $1/q < \underline{\alpha}_B \le \overline{\alpha}_B < 1/p$, *then* $L^q \hookrightarrow B \hookrightarrow L^p$.

For the proof, see [37, p. 132].

We now again consider a general Banach function space B. For each r.v. f we set

$$\|f\|_{B'} = \sup\{\, E[fg] \colon g \in B,\ \|g\|_B \le 1 \,\};$$
$$B' = \{f \colon \|f\|_{B'} < \infty\}.$$

It is easily checked that B' is a Banach function space again, which is called the *associate space* of B. For instance, if $B = L^p$ $(1 \le p \le \infty)$, then $B' = L^{p'}$, where p' is the exponent conjugate to p, and if (Φ, Ψ) is a complementary pair of Young functions and $B = L^{\Phi}$, then $B' = L^{\Psi}$ (in the sense that the norms of these spaces are equivalent). By definition, we see that $g \in B'$ if and only $fg \in L^1$ for all $f \in B$ (cf. [8, p. 10]) and

$$\int_\Omega |fg|\, dP \le \|f\|_B\, \|g\|_{B'}, \qquad f \in B,\ g \in B'. \tag{1.12}$$

In general, B' is not the dual of B: indeed, $L^1 = (L^\infty)'$ is not the dual of L^∞. We have $B'' = (B')' = B$ for any Banach function space B however (cf. [8, p. 10]).

Definition 1.6. A random variable f in a Banach function space B is said to *have absolutely continuous norm* or *be of absolutely continuous norm* in B if $\|f 1_{A_n}\|_B \to 0$ as $n \to \infty$ for every sequence $\{A_n\}_{n=1}^\infty$ of sets in $\mathcal{F}$ such that $A_n \downarrow \varnothing$ a.s.

If every $f \in B$ has absolutely continuous norm in B, then B itself is said to *have absolutely continuous norm* or *be of absolutely continuous norm.*

For instance, if $1 \le p < \infty$, L^p has absolutely continuous norm, while there is no non-zero r.v. which has absolutely continuous norm in L^∞. We often use the absolute continuity in the form of dominated convergence.

Theorem 1.7. *Suppose that $f \in B$ has absolutely continuous norm in B. If $\{g_n\}_{n=0}^\infty$ is a sequence of r.v.'s such that $|g_n| \le |f|$ a.s. and $g_n \to g$ a.s., then $\|g_n - g\|_B \to 0$.*

For the proof, see [8, p. 16]. Suppose that B has absolutely continuous norm. If $f \in B$, then there is a sequence of r.v.'s $\{f_n\}_{n \ge 0}$ in L^∞ such that $f_n \to f$ a.s. and $|f_n| \le |f|$. Hence Theorem 1.7 gives:

Corollary 1.8. *If B has absolutely continuous norm, then L^∞ is dense in B.*

2. Norm inequalities for processes

In this chapter, we shall give a norm inequality in a r.i. function space for some increasing processes. Using this, we shall establish the Burkholder–Davis–Gundy type inequalities in u.r.i. spaces, and prove that the BMO_1-norm and BMO_Φ-norm for martingales are equivalent if Φ is a Young function satisfying $\int_0^\infty \Phi(ct)e^{-t}\,dt < \infty$ for some $c > 0$.

We shall also establish some martingale inequalities involving the ratios of the maximal process and the quadratic variation process of the martingale. Our result, which is a probabilistic analog of the work by Murai and Uchiyama [44], extends the Burkholder–Davis–Gundy inequality and the inequalities established by Gundy [23] and Yor [56] independently. To prove them, we will use the inequality for increasing processes mentioned above.

§ 1. Norm inequalities for increasing processes

Throughout this section, we assume that $P(\Omega_0) > 0$, where Ω_0 denotes the non-atomic part of Ω, see §2 of Chapter 1.

Definition 2.1. We denote by $\mathcal{P}$ and $\mathcal{P}'$ the linear operators defined as follows: for each $\varphi \in L^1(I)$, we set

$$\mathcal{P}\varphi(t) = \frac{1}{t}\int_0^t \varphi(s)\,ds, \qquad t \in I; \tag{2.1}$$

$$\mathcal{P}'\varphi(t) = \int_t^1 \frac{\varphi(s)}{s}\,ds, \qquad t \in I, \tag{2.2}$$

whenever the integrals exist.

These operators are called *Hardy's averaging operators.*

We begin with some preliminaries. For each $\varphi \in L^1(I)$, we put $\varphi^\# = \mathcal{P}\varphi - \varphi$ (cf. [30]).

Lemma 2.2. (i) *If $\varphi \in L^1(I)$, then*

$$(\mathcal{P}'\varphi)^\#(t) = \mathcal{P}\varphi(t) \qquad \textit{for every } t \in I. \tag{2.3}$$

(ii) *If $\varphi \in \cup_{p>1} L^p(I)$ and $\psi \in L^\infty$, then*

$$\int_0^1 \varphi(s)\,\psi(s)\,ds = \int_0^1 \varphi^\#(s)\,\psi^\#(s)\,ds + I_\varphi\, I_\psi, \tag{2.4}$$

where $I_\varphi = \int_0^1 \varphi\, ds$ *and* $I_\psi = \int_0^1 \psi\, ds$. *Furthermore if* φ *is nonnegative and decreasing, then we may replace the condition* $\varphi \in \cup_{p>1} L^p(I)$ *by* $\varphi \in L^1(I)$.

Proof. Since $\mathcal{P}\mathcal{P}'\varphi = \mathcal{P}\varphi + \mathcal{P}'\varphi$, we have $(\mathcal{P}'\varphi)^\#(t) = \mathcal{P}\varphi + \mathcal{P}'\varphi - \mathcal{P}'\varphi = \mathcal{P}\varphi$. Note that if $\varphi \in \cup_{p>1} L^p(I)$, then $\mathcal{P}\varphi \in L^1(I)$; indeed we have

$$\int |\mathcal{P}\varphi|\, ds \le -\int |\varphi| \log s\, ds \le \Gamma(p'+1)\, \|\varphi\|_p < \infty$$

by Hölder's inequality, where Γ stands for the gamma function. Hence the function $\mathcal{P}'\mathcal{P}\varphi$ can be defined and satisfies

$$\mathcal{P}'\mathcal{P}\varphi + \int_0^1 \varphi(s)\, ds = \mathcal{P}\varphi + \mathcal{P}'\varphi \qquad \text{on } I.$$

From this formula, it follows that

$$\begin{aligned}
&\int \varphi^\#\, \psi^\#\, ds = \int (\mathcal{P}\varphi - \varphi)(\mathcal{P}\psi - \psi)\, ds \\
&= \int (\mathcal{P}'\mathcal{P}\varphi)\, \psi\, ds - \int (\mathcal{P}\varphi)\, \psi\, ds - \int \varphi\, (\mathcal{P}\psi)\, ds + \int \varphi\, \psi\, ds \\
&= \int (-I_\varphi + \mathcal{P}\varphi + \mathcal{P}'\varphi)\, \psi\, ds - \int (\mathcal{P}\varphi)\, \psi\, ds - \int \varphi\, (\mathcal{P}\psi)\, ds + \int \varphi\, \psi\, ds \\
&= -I_\varphi\, I_\psi + \int \varphi\, \psi\, ds.
\end{aligned}$$

Thus (2.4) is proved.

Now suppose that φ is decreasing and integrable over I. For each positive integer n, define φ_n by $\varphi_n(s) = \varphi(s \vee n^{-1})$. As $\varphi \ge 0$ is decreasing, we have $\varphi_n \uparrow \varphi$ and $\varphi_n^\# \uparrow \varphi^\#$ as $n \to \infty$. Since φ_n is bounded, (2.4) is valid for φ_n:

$$\int \varphi_n\, \psi\, ds = \int \varphi_n^\#\, \psi^\#\, ds + I_{\varphi_n}\, I_\psi\,.$$

The dominated convergence theorem gives (2.4) for this φ. □

Lemma 2.3. *Let* φ, $\psi \in L^1(I)$ *be nonnegative decreasing functions and* I_φ, I_ψ *be as in the preceding lemma. If* $\varphi^\# \le \psi^\#$ *on* I *and* $I_\varphi \le I_\psi$, *then* $\varphi \prec \psi$.

Proof. For each $t \in I$, we have $1_{[0,t[}^\# \ge 0$ on I and therefore $\varphi^\#\, 1_{[0,t[}^\# \le \psi^\#\, 1_{[0,t[}^\#$ on I. Hence the result follows immediately from (2.4). □

The following theorem is an extension of Garsia's lemma, which was established independently by Garsia [21] and Neveu [45] for processes with discrete parameter. Garsia's lemma for continuous parameter processes is due to Chou [14, p. 216]. See also Stroock [53].

Theorem 2.4. *Let B be a u.r.i. space over Ω and $\hat{B}$ be the r.i. space over I satisfying* (1.10). *If $\gamma \in L^1(\Omega)$ and $A = (A_t)_{t \geq 0}$ is an adapted increasing process satisfying*

$$E[A_\infty - A_{T-} \,|\, \mathcal{F}_T\,] \leq E[\,\gamma \,|\, \mathcal{F}_T\,] \tag{2.5}$$

for every stopping time T, then $A^*_\infty \prec \mathcal{P}'\gamma^*$. Therefore if $\mathcal{P}' \in \mathfrak{L}[\hat{B}]$, then $\|A_\infty\|_B \leq \|\mathcal{P}'\|_{\mathfrak{L}[\hat{B}]}\, \|\gamma\|_B$.

Let $A = (A_t)_{0 \leq t \leq \infty}$ be an adapted increasing process[‡] such that $A_\infty \in L^1$. Recall that a process $Z = (Z_t)_{t \geq 0}$ defined by

$$Z_t = E[A_\infty - A_{t-} \,|\, \mathcal{F}_t\,] = E[A_\infty \,|\, \mathcal{F}_t\,] - A_{t-} \quad (t \geq 0)$$

is called the *left potential* generated by the increasing process A. Hence (2.5) shows that the left potential generated by A is dominated by the martingale $\gamma_t = E[\,\gamma \,|\, \mathcal{F}_t\,]$. Furthermore the process defined by

$$Z_t = E[A_\infty - A_t \,|\, \mathcal{F}_t\,] = E[A_\infty \,|\, \mathcal{F}_t\,] - A_t \quad (t \geq 0)$$

is called the *potential* generated by A. The potential is a supermartingale. For details, see [18, p. 165].

Proof of Theorem 2.4. Let $\lambda > 0$ and set $T = \inf\{\,t \geq 0 \colon A_t > \lambda\,\}$. Then (2.5) implies that

$$\begin{aligned} E\big[(A_\infty - \lambda)^+\big] &\leq E\big[(A_\infty - A_{T-})\, 1_{\{T<\infty\}}\,\big] \\ &\leq E[\,\gamma\, 1_{\{T<\infty\}}] = E[\,\gamma\, 1_{\{A_\infty > \lambda\}}], \qquad \lambda > 0. \end{aligned}$$

Setting $\lambda = A^*_\infty(t)$, we have

$$\begin{aligned} \int_0^t \big(A^*_\infty(s) - A^*_\infty(t)\big)\, ds &\leq \int_0^1 \big(A^*_\infty(s) - A^*_\infty(t)\big) 1_{\{A^*_\infty \geq A^*_\infty(t)\}}\, ds \\ &= E\Big[\big(A_\infty - A^*_\infty(t)\big)^+\Big] \\ &\leq E\big[\,\gamma\, 1_{\{A_\infty > A^*_\infty(t)\}}\big]. \end{aligned} \tag{2.6}$$

‡ A may have a jump at 0 and at ∞, but $A_{0-} = 0$ by convention.

On the other hand, Hardy's inequality (1.3) implies that

$$E\big[\,\gamma\, 1_{\{A_\infty > A^*_\infty(t)\}}\big] \le \int_0^t \gamma^*(s)\, ds, \tag{2.7}$$

since $P\{A_\infty > A^*_\infty(t)\} \le t$. From (2.6), (2.7), and Lemma 2.2 (i), it follows that $(A^*_\infty)^\#(t) \le \mathcal{P}\gamma^*(t) = (\mathcal{P}'\gamma^*)^\#(t)$ for all $t \in I$. Since (2.5) yields that

$$\int_0^t A^*_\infty(s)\, ds = E[A_\infty] \le E[\,\gamma\,] = \int_0^1 \mathcal{P}'\gamma^*(s)\, ds,$$

we have $A^*_\infty \prec \mathcal{P}'\gamma^*$ by Lemma 2.3. It then follows that

$$\|A_\infty\|_B = \|A^*_\infty\|_{\hat{B}} \le \|\mathcal{P}'\gamma^*\|_{\hat{B}} \le \|\mathcal{P}'\|_{\mathfrak{L}[\hat{B}]}\, \|\gamma^*\|_{\hat{B}} = \|\mathcal{P}'\|_{\mathfrak{L}[\hat{B}]}\, \|\gamma\|_B\,,$$

which completes the proof. □

Corollary 2.5. *Let B be a u.r.i. space over Ω, $\gamma \in L^1(\Omega)$, and $A = (A_t)_{t\ge 0}$ be an adapted increasing process. If $A = (A_t)$ satisfies (2.5) and $\underline{\alpha}_B > 0$, then $\|A_\infty\|_B \le C\,\|\gamma\|_B$, where $C > 0$ is a constant depending on B only.*

Proof. The corollary follows from the following well-known theorem due to Shimogaki.

Theorem 2.6 (Shimogaki [51]). *Let $\mathfrak{X}$ be a r.i. space over I, and $\mathcal{P}$ and $\mathcal{P}'$ be averaging operators given by (2.1) and (2.2). Then:*

(i) *$\mathcal{P} \in \mathfrak{L}[\mathfrak{X}]$ if and only if $\overline{\alpha}_{\mathfrak{X}} < 1$;*

(ii) *$\mathcal{P}' \in \mathfrak{L}[\mathfrak{X}]$ if and only if $\underline{\alpha}_{\mathfrak{X}} > 0$.*

We now recall (the conditional form of) Davis's inequality: for every martingale $X = (X_t)_{t\ge 0}$ and stopping time T, we have

$$E[\,X^\circledast_\infty - X^\circledast_{T-} \,|\, \mathcal{F}_T\,] \le C\, E\big[\,[X,X]^{1/2}_\infty \,|\, \mathcal{F}_T\,\big], \tag{2.8}$$

$$E\big[\,[X,X]^{1/2}_\infty - [X,X]^{1/2}_{T-} \,|\, \mathcal{F}_T\big] \le C' E[\,X^\circledast_\infty \,|\, \mathcal{F}_T\,]. \tag{2.9}$$

Theorem 2.7. *Let B be a r.i. space over Ω. Then the inequalities*

$$c_B\, \big\|[X,X]^{1/2}_\infty\big\|_B \le \big\|X^\circledast_\infty\big\|_B \le C_B\, \big\|[X,X]^{1/2}_\infty\big\|_B \tag{2.10}$$

hold for every martingale $X = (X_t)$ with respect to arbitrary filtration if and only if $\mathcal{P}' \in \mathfrak{L}[\hat{B}]$, where c_B and C_B denote the positive constants depending on B only.

From (2.8), (2.9) and Theorem 2.4, we easily obtain (2.10) if $\mathcal{P}' \in \mathfrak{L}[\hat{B}]$. To prove the converse, we need some lemmas.

Lemma 2.8. *Let $\mathfrak{X}$ be a r.i. function space over I and $0 < t_0 \leq 1$. If the inequality*

$$\left\| (\mathcal{P}\varphi)\, 1_{[0,\, t_0[} \right\|_{\mathfrak{X}} \leq c \left(\|\varphi^{\#}\|_{\mathfrak{X}} + \|\varphi\|_1 \right) \tag{2.11}$$

holds for every $\varphi \in L^1(I)$, then $\mathcal{P}' \in \mathfrak{L}[\mathfrak{X}]$.

Proof. Without loss of generality, we may assume that $\|1\|_{\mathfrak{X}} = 1$. It suffices to show that $\|\mathcal{P}'\varphi\|_{\mathfrak{X}} \leq C\, \|\varphi\|_{\mathfrak{X}}$ for every positive $\varphi \in L^1(I)$. Put $\psi = \mathcal{P}'\varphi - \varphi$. Clearly we have $\psi \in L^1(I)$ and $\|\psi\|_1 \leq \|\mathcal{P}'\varphi\|_1 + \|\varphi\|_1 \leq 2\, \|\varphi\|_1$. Since $\mathcal{P}\mathcal{P}' = \mathcal{P} + \mathcal{P}'$, we get $\mathcal{P}\psi = \mathcal{P}'\varphi$ and $\psi^{\#} = \varphi$. As $|\mathcal{P}'\varphi(t)| \leq t_0^{-1}\, \|\varphi\|_1$ for $t \in [\, t_0,\ 1]$, (2.11) gives that

$$\begin{aligned}
\|\mathcal{P}'\varphi\|_{\mathfrak{X}} &\leq \left\| (\mathcal{P}'\varphi)\, 1_{[0,\, t_0[} \right\|_{\mathfrak{X}} + \left\| (\mathcal{P}'\varphi)\, 1_{[t_0,\, 1]} \right\|_{\mathfrak{X}} \\
&\leq \left\| (\mathcal{P}\psi)\, 1_{[0,\, t_0]} \right\|_{\mathfrak{X}} + t_0^{-1}\, \|\varphi\|_1 \\
&\leq c\big(\|\psi^{\#}\|_{\mathfrak{X}} + \|\psi\|_1 \big) + t_0^{-1}\, \|\varphi\|_1 \\
&\leq c\, \|\varphi\|_{\mathfrak{X}} + (2c + t_0^{-1})\, \|\varphi\|_1 \leq C\, \|\varphi\|_{\mathfrak{X}}\,,
\end{aligned}$$

which completes the proof. □

Lemma 2.9. *Let $t_0 = P(\Omega_0)$, where Ω_0 is the non-atomic part of Ω. Then, for each $\varphi \in L^1(I)$, there exists a uniformly integrable martingale $X = (X_t)_{t \geq 0}$ possessing the following properties*:

(i) $|X_0| \leq t_0^{-1}\, \|\varphi\|_1$,

(ii) $X_\infty^*(t) = \big(\varphi\, 1_{[0,\, t_0[} \big)^*(t), \quad t \in I$,

(iii) $\big\{ (\mathcal{P}\varphi)\, 1_{[0,\, t_0[} \big\}^*(t) \leq (X_\infty^{\circledast})^*(t), \quad t \in I$,

(iv) $\Big\{ \big([X,\, X]_\infty - X_0^2 \big)^{1/2} \Big\}^*(t) = \big(\varphi^{\#}\, 1_{[0,\, t_0[} \big)^*(t), \quad t \in I$.

Proof. Since Ω_0 contains no atom, there exists a family of measurable sets $\{A(t)\colon\ 0 \leq t \leq t_0\}$ satisfying the following conditions:

(a) $A(t) \subset A(s) \subset \Omega_1$ if $0 \leq s \leq t \leq t_0$;

(b) $P\,(A(t)) = t_0 - t$ for every $0 \leq t \leq t_0$.

For the proof, see [13, p. 44]. For each $0 \le t \le t_0$, let $\mathcal{F}_t$ be the σ-field generated by all measurable subsets of $\Omega \setminus A(t)$ and P-negligible sets, and for each $t \ge t_0$, set $\mathcal{F}_t = \mathcal{F}_{t_0}$. Clearly $(\mathcal{F}_t)_{t\ge 0}$ satisfies the usual hypotheses, and for every $0 \le t \le t_0$, $A(t)$ is an $\mathcal{F}_t$-atom.

Now for each $\omega \in \Omega$, put

$$T(\omega) = \begin{cases} \sup\{s \in [\,0,\, t_0\,] : \omega \in A(s)\}, & \text{if} \quad \omega \in A(0), \\ 0, & \text{otherwise.} \end{cases}$$

It easy to see that $\{T > t\} = A(t)$ a.s. for every $t \in [\,0\,,\, t_0\,]$. Hence T is an $(\mathcal{F}_t)$-stopping time, and $T^*(s) = (t_0 - s)^+ = (t_0 - s) \vee 0\,,\ s \in I$. Since $\varphi(t_0 - T)\, 1_{\{T>0\}}$ and $\varphi\, 1_{[\,0,\, t_0\,[}$ have the same distribution, $\varphi(t_0 - T)\, 1_{\{T>0\}}$ is integrable over Ω. Let $X = (X_t)$ be the martingale given by

$$\text{(2.12)} \qquad \begin{aligned} X_t &= E[\varphi(t_0 - T) \mid \mathcal{F}_t]\, 1_{\{T>0\}} \\ &= \varphi(t_0 - T)\, 1_{\{0<T\le t\}} + \mathcal{P}\varphi(t_0 - t)\, 1_{\{t<T\}}. \end{aligned}$$

Note that the processes on both sides of (2.12) are indistinguishable, that is, (2.12) holds for every $t \ge 0$ on a set Ω' of probability one.

We show that X satisfies the required conditions. In fact, (i) and (ii) are straightforward consequence of the definition. Since $T^*(s) = (t_0 - s)^+$, we have $|X_0| \le |\mathcal{P}\varphi(t_0)| \le t_0^{-1}\, \|\varphi\|_1$ and $X^*_\infty = \{\varphi(t_0 - T)\, 1_{\{T>0\}}\}^* = (\varphi\, 1_{[\,0,\, t_0\,[})^*$.

From (2.12) we see easily that $|\mathcal{P}\varphi(t_0 - T)|\, 1_{\{T>0\}} = |X_{T-}| \le X^{\circledast}_\infty$, which implies (iii).

Now it remains to prove (iv). Again from (2.12) we see that the path $t \mapsto X_t(\omega)$ is of bounded variation on $[\,0,\, \infty\,[$, continuous on $[\,0,\, T(\omega)[$ and constant on $[\,T(\omega),\, \infty\,[$, provided that $\omega \in \Omega'$. Therefore we have $\Delta X_T\, 1_{\{T>0\}} = -\varphi^\#(t_0 - T)\, 1_{\{T>0\}}$ and the continuous martingale part X^c of X is equal to zero. This implies that

$$\left([X,\, X]_\infty - X_0^2\right)^{1/2} = |\varphi^\#(t_0 - T)|\, 1_{\{T>0\}}.$$

Thus (iv) is obtained and the lemma is established. □

We are now in a position to prove Theorem 2.7.

Proof of Theorem 2.7. As mentioned above, (2.10) is true if $\mathcal{P}' \in \mathfrak{L}[\hat{B}]$. We prove that $\mathcal{P}' \in \mathfrak{L}[\hat{B}]$ if (2.10) is true. Suppose that (2.10) holds for any martingale. For each $\varphi \in L^1(I)$, there exits a martingale $X = (X_t)$ which satisfies the conditions of Lemma 2.9. Assuming $\|1\|_{\hat{B}} = 1$ for simplicity, we have

$$\begin{aligned}\|(\mathcal{P}\varphi)\,1_{[0,\,t_0[}\|_{\hat{B}} &\le \|X^{\circledast}_\infty\|_B \le C_B\,\|[X,\,X]^{1/2}_\infty\|_B \\ &\le C_B\|\big([X,\,X]_\infty - X_0^2\big)^{1/2}\|_B + C_B\,\|X_0\|_B \\ &\le C_B\,\|\varphi^{\#}\,1_{[0,\,t_0[}\|_{\hat{B}} + C_B\,t_0^{-1}\,\|\varphi\|_1 \\ &\le C_B\,t_0^{-1}\big(\|\varphi^{\#}\,1_{[0,\,t_0[}\|_{\hat{B}} + \|\varphi\|_1\big).\end{aligned}$$

Lemma 2.8 implies that $\mathcal{P}' \in \mathcal{L}[\hat{B}]$. The theorem is established. □

Corollary 2.10. *Let B be a r.i. space over Ω. Then the inequalities*

$$c_B\,\|[X,\,X]^{1/2}_\infty\|_B \le \|X^{\circledast}_\infty\|_B \le C_B\,\|[X,\,X]^{1/2}_\infty\|_B \tag{2.13}$$

hold for every martingale $X = (X_t)$ with respect to arbitrary filtration if and only if $\underline{\alpha}_B > 0$.

Remark. The above corollary is deduced from Shimogaki's theorem (Theorem 2.6). On the other hand, Theorem 2.7 shows that Shimogaki's theorem is deduced from the statement of the above corollary.

§ 2. *BMO*$_\Phi$–spaces for martingales

Needless to say, *BMO*–martingales play essential role in the development of theory of martingales. Recall that a uniformly integrable martingale $X = (X_t)_{t\ge 0}$ is said to be a *BMO–martingale* if

$$\sup_T E\big[\,|X_\infty - X_{T-}|\,\big|\,\mathcal{F}_T\,\big] \le c \quad \text{a.s.}, \tag{2.14}$$

where the supremum is taken over all stopping times T. The infimum of c appearing in (2.14) is denoted by $\|X\|_{BMO_1}$ and called BMO_1-norm of X. It is well-known that $X = (X_t)$ is a *BMO*–martingale if and only if

$$\sup_T E\big[\,|X_\infty - X_{T-}|^p\,\big|\,\mathcal{F}_T\,\big]^{1/p} \le c \quad \text{a.s.}, \tag{2.15}$$

where $1 \le p < \infty$. Denoting by $\|X\|_{BMO_p}$ the infimum of c in (2.15), we have

$$\|X\|_{BMO_1} \le \|X\|_{BMO_p} \le C_p\,\|X\|_{BMO_1}\,.$$

For the proof, see [26, p. 28].

On the other hand, Bassily and Mogyródi [6] introduced the BMO_Φ-norm corresponding to a general Young function Φ: with each uniformly integrable martingale $X = (X_t)$ we associate the value $\|X\|_{BMO_\Phi}$ given by

$$\begin{aligned} \|X\|_{BMO_\Phi} & \\ &= \inf\{\lambda > 0 : \sup_T \|E[\Phi(\lambda^{-1}|X_\infty - X_{T-}|) \mid \mathcal{F}_T]\|_\infty \le 1\}, \end{aligned} \tag{2.16}$$

where $\inf \varnothing = \infty$ as usual. The original definition of $\|X\|_{BMO_\Phi}$ is slightly different from our definition, but they coincide in fact.

Recall that Φ is said to satisfy the Δ_2-condition if there are constants $t_0 > 0$ and $C > 0$ such that $\Phi(2t) \le C\,\Phi(t)$ for every $t \ge t_0$.

Bassily and Mogyródi proved that if Φ satisfies the Δ_2-condition, then the BMO_1-norm and BMO_Φ-norm are equivalent. Their proof is elementary but somewhat complicated. We shall give a more simple proof, using the results in the previous section.

We begin with establishing a new inequality for increasing process with bounded left potential.

Theorem 2.11. *Let Φ be an increasing convex function on $[0, \infty[$ such that $\Phi(0+) = \Phi(0) = 0$. If $A = (A_t)_{t\ge 0}$ is an adapted increasing process satisfying*

$$E[A_\infty - A_{T-} \mid \mathcal{F}_T] \le c \quad a.s., \tag{2.17}$$

then

$$E[\Phi(A_\infty)] \le c^{-1}\Big(\int_0^\infty \Phi(ct)\, e^{-t}\, dt\Big) E[A_\infty]. \tag{2.18}$$

The inequality (2.18) extends the John–Nirenberg inequality and the so-called energy inequality for increasing processes.

Corollary 2.12. *Let $A = (A_t)_{t\ge 0}$ be as in Theorem 2.11. Then:*

$$E[A_\infty^p] \le c^{p-1}\Gamma(p+1)E[A_\infty] \le c^p\Gamma(p+1), \qquad (1 \le p < \infty);$$

$$E[\exp(\alpha A_\infty)] \le \frac{\alpha}{1 - c\alpha}E[A_\infty] + 1 \le \frac{1}{1 - c\alpha}, \qquad (0 < \alpha < 1/c).$$

To prove Theorem 2.11, we need the following lemma which shows that L^Φ is a u.r.i. space.

Lemma 2.13. *Let Φ be as in Theorem 2.11 and let x, $y \in L^1([0,1])$ be nonnegative. If $x \prec y$, then*

$$\int_0^1 \Phi\big(x(t)\big)\,dt \le \int_0^1 \Phi\big(y(t)\big)\,dt. \tag{2.19}$$

Proof. Let φ be the right-derivative of Φ, and set $\Phi_n(t) = \int_0^t \varphi(s) \wedge n\,ds$ for each integer $n \ge 1$. Then each Φ_n is an increasing convex function with $\Phi(0) = 0$. To prove the lemma, it suffices to verify (2.19) for Φ_n instead of Φ by the monotone convergence theorem. Therefore we prove (2.19) for Φ under the assumption that φ is *bounded.* Clearly we may assume also that the right-hand side of (2.19) is finite, and hence that x, $y \in L^1([0,\,1])$. Since $x \prec y$, we have

$$\int_0^1 x^*(t)\psi(t)\,dt \le \int_0^1 y^*(t)\psi(t)\,dt$$

for every nonnegative decreasing function ψ (Hardy's lemma: cf. [8, p. 56]). In particular, we obtain

$$\int_0^1 x^*(t)\varphi\circ x^*(t)\,dt \le \int_0^1 y^*(t)\varphi\circ x^*(t)\,dt. \tag{2.20}$$

Recall the formula

$$v\varphi(u) \le \int_{]0\,,\,u]} s\,d\varphi(s) + \Phi(v),$$

where the equality holds if $u = v$. From this fact, it follows that

$$\int_0^1 x^*(t)\varphi\circ x^*(t)\,dt = \int_0^1 dt\Big[\int_0^\infty s1_{\{x^*(t)\ge s\}}\,d\varphi(s)\Big] + \int_0^1 \Phi\big(x^*(t)\big)\,dt;$$

$$\int_0^1 y^*(t)\varphi\circ x^*(t)\,dt \le \int_0^1 dt\Big[\int_0^\infty s1_{\{x^*(t)\ge s\}}\,d\varphi(s)\Big] + \int_0^1 \Phi\big(y^*(t)\big)\,dt.$$

Since φ is bounded and x is integrable, the iterated integral is finite. Hence (2.19) follows from (2.20) and the above inequalities, since x and x^* have the same distribution. □

Proof of Theorem 2.11. Without loss of generality, we may assume that $c = 1$. From (2.6) and (2.17) with $\gamma = c = 1$, it follows that

$$(A^*_\infty)^\#(t) = \frac{1}{t}\int_0^t \big(A^*_\infty(s) - A^*_\infty(t)\big)\,ds \le 1.$$

On the other hand, easily we have $(A^*_\infty)^\#(t) \leq t^{-1}E[A_\infty]$: thus

$$(A^*_\infty)^\#(t) \leq 1 \wedge (a/t) = \mathcal{P}1_{[0,\,a]}(t) = \big(\mathcal{P}'1_{[0,\,a]}\big)^\#(t),$$

where $a = E[A_\infty]$. Since $a = \int_0^1 \mathcal{P}'1_{[0,\,a]}\,ds$, Lemma 2.3 gives that $A^*_\infty \prec \mathcal{P}'1_{[0,\,a]}(t) = \log^+(a/t)$. From Lemma 2.13, we obtain

$$E\big[\Phi(A_\infty)\big] = \int_0^1 \Phi(A^*_\infty)\,ds \leq \int_0^1 \Phi\big(\log^+(a/t)\big)\,dt = a\int_0^\infty \Phi(s)e^{-s}\,ds,$$

which completes the proof. □

Now let $X = (X_t)_{t\geq 0}$ be a uniformly integrable martingale such that $\|X\|_{BMO_\Phi} < \infty$. Then Jensen's inequality gives that

$$\Phi\bigg(\frac{E\big[|X_\infty - X_{T-}|\,\big|\,\mathcal{F}_T\big]}{\|X\|_{BMO_\Phi}}\bigg) \leq E\bigg[\Phi\bigg(\frac{|X_\infty - X_{T-}|}{\|X\|_{BMO_\Phi}}\bigg)\bigg|\,\mathcal{F}_T\bigg] \leq 1 \quad \text{a.s.},$$

for every stopping time T. Let $\Phi^{-1}(t)$ denote the right-continuous generalized inverse of Φ: $\Phi^{-1}(t) = \sup\{s \geq 0\colon \Phi(s) \leq t\}$. Then the above inequality shows that

$$E\big[|X_\infty - X_{T-}|\,\big|\,\mathcal{F}_T\big] \leq \Phi^{-1}(1)\,\|X\|_{BMO_\Phi} \quad \text{a.s.}$$

for every T. Thus X is a BMO–martingale and we obtain

$$c_\Phi\,\|X\|_{BMO_1} \leq \|X\|_{BMO_\Phi}\,.$$

This inequality holds for arbitrary Φ. On the other hand, we have:

Theorem 2.14. *Let Φ be an increasing convex function on $[\,0,\,\infty\,[$ such that $\Phi(0+) = \Phi(0) = 0$. If $\int_0^\infty \Phi(ct)e^{-t}\,dt < \infty$ with some constant $c > 0$, then*

$$c_\Phi\,\|X\|_{BMO_1} \leq \|X\|_{BMO_\Phi} \leq C_\Phi\,\|X\|_{BMO_1} \tag{2.21}$$

for every uniformly integrable martingale $X = (X_t)_{t\geq 0}$.

Proof. The left-hand side of (2.21) has been proved above. To prove the right-hand side, let $X = (X_t)$ be a BMO–martingale. It is well-known that

$$E[X^\circledast_\infty - X^\circledast_{T-}\,|\,\mathcal{F}_T] \leq 4\,\|X\|_{BMO_1}\,.$$

For the proof, see [18, p. 193]. Let $C_\Phi^{-1} = \sup\{c > 0 : \int_0^\infty \Phi(ct)e^{-t}\,dt \le 1\}$: then $0 < C_\Phi < \infty$. Setting $\beta = 4\,\|X\|_{BMO_1}$, we have by Theorem 2.11

$$E\Big[\Phi\Big(\frac{X_\infty^\circledast}{\beta C_\Phi}\Big)\Big] \le \frac{1}{\beta}\Big(\int_0^\infty \Phi(C_\Phi^{-1}t)\,e^{-t}\,dt\Big)E\big[X_\infty^\circledast\big] \tag{2.22}$$

$$\le \beta^{-1}E\big[X_\infty^\circledast\big] \le 1.$$

We can obtain the conditional form of this inequality: for every stopping time T,

$$E\Big[\Phi\Big(\frac{1}{\beta C_\Phi}\sup_{t\ge 0}\big|X_{T+t} - X_{T-}\big|\Big)\Big|\,\mathcal{F}_T\Big] \le 1 \quad \text{a.s.}, \tag{2.23}$$

This will imply that $\|X\|_{BMO_\Phi} \le \beta C_\Phi = 4C_\Phi\,\|X\|_{BMO_1}$ and we will finish the proof.

To prove (2.23), let T be an arbitrary stopping time with $P(T < \infty) > 0$. Let $\Omega' = \{T < \infty\}$, $dP' := 1_{\Omega'}P(\Omega')^{-1}\,dP$ and $(\mathcal{F}_t')_{t\ge 0} = (\mathcal{F}_{T+t})_{t\ge 0}$. If we denote by (X_t') the process defined by $X_t' = X_{T+t} - X_{T-}$, then (X_t') is a uniformly integrable martingale on Ω' with respect to P' and $(\mathcal{F}_t')$. If S is an $(\mathcal{F}_t')$–stopping time, then $T + S$ is an $(\mathcal{F}_t)$–stopping time and $\mathcal{F}_S' \subset \mathcal{F}_{T+S}$. Furthermore we have

$$E'\big[|X_\infty' - X_{S-}'|\,\big|\,\mathcal{F}_S'\big] \le \|X\|_{BMO_1}\,,$$

where E' stands for the expectation with respect to P'. Therefore, by (2.22), we have

$$P(\Omega')^{-1}E\Big[\Phi\Big(\frac{1}{\beta C_\Phi}\sup_{t\ge 0}\big|X_{T+t} - X_{T-}\big|\Big)\,1_{\Omega'}\Big] = E'\Big[\Phi\Big(\frac{X_\infty'^\circledast}{\beta C_\Phi}\Big)\Big] \le 1.$$

Now let $F \in \mathcal{F}_T$ and T_F be the stopping time defined by

$$T_F = \begin{cases} T, & \text{on } F, \\ \infty, & \text{on } \Omega \setminus F. \end{cases} \tag{2.24}$$

Using T_F instead of T, we obtain

$$E\Big[\Phi\Big(\frac{1}{\beta C_\Phi}\sup_{t\ge 0}\big|X_{T+t} - X_{T-}\big|\Big)\,1_F\Big] \le P(T_F < \infty) \le P(F),$$

which implies (2.23). The theorem is established. □

§ 3. Ratio inequalities for martingales

In this section, we shall prove some ratio inequalities for continuous martingales. Theorem 2.11 plays an essential role in our argument.

The following lemma is a variant of the result due to Murai and Uchiyama [44]: we shall give here an alternative proof established in [29].

Lemma 2.15. *Let f and g be nonnegative random variables on Ω, and $a > 0$ and $b > 0$ be constants. Suppose that*

$$P(f > \gamma\lambda,\ g \le \lambda) \le b\, e^{-a\gamma} P(f > \lambda) \tag{2.25}$$

holds for every $\lambda > 0$ and $\gamma > 1$. Then the inequalities

$$E\big[\, f^p \exp(\alpha f/g)\big] \le C E[\, f^p\,], \tag{2.26}$$

$$E\big[\, g^p \exp(\alpha f/g)\big] \le C E[\, g^p\,] \tag{2.27}$$

hold for every $0 \le \alpha < a$ and $0 < p < \infty$, where $C > 0$ is a constant depending on a, b, and α only. Furthermore $E[\, f^p\,]$ can be replaced by $E[\, g^p\,]$ in (2.26).

Proof. Letting $\lambda \downarrow 0$ and then $\gamma \to \infty$ in (2.21), we have $P(f > 0,\ g = 0) = 0$. Hence the ratio f/g is meaningful a.s. Let $\alpha/a < \delta < 1$. Integrating the both sides of (2.25) with respect to the measure $d\lambda^p$ and using Fubini's theorem, we have

$$E\big[\{(f/\gamma)^p - g^p\}1_{\{\delta f/g \ge \gamma\}}\big] \le b\, e^{-a\gamma} E[\, f^p\,]$$

for every $\gamma > 1$. Again we integrate the both sides of this inequality over $[1, \infty[$ with respect to the measure $\gamma^p \exp\big((\alpha/\delta)\gamma\big)\, d\gamma$. It then follows that

$$E\Big[\int_1^{\delta f/g} (f^p - \gamma^p g^p)\ \exp\big((\alpha/\delta)\gamma\big)\, d\gamma\ 1_{\{f/g \ge 1/\delta\}}\Big] \le C' E[\, f^p\,],$$

where $C' = b \int_0^\infty \gamma^p \exp\{-(a - \alpha/\delta)\gamma\}\, d\gamma$ is a finite constant depending on a, b, and α. For every γ with $1 \le \gamma < \delta f/g$, we have $f^p - \gamma^p g^p \ge (1 - \delta^p) f^p$. Hence the above inequality gives that

$$E\big[f^p\{\exp(\alpha f/g) - \exp(\alpha/\delta)\}\ 1_{\{f/g \ge 1/\delta\}}\big] \le \frac{\alpha C'}{(1 - \delta^p)\delta}\, E[\, f^p\,].$$

From this we see that

$$E\big[f^p \exp(\alpha f/g)\, 1_{\{f/g \geq 1/\delta\}}\big] \leq \frac{\alpha C'}{(1-\delta^p)\delta}\, E[\, f^p\,] + e^{\alpha/\delta} E\big[f^p\, 1_{\{f/g \geq 1/\delta\}}\big],$$

and therefore that

$$\begin{aligned} E\big[f^p \exp(\alpha f/g)\big] &\leq E\big[f^p \exp(\alpha f/g)\, 1_{\{f/g \geq 1/\delta\}}\big] + e^{\alpha/\delta} E\big[f^p\, 1_{\{f/g < 1/\delta\}}\big] \\ &\leq \Big\{\frac{\alpha C'}{(1-\delta^p)\delta} + 1\Big\} E[\, f^p\,]. \end{aligned}$$

Thus (2.26) is established. Moreover it is well-known that (2.25) implies that $E[\, f^p\,] \leq K_p E[\, g^p\,]$ for some constant $K_p > 0$: for the proof, see e.g. [7]. Therefore we may replace $E[\, f^p\,]$ by $E[\, g^p\,]$ in (2.26). Finally we prove (2.27). We have easily that

$$\begin{aligned} E\big[g^p \exp(\alpha f/g)\big] &\leq \delta^p E\big[f^p \exp(\alpha f/g)\, 1_{\{f/g \geq 1/\delta\}}\big] + e^{\alpha/\delta} E[\, g^p\,] \\ &\leq \Big\{\frac{\alpha C'}{(1-\delta^p)\delta} + 1\Big\} E[\, f^p\,] + e^{\alpha/\delta} E[\, g^p\,] \\ &\leq C E[\, g^p\,]. \end{aligned}$$

Thus the lemma is established. □

The following theorem is applicable to many increasing processes associated with martingales.

Theorem 2.16. *Let $U = (U_t)_{t \geq 0}$ and $V = (V_t)_{t \geq 0}$ be adapted increasing processes. Suppose that*

$$E[\, U_\infty^\sigma - U_{T-}^\sigma \,|\, \mathcal{F}_T\,] \leq \kappa E[\, V_{\sigma-} \,|\, \mathcal{F}_T\,] \tag{2.28}$$

holds for all stopping times σ and T with some constant $\kappa > 0$, where U^σ denotes the stopped process $(U_{t\wedge\sigma})$. Then the inequalities

$$E\big[\, U_\infty^p \exp(\alpha\, U_\infty / V_\infty)\big] \leq C E[\, U_\infty^p], \tag{2.29}$$

$$E\big[\, V_\infty^p \exp(\alpha\, U_\infty / V_\infty)\big] \leq C E[\, V_\infty^p] \tag{2.30}$$

hold for every $0 < p < \infty$ and $0 < \alpha < 1/\kappa$, where $C > 0$ is a constant depending on α, p and κ only.

Proof. For each fixed $\lambda > 0$, let τ and σ denote the stopping times given by $\tau = \inf\{t \geq 0 : U_t > \lambda\}$ and $\sigma = \inf\{t \geq 0 : V_t > \lambda\}$. Since $V_{\sigma-} \leq \lambda$ a.s., (2.28) gives that

$$E[\,U^\sigma_\infty - U^\sigma_{T-}\,|\,\mathcal{F}_T\,] \leq \kappa\lambda$$

for every stopping time T. From Corollary 2.12, we see that

$$E\big[\exp(\delta\kappa^{-1}\lambda^{-1}U^\sigma_\infty)\big] \leq (1-\delta)^{-1}$$

for every $0 < \delta < 1$. As in the proof of Theorem 2.14, we obtain the conditional form of this inequality:

$$E\big[\exp\big\{\delta\kappa^{-1}\lambda^{-1}(U^\sigma_\infty - U^\sigma_{T-})\big\}\,\big|\,\mathcal{F}_T\,\big] \leq (1-\delta)^{-1}.$$

From this and the fact that $U_{\tau-} \leq \lambda$, we obtain

$$\begin{aligned}
&P\{\,U_\infty > \gamma\lambda, V_\infty \leq \lambda\,\}\\
&\qquad\leq P\big\{\,U_\infty - U_{\tau-} > (\gamma-1)\lambda,\ \tau < \infty,\ \sigma = \infty\,\big\}\\
&\qquad\leq P\big\{\,\delta\kappa^{-1}\lambda^{-1}(U^\sigma_\infty - U^\sigma_{\tau-}) > \delta\kappa^{-1}(\gamma-1),\ \tau < \infty\big\}\\
&\qquad\leq \exp\big\{-\delta\kappa^{-1}(\gamma-1)\big\}E\big[\exp\big\{\delta\kappa^{-1}\lambda^{-1}(U^\sigma_\infty - U^\sigma_{\tau-})\big\}\,1_{\{\tau<\infty\}}\big]\\
&\qquad\leq \exp\big\{-\delta\kappa^{-1}(\gamma-1)\big\}(1-\delta)^{-1}P(\,\tau < \infty\,)\\
&\qquad= C_\delta \exp(-\delta\kappa^{-1}\gamma\,)\,P(\,U_\infty > \lambda\,),
\end{aligned}$$

where $C_\delta = (1-\delta)^{-1}e^{\delta/\kappa}$. Now let $0 < \alpha < 1/\kappa$ and choose $0 < \delta < 1$ such that $\alpha < \delta/\kappa$. From the above inequalities and Lemma 2.15, we obtain (2.29) and (2.30). Thus the theorem is established. □

Note that, for a continuous martingale $X = (X_t)$, the conditional form of Davis's inequality (2.8) can be rewritten as

$$E\big[X^\circledast_\sigma - X^\circledast_{\sigma\wedge T}\,\big|\,\mathcal{F}_T\,\big] \leq 4\sqrt{2}\,E\big[{<}X{>}^{1/2}_\sigma\,\big|\,\mathcal{F}_T\,\big] \tag{2.31}$$

for every stopping time σ and T, where $({<}X{>}_t) = ({<}X,\,X{>}_t)$. Furthermore, from the definition of $({<}X{>}_t)$, it follows immediately that

$$E\big[{<}X{>}_\sigma - {<}X{>}_{\sigma\wedge T}\,\big|\,\mathcal{F}_T\,\big] \leq E\big[X^{\circledast 2}_\sigma\,\big|\,\mathcal{F}_T\,\big]. \tag{2.32}$$

From the above two inequalities and Theorem 2.16, we have:

Theorem 2.17. (i) *If* $0 \leq \alpha < \left(4\sqrt{2}\right)^{-1}$ *and* $0 < p < \infty$, *then the inequalities*

$$E\big[X_\infty^{\circledast\,p} \exp(\alpha X_\infty^{\circledast} / \langle X\rangle_\infty^{1/2}\,)\big] \leq C_{\alpha,p}\, E[X_\infty^{\circledast\,p}],$$

$$E\big[\langle X\rangle_\infty^{p} \exp(\alpha X_\infty^{\circledast} / \langle X\rangle_\infty^{1/2}\,)\big] \leq C_{\alpha,p}\, E[\langle X\rangle_\infty^{p}]$$

hold for every continuous martingale $X = (X_t)$ *with* $X_0 = 0$.

(ii) *If* $0 \leq \alpha < 1$ *and* $0 < p < \infty$, *then the inequalities*

$$E\big[\langle X\rangle_\infty^{p} \exp(\alpha \langle X\rangle_\infty / X_\infty^{\circledast\,2}\,)\big] \leq C_{\alpha,p}\, E[\langle X\rangle_\infty^{p}],$$

$$E\big[X_\infty^{\circledast\,p} \exp(\alpha \langle X\rangle_\infty / X_\infty^{\circledast\,2}\,)\big] \leq C_{\alpha,p}\, E[\langle X\rangle_\infty^{p}]$$

hold for every continuous martingale $X = (X_t)$ *with* $X_0 = 0$.

Here $C_{\alpha,p}$ *denotes the constant depending on* α *and* p *only, and it is not the same from line to line.*

Remark. More sharp estimates have been obtained in [29]. If $0 \leq \alpha < 1/2$, then the inequality

$$E\big[X_\infty^{\circledast\,p} \exp(\alpha X_\infty^{\circledast\,2} / \langle X\rangle_\infty)\big] \leq C_{\alpha,p}\, E[X_\infty^{\circledast\,p}]$$

is valid for every $0 < p < \infty$ and continuous martingale $X = (X_t)$ with $X_0 = 0$: it is not valid for any $p > 0$ when $\alpha \geq 1/2$.

If $0 \leq \alpha < \pi^2/8$, then the inequality

$$E\big[\langle X\rangle_\infty^{p} \exp(\alpha \langle X\rangle_\infty / X_\infty^{\circledast\,2}\,)\big] \leq C_{\alpha,p}\, E[\langle X\rangle_\infty^{p}]$$

is valid for every $0 < p < \infty$ and continuous martingale $X = (X_t)$ with $X_0 = 0$: it is not valid for any $p > 0$ when $\alpha \geq \pi^2/8$.

We conclude this section by mentioning inequalities involving a ratio of local times and maximal processes (or quadratic variations) of martingales.

We recall first the Barlow–Yor inequality for local times. Let $(L_t^a)_{t\geq 0} = \big(L^a(X)_t\big)_{t\geq 0}$ be the local time of a continuous martingale $X = (X_t)$ at a point $a \in \mathbb{R}$. Barlow and Yor proved in [5] that the inequalities

$$c_p\, E[\, X_\infty^{\circledast\,p}\,] \leq E[\, L_\infty^{*\,p}\,] \leq C_p\, E[\, X_\infty^{\circledast\,p}\,] \quad (0 < p < \infty)$$

hold for every continuous martingale $X = (X_t)$, where $L_t^* = \sup_{a\in\mathbb{R}} L_t^a$. From these inequalities, we obtain

$$E\big[X_\sigma^{\circledast} - X_{\sigma\wedge T}^{\circledast} \,\big|\, \mathcal{F}_T\big] \leq C E[L_\sigma^* \,|\, \mathcal{F}_T\,];$$

$$E\big[L_\sigma^* - L_{\sigma\wedge T}^* \,\big|\, \mathcal{F}_T\big] \leq C E[X_\sigma^{\circledast} \,|\, \mathcal{F}_T\,].$$

Therefore, by Theorem 2.16, we have:

Theorem 2.18. *If $\alpha \geq 0$ is sufficiently small, then the inequalities*

$$E\big[L_\infty^{*p} \exp(\alpha L_\infty^* / X_\infty^{\circledast})\big] \leq C_{\alpha,p}\, E[L_\infty^{*p}];$$
$$E\big[X_\infty^{\circledast p} \exp(\alpha L_\infty^* / X_\infty^{\circledast})\big] \leq C_{\alpha,p}\, E[X_\infty^{\circledast p}];$$
$$E\big[X_\infty^{\circledast p} \exp(\alpha X_\infty^{\circledast} / L_\infty^*)\big] \leq C_{\alpha,p}\, E[X_\infty^{\circledast p}];$$
$$E\big[L_\infty^{*p} \exp(\alpha X_\infty^{\circledast} / L_\infty^*,)\big] \leq C_{\alpha,p}\, E[L_\infty^{*p}],$$

hold for every $0 < p < \infty$ and continuous martingales $X = (X_t)$ with $M_0 = 0$.

3. Doob's inequality in function spaces

§ 1. Doob's inequality and filtration

Throughout this short chapter we shall work with a fixed probability space $(\Omega, \mathcal{F}, P)$ which contains no atom, and for the sake of convenience, we shall consider discrete parameter martingales only. Note that a martingale $X = (X_n)_{n \geq 0}$ with discrete parameter can be regarded as a martingale with continuous parameter: for each $t \geq 0$, we set $X_t = X_{[t]}$, where $[t]$ denotes the integer part of t.

Let B be a r.i. space. Recently Antipa [1] and Novikov [46] proved independently that the Doob type inequality

$$(3.1) \qquad \|X^{\circledast}_{\infty}\|_B \leq C_B \|X_{\infty}\|_B$$

holds for all uniformly integrable martingales $X = (X_n)_{n\geq 0}$ if and only if $\overline{\alpha}_B < 1$, where C_B is a positive constant depending on the space B only. Note that the condition $\overline{\alpha}_B < 1$ is independent of the filtration $(\mathcal{F}_n)_{n\geq 0}$ with respect to which $X = (X_n)$ is a martingale. In this section, we shall show that if B is a Banach function space and the Doob type inequality (3.1) is valid for any filtration, then B is r.i. Thus the Doob type inequality is a characteristic property of r.i. space.

Theorem 3.1. *Let $(B, \|\cdot\|_B)$ be a Banach function space over Ω. Then the following statements are equivalent:*

(i) *The inequality* (3.1) *holds for all uniformly integrable martingale $X = (X_n)_{n\geq 0}$ with respect to an arbitrary filtration $(\mathcal{F}_n)_{n\geq 0}$.*

(ii) *There exists a norm $|||\cdot|||_B$ which is equivalent to $\|\cdot\|_B$ and such that $(B, |||\cdot|||_B)$ is a rearrangement invariant space with $\overline{\alpha}_B < 1$.*

To prove the theorem, we shall use the following two lemmas. The first one is a well-known and powerful lemma which characterizes r.i. spaces. The other is a simple modification of the first one.

Lemma 3.2. *Let $(B, \|\cdot\|_B)$ be a Banach function space over Ω. The following statements are equivalent:*

(i) *There exists a norm $|||\cdot|||_B$ which is equivalent to $\|\cdot\|_B$ and such that $(B, |||\cdot|||_B)$ is rearrangement invariant.*

(ii) *If $f \in B$ and $g \in L^1(\Omega)$ are identically distributed, then $g \in B$.*

Condition (ii) is the original definition of r.i. space by Luxemburg [41]. We leave details to sections 11 and 16 of [41], or sections 16 and 17 of [13].

Lemma 3.3. *Let $(B, \|\cdot\|_B)$ be a Banach function space over Ω. The following statements are equivalent:*

(i) *There exists a norm $|||\cdot|||_B$ which is equivalent to $\|\cdot\|_B$ and such that $(B, |||\cdot|||_B)$ is rearrangement invariant;*

(ii) *If $f \in B$ and $g \in L^1(\Omega)$ are nonnegative, identically distributed r.v.'s such that $\{ f > 0 \} \cap \{ g > 0 \} = \varnothing$ a.s., then $g \in B$.*

Proof. It suffices to prove that (ii) implies (i), since the converse follows from Lemma 3.2. Suppose that f and g are identically distributed nonnegative r.v.'s and $f \in B$. In view of Lemma 3.2, it suffices to show that $g1_{\{g>\lambda\}} \in B$ for some constant $\lambda > 0$, since $g1_{\{g \le \lambda\}} \in B$.

We denote by f_λ and g_λ the r.v.'s $f1_{\{f>\lambda\}}$ and $g1_{\{g>\lambda\}}$, respectively. Since $g \in L^1$, we can find a $\lambda > 0$ such that $P\{f > \lambda\} = P\{g > \lambda\} < 1/3$. Then we have $P\{f \le \lambda,\ g \le \lambda\} > 1/3$. Since Ω is non-atomic, there exists a random variable h such that $h^* = f^*_\lambda$ and $\{h > 0\} \subset \{f \le \lambda,\ g \le \lambda\}$ (cf. [13, p. 44]). Condition (ii), together with the fact that $f_\lambda \in B$, implies that $h \in B$, since $\{f_\lambda > 0\} \cap \{h > 0\} = \varnothing$. Since $h^* = g_\lambda$ and $\{g_\lambda > 0\} \cap \{h > 0\} = \varnothing$, we obtain $g_\lambda \in B$ again by (ii). The lemma is proved. □

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. (i) $\Rightarrow$ (ii). Suppose that f and g have the same distribution, $0 \le f \in B$ and $\{f > 0\} \cap \{g > 0\} = \varnothing$ a.s. We show that $g \in B$; then it will be follow from Lemma 3.3 that $(B, |||\cdot|||_B)$ is a r.i. space with some equivalent norm $|||\cdot|||_B$. For integers $n \ge 0$ and $k \ge 0$, we set

$$A_k^{(n)} = \left\{ \frac{k}{2^n} < f \le \frac{k+1}{2^n} \right\} \quad \text{and} \quad B_k^{(n)} = \left\{ \frac{k}{2^n} < g \le \frac{k+1}{2^n} \right\}.$$

For an arbitrarily fixed integer $m > 0$, we define filtrations $(\mathcal{F}_n)_{n \ge 0}$ and

$(\mathcal{G}_n)_{n\geq 0}$ as follows:

$$\mathcal{F}_n = \begin{cases} \sigma\big(A_k^{(n)} \cup B_k^{(n)},\ k \geq 0\big), & \text{if } 0 \leq n \leq m, \\ \sigma\big(A_k^{(n)},\ B_k^{(n)},\ k \geq 0\big), & \text{if } n > m, \end{cases}$$

$$\mathcal{G}_n = \sigma\big(B_k^{(n)},\ k \geq 0\big).$$

More precisely, each $\mathcal{F}_n$ (or $\mathcal{G}_n$) is the completion of the σ-field defined above. Let $X = (X_n)_{n\geq 0}$ and $Y = (Y_n)_{n\geq 0}$ be the martingales defined by

$$X_n = E[\,f\,|\,\mathcal{F}_n] \quad \text{and} \quad Y_n = E[\,g\,|\,\mathcal{G}_n\,].$$

Note that $Y_n \to g$ a.s. and $X_n \to f$ a.s. For each $n \leq m$, we have

$$\begin{aligned} X_n &= \sum_{k=0}^{\infty} \frac{1_{A_k^{(n)} \cup B_k^{(n)}}}{P\Big(A_k^{(n)} \cup B_k^{(n)}\Big)} \int_{A_k^{(n)} \cup B_k^{(n)}} f\,dP \\ &\geq \sum_{k=0}^{\infty} \frac{1_{B_k^{(n)}}}{2\,P\Big(B_k^{(n)}\Big)} \int_{A_k^{(n)}} f\,dP = \sum_{k=0}^{\infty} \frac{1_{B_k^{(n)}}}{2\,P\Big(B_k^{(n)}\Big)} \int_{B_k^{(n)}} g\,dP = \frac{1}{2}\,Y_n, \end{aligned}$$

where we have used the convention that $0/0 = 0$. The above inequalities show that $Y_m^{\circledast} \leq X_\infty^{\circledast} \in B$. From (3.1) it follows that $Y_m^{\circledast} \in B$ and

$$\left\|Y_m^{\circledast}\right\|_B \leq C_B \left\|f\right\|_B .$$

From the Fatou property, we see that $Y_\infty^{\circledast} \in B$ and hence $g \in B$. Thus B is renormed so as to be r.i.

Now we must prove that $\overline{\alpha}_B = \overline{\alpha}_{\hat{B}} < 1$, where $\hat{B}$ is a r.i. space over I satisfying (1.10). In [1], Antipa proved that the Doob type inequality holds in a r.i. space $\mathfrak{X}$ if and only if $\overline{\alpha}_{\mathfrak{X}} < 1$. We give a new proof of this fact, which is valid for general probability space Ω.

In view of Theorem 2.6, it suffices to show that $\mathcal{P} \in \mathfrak{L}[\hat{B}]$. Let $f \in L^1(\Omega)$ be a nonnegative r.v. Since Ω is non-atomic, there exists a family of measurable sets $\{A(t)\colon t \in I\,\}$ possessing the following properties:

(a) $A(s) \subset A(t)$ if $s \leq t$;

(b) $P\big(A(t)\big) = t$ for every $t \in I$;

(c) $\displaystyle\int_{A(t)} f\,dP = \int_0^t f^*(s)\,ds$ for every $t \in I$.

For the proof, see [8, p. 46]. For each integer $n \geq 1$ and $k = 0, 1, \dots, 2^n$, let $A_{n,k} = A(1 - k2^{-n})$ and $\mathcal{F}_{n,k}$ denote the sub-σ-field of $\mathcal{F}$ generated by $\{B \setminus A_{n,k} : B \in \mathcal{F}\}$. Consider the martingale $X^{(n)} = \left(X_k^{(n)}\right)_{0 \leq k \leq 2^n}$ given by $X_k^{(n)} = E[\, f \mid \mathcal{F}_{n,k}\,]$. We have

$$X_k^{(n)} = \left\{\frac{1}{1 - k2^{-n}} \int_0^{1-k2^{-n}} f^*(s)\, ds\right\} 1_{A_{n,k}} + f\, 1_{\Omega \setminus A_{n,k}}.$$

Therefore if we set $B_{n,k} = A_{n,k} \setminus A_{n,k+1}$ and

$$\xi_n := \sum_{k=0}^{2^n-1} \left\{\frac{1}{1 - k2^{-n}} \int_0^{1-k2^{-n}} f^*(s)\, ds\right\} 1_{B_{n,k}} = \sum_{k=0}^{2^n-1} X_k^{(n)} 1_{B_{n,k}},$$

then $\xi_n \leq \left(X^{(n)}\right)_{2^n}^{\circledast} = \sup_{0 \leq k \leq 2^n} |X_k^{(n)}|$ for all $n \geq 0$.

On the other hand, we claim that $\xi_n^*(t) \to \mathcal{P} f^*(t)$ as $n \to \infty$ for every $t \in I$. To see this, let $J_{n,k}$ denote the interval $\left[1 - \frac{k+1}{2^n},\, 1 - \frac{k}{2^n}\right[$. Then clearly we have

$$\xi_n^*(t) = \sum_{k=0}^{2^n-1} \left\{\frac{1}{1 - k2^{-n}} \int_0^{1-k2^{-n}} f^*(s)\, ds\right\} 1_{J_{n,k}}(t).$$

It follows immediately that $\xi_n^*(t) \to \frac{1}{t}\int_0^t f^*(s)\, ds = \mathcal{P} f^*(t)$. Thus the claim is proved. By (1.7), (1.10) and (3.1), we have

$$\begin{aligned} \|\mathcal{P} f^*\|_{\hat{B}} &\leq \varliminf_{n\to\infty} \|\xi_n^*\|_{\hat{B}} = \varliminf_{n\to\infty} \|\xi_n\|_B \\ &\leq \varliminf_{n\to\infty} \left\| \left(X^{(n)}\right)_{2^n}^{\circledast} \right\|_B \leq C_B \|f\|_B = C_B \|f^*\|_{\hat{B}}\,. \end{aligned}$$

Now let $\varphi \in L^1(I)$ be an arbitrary nonnegative function. Then there is a r.v. f such that $f^* = \varphi^*$. Since $\mathcal{P}\varphi \leq \mathcal{P}\varphi^*$, the above inequalities show that $\mathcal{P} \in \mathfrak{L}[\hat{B}]$ as desired.

(ii) $\Rightarrow$ (i). Let $X = (X_n)$ be an arbitrary uniformly integrable martingale. Recall Doob's week type inequality

$$\lambda P\{X_\infty^{\circledast} \geq \lambda\} \leq \int_{\{X_\infty^{\circledast} \geq \lambda\}} |X_\infty|\, dP.$$

Substituting $\lambda = \left(X_\infty^{\circledast}\right)^*(t)$ in this inequality, we have

$$\left(X_\infty^{\circledast}\right)^*(t) \leq \frac{1}{P\left\{X_\infty^{\circledast} \geq \left(X_\infty^{\circledast}\right)^*(t)\right\}} \int_0^{P\{X_\infty^{\circledast} \geq (X_\infty^{\circledast})^*(t)\}} X_\infty^*(s)\, ds \leq \mathcal{P} X_\infty^*(t),$$

where we have used Hardy's inequality (1.3) and the fact that $\mathcal{P}X^*_\infty(t)$ is decreasing in t. Since $\overline{\alpha}_B < 1$, Theorem 2.6 implies that

$$\|X^{\circledast}_\infty\|_B = \|(X^{\circledast}_\infty)^*\|_{\hat{B}} \le \|\mathcal{P}X^*_\infty\|_{\hat{B}} \le \|\mathcal{P}\|_{\mathcal{L}[\hat{B}]} \|X^*_\infty\|_{\hat{B}} \le \|\mathcal{P}\|_{\mathcal{L}[\hat{B}]} \|X_\infty\|_B .$$

Thus we obtain (3.1) and the theorem is proved completely. □

§ 2. Condition A_p and filtrations

In this short section, we shall consider weighted L^p-spaces. For the sake of convenience, we assume that $\mathcal{F}_0$ is (the completion of) the trivial σ-field $\{\varnothing, \Omega\}$. Let $\hat{P}$ be a probability measure which is equivalent to P and W_∞ be the Radon–Nikodym derivative $d\hat{P}/dP$. The martingale $W = (W_n)_{n\ge0}$ given by $W_n = E[W_\infty \,|\, \mathcal{F}_n]$, $n \ge 0$, is called the *weight martingale.*

Let $1 < p < \infty$. We say that W_∞ or $W = (W_n)$ *satisfies the condition* (A_p) *with respect to* P *and* $(\mathcal{F}_n)$ and write $W \in A_p(K, (\mathcal{F}_n))$ if

$$\sup_{n\ge0} \Big\| E\Big[(W_n/W_\infty)^{\frac{1}{p-1}} \,\Big|\, \mathcal{F}_n \Big] \Big\|_\infty =: K < \infty. \tag{3.2}$$

When $p = 1$, we say that W_∞ or $W = (W_n)$ *satisfies the condition* A_1 and write $W \in A_1(K, (\mathcal{F}_n))$ if

$$\sup_{n\ge0} \| W_n/W_\infty \|_\infty =: K < \infty. \tag{3.3}$$

Suppose that $W = (W_n) \in A_p(K, (\mathcal{F}_n))$; then

$$E\Big[W_\infty^{-\frac{1}{p-1}} \Big] < \infty \quad \text{or} \quad W_\infty^{-1} \in L^\infty(P) \tag{3.4}$$

according as $1 < p < \infty$ or $p = 1$, since $\mathcal{F}_0$ is the trivial σ-field. Using this fact and Hölder's inequality, we have

$$L^\infty(P) \hookrightarrow L^p(\hat{P}) \hookrightarrow L^1(P).$$

Clearly $L^p(\hat{P})$ satisfies (1.5) and (1.6). Therefore $L^p(\hat{P})$ is a Banach function space if $W \in A_p(K, (\mathcal{F}_n))$.

Condition (A_p) is introduced by Izumisawa and Kazamaki [24]. For details on the weight theory of martingales, see Kazamaki [26] or Long [40]. It was proved by Tsuchikura [54] and Uchiyama [55] that $W \in A_p(K, (\mathcal{F}_t))$ for some $K > 0$ if and only if

$$\lambda^p \hat{P}\{X_\infty^\circledast \geq \lambda\} \leq C_p \int_{\{X_\infty^\circledast \geq \lambda\}} |X_\infty|^p \, d\hat{P}, \qquad \lambda > 0,$$

holds for every uniformly integrable martingale $X = (X_n)$.

A very important result on the A_p conditions is the following.

Theorem 3.4 (Izumisawa and Kazamaki [24]). *Let $W = (W_n)$ be the weight martingale with $W_\infty = d\hat{P}/dP$. If $W = (W_n) \in A_{p_0}(K, (\mathcal{F}_t))$ for some $1 < p_0 < \infty$, then*

$$\|X_\infty^\circledast\|_{L^p(\hat{P})} \leq C_{p,K} \, \|X_\infty\|_{L^p(\hat{P})} \tag{3.5}$$

holds for every $p > p_0$ and every uniformly integrable martingale $X = (X_n)_{n \geq 0}$ with respect to P and $(\mathcal{F}_n)$.

Combining this result and Theorem 3.1, we can prove the following theorem.

Theorem 3.5. *Let $W = (W_n)$ be as in Theorem 3.4. The following statements are equivalent:*

(i) *$W = (W_n) \in A_{p_0}(K, (\mathcal{F}_n))$ for any filtration $(\mathcal{F}_n)$ and* some *fixed $1 < p_0 < \infty$.*

(ii) *$W = (W_n) \in A_p(K, (\mathcal{F}_n))$ for any filtration $(\mathcal{F}_n)$ and* any *$1 < p < \infty$.*

(iii) *There exist constants α and β such that $0 < \alpha \leq W_\infty \leq \beta$ a.s.*

(iv) *There exists a norm $|||\cdot|||_{L^p(\hat{P})}$ which is equivalent to the original norm of $L^p(\hat{P})$ such that $\left(L^p(\hat{P}), |||\cdot|||_{L^p(\hat{P})}\right)$ is rearrangement invariant with respect to P, for* any *$1 \leq p < \infty$.*

(v) *There exists a norm $|||\cdot|||_{L^p(\hat{P})}$ which is equivalent to the original norm of $L^p(\hat{P})$ such that $\left(L^p(\hat{P}), |||\cdot|||_{L^p(\hat{P})}\right)$ is rearrangement invariant with respect to P, for* some *$1 \leq p < \infty$.*

Proof. It is clear that (iii) implies (ii), (ii) implies (i) and that (iv) implies (v). Thus it suffices to show that, (iii) implies (iv), (i) implies (v) and that (v) implies (iii).

(iii) $\Rightarrow$ (iv). From (iii) we see that, if $f \in L^p(\hat{P})$ and f, g are identically distributed, then $g \in L^p(\hat{P})$. Hence (iv) follows from Lemma 3.2.

(i) $\Rightarrow$ (v). According to Theorem 3.4, (3.1) holds for every $p > p_0$ and every uniformly integrable martingale with respect to P and $(\mathcal{F}_n)$, by hypothesis. Then (v) follows from Theorem 3.1.

(v) $\Rightarrow$ (iii). Suppose that A, $\tilde{A} \in \mathcal{F}$ and $P(A) = P(\tilde{A})$. Since $\|\cdot\|_{L^p(\hat{P})}$ and $|||\cdot|||_{L^p(\hat{P})}$ are equivalent, we get

$$\begin{aligned}\alpha \int_A W_\infty \, dP &= \alpha \, \|1_A\|_{L^p(\hat{P})} \\ &\leq |||1_A|||_{L^p(\hat{P})} = |||1_{\tilde{A}}|||_{L^p(\hat{P})} \\ &\leq \beta \, \|1_{\tilde{A}}\|_{L^p(\hat{P})} = \beta \int_{\tilde{A}} W_\infty \, dP,\end{aligned}$$

with some constants $\alpha > 0$ and $\beta > 0$ independent of A and $\tilde{A}$. As Ω is non-atomic, we have for each $t \in I$,

$$\begin{aligned}\alpha \, t \, W_\infty^*(t) &\leq \alpha \int_0^t W_\infty^*(s) \, ds = \alpha \sup_{P(A)=t} \int_A W_\infty \, dP \\ &\leq \beta \inf_{P(\tilde{A})=t} \int_{\tilde{A}} W_\infty \, dP = \beta \int_{1-t}^1 W_\infty^*(s) \, ds \leq \beta \, t \, W_\infty^*(1-t),\end{aligned}$$

where we have used elementary inequalities for integrals of decreasing rearrangements; see [13, p. 36] (cf. (1.4)). Thus $\alpha \, W_\infty^*(t) \leq \beta \, W_\infty^*(1-t)$ for every $t \in I$. Letting $t \downarrow 0$, we obtain

$$\alpha \operatorname*{ess\,sup}_{\omega \in \Omega} W_\infty(\omega) \leq \beta \operatorname*{ess\,inf}_{\omega \in \Omega} W_\infty(\omega),$$

which implies (iii). The theorem is proved. $\square$

4. Convergence of martingales

In this chapter we consider the convergence of martingales relative to the norm topology of Banach function spaces. It is well-known that if $X_\infty \in L^p$ for some $1 \le p < \infty$, then the uniformly integrable martingale $X = (X_n)_{n\ge 0}$ converges with respect to the norm topology of L^p. However, it seems not familiar that if $X_\infty \in L\log L$, then $X = (X_n)$ converges with respect to the norm of $L\log L$. We shall investigate such a problem in general Banach function spaces.

In this chapter we shall consider martingales with respect to a fixed filtration $(\mathcal{F}_n)_{n\ge 0}$ with discrete parameter, unless otherwise stated. Throughout the chapter, we assume that $\mathcal{F} = \bigvee_{n\ge 0} \mathcal{F}_n$ and $\mathcal{F}_n \subsetneqq \mathcal{F}$ for every $n \ge 0$. It then follows that if $X = (X_n)_{n\ge 0}$ is a martingale such that $X_n = E[\, f \mid \mathcal{F}_n\,]$ for all $n \ge 0$, then $X_n \to f$ a.s. and in L^1.

§ 1. Boundedness of conditional expectation operator

Note that every conditional expectation operator lies in the unit sphere of $\mathfrak{L}[L^p]$, but in general Banach function space B, the analogue is not true. For instance, let Ω be the probability space $I \times I (= [0,1] \times [0,1])$ with Lebesgue measure m, $\mathcal{G}$ the σ-field consisting of all sets of the form $F \times I$, where F is Lebesgue measurable, and X a Banach function space with the norm given by

$$\|f\|_X = \int_{I\times[0,\,a]} |f|\, dm + \left\{ \int_{I\times]a,\,1]} |f|^2\, dm \right\}^{1/2},$$

where $0 < a < 1$ is a constant. Choose $\varphi \in L^1(I) \setminus L^2(I)$ and set $f(s,t) = \varphi(s) 1_{I\times[0,\,a]}(s,t)$. Then we have $f \in X$, while $E[\, f \mid \mathcal{G}\,](s,t) = a\,\varphi(s) \notin X$.

Let now $\mathcal{G}$ be an arbitrary sub-σ-field of $\mathcal{F}$ and B an arbitrary Banach function space. Note that

$$E[\,\cdot \mid \mathcal{G}\,] \in \mathfrak{L}[B] \iff E[\, f \mid \mathcal{G}\,] \in B \ \text{ for every } f \in B.$$

To see this, in view of the closed graph theorem, it suffices to show that $\Gamma = \{(f,\, E[f \mid \mathcal{G}]) \in B \times B : f \in B\}$ is closed in $B \times B$. Suppose that $\Gamma \ni (f_n,\, E[\, f_n \mid \mathcal{G}\,]) \to (f,\, g)$ in $B \times B$. From (1.5), we see that $f_n \to f$ in L^1. Hence we get $E[\, f_n \mid \mathcal{G}\,] \to E[\, f \mid \mathcal{G}\,]$ in L^1, which implies that $g = E[\, f \mid \mathcal{G}\,]$. Thus Γ is closed in $B \times B$ as desired.

We denote by E_n the conditional expectation operator $E[\,\cdot\,|\,\mathcal{F}_n\,]$ for each $n \geq 0$. Our main result in this section is the following:

Theorem 4.1. *Suppose that a Banach function space B has absolutely continuous norm. Then the following statements are equivalent:*

(i) *If $X = (X_n)_{n\geq 0}$ is a uniformly integrable martingale and $X_\infty \in B$, then $X_n \in B$ for every $n \geq 0$ and $\lim_{n\to\infty} \|X_n - X_\infty\|_B = 0$.*

(ii) *For every $n \geq 0$, $E_n \in \mathfrak{L}[B]$ and*

$$\sup_{n\geq 0} \|E_n\|_{\mathfrak{L}[B]} < \infty. \tag{4.1}$$

Remark. As the following example shows, there exists a Banach function space B and a filtration $(\mathcal{F}_n)_{n\geq 0}$ such that $E_n \in \mathfrak{L}[B]$ for each $n \geq 0$ but $\sup_{n\geq 0} \|E_n\|_{\mathfrak{L}[B]} = +\infty$. The example shows that we cannot remove hypothesis (4.1) in Theorem 4.1.

Example 4.2. Let Ω denote the interval I, $\mathcal{F}$ the σ-field of the Lebesgue measurable subsets of I, and P the Lebesgue measure on I. For each integer $n \geq 1$, put $A_n = [0, 1/n]$, $B_n = [1 - 1/n, 1]$ and $C_n = A_n \cup B_n$, and let $\mathcal{F}_n$ denote the σ-field generated by C_n and the measurable subsets of $\Omega \setminus C_n$. Let $1 \leq p < \infty$ and B denote the Banach space of measurable functions f with norm

$$\|f\|_B = \Big\{ \int_0^1 |f(s)|^p \, d\sqrt{s} \,\Big\}^{1/p}.$$

By Hölder's inequality, we see that $L^\infty(P) \hookrightarrow B \hookrightarrow L^1(P)$ and B is a Banach function space.

Let $X = (X_n)_{n\geq 1}$ be a uniformly integrable martingale given by $X_n = E[X_\infty\,|\,\mathcal{F}_n\,]$ with $X_\infty \in B$. We have

$$X_n = \frac{n}{2}\Big(\int_{C_n} X_\infty \, dP \Big) 1_{C_n} + X_\infty \, 1_{\Omega\setminus C_n}$$

and hence

$$\|X_n\|_B \leq \|X_\infty\|_B + \frac{n}{2} \int_{C_n} |X_\infty| \, dP \leq c_n \, \|X_\infty\|_B$$

for some constant $c_n > 0$ depending only on n. Thus $E_n \in \mathcal{L}[B]$ for every $n \geq 1$. However (i) of Theorem4.1 is false in this case. To see this, let

$$Y_\infty(t) = \begin{cases} (1-t)^{-\frac{1}{2p}}, & \text{if } \frac{1}{2} \leq t \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $Y = (Y_n)_{n\geq 1}$ be the martingale given by $Y_n = E[\,Y_\infty \,|\, \mathcal{F}_n\,]$. Then we have $Y_\infty \in B$ and $\|Y_\infty\|_B = (\pi/4)^{1/p}$. If $n \geq 2$, then

$$Y_n = \frac{p\, n^{\frac{1}{2p}}}{2p-1}\, 1_{C_n} + Y_\infty\, 1_{\Omega \setminus C_n},$$

and therefore it follows that for each $n \geq 2$,

$$\|Y_\infty - Y_n\|_B = \|\,(Y_\infty - Y_n)\, 1_{C_n}\|_B \geq \frac{p\, n^{\frac{1}{2p}}}{2p-1}\, \|1_{A_n}\|_B = \frac{p}{2p-1}.$$

Thus (i) of Theorem 4.1 (and consequently (ii)) is false. In fact, it is easy to see directly that (4.1) is false: we have

$$\frac{\|E[\,1_{B_n} \,|\, \mathcal{F}_n\,]\|_B}{\|1_{B_n}\|_B} = \frac{1}{2}\big(1 + \sqrt{n} + \sqrt{n-1}\,\big)^{1/p} \to \infty \quad \text{as} \quad n \to \infty.$$

Proof of Theorem 4.1. (i) $\Rightarrow$ (ii). Suppose that (i) is true. Then $E[\,f \,|\, \mathcal{F}_n\,] \in B$ for all $f \in B$, and hence $E_n \in \mathcal{L}[B]$ as mentioned before the statement of Theorem 4.1. Now, (4.1) follows immediately from the Banach–Steinhaus theorem.

(ii) $\Rightarrow$ (i). Since B has absolutely continuous norm: then L^∞ is dense in B by Corollary 1.8. Given an arbitrary $\varepsilon > 0$, choose $Y_\infty \in L^\infty$ such that $\|X_\infty - Y_\infty\|_B < \varepsilon$, and define a martingale $Y = (Y_n)$ by $Y_n = E[\,Y_\infty \,|\, \mathcal{F}_n\,]$. Then, by (4.1), we have

$$\begin{aligned}
&\|X_n - X_\infty\|_B \\
&\quad \leq \|E[\,X_\infty - Y_\infty \,|\, \mathcal{F}_n\,]\|_B + \|Y_n - Y_\infty\|_B + \|Y_\infty - X_\infty\|_B \\
&\quad \leq (C+1)\,\varepsilon + \big\|(Y_n - Y_\infty) 1_{\{|Y_n - Y_\infty| \leq \varepsilon\}}\big\|_B + \big\|(Y_n - Y_\infty) 1_{\{|Y_n - Y_\infty| > \varepsilon\}}\big\|_B \\
&\quad \leq (C + 1 + \|1\|_B)\,\varepsilon + 2\,\|Y_\infty\|_\infty\, \big\|1_{\{|Y_n - Y_\infty| > \varepsilon\}}\big\|_B\,,
\end{aligned}$$

where $C = \sup_n \|E_n\|_{\mathcal{L}[B]}$ and $\|1\|_B$ stands for the norm of the constant function 1. Since $Y_n \to Y_\infty$ a.s. and B has absolutely continuous norm, the last term on the right-hand side tends to zero as $n \to \infty$. This shows that $X = (X_n)$ is convergent in B as required. □

An alternative proof of Theorem 4.1 was given in [35].

Now suppose that B is u.r.i. (or that B is r.i. and Ω is non-atomic). Then every conditional expectation operator lies in the unit sphere of $\mathfrak{L}[B]$. In fact, from the equality

$$\int_0^t f^*(s)\,ds = \inf\{\|f_1\|_1 + t\,\|f_2\|_\infty : f = f_1 + f_2,\ f_1 \in L^1,\ f_2 \in L^\infty\},$$

we see that

$$\int_0^t E[\,f\,|\,\mathcal{G}\,]^*(s)\,ds \le \|E[\,f_1\,|\,\mathcal{G}\,]\|_1 + t\,\|E[\,f_2\,|\,\mathcal{G}\,]\|_\infty \le \|f_1\|_1 + t\,\|f_2\|_\infty\,,$$

where $f = f_1 + f_2$, $f_1 \in L^1$ and $f_2 \in L^2$. Taking the infimum of the right-hand side, we have $E[\,f\,|\,\mathcal{G}\,] \prec f$, and hence $\|E[f\,|\,\mathcal{G}\,]\|_B \le \|f\|_B$. For the proof of the above equality, see [8, p. 74]. Thus we have:

Corollary 4.3. *Suppose that B is universally rearrangement invariant and has absolutely continuous norm. If $X = (X_n)_{n\ge 0}$ is a uniformly integrable martingale and $X_\infty \in B$, then $X_n \in B$ for every $n \ge 0$ and*

$$\lim_{n\to\infty} \|X_n - X_\infty\|_B = 0.$$

Note that in Theorem 4.1 and Corollary 4.3, we cannot remove the hypothesis of the absolute continuity of the norm of B. Furthermore, we have:

Theorem 4.4. *Suppose that Ω is non-atomic and B is rearrangement invariant. Then the following statements are equivalent:*

(i) *If $X = (X_n)_{n\ge 0}$ is a uniformly integrable martingale such that $X_\infty \in B$, then $\lim_{n\to\infty} \|X_n - X_\infty\|_B = 0$.*

(ii) *B has absolutely continuous norm.*

To prove Theorem 4.4, we need the following lemma.

Lemma 4.5. *If A_1, $A_2 \in \mathcal{F}$ and $P(A_1) = P(A_2) > 0$, then for every random variable f, there is a random variable g such that*

$$P\{\omega \in A_1 : |f(\omega)| > \lambda\} = P\{\omega \in A_2 : |g(\omega)| > \lambda\}, \quad \lambda > 0.$$

Proof. Let f_1^* denote the decreasing rearrangement of f with respect to the conditional probability measure $P(\cdot\,|A_1)$. Since there is no $P(\cdot\,|A_2)$-atom in Ω, there is a random variable g such that $f_1^* = g_2^*$, where g_2^* denotes the decreasing rearrangement of g with respect to $P(\cdot\,|A_2)$ (cf. [13, p. 44]). From the choice of g, we see that $|f|$ and $|g|$ have the same distribution with respect to $P(\cdot\,|A_1)$ and $P(\cdot\,|A_2)$, respectively. Since $P(A_1) = P(A_2)$, we obtain the lemma. $\square$

Proof of Theorem 4.4. Corollary 4.3 asserts that (ii) implies (i).

We prove that, if (ii) is false, then (i) is false. Suppose that B dose not have absolutely continuous norm. Then there is an $f \in B$ and a sequence $\{A_n\}_{n\geq 0}$ of sets in $\mathcal{F}$ such that $A_n \downarrow \varnothing$ a.s. and $\|f1_{A_n}\|_B \geq \delta > 0$ for all $n \geq 0$. We may assume that $f \geq 0$, $\{f > 0\} \subset A_0$ and $P(A_0) \leq 1/2$. As Ω is non-atomic, there is a sequence $(\tilde{A}_n)_{n\geq 0}$ such that $\tilde{A}_n \downarrow \varnothing$ a.s., $A_0 \cap \tilde{A}_0 = \varnothing$ and $P(\tilde{A}_n) = P(A_n)$ for each $n \geq 0$. According to Lemma 4.5, for each $n \geq 0$, there is a r.v. g_n such that

$$P\{\omega \in A_n \setminus A_{n+1} \colon f(\omega) > \lambda\} = P\{\omega \in \tilde{A}_n \setminus \tilde{A}_{n+1} \colon g_n(\omega) > \lambda\}, \quad \lambda > 0.$$

Let g be the random variable defined by $g = \sum_{n=0}^{\infty} g_n 1_{\tilde{A}_n \setminus \tilde{A}_{n+1}}$. Then $f1_{A_n}$ and $g1_{\tilde{A}_n}$ have the same distribution. In fact, since $\{f > 0\} \cap \{g > 0\} = \varnothing$, we have

$$\begin{aligned}
(4.2) \qquad P\{\omega \in A_n \colon f(\omega) > \lambda\} &= \sum_{k=n}^{\infty} P\{\omega \in A_k \setminus A_{k+1} \colon f(\omega) > \lambda\} \\
&= \sum_{k=n}^{\infty} P\{\omega \in \tilde{A}_k \setminus \tilde{A}_{k+1} \colon g_k(\omega) > \lambda\} \\
&= P\{\omega \in \tilde{A}_n \colon g(\omega) > \lambda\}, \qquad \lambda > 0.
\end{aligned}$$

Let $\mathcal{F}_n$ be the σ-field generated by $A_n \cup \tilde{A}_n$ and the subsets of $\Omega \setminus (A_n \cup \tilde{A}_n)$ in $\mathcal{F}$: then $A_n \cup \tilde{A}_n$ is a single atom with respect to $\mathcal{F}_n$ and P. Let $X_\infty = f - g$ and $X = (X_n)_{n\geq 0}$ be the martingale defined by $X_n = E[\,X_\infty \,|\, \mathcal{F}_n\,]$ for each $n \geq 0$. Since $X_\infty \in B$ and B is r.i., we have $\sup_n \|X_n\|_B \leq \|X_\infty\|_B < \infty$. We claim that $X = (X_n)$ is not convergent in B. To see this, observe that $E[\,f1_{A_n}\,] = E[\,g1_{\tilde{A}_n}\,]$ by (4.2), and that for every $n \geq 0$,

$$X_n = \frac{1_{A_n \cup \tilde{A}_n}}{P(A_n \cup \tilde{A}_n)} E\big[\,(f-g)1_{A_n \cup \tilde{A}_n}\,\big] + X_\infty 1_{\Omega \setminus (A_n \cup \tilde{A}_n)} = X_\infty 1_{\Omega \setminus (A_n \cup \tilde{A}_n)}.$$

Hence we have

$$|X_\infty - X_n| = |X_\infty|\, 1_{A_n \cup \tilde{A}_n} = f 1_{A_n} + g 1_{\tilde{A}_n} \geq f 1_{A_n},$$

which implies that $\|X_\infty - f_n\|_X \geq \|f 1_{A_n}\|_X \geq \delta > 0$ for every $n \geq 0$. Thus the claim is verified. □

§ 2. Condition A_p and convergence of martingales

In this section we consider the convergence of martingales in a weighted L^p-space. Let $\hat{P}$ be a probability measure equivalent to P, and $W = (W_n)_{n \geq 0}$ denote the weight martingale with $W_\infty = d\hat{P}/dP$. As in Section 2 of Chapter 3, we assume that $\mathcal{F}_0$ is the completion of trivial σ-field $\{\Omega, \varnothing\}$ in this section.

Condition A_p, which was introduced by Izumisawa and Kazamaki, is the probabilistic analogue of the Muckenhoupt A_p-condition in classical analysis. In the martingale theory, condition A_p has been considered only for weighted norm inequalities, while Muckenhoupt proved in [43] that a weight function U satisfies the classical (A_p) if and only if

$$\lim_{r \uparrow 1} \int_0^{2\pi} |f(r, \theta) - f(\theta)|^p \, U(\theta)\, d\theta = 0$$

for every function f of period 2π such that $\int_0^{2\pi} |f(\theta)|^p \, U(\theta)\, d\theta < \infty$, where $f(r, \theta)$ denotes the Poisson integral of f. The following theorem is a probabilistic analogue of his result.

Theorem 4.6. *The following statements are equivalent:*

(i) *If $X = (X_n)_{n \geq 0}$ is a uniformly integrable martingale with respect to P and $X_\infty \in L^p(\hat{P})$, then $X_n \in L^p(\hat{P})$ for every $n \geq 0$ and*

$$\lim_{n \to \infty} \|X_n - X_\infty\|_{L^p(\hat{P})} = 0.$$

(ii) *$W = (W_n)_{n \geq 0} \in A_p\big(K, (\mathcal{F}_n)\big)$ for some $K > 0$.*

In view of Theorem 4.1, Theorem 4.6 follows from the following lemma due to Doléans-Dade and Meyer (Proposition 1' of [19]; see also Proposition 6.2.2 of [40]).

Lemma 4.7 (Doléans-Dade and Meyer). *Let $1 \le p < \infty$. The following statements are equivalent:*

(i) $E_n \in \mathcal{L}[L^p(\hat{P})]$ *for all* $n \ge 0$ *and* $\sup_{n\ge 0} \|E_n\|_{\mathcal{L}[L^p(\hat{P})]} < \infty$.

(ii) $W = (W_n) \in A_p\big(K, (\mathcal{F}_n)\big)$ *for some* $K > 0$.

Proof. (ii) $\Rightarrow$ (i). We assume first that $1 < p < \infty$ and $W = (W_n) \in A_p\big(K, (\mathcal{F}_n)\big)$. Then we have for every $f \in L^p(\hat{P})$ and every $n \ge 0$,

$$\begin{aligned} |\, E[\, f \,|\, \mathcal{F}_n \,]\,|^p &\le E\left[\left(\frac{W_n}{W_\infty}\right)^{1/p} \left(\frac{W_\infty}{W_n}\right)^{1/p} |f| \,\middle|\, \mathcal{F}_n \right]^p \\ &\le E\left[(W_n/W_\infty)^{\frac{1}{p-1}} \,\middle|\, \mathcal{F}_n \right]^{p-1} E\left[\, |f|^p\, W_\infty \,|\, \mathcal{F}_n \,\right] W_n^{-1} \\ &\le K^{p-1} E\left[\, |f|^p\, W_\infty \,|\, \mathcal{F}_n \,\right] W_n^{-1}, \end{aligned}$$

where we have used Hölder's inequality. Hence it follows that

$$\|E[\, f \,|\, \mathcal{F}_n \,]\|^p_{L^p(\hat{P})} \le K^{p-1} \hat{E}\Big[E\left[\, |f|^p\, W_\infty \,|\, \mathcal{F}_n \,\right] W_n^{-1} \Big] = K^{p-1}\, \|f\|^p_{L^p(\hat{P})},$$

where $\hat{E}$ stands for the expectation with respect to $\hat{P}$. This implies that $E_n \in \mathcal{L}[L^p(\hat{P})]$ and $\|E_n\|_{\mathcal{L}[L^p(\hat{P})]} \le K^{1/p'}$ for every $n \ge 0$, with $p' = p/(p-1)$. Now assume that $p = 1$ and $W = (W_n) \in A_1\big(K, (\mathcal{F}_n)\big)$. Then we have

$$\big|E[\, f \,|\, \mathcal{F}_n \,]\big| \le K E\big[\, |f|\, W_\infty \,|\, \mathcal{F}_n \,\big]\, W_n^{-1}.$$

It follows that $\|E[\, f \,|\, \mathcal{F}_n \,]\|_{L^1(\hat{P})} \le K\, \|f\|_{L^1(\hat{P})}$. Thus (ii) implies (i).

Suppose now conversely that (i) is true: then for every $n \ge 0$ and every random variable $f \ge 0$, we have

$$E\big[E[\, f \,|\, \mathcal{F}_n \,]^p\, W_n \,\big] \le K^p E[\, f^p\, W_\infty \,], \tag{4.3}$$

where K is the supremum appearing in (i). Assume that $1 < p < \infty$. Taking the random variable $W_\infty^{-\frac{1}{p-1}}\, 1_{\{W_\infty > a\} \cap A}$ for f, where $A \in \mathcal{F}_n$ and $a > 0$ is a constant, we obtain

$$E\left[E\Big[W_\infty^{-\frac{1}{p-1}}\, 1_{\{W_\infty > a\}} \,\Big|\, \mathcal{F}_n \Big]^p W_n\, 1_A \right] \le K^p E\Big[W_\infty^{-\frac{1}{p-1}}\, 1_{\{W_\infty > a\} \cap A} \Big].$$

Since $A \in \mathcal{F}_n$ is arbitrary, it follows that

$$E\Big[W_\infty^{-\frac{1}{p-1}}\, 1_{\{W_\infty > a\}} \,\Big|\, \mathcal{F}_n \Big]^p W_n \le K^p E\Big[W_\infty^{-\frac{1}{p-1}}\, 1_{\{W_\infty > a\}} \,\Big|\, \mathcal{F}_n \Big] \quad \text{a.s.}$$

Thus we obtain

$$E\Big[(W_n/W_\infty)^{\frac{1}{p-1}}1_{\{W_\infty>a\}}\,\Big|\,\mathcal{F}_n\Big]^{p-1} \le K^p \quad \text{a.s.},$$

for every $n \ge 0$. Letting $a \downarrow 0$ we see that $W \in A_p\big(K^{p'}, (\mathcal{F}_n)\big)$. If $p = 1$, then (4.3) is rewritten as

$$E[\,fW_n\,] \le KE[\,fW_\infty\,].$$

Taking fW_∞^{-1} for f, we have $E\big[(W_n/W_\infty)f\,\big] \le KE[\,f\,]$ for all $n \ge 0$ and all nonnegative f. It then follows that $W \in A_1\big(K, (\mathcal{F}_n)\big)$. The lemma is established. □

Corollary 4.8. *Suppose that* $1 < p < \infty$ *and* $W_{n-1} \le c\,W_n$ *for every* n *with some positive constant* c*. Then the following statements are equivalent:*

(i) *If* $X = (X_n)_{n\ge 0}$ *be a uniformly integrable martingale with respect to* P *such that* $X_\infty \in L^p(\hat{P})$*, then* $X_n \in L^p(\hat{P})$ *for all* $n \ge 0$ *and*

$$\lim_{n\to\infty} \|X_n - X_\infty\|_{L^p(\hat{P})} = 0.$$

(ii) *If* $X = (X_n)_{n\ge 0}$ *be a uniformly integrable martingale with respect to* P *such that* $X_\infty \in L^p(\hat{P})$*, then* $X_\infty^{\circledast} \in L^p(\hat{P})$*.*

Proof. Assume that (i) is true. Then, according to Theorem 4.6, $W = (W_n) \in A_p\big(K, (\mathcal{F}_n)\big)$. By the result of Doléans-Dade and Meyer, $W \in A_q\big(K', (\mathcal{F}_n)\big)$ for some $1 < q < p$ and $K' > 0$, since $W_{n-1} \le c\,W_n$ for all n (cf. [19, p. 323] and [40, p. 238]). By Theorem 2 of Izumisawa and Kazamaki [24], we have

$$\hat{E}\left[X_\infty^{\circledast p}\right] \le C\hat{E}\big[|X_\infty|^p\big] < \infty$$

with some constant $C > 0$ independent of X.

On the other hand, (i) immediately follows from (ii) and the dominated convergence theorem. □

Corollary 4.9. *In the statement of Corollary 4.8, we may replace the hypothesis "*$W_{n-1} \le c\,W_n$*" by the following condition:*

$$E[\,f\,|\,\mathcal{F}_n\,] \le c\,E[\,f\,|\,\mathcal{F}_{n-1}\,], \quad n \ge 1, \tag{4.4}$$

for every nonnegative random variable f.

Proof. To prove the corollary, it is sufficient to show that if (i) of Corollary 4.8 is true, then (4.4) implies $W_{n-1} \le C\,W_n$ for every n.

Let $Z_\infty = W_\infty^{-\frac{1}{p-1}}$ and $Z_n = E[Z_\infty \,|\, \mathcal{F}_n]$. Then $Z_n \le c\,Z_{n-1}$ for all $n \ge 1$ by (4.4). Since $W = (W_n) \in A_p\big(K, (\mathcal{F}_n)\big)$ for some $K > 0$, Hölder's inequality gives that

$$1 = E\big[\,W_\infty^{1/p} W_\infty^{-1/p} \,|\, \mathcal{F}_n\,\big]^p \le W_n\, E\Big[\,W_\infty^{-\frac{1}{p-1}} \,\Big|\, \mathcal{F}_n\,\Big]^{p-1} \le W_n Z_n^{p-1} \le K^{p-1}$$

for every $n \ge 0$. Hence we have

$$W_{n-1} \le K^{p-1} Z_{n-1}^{1-p} \le K^{p-1} c^{p-1} Z_n^{1-p} \le K^{p-1} c^{p-1} W_n$$

as desired. □

§ 3. Convergence of conditional expectations

Let $\{\mathcal{G}_n\}_{n\ge0}$ be a sequence of sub-σ-fields of $\mathcal{F}$ which is not necessarily increasing. In this section we consider the convergence of a sequence $\{E[\,f \,|\, \mathcal{G}_n\,]\}_{n\ge0}$ in a Banach function space B.

In [2], Alonso and Brambila-Paz gave recently a necessary and sufficient condition on $\{\mathcal{G}_n\}$ in order that the sequence $\{E[\,f \,|\, \mathcal{G}_n\,]\}_{n\ge0}$ converges in L^p for every $f \in L^p$. We use the notation in [2].

Definition 4.10. Let $\{\mathcal{G}_n\}_{n\ge0}$ be a sequence of sub-σ-fields of $\mathcal{F}$ and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$.

(i) We write $\mathcal{G}_n \xrightarrow{P} \mathcal{G}$ if, for every $A \in \mathcal{G}$, there exists a sequence $\{A_n\}_{n\ge0}$ of sets such that, $A_n \in \mathcal{G}_n$ for each $n \ge 0$, and $1_{A_n} \to 1_A$ in L^1.

(ii) We write $\mathcal{G}_n \xrightarrow{\perp} \mathcal{G}$ if $1_{A_n} - E[\,1_{A_n} \,|\, \mathcal{G}_n\,] \to 0$ weakly in L^2 for any sequence $\{A_n\}_{n\ge0}$ such that $A_n \in \mathcal{F}_n$ for all $n \ge 0$.

Theorem 4.11 (Alonso and Brambila-Paz [2]). *Let $\{\mathcal{G}_n\}_{n\ge0}$ and $\mathcal{G}$ be as in Definition 4.10. The following statements are equivalent:*

(i) *If $1 \le p < \infty$, then $E[\,f \,|\, \mathcal{G}_n\,] \to E[\,f \,|\, \mathcal{G}\,]$ in L^p for every $f \in L^p$.*

(ii) *$\mathcal{G}_n \xrightarrow{P} \mathcal{G}$ and $\mathcal{G}_n \xrightarrow{\perp} \mathcal{G}$.*

However, the above theorem is not completely new: an analogous results are known in Rao [47, p. 110–111]. See Math. Rev. 99d:60047. We shall give an alternative proof of the above theorem in Appendix.

Our main result in this section is the following theorem, the proof of which was given in [33] in more general setting.

Theorem 4.12. *Let B be a Banach function spaces of absolutely continuous norm, and $\{\mathcal{G}_n\}_{n\geq 0}$ be a sequence of sub-σ-fields of $\mathcal{G}$.*

(a) *If $E[\,\cdot\,|\,\mathcal{G}_n\,] \in \mathcal{L}[B]$ for all $n \geq 0$ and $\sup_n \|E[\,\cdot\,|\,\mathcal{G}_n]\|_{\mathcal{L}[B]} < \infty$, then the following statements are equivalent:*

(i) $\mathcal{G}_n \xrightarrow{P} \mathcal{G}$ *and* $\mathcal{G}_n \xrightarrow{\perp} \mathcal{G}$.

(ii) $E[\,f\,|\,\mathcal{G}_n\,] \to E[\,f\,|\,\mathcal{G}\,]$ *in probability for every* $f \in L^1$.

(iii) $E[\,\cdot\,|\,\mathcal{G}\,] \in \mathcal{L}[B]$ *and* $E[\,f\,|\,\mathcal{G}_n\,] \to E[\,f\,|\,\mathcal{G}\,]$ *in B for every $f \in B$.*

(b) *If $E[\,\cdot\,|\,\mathcal{G}_n\,] \in \mathcal{L}[B]$ for all $n \geq 0$ and $\{E[\,f\,|\,\mathcal{G}_n\,]\}$ converges in B for any $f \in B$, then $\sup_n \|E[\,\cdot\,|\,\mathcal{G}_n\,]\|_{\mathcal{L}[B]} < \infty$ and one of (and hence all of)* (i)–(iii) *is true for some $\mathcal{G}$.*

Proof. (a). Note that the family $\{E[\,f\,|\,\mathcal{G}_n\,] : n \geq 0\}$ is uniformly integrable. Therefore $\{E[\,f\,|\,\mathcal{G}_n\,]\}_{n\geq 0}$ converges in probability if and only if it converges in L^1 (cf. [48, p. 116]). Thus the equivalence between (i) and (ii) follows from Theorem 4.11.

(iii) $\Rightarrow$ (ii) (or (i)). Suppose that (iii) is true. It is clear from (1.5) that $E[\,f\,|\,\mathcal{G}_n\,] \to E[\,f\,|\,\mathcal{G}\,]$ in L^1 for every $f \in L^\infty$. This remains valid for every $f \in L^1$, since L^∞ is dense in L^1 and

$$\|E[\,f\,|\,\mathcal{G}_n\,] - E[\,f\,|\,\mathcal{G}\,]\|_1 \leq 2\,\|f - g\|_1 + \|E[\,g\,|\,\mathcal{G}_n\,] - E[\,g\,|\,\mathcal{G}\,]\|_1 \,.$$

Thus we obtain that $E[\,f\,|\,\mathcal{G}_n\,] \to E[\,f\,|\,\mathcal{G}\,]$ in L^1 for every $f \in L^1$, and hence (ii) is true. We also obtain that (iii) $\Rightarrow$ (i) (cf. Theorem 4.11).

(ii) $\Rightarrow$ (iii). Assume that (ii) is true. Then, for any $f \in B$, there exists a subsequence $\{E[\,f\,|\,\mathcal{G}_{n_k}\,]\}_{k\geq 0}$ which converges to $E[\,f\,|\,\mathcal{G}\,]$ a.s. It follows from the Fatou property that

$$\|E[\,f\,|\,\mathcal{G}\,]\|_B \leq \varliminf_{k\to\infty} \|E[\,f\,|\,\mathcal{G}_{n_k}\,]\|_B \leq C\,\|f\|_B \,,$$

where $C = \sup_n \|E[\,\cdot\,|\,\mathcal{G}_n]\|_{\mathcal{L}[B]} < \infty$. This shows that $E[\,\cdot\,|\,\mathcal{G}\,] \in \mathcal{L}[B]$ and $\|E[\,\cdot\,|\,\mathcal{G}\,]\|_{\mathcal{L}[B]} \le C$. Now, for any given $\varepsilon > 0$, choose $g \in L^\infty$ such that $\|f - g\|_B < \varepsilon$. This is possible, since L^∞ is dense in B by Corollary 1.8. Then, as in the proof of Theorem 4.1, we have

$$\begin{aligned}\|E[\,f\,|\,\mathcal{G}_n\,] - E[\,f\,|\,\mathcal{G}\,]\|_B &\le 2C\varepsilon + \|g_n - g_\infty\|_B \\ &\le \big(2C + \|1\|_B\big)\,\varepsilon + 2\,\|g\|_\infty \left\|1_{\{|g_n - g_\infty| > \varepsilon\}}\right\|_B,\end{aligned}$$

where $g_n = E[\,g\,|\,\mathcal{G}_n\,]$ and $g_\infty = E[\,g\,|\,\mathcal{G}\,]$. Since $1_{\{|g_n - g_\infty| > \varepsilon\}} \to 0$ in L^1 and B has absolutely continuous norm, the last term on right-hand side tends to zero. As $\varepsilon >0$is arbitrary, we obtain (iii).

(b) Suppose that $E[\,\cdot\,|\,\mathcal{G}_n\,] \in \mathcal{L}[B]$ for all $n \ge 0$ and $\{E[\,f\,|\,\mathcal{G}_n\,]\}_{n\ge 0}$ converges in B for all $f \in B$. Then the Banach–Steinhaus theorem shows that $\sup_n \|E[\,\cdot\,|\,\mathcal{G}_n\,]\|_{\mathcal{L}[B]} < \infty$. Let T denote the operator given by

$$Tf = \lim_{n\to\infty} E[\,f\,|\,\mathcal{G}_n\,] \;\text{ in } B, \quad f \in X.$$

We must prove that T is a conditional expectation operator. To this end, it suffices to show that T extends to an idempotent linear contraction on L^1 (see [45, p. 14]). A standard argument shows that T extends to a linear contraction on L^1 and $Tf = \lim_n E[\,f\,|\,\mathcal{F}_n\,]$ in L^1, since $B \hookrightarrow L^1$ and B is dense in L^1. Furthermore we have $T(Tf) = Tf$, because

$$\begin{aligned}\|E[\,f\,|\mathcal{F}_n\,] - E[\,Tf\,|\,\mathcal{F}_n\,]\|_1 &= \big\|E\big[E[\,f\,|\mathcal{F}_n\,] - Tf\,|\,\mathcal{F}_n\,\big]\big\|_1 \\ &\le \|E[\,f\,|\,\mathcal{F}_n\,] - Tf\|_1 \to 0 \quad (n \to \infty).\end{aligned}$$

This completes the proof. □

Now let W_∞ be a positive random variable satisfying (3.4) and $W_n = E[\,W_\infty\,|\,\mathcal{G}_n\,]$, where $\{\mathcal{G}_n\}_{n\ge 0}$ is a sequence of sub-σ-fields of $\mathcal{F}$. We say that $W = \{W_n\}$ satisfies A_p and write $A_p(K, (\mathcal{G}_n))$ if $W = (W_n)$ satisfies (3.2) or (3.3) according as $1 < p < \infty$ or $p = 1$, with $\{\mathcal{G}_n\}$ instead of $(\mathcal{F}_n)$. Note that in the proof of Lemma 4.7, we have not used the fact that $(\mathcal{F}_n)$ is increasing in n. Hence we have

Corollary 4.13. *Suppose that B is a rearrangement invariant space or $B = L^p(\hat{P})$, where $d\hat{P} = W_\infty\, dP$ with $W = \{W_n\} \in A_p\big(K, (\mathcal{G}_n)\big)$. Then* (i) – (iii) *of Theorem 4.12 are equivalent.*

5. Continuous linear functionals on martingale spaces

Let $1 \leq p \leq \infty$. We denote by H^p the linear space consisting of all martingales $X = (X_t)_{t \geq 0}$ such that $X_\infty^\circledast \in L^p$, where we consider martingales with continuous parameter. If we set $\|X\|_{H^p} = \|X^\circledast\|_p$ for $X = (X_t) \in H^p$, $(H^p, \|\cdot\|)$ forms a Banach space, which is called the (martingale) Hardy space. Doob's inequality shows that H^p is isomorphic to the Lebesgue space L^p for $1 < p \leq \infty$. This is not true, however, for $p = 1$: Getoor and Sharpe proved in [22] that the dual of H^1 is BMO as the space of continuous martingales, and Meyer [42] extends this result to the spaces of càdlàg martingales.

More generally, for a Young function Φ, we denote by H^Φ the Banach space of martingales X such that $X_\infty^\circledast \in L^\Phi$, where L^Φ stands for the Orlicz space as usual. Doob's inequality for the norm of Orlicz space shows that, if Ψ is the complementary function to Φ and both Φ and Ψ satisfies the Δ_2-condition, then the dual of H^Φ is H^Ψ. However any representation of the dual of H^Φ with Φ such as $\Phi(t) = (1+t)\log(1+t) - t$ is not known.

On the other hand, Garsia [21] introduced the spaces $\mathcal{K}^p$ $(2 \leq p \leq \infty)$ and proved that the dual of H^p is $\mathcal{K}^{p'}$. These facts suggest that the dual of H^Φ may be represented by a suitable extension of $\mathcal{K}^p$. In fact, this is possible for some filtrations.

In this chapter we consider such a problem for the spaces of martingales associated with Banach function spaces.

§ 1. Linear functionals on spaces of adapted processes

We begin with a definition.

Definition 5.1. Let B be a Banach function space over Ω. We denote by $\mathfrak{R}_{\mathrm{r}}(B)$ the linear space consisting of all not necessarily adapted, càdlàg processes $X = (X_t)_{t \geq 0}$ such that $X_\infty^\circledast \in B$ with a finite limit $X_\infty = \lim_{t \to \infty} X_t$. We denote by $\mathfrak{R}(B)$ the linear subspace of $\mathfrak{R}_{\mathrm{r}}(B)$ consisting of all adapted processes in $\mathfrak{R}_{\mathrm{r}}(B)$. We put

$$\|X\|_{\mathfrak{R}_{\mathrm{r}}(B)} = \|X_\infty^\circledast\|_B \quad \text{for} \quad X \in \mathfrak{R}_{\mathrm{r}}(B);$$
$$\|X\|_{\mathfrak{R}(B)} = \|X_\infty^\circledast\|_B \quad \text{for} \quad X \in \mathfrak{R}(B).$$

The letter 'r' in the symbol $\mathfrak{R}_{\mathrm{r}}(B)$ means that $\mathfrak{R}_{\mathrm{r}}(B)$ is a linear space of "raw" (i.e. not necessarily adapted) processes. In this section we consider the linear functionals on $\mathfrak{R}(B)$ and $\mathfrak{R}_{\mathrm{r}}(B)$.

Lemma 5.2. *$\mathfrak{R}_{\mathrm{r}}(B)$ is a Banach space and $\mathfrak{R}(B)$ is a closed linear subspace of $\mathfrak{R}_{\mathrm{r}}(B)$.*

Proof. It is clear that $\mathfrak{R}(B)$ is a closed linear subspace of $\mathfrak{R}_{\mathrm{r}}(B)$. Thus it suffices to prove the completeness of $\mathfrak{R}_{\mathrm{r}}(B)$. Let $\{X^{(n)}\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathfrak{R}_{\mathrm{r}}(B)$. To prove that $\{X^{(n)}\}$ converges in $\mathfrak{R}_{\mathrm{r}}(B)$, we may assume that $\sum_{n=1}^{\infty} \left\|X^{(n)} - X^{(n+1)}\right\|_{\mathfrak{R}_{\mathrm{r}}(B)} < \infty$ by choosing a subsequence, if necessary. Since $B \hookrightarrow L^1$, the series $\sum_n \left(X^{(n)} - X^{(n+1)}\right)^{\circledast}_{\infty}$ converges in L^1, and hence the series converges a.s. If we set $X_t = \sum_n \left(X_t^{(n)} - X_t^{(n+1)}\right)$ for each $t \geq 0$, then we easily obtain that $X = (X_t) \in \mathfrak{R}_{\mathrm{r}}(B)$ and $X^{(n)} \to X$ in $\mathfrak{R}_{\mathrm{r}}(B)$. This completes the proof. □

Recall that a process $A = (A_t)_{0 \leq t \leq \infty}$ is said to be of *finite variation* if almost every path is of bounded variation on each compact interval. We denote by $\int_{[0,\,t]} |dA_s|$ the total variation on the interval $[\,0,\,t\,]$.

Our first result is a representation of linear functionals on $\mathfrak{R}_{\mathrm{r}}(B)$.

Theorem 5.3. (a) *Let $A^{\pm} = (A_t^{\pm})_{0 \leq t \leq \infty}$ be processes of finite variation which are not necessarily adapted. Suppose that*[§] *$A_0^- = 0, A_{\infty-}^+ = A_{\infty}^+$ and $\int_{[0,\,\infty]} |dA_s^{\pm}| \in B'$. Then the expression*

$$(5.1) \qquad J(X) = E\left[\int_{]\,0,\,\infty\,]} X_{s-}\, dA_s^- + \int_{[\,0,\,\infty\,[} X_s\, dA_s^+\right], \qquad X \in \mathfrak{R}_{\mathrm{r}}(B),$$

is meaningful and defines a continuous linear functional on $\mathfrak{R}_{\mathrm{r}}(B)$. Furthermore we have

$$\|J\| \leq \left\| \int_{]\,0,\,\infty\,]} |dA_s^-| + \int_{[\,0,\,\infty\,[} |dA_s^+| \right\|_{B'}.$$

(b) *Suppose that B has absolutely continuous norm. Then every continuous linear functional J on $\mathfrak{R}_{\mathrm{r}}(B)$ has a representation of the form* (5.1) *with $A^{\pm}$ satisfying*

$$\left\| \int_{]\,0,\,\infty\,]} |dA_s^-| + \int_{[\,0,\,\infty\,[} |dA_s^+| \right\|_{B'} \leq 4\,\|J\|.$$

[§] We do not assume the continuity at ∞ in general.

To prove the theorem we shall use the following representation theorem for linear functionals (applied to the case where $\mathcal{F}_t = \mathcal{F}$ for all $t \geq 0$), due to Dellacherie and Meyer [18, p. 201].

Theorem 5.4 (Dellacherie and Meyer). *Let $\mathfrak{D}$ denote the linear space of all bounded processes and J be a positive linear functional on $\mathfrak{D}$. If $\lim_{n\to\infty} J(X^{(n)}) = 0$ for any sequence $\{X^{(n)}\}_{n\geq 1}$ in $\mathfrak{D}$ satisfying $(X^{(n)})^{\circledast}_{\infty} \downarrow 0$ a.s., then there exist increasing processes $A^{\pm} = (A^{\pm}_t)_{0\leq t\leq\infty}$ which are not necessarily adapted and satisfying the following conditions:*

(i) $A^{+}_{\infty-} = A^{+}_{\infty}$ *and* $A^{-}_0 = 0$;

(ii) $J(X) = E\left[\int_{]0,\infty]} X_{s-}\, dA^-_s + \int_{[0,\infty[} X_s\, dA^+_s\right]$ *for all* $X \in \mathfrak{D}$.

Furthermore we can choose A^+ so as to be purely discontinuous: then the representation is unique.

Recall that if $A = (A_t)_{0\leq t\leq\infty}$ is an increasing process, then there exists a continuous increasing process $A^c = (A^c_t)$, a sequence $(T_n)_{n\geq 1}$ of stopping times and a sequence $(\lambda_n)_{n\geq 1}$ of positive numbers such that

$$A_t = A^c_t + \sum_{n=1}^{\infty} \lambda_n 1_{\{T_n\leq t\}}. \tag{5.2}$$

If A is predictable, each T_n can be chosen predictable. In the above representation, A^c is unique and called the *continuous part* of A. The process $A - A^c$ is denoted by A^d and is called the *(purely) discontinuous part* or *jump part* of A. If $A = A^d$, then A is said to be *purely discontinuous*. Clearly we may define the continuous part and purely discontinuous part of processes of finite variation. Furthermore, we can define the continuous part and discontinuous part of non-adapted process A: each T_n in (5.2) is not a stopping time but merely a random variable. For details, see [18, p. 127].

Proof of Theorem 5.3. Part (a) follows from the inequalities:

$$\begin{aligned}|J(X)| &\leq E\left[X^{\circledast}_{\infty}\left(\int_{]0,\infty]} |dA^-_s| + \int_{[0,\infty[} |dA^+_s|\right)\right] \\ &\leq \|X\|_{\mathfrak{R}_r(B)} \left\| \int_{]0,\infty]} |dA^-_s| + \int_{[0,\infty[} |dA^+_s| \right\|_{B'}.\end{aligned}$$

To prove (b), assume first that J is positive and continuous. Let $\{X^{(n)}\}_{n\geq 0}$ be a sequence of nonnegative processes in $\mathfrak{D}$ such that $(X^{(n)})^{\circledast}_{\infty} \downarrow 0$ a.s. As B has absolutely continuous norm, we have $\lim_{n\to\infty} \|X^{(n)}\|_{\mathfrak{R}_r(B)} = 0$ and hence $\lim_{n\to\infty} J(X^{(n)}) = 0$. So there exist increasing processes $A^{\pm}$ satisfying (i) and (ii) of Theorem 5.4. We shall prove that $A^{\pm}_{\infty} \in B'$. To this end, let $\xi \geq 0$ be a bounded r.v. with $\|\xi\|_B \leq 1$ and set $X_t = \xi$ for every $t \geq 0$. Since $\|X\|_{\mathfrak{R}_r(B)} = \|\xi\|_B \leq 1$, we have

$$E\big[(A^+_{\infty} + A^-_{\infty})\,\xi\,\big] = E\left[\int_{]\,0,\infty\,]} X_{s-}\,dA^-_s + \int_{[\,0,\infty\,[} X_s\,dA_s\right] = J(X) \leq \|J\|\,.$$

Clearly this remains valid for every nonnegative $\xi \in B$ such that $\|\xi\|_B \leq 1$. Thus we see that $A^{\pm}_{\infty} \in B'$ and $\|A^-_{\infty} + A^+_{\infty}\|_{B'} \leq \|J\|$. Therefore, by part (a) of the theorem, the functional

$$X \longmapsto E\left[\int_{]\,0,\infty\,]} X_{s-}\,dA^-_s + \int_{[\,0,\infty\,[} X_s\,dA^+_s\right],$$

is continuous and linear on $\mathfrak{R}_r(B)$. To prove that this functional is equal to J, it remains to show that $\mathfrak{D}$ is dense in $\mathfrak{R}_r(B)$. For each $X \in \mathfrak{R}_r(B)$, the process $X^{(k)} = \big(X^{(k)}_t\big)_{t\geq 0}$ defined by

$$X^{(k)}_t = X_t\,1_{\{|X_t|\leq k\}} + k\,1_{\{|X_t|>k\}}$$

tends to X in $\mathfrak{R}_r(B)$, since $\big(X - X^{(k)}\big)^{\circledast}_{\infty} \leq 2X^{\circledast}_{\infty}\,1_{\{X^{\circledast}_{\infty}>k\}} \downarrow 0$ as $k \to \infty$. Thus $\mathfrak{D}$ is dense in $\mathfrak{R}_r(B)$.

Next let J be an arbitrary continuous linear functional on $\mathfrak{R}_r(B)$. Then there exist positive linear functionals $J^{\pm}$ such that $J = J^+ - J^-$ and $\|J^{\pm}\| \leq 2\,\|J\|$: they are given by

$$J^+(X) = \sup\{J(Y) : Y \in \mathfrak{D},\ 0 \leq Y \leq X\};$$
$$J^-(X) = J(X) - J^+(X),$$

for nonnegative $X = (X_t)$. Applying the result just have been proved above, we obtain (b) of Theorem 5.3. □

We now pass to the study of linear functionals on $\mathfrak{R}(B)$. In what follows we frequently use the letter γ to denote a r.v. in B (or B'). For such a γ we denote by $(\gamma_t)_{t\geq 0}$ the martingale induced by γ : $\gamma_t = E[\gamma\,|\,\mathcal{F}_t]$, $t \geq 0$.

Theorem 5.5. (a) *Let* $A^{\pm} = (A^{\pm}_t)_{0 \le t \le \infty}$ *be processes of finite variation satisfying the following conditions:*

(i) A^+ *is adapted, purely discontinuous, and satisfies* $A^+_{\infty-} = A^+_{\infty}$;

(ii) A^- *is predictable and* $A^-_0 = 0$;

(iii) *there exists a* $\gamma \in B'$ *such that for every stopping time* T,

$$E\left[\int_{]T,\infty]} |dA^-_s| + \int_{[T,\infty[} |dA^+_s| \;\Big|\; \mathcal{F}_T\right] \le \gamma_T . \tag{5.3}$$

Then the linear functional J *defined by*

$$J(X) = E\left[\int_{]0,\infty]} X_{s-}\, dA^-_s + \int_{[0,\infty[} X_s\, dA^+_s\right], \quad X \in \mathfrak{R}(B), \tag{5.4}$$

is continuous on $\mathfrak{R}(B)$ *and* $\|J\| \le \|\gamma\|_{B'}$.

(b) *If* B *has absolutely continuous norm, then every continuous linear functional* J *on* $\mathfrak{R}(B)$ *has a representation of the form* (5.4) *with* $A^{\pm}$ *and* γ *satisfying* $\|\gamma\|_{B'} \le 12\, \|J\|$ *and* (i)–(iii) *of* (a). *Moreover the representation is unique.*

Proof. (a) We first point out that A^+ does not need to be purely discontinuous. Suppose that J is given by (5.4). For the sake of simplicity, we set $B^{\pm}_t = \int_{[0,t]} |dA^{\pm}_s|$ for each $t \ge 0$. Let Z^+ (resp. Z^-) denote the left potential (resp. potential) generated by the process B^+ (resp. B^-). Then by (5.3), we have $Z^+_t + Z^-_t \le \gamma_t$ outside of an evanescent set. Using integration by parts, we have

$$\begin{aligned}
|J(X)| &\le E\left[\int_{]0,\infty]} X^{\circledast}_{s-}\, dB^-_s + \int_{[0,\infty[} X^{\circledast}_s\, dB^+_s\right] \\
&= E\left[\int_{[0,\infty[} (B^-_{\infty} - B^-_s)\, dX^{\circledast}_s + \int_{[0,\infty[} (B^+_{\infty} - B^+_{s-})\, dX^{\circledast}_s\right] \\
&= E\left[\int_{[0,\infty[} (Z^-_s + Z^+_s)\, dX^{\circledast}_s\right] \\
&\le E\left[\int_{[0,\infty[} \gamma_s\, dX^{\circledast}_s\right] = E\left[\gamma\, X^{\circledast}_{\infty}\right] \le \|\gamma\|_{B'}\, \|X\|_{\mathfrak{R}(B)},
\end{aligned}$$

where we have used the fact that the optional projection of the process $Y_t = \gamma$ $(t \ge 0)$ is equal to the martingale (γ_t). Thus J is well-defined, continuous on $\mathfrak{R}(B)$ and satisfies $\|J\| \le \|\gamma\|_{B'}$.

(b) Let J be a continuous linear functional on $\mathfrak{R}(B)$. Then by the Hahn-Banach theorem, some $\bar{J} \in (\mathfrak{R}_{\mathrm{r}}(B))^*$ extends J without increasing norm. According to Theorem 5.3 (b), there exist processes $B^{\pm} = (B_t^{\pm})_{0 \le t \le \infty}$ of finite variation such that $B_0^- = 0$, $B_{\infty-}^+ = B_\infty^+$, $\int_{[0,\infty]} |dB_s^{\pm}| \in B'$ and

$$\bar{J}(X) = E\left[\int_{]0,\infty]} X_{s-}\, dB_s^- + \int_{[0,\infty[} X_s\, dB_s^+\right] \quad \text{for all } X \in \mathfrak{R}_{\mathrm{r}}(B).$$

In addition, setting $\gamma' := \int_{]0,\infty]} |dB_s^-| + \int_{[0,\infty[} |dB_s^+| \in B'$, we have

$$\|\gamma'\|_{B'} \le 4\,\|J\|\,. \tag{5.5}$$

Now let A^+ and A^- denote respectively the dual optional and dual predictable projections of B^+ and B^-. Then we obtain (5.4), and $A^{\pm}$ satisfy $A_{\infty-}^+ = A_\infty^+$ and $A_0^- = 0$. Moreover we claim that

$$E\left[\int_{]T,\infty]} |dA_s^-| + \int_{[T,\infty[} |dA_s^+| \;\middle|\; \mathcal{F}_T\right] \le \gamma'_T \tag{5.3}$$

holds for all stopping times T. To see this, let $H^+ = (H_t^+)_{0 \le t \le \infty}$ and $H^- = (H_t^-)_{0 \le t \le \infty}$ be densities $dA_s^+/|dA_s^+|$ and $dA_s^-/|dA_s^-|$ with values in $\{-1, 1\}$, respectively. Then, for every stopping time T, we have

$$\begin{aligned}
E\left[\int_{]T,\infty]} |dA_s^-| + \int_{[T,\infty[} |dA_s^+|\right] &= E\left[\int_{]T,\infty]} H_s^-\, dA_s^- + \int_{[T,\infty[} H_s^+\, dA_s^+\right] \\
&= E\left[\int_{]T,\infty]} H_s^-\, dB_s^- + \int_{[T,\infty[} H_s^+\, dB_s^+\right] \\
&= E\left[\int_{]T,\infty]} |dB_s^-| + \int_{[T,\infty[} |dB_s^+|\right] \\
&\le E\left[\gamma'\, 1_{\{T<\infty\}}\right] = E\left[\gamma'_T\, 1_{\{T<\infty\}}\right],
\end{aligned}$$

where the second equality follows from the fact that the process $(H_t^-\, 1_{\{T<t\}})$ (resp. $(H_t^+\, 1_{\{T \le t\}})$) is predictable (resp. optional) and A^- (resp. A^+) is the dual predictable (resp. optional) projection of B^- (resp. B^+). Replacing T by T_F with $F \in \mathcal{F}_T$ defined by (2.24), we get (5.3). Thus we obtain the representation of J, provided that A^+ is not required to be purely discontinuous. Now we show that A^+ can be chosen purely discontinuous. Let A^{+c} be the continuous part of A^+. Since the set of s such that $\triangle X_s \ne 0$ is at most countable, we have

$$E\left[\int_{[0,\infty[} X_s\, dA_s^{+c}\right] = E\left[\int_{]0,\infty]} X_{s-}\, dA_s^{+c}\right],$$

and hence

$$(5.6)\qquad J(X)=E\left[\int_{]0,\infty]} X_{s-}\,d(A_s^- + A_s^{+c}) + \int_{[0,\infty[} X_s\,dA_s^{+d}\right],$$

where $A^{+d} = A^+ - A^{+c}$ is the purely discontinuous part of A^+. Hence the processes $A^- + A^{+c}$ and A^{+d} give a representation of J. Observe that

$$\begin{aligned}
\int_{]T,\infty]} |d(A_s^- + A_s^{+c})| + \int_{[T,\infty[} |dA_s^{+d}| &\\
\le \int_{]T,\infty]} |dA_s^-| + \int_{]T,\infty]} |dA_s^{+c}| + \int_{[T,\infty[} |dA_s^{+d}| &\\
\le \int_{]T,\infty]} |dA_s^-| + \int_{[T,\infty[} |dA_s^+| + 2\int_{[T,\infty[} |dA_s^{+d}| &\\
\le \int_{]T,\infty]} |dA_s^-| + 3\int_{[T,\infty[} |dA_s^+| \,. &
\end{aligned}$$

It then follows that

$$(5.7)\qquad E\left[\int_{]T,\infty]} |d(A_s^- + A_s^{+c})| + \int_{[T,\infty[} |dA_s^{+d}| \,\middle|\, \mathcal{F}_T\right] \le 3\,\gamma'_T\,.$$

If we set $\tilde{A}^- = A^- + A^{+c}$, $\tilde{A}^+ = A^{+d}$ and $\gamma = 3\gamma'$, then we obtain a representation of J with $\tilde{A}^\pm$ satisfying (5.3) and (5.4). Furthermore from (5.5) and (5.7), we see that $\|J\| \le 4\,\|\gamma'\|_{B'} \le 12\,\|\gamma\|_{B'}$.

Finally we give here the proof of uniqueness of the representation due to Dellacherie and Meyer (cf. [18, p. 204]). Suppose that two pairs of process $A^\pm$ and $B^\pm$ satisfy

$$\begin{aligned}
E\left[\int_{]0,\infty]} X_{s-}\,dA_s^- + \int_{[0,\infty[} X_s\,dA_s^+\right] &\\
= E\left[\int_{]0,\infty]} X_{s-}\,dB_s^- + \int_{[0,\infty[} X_s\,dB_s^+\right] &
\end{aligned}$$

for every $X \in \mathfrak{R}(B)$. We take the process $\left(1_{\{T\le t\}}\right)$ for X. It follows that

$$(5.8)\qquad E[A_\infty^- - A_T^- + A_\infty^+ - A_{T-}^+] = E[B_\infty^- - B_T^- + B_\infty^+ - B_{T-}^+].$$

Taking $T+\frac{1}{n}$ instead of T, we have

$$E[A_{T+\frac{1}{n}}^- - A_T^- + A_{(T+\frac{1}{n})-}^+ - A_{T-}^+] = E[B_{T+\frac{1}{n}}^- - B_T^- + B_{(T+\frac{1}{n})-}^+ - B_{T-}^+].$$

Letting $n \to \infty$, we obtain $E[\Delta A_T^+ 1_{\{T<\infty\}}] = E[\Delta B_T^+ 1_{\{T<\infty\}}]$, since A^+ and B^+ do not have jump at infinity. Replacing T by T_F with $F \in \mathcal{F}_T$, we get $E[\Delta A_T^+ 1_F] = E[\Delta B_T^+ 1_F]$. As ΔA_T^+ and ΔB_T^+ are $\mathcal{F}_T$-measurable, we have $\Delta A_T^+ = \Delta B_T^+$. Since A^+ and B^+ are purely discontinuous and T is arbitrary, we obtain $A^+ = B^+$ (indistinguishable).

Then (5.8) gives that $E[A_\infty^- - A_T^-] = E[B_\infty^- - B_T^-]$ for any stopping time T. Replacing T by T_F as before, we get $E[A_\infty^- - A_T^- \mid \mathcal{F}_T] = E[B_\infty^- - B_T^- \mid \mathcal{F}_T]$. This implies $A^- = B^-$, since A^- and B^- are predictable: see [18, p. 145]. □

§ 2. Linear functionals on martingale Hardy spaces

Let $H(B)$ denote the linear subspace of $\mathfrak{R}(B)$ consisting of all martingales in $\mathfrak{R}(B)$. If $X^{(n)} = \big(X_t^{(n)}\big)_{t\ge 0} \in H(B)$ converges to $X = (X_t)$ as $n \to \infty$ in $\mathfrak{R}(B)$, then $E\big[\big(X - X^{(n)}\big)^{\circledast}_{\infty}\big] = E\big[\sup_{t\ge0} |X_t - X_t^{(n)}|\big] \to 0$, since $B \hookrightarrow L^1$. It follows that

$$X_t = \lim_n X_t^{(n)} = \lim_n E\big[X_\infty^{(n)} \,\big|\, \mathcal{F}_t\big] = E\big[X_\infty \,\big|\, \mathcal{F}_t\big]$$

holds for all $t \ge 0$, where the limits are taken in L^1. Thus X is a martingale and hence $H(B)$ is a closed linear subspace of $\mathfrak{R}(B)$.

In this section we shall consider linear functionals on $H(B)$. To describe our results, we shall introduce some new spaces of processes. Recall that a process $A = (A_t)_{0\le t\le\infty}$ is said to be of *integrable variation* if it is of finite variation and $\int_{[0,\infty[} |dA_s|$ is integrable.

Definition 5.6. Let B be a Banach function space.

(a) We denote by $\mathcal{V}(B)$ the space of all processes $A = (A_t)_{0\le t\le\infty}$ with the following properties:

(i) $A = (A_t)_{0\le t\le\infty}$ is of integrable variation;

(ii) there exists a $\gamma \in B$ such that, for every stopping time T,

$$E\left[\int_{[T,\infty]} |dA_s| \,\middle|\, \mathcal{F}_T\right] \le \gamma_T . \tag{5.9}$$

For each $A \in \mathcal{V}(B)$, we set

$$\|A\|_{\mathcal{V}(B)} = \inf\{\|\gamma\|_B : \gamma \in B \text{ satisfies } (5.9)\}.$$

(b) We denote by $\mathcal{K}(B)$ the space of all uniformly integrable martingales Y such that there exists a $\gamma \in B$ satisfying

$$E\big[\,|Y_\infty - Y_{T-}|\,\big|\,\mathcal{F}_T\,\big] \le \gamma_T \tag{5.10}$$

for every stopping time T. For each $Y \in \mathcal{K}(B)$, we set

$$\|Y\|_{\mathcal{K}(B)} = \inf\{\|\gamma\|_B : \gamma \text{ satisfies } (5.10)\}.$$

Note that (5.10) implies $|\Delta Y_T| \le \gamma_T$ for every stopping time T: this fact will be used frequently in what follows.

Suppose that Doob's inequality holds in B: i.e., $\|X_\infty^{*}\|_B \le C_B\,\|X_\infty\|_B$ folds for every uniformly integrable martingale $X = (X_t)$. Let $\gamma \in B$ and $\bar{\gamma} = \sup_{t\ge 0}|\gamma_t| = \gamma^{*}$: then we have

$$\gamma_T^{*} = \sup_{0\le t\le T}|\gamma_t| \le E[\,\bar{\gamma}\,|\,\mathcal{F}_T\,] = \bar{\gamma}_T \tag{5.11}$$

for every stopping time T, and $\|\bar{\gamma}\|_B \le C_B\,\|\gamma\|_B$.

Theorem 5.7. (a) *If $A = (A_t)_{0\le t\le\infty} \in \mathcal{V}(B')$, then the expression*

$$J(X) = E\left[\int_{[0,\infty]} X_s\,dA_s\right], \qquad X \in H(B), \tag{5.12}$$

defines a continuous linear functional on $H(B)$ satisfying $\|J\| \le 2\,\|A\|_{\mathcal{V}(B')}$.

(b) *Assume that B has absolutely continuous norm and that Doob's inequality holds in B'. If J is a continuous linear functional on $H(B)$, then there exists a process $A = (A_t)_{0\le t\le\infty} \in \mathcal{V}(B')$ satisfying* (5.12) *and*

$$\|A\|_{\mathcal{V}(B')} \le 24\,C_{B'}\,\|J\|. \tag{5.13}$$

Moreover, if X is bounded, then $J(X) = E\big[X_\infty A_\infty\big]$.

If, in addition, every $(\mathcal{F}_t)$–martingale is continuous, then we can remove the hypothesis that Doob's inequality holds in B', and we can replace the constant $24\,C_{B'}$ by 12 in (5.13).

Proof. (a) Assume that $\gamma \in B'$ satisfies (5.9). When $T = \infty$ a.s., (5.9) is reduced to the inequality $|\Delta A_\infty| \leq \gamma$ a.s., and therefore the linear functional $H(B) \ni X \mapsto E[X_\infty \Delta A_\infty]$ is continuous on $H(B)$ with norm less than $\|\gamma\|_{B'}$. On the other hand, by virtue of Theorem 5.5, the linear functional j given by

$$j(X) = E\left[\int_{[0,\infty[} X_s \, dA_s\right], \qquad X \in H(B),$$

is continuous and $\|j\| \leq \|\gamma\|_{B'}$. Therefore J is a continuous linear functional on $H(B)$ and satisfies $\|J\| \leq 2\|\gamma\|_{B'}$. Taking the infimum over such γ we have $\|J\| \leq 2\|A\|_{\mathcal{V}(B')}$.

(b) We assume nothing on the continuity of $(\mathcal{F}_t)$–martingales, but assume the validity of Doob's inequality in B'. Let J be a continuous linear functional on $H(B)$. We may extend J to a linear functional $\bar{J}$ on $\mathfrak{R}(B)$ such that $\|J\| = \|\bar{J}\|$ by the Hahn-Banach theorem. According to Theorem 5.5 (b), there exist processes $A^\pm = (A^\pm_t)_{0 \leq t \leq \infty}$ of integrable variation satisfying (i)–(iii) of Theorem 5.5 and

$$\bar{J}(X) = E\left[\int_{]0,\infty]} X_{s-} \, dA^-_s + \int_{[0,\infty[} X_s \, dA^+_s\right], \qquad X \in \mathfrak{R}^\rho, \tag{5.14}$$

with some $\gamma \in B'$ such that $\|\gamma\|_{B'} \leq 12\|J\|$. If every $(\mathcal{F}_t)$–martingale is continuous, then this implies (5.12) and (5.13) with 12 instead of $24\,C_{B'}$.

We claim that $|\Delta A^-_T| \leq \gamma_{T-}$ holds for every stopping time T. If T is predictable, this follows immediately from the inequality

$$E\left[\,|\Delta A^-_T| \,\middle|\, \mathcal{F}_{T_n}\right] \leq E\left[\int_{]T_n,\infty]} |dA^-_s| \,\middle|\, \mathcal{F}_{T_n}\right] \leq \gamma_{T_n},$$

where (T_n) is an announcing sequence of T: for the definition of announcing sequence of T and the theory of predictable stopping times, see [17, p. 202: p. 214]. Letting $n \to \infty$ in the above inequalities, we have $|\Delta A^-_T| \leq \gamma_{T-}$ a.s., since ΔA^-_T is $\mathcal{F}_{T-}$-measurable and $\cup_{n\geq 1}\mathcal{F}_n$ generates $\mathcal{F}_{T-}$ (cf. [17, p. 200]). Next let T be an arbitrary stopping time. Since A^- is predictable, the set $\{\Delta A^- \neq 0\} \subset \mathbb{R} \times \Omega$ is indistinguishable from the countable union of a sequence of disjoint graphs of predictable stopping times S_n (see [17, p. 261] or [16, p. 138]). Hence we obtain

$$|\Delta A^-_T| = \sum_{n=1}^{\infty} |\Delta A^-_{S_n}|\, 1_{\{T=S_n\}} \leq \sum_{n=1}^{\infty} \gamma_{S_n-}\, 1_{\{T=S_n\}} \leq \gamma_{T-}\,.$$

Since Doob's inequality holds in B', there exits a $\bar{\gamma} \in B'$ satisfying (5.11) and $\|\bar{\gamma}\|_{B'} \leq C_{B'} \, \|\gamma\|_{B'}$. Therefore, if we set $A = A^+ + A^-$, then

$$
\begin{aligned}
E\Big[\int_{[T,\infty]} |dA_s| \,\Big|\, \mathcal{F}_T\Big]
&\leq E\Big[\int_{[T,\infty[} |dA_s^+| + \int_{]T,\infty]} |dA_s^-| \,\Big|\, \mathcal{F}_T\Big] + |\Delta A_T^-| \\
&\leq \gamma_T + \gamma_{T-} \leq 2\,\gamma_T^{\circledast} \leq 2\,\bar{\gamma}_T =: \xi_T.
\end{aligned}
$$

From the above inequalities, we see that

$$\|A\|_{\mathcal{V}(B')} \leq 2\,\|\bar{\gamma}\|_{B'} \leq 2\,C_{B'}\,\|\gamma\|_{B'} \leq 24\,C_{B'}\,\|J\|.$$

Now we show that J is written as (5.12). As in the proof of Theorem 5.5 (a), we can prove that

$$\text{(5.15)} \qquad E\Big[\int_{[0,\infty]} X_s^{\circledast}\,|dA_s^-|\Big] \leq 2\,\|\bar{\gamma}\|_{B'}\,\|X\|_{H(B)} < \infty.$$

Since the left-continuous process $X_- = (X_{t-})$ is the predictable projection of X, we have

$$E\Big[\int_{]0,\infty]} X_{s-}\,dA_s^-\Big] = E\Big[\int_{]0,\infty]} X_s\,dA_s^-\Big],$$

where these expectations are finite by (5.15). This, together with (5.14), implies (5.12).

Finally suppose that X is bounded. Then we see that $|X_\infty| \int_{[0,\infty]} |dA_s|$ is integrable and hence

$$J(X) = E\Big[\int_{[0,\infty]} X_s\,dA_s\Big] = E\,[X_\infty A_\infty],$$

since $X = (X_t)$ is the optional projection of the process Y defined by $Y_t = X_\infty$ for all $t \geq 0$. Thus the theorem is established. □

When B is equal to an Orlicz space L^Φ, we write $H(\Phi) = H(L^\Phi)$, $\mathcal{K}(\Phi) = K(L^\Phi)$, and $\mathcal{V}(\Phi) = \mathcal{V}(L^\Phi)$.

Corollary 5.8. (a) *Let B be a r.i. space which has absolutely continuous norm. If $\underline{\alpha}_B > 0$, then the dual of $H(B)$ is $\mathcal{V}(B')$.*

(b) *Let B be as in* (a). *If every $(\mathcal{F}_t)$–martingale is continuous, then the dual of $H(B)$ is $\mathcal{V}(B')$.*

(c) *If Φ is a Young function satisfying the Δ_2-condition, then the dual of $H(\Phi)$ is $\mathcal{V}(\Psi)$, where Ψ stands for the complementary Young function to Φ.*

Proof. (a) Let B be a r.i. space with absolutely continuous norm. Antipa [1] proved that Doob's inequality holds in B' if and only if $\overline{\alpha}_{B'} < 1$. (Note that in his paper the definition of Boyd indices is different from ours). As $\underline{\alpha}_B = 1 - \overline{\alpha}_{B'}$, we obtain (a) from the preceding theorem.

(b) This is an immediate consequence of Theorem 5.7.

(c) Let Φ be a Young function satisfying the Δ_2-condition and Ψ be the Young function complementary to Φ. Note that the associate space of L^Φ is isomorphic to L^Ψ and that Doob's inequality holds in L^Ψ (cf. [18, p. 186]). On the other hand, Φ satisfies the Δ_2-condition if and only if L^Φ has absolutely continuous norm (cf. [36, pp. 87–88]) . Therefore (c) follows from the preceding theorem. $\square$

Remark. It is clear that $\underline{\alpha}_{L^p} = \overline{\alpha}_{L^p} = 1/p$. Hence L^p has absolutely continuous norm if and only if $\underline{\alpha}_{L^p} > 0$. However, the similar statement is false for general r.i. function space. In fact, let $1 < p < \infty$ and B be the Lorentz space $L^{p,\infty}$: $f \in L^{p,\infty}$ if and only if

$$\|f\|_{p,\infty} = \sup_{0<t\le 1} t^{1/p} f^{**}(t) = \sup_{0<t\le 1} \frac{1}{t^{1/p'}} \int_0^t f^*(s)\, ds < \infty,$$

where $p' = p/(1-p)$. For details, see [52, pp. 188–205]. Assume that Ω is non-atomic. Then there exists a random variable f such that $f^*(s) = s^{-1/p}$, $s \in I$ (cf. [13, p. 44]). It is easy to see that

$$\left(|f|\, 1_{\{|f|>a\}}\right)^*(s) = f^*(s)\, 1_{\{0\le s<\varepsilon_a\}}, \qquad \varepsilon_a = P(|f| > a).$$

It follows that for each $a > 0$

$$\left\|f\, 1_{\{|f|>a\}}\right\|_{p,\infty} = \sup_{0<t\le 1} \frac{1}{t^{1/p'}} \int_0^{t\wedge\varepsilon_a} f^*(s)\, ds = p'.$$

This means that $L^{p,\infty}$ does not have absolutely continuous norm. However we have $\underline{\alpha}_{L^{p,\infty}} = \overline{\alpha}_{L^{p,\infty}} = 1/p > 0$, since $\|D_s\|_{L^{p,\infty}} = (1/s)^{1/p}$ for all $s > 0$.

We have established the duality between $H(B)$ and $\mathcal{V}(B')$. It is natural, however, to ask whether $(H^\rho)^*$ is isomorphic to a martingale space. The answer is affirmative if every $(\mathcal{F}_t)$–martingale is continuous. We begin with the following lemma.

Lemma 5.9. *Suppose that $Y = (Y_t) \in \mathcal{K}(B')$ and $\gamma \in B'$ satisfies (5.10). If $\gamma_{T-} \le \kappa\,\gamma_T$ holds for every stopping time T, then*

$$|E[\,X_\infty Y_\infty]| \le \big(3 + 2\sqrt{2}\big)\kappa\, E[\gamma\, X^{\circledast}_\infty] \tag{5.16}$$

holds for every bounded martingale $X = (X_t)$, where $\kappa \ge 1$ is a constant.

Proof. Without loss of generality, we may assume that $\gamma \ge \varepsilon$ a.s. for some constant $\varepsilon > 0$. Let $\delta > 1$ be arbitrary and define a sequence of stopping times (T_n) inductively by

$$T_0 = 0\,; \qquad T_{n+1} = \inf\{\, t \ge T_n \colon |Y_t - Y_{T_n-}| > \delta\gamma_t\,\}, \quad n \ge 0.$$

As $\gamma \ge \varepsilon$ a.s. and $|\Delta Y_{T_n}| \le \gamma_{T_n}$ by (5.10), we have $T_n < T_{n+1}$ on $\{T_n < \infty\}$ and $T_n(\omega) \uparrow \infty$, if $T_n(\omega) < \infty$ for all $n \ge 0$. We set

$$A_t = \sum_{n=0}^{\infty} \gamma_{T_n} 1_{\{T_n \le t\}} \quad (0 \le t < \infty)\,; \qquad A_\infty = \lim_{t\to\infty} A_t\,. \tag{5.17}$$

Then the increasing process $A = (A_t)$ belongs to $\mathcal{V}(B')$ and, in particular, A_∞ is integrable. To see this, observe first that, for every $n \ge 0$,

$$\begin{aligned} \delta\, E[\gamma_{T_{n+1}} 1_{\{T_{n+1}<\infty\}} \,|\, \mathcal{F}_{T_n}] &\le E\big[\,|Y_{T_{n+1}} - Y_{T_n-}|\,\big|\, \mathcal{F}_{T_n}\,\big]\, 1_{\{T_n<\infty\}} \\ &\le \gamma_{T_n} 1_{\{T_n<\infty\}}. \end{aligned}$$

Let T be an arbitrary stopping time and set $N = \inf\{n \ge 0 \colon T_n \ge T\,\}$; then N is $\mathcal{F}_T$-measurable. Hence, by the above inequalities, we have

$$\begin{aligned} E[\,A_\infty - A_{T-} \,|\, \mathcal{F}_T\,] &= \sum_{0\le k\le n} E[\,\gamma_{T_n} 1_{\{T_n<\infty\}} \,|\, \mathcal{F}_T\,]\, 1_{\{N=k\}} \\ &\le \sum_{0\le k\le n} \delta^{-n+k} E[\,\gamma_{T_k} 1_{\{T_k<\infty\}} \,|\, \mathcal{F}_T\,]\, 1_{\{N=k\}} \\ &\le \sum_{0\le k\le n} \delta^{-n+k} \gamma_T\, 1_{\{T<\infty\}} 1_{\{N=k\}} \\ &= \frac{\delta}{\delta-1}\gamma_T\, 1_{\{T<\infty\}}, \end{aligned} \tag{5.18}$$

from which it follows that $A \in \mathcal{V}(B')$. Now let U be a random variable defined by $U = \sup\{T_n \colon T_n < \infty\}$; it is not necessarily a stopping time. Then we have

$$Y_\infty = \sum_{n=0}^{\infty} (Y_{T_{n+1}-} - Y_{T_n-})\, 1_{\{T_{n+1}<\infty\}} + Y_\infty - Y_{U-}\,, \tag{5.19}$$

where $Y_{U-} = Y_\infty$ if $U = \infty$. Note that the series converges in L^1: for we have $|Y_{T_{n+1}-} - Y_{T_n-}| \le \delta\gamma_{T_{n+1}-} \le \delta\kappa\,\gamma_{T_{n+1}}$ and hence

$$\sum_{n=0}^{\infty} \big\| (Y_{T_{n+1}-} - Y_{T_n-})\, 1_{\{T_{n+1}<\infty\}} \big\|_1 \le \delta\kappa\, \|A_\infty\|_1 < \infty.$$

For each $t \ge 0$, put $\tilde{A}_t = \sum_{n=0}^{\infty} (Y_{T_{n+1}-} - Y_{T_n-})\, 1_{\{T_{n+1}\le t\}}$. Then $\tilde{A} = (\tilde{A}_t)$ is a process of integrable variation and

$$E[\,X_\infty Y_\infty\,] = E[\,X_\infty \tilde{A}_\infty\,] + E[\,X_\infty\, (Y_\infty - Y_{U-})\,]. \tag{5.20}$$

We estimate the right-hand side. On one hand, using (5.18) we have

$$\begin{aligned} \big|E[\,X_\infty \tilde{A}_\infty]\big| &= \Big| \sum_{n=0}^{\infty} E\big[\,X_\infty\, (Y_{T_{n+1}-} - Y_{T_n-})\, 1_{\{T_{n+1}<\infty\}}\big]\Big| \\ &\le \sum_{n=0}^{\infty} \Big| E[\,X_{T_{n+1}}\, (Y_{T_{n+1}-} - Y_{T_n-})\, 1_{\{T_{n+1}<\infty\}}]\Big| \\ &\le \delta\kappa \sum_{n=0}^{\infty} E\big[\,X^{\circledast}_{T_{n+1}}\, \gamma_{T_{n+1}}\, 1_{\{T_{n+1}<\infty\}}\big] \\ &\le \delta\kappa\, E\Big[\int_{[\,0,\,\infty[} X^{\circledast}_s\, dA_s\Big] \\ &= \delta\kappa\, E\Big[\int_{[\,0,\,\infty[} (A_\infty - A_{s-})\, dX^{\circledast}_s\Big] \\ &\le \frac{\delta^2\kappa}{\delta - 1}\, E\Big[\int_{[\,0,\,\infty[} \gamma_s\, dX^{\circledast}_s\Big] = \frac{\delta^2\kappa}{\delta - 1}\, E[\,\gamma X^{\circledast}]. \end{aligned} \tag{5.21}$$

On the other hand, we have $|Y_\infty - Y_{U-}| \le \delta\gamma$ and therefore

$$\big|E[\,X_\infty\, (Y_\infty - Y_{U-})]\big| \le \delta E\big[\,\gamma X^{\circledast}\,\big].$$

From (5.20), (5.21) and the above inequality, it follows that

$$\big|E[X_\infty Y_\infty]\big| \le \frac{2\delta^2-\delta}{\delta-1}\kappa\, E\big[\gamma X^{\circledast}\big]\,.$$

Since the smallest value of $(2\delta^2-\delta)/(\delta-1)$ for $\delta>1$ is equal to $3+2\sqrt{2}$, we obtain (5.16) and the lemma is established. □

Lemma 5.10. *Suppose that B has absolutely continuous norm and Doob's inequality holds in B'. Then for each $J \in (H(B))^*$, there exists a martingale $Y=(Y_t)\in \mathcal{K}(B')$ such that $\|Y\|_{\mathcal{K}(B')} \le 48C_{B'}^2\,\|J\|$ and*

$$J(X) = E[X_\infty Y_\infty] \tag{5.22}$$

holds for every bounded martingale X, where $C_{B'}>0$ is a constant appearing in Doob's inequality. Moreover if every $(\mathcal{F}_t)$-martingale is continuous, then the hypothesis that Doob's inequality holds in B' can be removed, and 24 replaces $48C_{B'}^2$.

Proof. According to Theorem 5.7, there exists a process $A=(A_t)\in\mathcal{V}(B')$ such that $\|A\|_{\mathcal{V}(B')}\le 24C_{B'}\,\|J\|$ and $J(X)=E[X_\infty A_\infty]$ holds for every bounded martingale X. Let $Y=(Y_t)$ be the martingale induced by A_∞; then (5.22) is obvious. If γ satisfies (5.9), then

$$|\,Y_T - A_{T-}| \le E\big[\,|A_\infty - A_{T-}|\,\big|\,\mathcal{F}_T\big] \le \gamma_T$$

holds for every stopping time T. The optional section theorem therefore yields that $\big\{|\,Y_t - A_{t-}| > \gamma_t\big\} \subset \mathbb{R}\times\Omega$ is an evanescent set. Taking the left-hand limit of this inequality, we also see that $\big\{|\,Y_{t-}-A_{t-}|>\gamma_{t-}\big\}$ is an evanescent set: i.e., $|\,Y_{T-}-A_{T-}|\le\gamma_{T-}$ for every stopping time T. Hence we have

$$\begin{aligned} E\big[\,|\,Y_\infty - Y_{T-}|\,\big|\,\mathcal{F}_T\big] &\le E\big[\,|A_\infty - A_{T-}|\,\big|\,\mathcal{F}_T\big] + |A_{T-}-Y_{T-}| \\ &\le \gamma_T + \gamma_{T-} \le 2\,\bar{\gamma}_T, \end{aligned}$$

where $\bar{\gamma}_T$ satisfies (5.11) and $\|\bar{\gamma}\|_{B'}\le C_{B'}\,\|\gamma\|_{B'}$. From this it follows that $\|Y\|_{\mathcal{K}(B')}\le 2C_{B'}\,\|A\|_{\mathcal{V}(B')}\le 48C_{B'}^2\,\|J\|$. Thus Y has the required properties.

If every $(\mathcal{F}_t)$-martingale is continuous, then $\|Y\|_{\mathcal{K}(B')}\le 2\,\|A\|_{\mathcal{V}(B')}\le 24\,\|J\|$ without the hypothesis of Doob's inequality in B', and the lemma is established. □

Lemma 5.11. *Suppose that a Banach function space B has absolutely continuous norm. Then the linear space of all bounded martingales is dense in $H(B)$.*

Proof. Let X be a martingale in $H(B)$. For each integer $n \geq 1$, put $T_n = \{t \geq 0 \colon |X_t - X_0| > n\}$ and $X^{(n)} = X_t^{T_n} 1_{\{|X_0| \leq n\}}$. Then each $X^{(n)}$ is a bounded martingale, and $B \ni 2X^{\circledast} \geq \left(X^{(n)} - X\right)^{\circledast} \to 0$ a.s. as $n \to \infty$. Since B has absolutely continuous norm, we have $\|X^{(n)} - X\|_{H(B)} \to 0$ as $n \to \infty$. The lemma is proved. $\square$

Theorem 5.12. *Assume that every $(\mathcal{F}_t)$–martingale is continuous. If B has absolutely continuous norm, then $\big(H(B)\big)^* = \mathcal{K}(B')$.*

Proof. Let $Y = (Y_t) \in \mathcal{K}(B')$ and put $J_Y = E[X_\infty Y_\infty]$ for bounded martingales $X = (X_t)$. It is clear from Lemma 5.9 and the continuity of (γ_t) that

$$|J_Y(X)| \leq \left(3 + 2\sqrt{2}\right) \|X\|_{H(B)} \|Y\|_{\mathcal{K}(B')} \, .$$

Since the linear space of bounded martingales is dense in $H(B)$ by the preceding lemma, J_Y extends to a continuous linear functional on $H(B)$: we denote by J_Y again the extension. We have $\|J_Y\| \leq \left(3 + 2\sqrt{2}\right) \|Y\|_{\mathcal{K}(B')}$.

On the other hand, Lemma 5.10 shows that if $J \in \big(H(B)\big)^*$, then there exists a unique martingale $Y \in \mathcal{K}(B')$ such that $J = J_Y$ and $\|Y\|_{\mathcal{K}(B')} \leq 24\,\|J\|$. This completes the proof. $\square$

Remark. In the case where $B = L^1$ or equivalently $H(B) = H^1$, Theorem 5.12 is reduces to the well-known (H^1, BMO)–duality theorem, if we can remove the extra hypothesis of the continuity of $(\mathcal{F}_t)$–martingales. In fact, this is possible, since we may take a positive constant in place of γ in Lemmas 5.9 and 5.10.

§ 3. Martingale paces $\mathcal{K}_p(B)$

In the preceding section we have proved that $\big(H(B)\big)^* = \mathcal{K}(B')$, provided that every $(\mathcal{F}_t)$–martingale is continuous. On the other hand, Doob's inequality applied to the L^p-norm gives that, if $1 < p < \infty$ and $1/p + 1/p' = 1$, then $(H^p)^* = (L^p)^* = L^{p'} = H^{p'}$. Hence it is natural to ask when $H(B')$ and $\mathcal{K}(B')$ coincide. We shall study this problem in more general setting.

Throughout this section we assume that Ω is *non-atomic* and B is *rearrangement invariant.*

Definition 5.13. Let $1 \le p < \infty$. We denote by $\mathcal{K}_p(B)$ the space of uniformly integrable martingales $X = (X_t)$ such that there exists a $\gamma \in B$ satisfying

$$E\big[\,|X_\infty - X_{T-}|^p \,\big|\, \mathcal{F}_T\,\big] \le E[\,\gamma^p \,|\, \mathcal{F}_T\,] \tag{5.23}$$

for all stopping times T. The norm of $X \in \mathcal{K}_p(B)$ is given by

$$\|X\|_{\mathcal{K}_p(B')} = \inf\big\{\|\gamma\|_B : \gamma \text{ satisfies } (5.23)\big\}.$$

It is obvious that $\mathcal{K}(B) = \mathcal{K}_1(B)$. Garsia [21] studied the spaces $\mathcal{K}_2^p$ in detail, i.e., the case where $B = L^p$ and $p = 2$: he proved that if $1 \le p \le 2$, then $(H^p)^* = \mathcal{K}_2^{p'}$, where $1/p + 1/p' = 1$.

Suppose that $X \in H(B)$. Setting $\gamma = 2\,X^{\circledast}$, we have (5.23) and hence $\|X\|_{\mathcal{K}_p(B)} \le 2\,\|X^{\circledast}\|_B = 2\,\|X\|_{H(B)}$. Thus $H(B) \hookrightarrow \mathcal{K}_p(B)$ holds for every finite $p \ge 1$. To obtain more relations between $H(B)$ and $\mathcal{K}_p(B)$, we need some preliminaries.

Lemma 5.14. *Let $p > 1$ and $1/p + 1/p' = 1$. If φ is a nonnegative decreasing function on the interval $I = [\,0,\,1\,]$, then*

$$\left\{\int_0^t \varphi(s)^p\,ds\right\}^{1/p} \le p^{-1}\int_0^t \varphi(s)\,s^{-1/p'}\,ds$$

holds for every $t \in I$.

Proof. Suppose first that φ is of the form

$$\varphi(s) = \sum_{k=0}^{n} a_k\,1_{[\,0,\,t_k[}(s), \qquad s \in I, \tag{5.24}$$

where $a_k \ge 0$ and $0 \le t_0 < t_1 < \cdots < t_n \le 1$. Using Minkowski's inequality we have

$$\begin{aligned}\left\{\int_0^t \varphi(s)^p\,ds\right\}^{1/p} &\le \sum_{k=1}^{n} a_k \left\{\int_0^t 1_{[\,0,\,t_k[}(s)\,ds\right\}^{1/p} = \sum_{k=1}^{n} a_k\,(t \wedge t_k)^{1/p} \\ &= p^{-1}\sum_{k=1}^{n} a_k \int_0^{t\wedge t_k} s^{-1/p'}\,ds = p^{-1}\int_0^t \varphi(s)\,s^{-1/p'}\,ds.\end{aligned}$$

For an arbitrary decreasing function φ, we can find a sequence of functions (φ_n) of the form (5.24) such that $0 \le \varphi_n \uparrow \varphi$ a.e. Hence we obtain the lemma by the monotone convergence theorem. $\square$

Definition 5.15. Let $1 \le p < \infty$ and p' be the exponent conjugate to p. We denote by $\mathcal{P}_p$ and $\mathcal{P}'_p$ the linear operators on the space of functions on I defined by

$$\mathcal{P}_p\,\varphi(t) = t^{-1/p} \int_0^t \varphi(s)\, s^{-1/p'}\, ds\,, \qquad t \in I;$$

$$\mathcal{P}'_p\,\varphi(t) = t^{-1/p} \int_t^1 \varphi(s)\, s^{-1/p'}\, ds\,, \qquad t \in I,$$

provided that the integral exists.

Note that $\mathcal{P}_1 = \mathcal{P}$ and $\mathcal{P}'_\infty = \mathcal{P}'$: see Definition 2.1. The following theorem is due to Shimogaki for $p = 1$, and Boyd for $1 < p < \infty$: for the proof, see [10].

Theorem 5.16 (Shimogaki [51]; Boyd [10]). *Let $\mathfrak{X}$ be a r.i. space over the interval $I = [0, 1]$. Then:*

(i) $\mathcal{P}_p \in \mathcal{L}[\mathfrak{X}]$ *if and only if* $\overline{\alpha}_{\mathfrak{X}} < 1/p$;

(ii) $\mathcal{P}'_p \in \mathcal{L}[\mathfrak{X}]$ *if and only if* $\underline{\alpha}_{\mathfrak{X}} > 1/p$.

Lemma 5.17. *Let $1 \le p < \infty$ and B be a r.i. space such that $0 < \underline{\alpha}_B \le \overline{\alpha}_B < 1/p$. If $f \in L^1$, $g \in B$ are nonnegative and*

$$(5.25) \qquad (\lambda_1 - \lambda_2)^p\, P\{f \ge \lambda_1\} \le E\big[\, g^p\, 1_{\{f > \lambda_2\}}\big]$$

holds for all λ_1 and λ_2 with $\lambda_1 \ge \lambda_2 \ge 0$, then $f \in B$ and $\|f\|_B \le C_{B,p}\, \|g\|_B$, where $C_{B,p} > 0$ is a constant depending only on p and the space B.

Proof. Note that if f satisfies (5.25), then $f \wedge n$ satisfies (5.25) for all integers $n \ge 1$. Thus we may assume that f is bounded and hence that $f \in B$. Let $0 < t \le 1 < s$, $\lambda_1 = f^*(t)$, and $\lambda_2 = D_s f^*(t)$. Then $\lambda_1 \ge \lambda_2$ and (5.25) gives that

$$\begin{aligned}(f^*(t) - D_s f^*(t))^p P\{f \ge f^*(t)\} \\ \le E\big[\, g^p\, 1_{\{f > D_s f^*(t)\}}\big] \le \int_0^{P\{f > D_s f^*(t)\}} (g^*(u))^p\, du.\end{aligned}$$

Since $P\{f \geq f^*(t)\} \geq t$ and $P\{f > D_s f^*(t)\} \leq st \wedge 1$, we have

$$
\begin{aligned}
f^*(t) &\leq D_s f^*(t) + \left\{ \frac{1}{t} \int_0^{st\wedge 1} (g^*(u))^p \, du \right\}^{1/p} \\
&\leq D_s f^*(t) + \left\{ \frac{1}{t} \int_0^{t} (g^*(u))^p \, du \right\}^{1/p} + \left\{ \frac{1}{t} \int_t^{st\wedge 1} (g^*(u))^p \, du \right\}^{1/p} \\
&\leq D_s f^*(t) + p^{-1} t^{-1/p} \int_0^t g^*(u) u^{-1/p'} \, du + (s-1)^{1/p} g^*(t) \\
&\leq D_s f^*(t) + p^{-1} \mathcal{P}_p \, g^*(t) + (s-1)^{1/p} g^*(t),
\end{aligned}
$$

where the third inequality follows from Lemma 5.14. Let $\hat{B}$ be a r.i. space over I satisfying (1.10). As $\overline{\alpha}_{\hat{B}} = \overline{\alpha}_B < 1/p$, we have $\mathcal{P}_p \in \mathfrak{L}[\hat{B}]$ by Theorem 5.16. It then follows that

$$
\|f\|_B = \|f^*\|_{\hat{B}} \leq \|D_s\|_{\mathfrak{L}[\hat{B}]} \|f\|_B + p^{-1} \|\mathcal{P}_p\|_{\mathfrak{L}[\hat{B}]} \|g\|_B + (s-1)^{1/p} \|g\|_B \, .
$$

On the other hand, since $\underline{\alpha}_{\hat{B}} = \underline{\alpha}_B > 0$, we have $\log \|D_s\|_{\mathfrak{L}[\hat{B}]} < \alpha \log(1/s)$ or equivalently $\|D_s\|_{\mathfrak{L}[\hat{B}]} \leq (1/s)^\alpha$ for sufficiently large $s > 1$, where $\alpha > 0$ is a constant. Hence $\|D_s\|_{\mathfrak{L}[\hat{B}]} \to 0$ as $s \to \infty$. Choose an $s > 1$ such that $\|D_s\|_{\mathfrak{L}[\hat{B}]} < 1$: then we have

$$
\|f\|_B \leq \frac{1}{1 - \|D_s\|_{\mathfrak{L}[\hat{B}]}} \left\{ p^{-1} \|\mathcal{P}_p\|_{\mathfrak{L}[\hat{B}]} + (s-1)^{1/p} \right\} \|g\|_B \, .
$$

This completes the proof. □

The following result is an extension of Stroock's lemma (Lemma 1.1 in [53]).

Lemma 5.18. *Let B be a r.i. space satisfying $0 < \underline{\alpha}_B \leq \overline{\alpha}_B < 1/p$, where $1 \leq p < \infty$, and $X = (X_t)_{t \geq 0}$ be an adapted càdlàg process. If $\gamma \in B$ and*

$$
E\left[\, |X_T - X_{S-}|^p \,\middle|\, \mathcal{F}_S \right] \leq E[\, \gamma^p \,|\, \mathcal{F}_S\,], \tag{5.26}
$$

holds for all stopping times S and T such that $S \leq T$, then $X^* \in B$ and $\|X\|_{\mathfrak{R}(B)} = \|X^*\|_B \leq C_{B,p} \|\gamma\|_B$.

Proof. Let $\lambda_1 > \lambda_2 > 0$, and S and T be the stopping times defined by $S = \inf\{t \geq 0 \colon |X_t| > \lambda_2\}$ and $T = \inf\{t \geq 0 \colon |X_t| > \lambda_1\}$, respectively.

Then $|X_T| \geq \lambda_1$ a.s. on $\{T < \infty\}$, $|X_{S-}| \leq \lambda_2$ and $S \leq T$ a.s. on Ω. Therefore we have $(\lambda_1 - \lambda_2)^p \leq |X_T - X_{S-}|^p$ on $\{T < \infty\}$ and

$$\begin{aligned}(\lambda_1 - \lambda_2)^p P\{X^{\circledast} > \lambda_1\} &\leq E\big[|X_T - X_{S-}|^p 1_{\{T<\infty\}}\big] \\ &\leq E\big[|X_T - X_{S-}|^p 1_{\{S<\infty\}}\big] \\ &\leq E[\gamma^p 1_{\{S<\infty\}}] = E\big[\gamma^p 1_{\{X^{\circledast}>\lambda_2\}}\big].\end{aligned}$$

Thus the statement follows from Lemma 5.17. □

Now we can prove the following theorem.

Theorem 5.19. *If $1 \leq p < \infty$ and B is a r.i. space such that $0 < \underline{\alpha}_B \leq \overline{\alpha}_B < 1/p$, then $\mathcal{K}_p(B)$ and $H(B)$ coincide and have equivalent norms.*

Proof. As remarked before, we have always $H(B) \hookrightarrow \mathcal{K}_p(B)$. Let $X \in \mathcal{K}_p(B)$ and suppose that $\gamma \in B$ satisfies (5.23). Then by the preceding lemma, we have $X^{\circledast} \in B$ and $\|X\|_{H(B)} = \|X\|_{\mathfrak{R}(B)} \leq C_{B,p} \|\gamma\|_B$. Taking the infimum over all such γ, we obtain $\|X\|_{H(B)} \leq C_{B,p} \|X\|_{\mathcal{K}_p(B)}$ i.e., $\mathcal{K}_p(B) \hookrightarrow H(B)$. The theorem is proved. □

By Theorem 5.19 we have $\mathcal{K}(B) = H(B)$, if $0 < \underline{\alpha}_B \leq \overline{\alpha}_B < 1$. Furthermore, as the following theorem shows, the converse is also true. The rest of this paper is devoted to the proof of this result.

Theorem 5.20. *There exists a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which has the following properties: if B is a r.i. space and $\mathcal{K}(B) \hookrightarrow H(B)$, then $0 < \underline{\alpha}_B \leq \overline{\alpha}_B < 1$.*

To prove the above theorem, we shall use an excellent theorem due to Azéma and Yor [3, 4]. Let $\varphi \in L^1(I)$ be a decreasing function such that $\int_0^1 \varphi(s)\, ds = 0$, and $\psi_\varphi(s)$ be the function defined on $\mathbb{R}$ by

$$\psi_\varphi(s) = \begin{cases} \dfrac{1}{m(\varphi \geq s)} \displaystyle\int_{\{\varphi \geq s\}} \varphi(t)\, dt, & \text{if } m(\varphi \geq s) > 0, \\ s, & \text{otherwise,} \end{cases}$$

where m denotes the Lebesgue measure.

Theorem 5.21 (Azéma and Yor [3, 4]). *Let $X = (X_t)$ be a continuous martingale with $X_0 = 0$, φ as above, and T be the stopping time defined by*

$$T = \inf\{t \geq 0 : S_t \geq \psi_\varphi(X_t)\}, \quad \textit{where } S_t = \sup_{s \leq t} X_s.$$

Then we have:

(i) *the stopped martingale $(X_{T\wedge t})$ is uniformly integrable;*

(ii) *X_T and φ have the same distribution;*

(iii) *S_T and $\mathcal{P}\varphi$ have the same distribution.*

Proof of Theorem 5.20. Let (B_t) be a one dimensional standard Brownian motion on a probability space $(\Omega', \mathcal{F}', P')$, and $(\mathcal{F}'_t)$ be the filtration generated by B. Besides this, we take the probability space $I = [0, 1]$ with the σ-field $\mathcal{G}$ of Lebesgue measurable subsets of I and the Lebesgue measure m. For each $t \in I$, let $\mathcal{G}_t$ denote the sub-σ-field of $\mathcal{G}$ generated by the Lebesgue measurable subsets of $[1 - t, 1]$, and put $\mathcal{G}_t = \mathcal{G}_1$ for $t \geq 1$. We set

$$\Omega = \Omega' \times I, \quad \mathcal{F}^\circ_t = \mathcal{F}'_t \otimes \mathcal{G}_t \ (0 \leq t < \infty), \text{ and } P = P' \otimes m.$$

Let $(\mathcal{F}_t)$ be the augmentation of $(\mathcal{F}^\circ_t)$: $\mathcal{F}_t = \cap_{s>t}\sigma(\mathcal{F}_s \cup \mathcal{N})$, where $\mathcal{N}$ denotes the collection of all P-negligible sets. Then $(\mathcal{F}_t)$ satisfies the usual hypotheses. We prove now that, if $\mathcal{K}(B) \hookrightarrow H(B)$, then $0 < \underline{\alpha}_B \leq \overline{\alpha}_B < 1$. Assume that $\mathcal{K}(B) \hookrightarrow H(B)$ with an embedding constant C. Let $X = (X_t)$ be the process on Ω defined by

$$X_t(\omega', s) = B_{T\wedge t}(\omega'), \quad (\omega', s) \in \Omega = \Omega' \times I.$$

Then X is a continuous $((\mathcal{F}_t), P)$-martingale.

Let $\hat{B}$ be a r.i. space over I satisfying (1.10), and $\varphi \in \hat{B}$ be a decreasing function such that $\int_0^t \varphi(s)\, ds = 0$. Define the stopping time T and the process $S = (S_t)$ as in Theorem 5.21. Since the inequality

$$E[\,|X^T_\infty - X^T_\tau|\,|\,\mathcal{F}_\tau] \leq 2\,E[\,|X_T|\,|\,\mathcal{F}_\tau\,]$$

holds for all stopping times τ, we have $\|X^T\|_{\mathcal{K}(B)} \leq 2\,\|X_T\|_B$, where X^T denotes the stopped martingale $(X_{T\wedge t})$ as usual. This implies the inequality $\|S_T\|_B \leq 2C\,\|X_T\|_B$, since we have assumed $H(B) \hookrightarrow \mathcal{K}(B)$. According to

Theorem 5.21, S_T (resp. X_T) and $\mathcal{P}\varphi$ (resp. φ) have the same distribution. So we have $\|S_T\|_B = \|\mathcal{P}\varphi\|_{\hat{B}}$ (resp. $\|X_T\|_B = \|\varphi\|_{\hat{B}}$) and hence

$$\|\mathcal{P}\varphi\|_{\hat{B}} \le 2C\,\|\varphi\|_{\hat{B}}\,.$$

Now let $\varphi \in \hat{B}$ be an arbitrary decreasing function. Then the above inequality is rewritten as

$$\|\mathcal{P}\varphi - \varphi_I\|_{\hat{B}} \le 2C\,\|\,\varphi - \varphi_I\|_{\hat{B}}\,,$$

where φ_I denotes the integral of φ over I. As $\hat{B} \hookrightarrow L^1(I)$, we have

$$\|\mathcal{P}\varphi\|_{\hat{B}} \le 2C\,\|\varphi\|_{\hat{B}} + (2C+1)\,\|\varphi\|_1\,\|1\|_{\hat{B}} \le K\,\|\varphi\|_{\hat{B}}\,,$$

where K is a constant depending only on $\hat{B}$ and the embedding constant C. Thus $\|\mathcal{P}\varphi\|_{\hat{B}} \le 2C\,\|\varphi\|_{\hat{B}}$ holds for every decreasing function $\varphi \in \hat{B}$. Note that this remains valid for an arbitrary function $\varphi \in \hat{B}$. In fact, Hardy's inequality implies that $\mathcal{P}\varphi(t) \le \mathcal{P}\varphi^*(t)$ (cf. (1.3)) and hence $\|\mathcal{P}\varphi\|_{\hat{B}} \le \|\mathcal{P}\varphi^*\|_{\hat{B}} \le 2C\,\|\varphi^*\|_{\hat{B}} = 2C\,\|\varphi\|_{\hat{B}}$. According to Theorem 5.16 (a), we have $\overline{\alpha}_B = \overline{\alpha}_{\hat{B}} < 1$.

Finally, we prove that $\underline{\alpha}_B = \underline{\alpha}_{\hat{B}} > 0$ or equivalently that $\mathcal{P}' \in \mathfrak{L}[\hat{B}]$. Let $\varphi \in L^1(I)$, and $X = (X_t)$ be a process defined by

$$X_t\,(\omega', s) = \begin{cases} \mathcal{P}\varphi(1-t), & \text{if } s < 1 - t \wedge 1, \\ \varphi(s), & \text{if } 1 - t \wedge 1 \le s \le 1. \end{cases}$$

Then X is a uniformly integrable martingale with respect to $(\mathcal{F}_t)$ and P, and we have $X^{\circledast}_\infty(\omega', s) \ge |\mathcal{P}\varphi(s)|$. Note that almost every path of X is of bounded variation; hence $X^c = 0$. If we put $T(\omega', s) = 1 - s$ for each $(\omega', s) \in \Omega$, then T is an $(\mathcal{F}_t)$–stopping time, and the each path of X jumps at T only. This implies that

$$[X,\, X]^{1/2}_\infty(\omega', s) \le |\Delta X_T(\omega', s)| + |X_0| = |\varphi(s) - \mathcal{P}\varphi(s)| + \|\varphi\|_1\,,$$

and therefore that $\big\|[X,\, X]^{1/2}_\infty\big\|_B \le \|\varphi - \mathcal{P}\varphi\|_{\hat{B}} + \|\varphi\|_1$. On the other hand, we have

$$\begin{aligned} E\big[\,|X_\infty - X_{\tau-}|\,\big|\,\mathcal{F}_\tau\,\big] &\le E\big[\sup_{t\ge\tau}|X_t - X_{\tau-}|\,\big|\,\mathcal{F}_\tau\,\big] \\ &\le KE\big[\big([X,\, X]_\infty - [X,\, X]_{\tau-}\big)^{1/2}\,\big|\,\mathcal{F}_\tau\,\big] \\ &\le KE\big[\,[X,\, X]^{1/2}_\infty\,\big|\,\mathcal{F}_\tau\,\big] \end{aligned}$$

for every stopping time τ, where the middle inequality follows from the conditional form of Davis's inequality. Hence we have

$$\|X\|_{\mathcal{K}(B)} \le K \left\| [X, X]_\infty^{1/2} \right\|_B \le K\big(\|\varphi - \mathcal{P}\varphi\|_{\hat{B}} + \|\varphi\|_1 \big).$$

By assumption $\mathcal{K}(B) \hookrightarrow H(B)$, we have

$$\|\mathcal{P}\varphi\|_{\hat{B}} \le \|X\|_{H(B)} \le C\,\|X\|_{\mathcal{K}(B)} \le CK\big(\|\varphi - \mathcal{P}\varphi\|_{\hat{B}} + \|\varphi\|_1 \big).$$

Now let $\psi \in L^1(I)$ be arbitrary, and set $\varphi = \mathcal{P}'\psi^* - \psi^*$. Then we have $\mathcal{P}\varphi = (\mathcal{P}\psi^* + \mathcal{P}'\psi^*) - \mathcal{P}\psi^* = \mathcal{P}'\psi^* = \varphi + \psi^*$, because $\mathcal{P}\mathcal{P}' = \mathcal{P} + \mathcal{P}'$. From the above inequality, we obtain

$$\|\mathcal{P}'\psi^*\|_{\hat{B}} = \|\mathcal{P}\varphi\|_{\hat{B}} \le CK\big(\|\psi^*\|_{\hat{B}} + \|\varphi\|_1 \big).$$

Since $\|\varphi\|_1 \le \|\mathcal{P}'\psi^*\|_1 + \|\psi^*\|_1 = 2\,\|\psi^*\|_1$ and $\hat{B} \hookrightarrow L^1(I)$, we get

$$\|\mathcal{P}'\psi^*\|_{\hat{B}} \le C'\,\|\psi^*\|_{\hat{B}} = \|\psi\|_{\hat{B}}\,.$$

If we can prove $\|\mathcal{P}'\psi\|_{\hat{B}} \le \|\mathcal{P}'\psi^*\|_{\hat{B}}$, then it will follow that $\mathcal{P}' \in \mathcal{L}[\hat{B}]$ and the proof will be complete. We prove this inequality.

Note that $\mathcal{P}'\psi \le \mathcal{P}'|\psi|$ on I. So we may assume that ψ is nonnegative: it follows that $\mathcal{P}'\psi(s)$ is decreasing in s. Let $t \in I$. As the function $(u \wedge t)/u$ is decreasing in u, Hardy's inequality (1.2) (applied to the case where $\Omega = I$) gives that

$$\int_0^t \mathcal{P}'\psi(s)\,ds = \int_0^1 \frac{\psi(u)}{u}\,(u \wedge t)\,du \le \int_0^1 \frac{\psi^*(u)}{u}\,(u \wedge t)\,du = \int_0^t \mathcal{P}'\psi^*(s)\,ds,$$

which means that $\mathcal{P}'\psi \prec \mathcal{P}'\psi^*$. This implies $\|\mathcal{P}'\psi\|_{\hat{B}} \le \|\mathcal{P}'\psi^*\|_{\hat{B}}$. The theorem is proved completely. $\square$

Appendix

We shall give here an alternative proof of Theorem 4.11 (cf. [33]).

Step 1. *If $\mathcal{G}_n \xrightarrow{\perp} \mathcal{G}$, each $f_n \in L^\infty$ is $\mathcal{G}_n$-measurable and $\sup_n \|f_n\|_\infty < \infty$, then $f_n - E[\,f_n \,|\, \mathcal{G}\,] \to 0$ weakly in L^2.*

We may assume that $\sup_n \|f_n\|_\infty \leq 1$. For each pair of integers $n \geq 1$ and $k \geq 1$, define the function $f_{n,k}$ by

$$f_{n,k} = \sum_{j=-k}^{k} \frac{j}{k} 1_{\{\frac{j}{k} \leq f_n < \frac{j+1}{k}\}}.$$

Obviously each $f_{n,k}$ is a simple $\mathcal{G}_n$-measurable function and $\|f_{n,k} - f_n\|_\infty \leq 1/k$ for all $n \geq 1$. Since $\mathcal{G}_n \xrightarrow{\perp} \mathcal{G}$, we see that $f_{n,k} - E[\,f_{n,k} \,|\, \mathcal{G}\,] \to 0$ weakly in L^2 as $n \to \infty$. For every $g \in L^2$, we have

$$\begin{aligned}
&\big|E\big[(\,f_n - E[\,f_n \,|\, \mathcal{G}\,]\,)\,g\,\big]\big| \\
&\quad\leq \big|E\big[\,(f_n - f_{n,k})\,g\,\big]\big| + \big|E\big[\,\big(\,f_{n,k} - E[\,f_{n,k} \,|\, \mathcal{G}\,]\,\big)\,g\,\big]\big| \\
&\qquad\qquad + \big|E\big[E[\,f_{n,k} - f_n \,|\, \mathcal{G}\,]\,g\,\big]\big| \\
&\quad\leq 2\,\|f_{n,k} - f_n\|_2\,\|g\|_2 + \big|E\big[\,\big(\,f_{n,k} - E[\,f_{n,k} \,|\, \mathcal{G}\,]\,\big)\,g\,\big]\big|.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\varlimsup_{n\to\infty} \big|E\big[(\,f_n - E[\,f_n \,|\, \mathcal{G}\,]\,)\,g\,\big] \leq 2\,\|g\|_2\,/k,$$

for any $k \geq 1$, and hence the assertion.

Step 2. *If $\mathcal{G}_n \xrightarrow{P} \mathcal{G}$ and $\mathcal{G}_n \xrightarrow{\perp} \mathcal{G}$, then $E[\,f \,|\, \mathcal{G}_n\,] \to E[\,f \,|\, \mathcal{G}\,]$ weakly in L^2 for every $f \in L^\infty$.*

If $f \in L^\infty$, then $\{E[\,f \,|\, \mathcal{G}_n\,]\}_{n=1}^\infty$ is bounded in L^2 and hence there exists a subsequence $\{E[\,f \,|\, \mathcal{G}_{n_k}\,]\}$ which converges weakly in L^2; let h be the weak limit of $E[\,f \,|\, \mathcal{G}_{n_k}\,]$. Step 1 shows that $E[\,f \,|\, \mathcal{G}_{n_k}\,] - E\big[\,E[\,f \,|\, \mathcal{G}_{n_k}\,] \,\big|\, \mathcal{G}\,\big] \to 0$ weakly in L^2. It then follows that for every $A \in \mathcal{F}$,

$$\begin{aligned}
E[\,h\,1_A\,] &= \lim_{k\to\infty} E\big[\,E[\,f \,|\, \mathcal{G}_{n_k}\,]\,1_A\,\big] = \lim_{k\to\infty} E\Big[\,E\big[\,E[\,f \,|\, \mathcal{G}_{n_k}\,] \,\big|\, \mathcal{G}\,\big]\,1_A\Big] \\
&= \lim_{k\to\infty} E\big[\,E[\,1_A \,|\, \mathcal{G}\,]\,E[\,f \,|\, \mathcal{G}_{n_k}\,]\,\big] = E\big[\,E[\,1_A \,|\, \mathcal{G}\,]\,h\,\big] \\
&= E\big[\,E[\,h \,|\, \mathcal{G}\,]\,1_A\,\big],
\end{aligned}$$

and therefore $h = E[h \mid \mathcal{G}]$ is $\mathcal{G}$-measurable. Now let $B \in \mathcal{G}$ and choose a sequence $\{B_n\}$ so that $B_n \in \mathcal{G}_n$ for all $n \geq 1$ and $P(B \triangle B_n) \to 0$. This is possible, since $\mathcal{G}_n \xrightarrow{P} \mathcal{G}$. Then we have

$$\begin{aligned}
&\left|E[f 1_{B_{n_k}}] - E[h 1_B]\right| \\
&\qquad \leq \left|E\left[E[f \mid \mathcal{F}_{n_k}] 1_{B_{n_k} \triangle B}\right]\right| + \left|E\left[\left(E[f \mid \mathcal{F}_{n_k}] - h\right) 1_B\right]\right| \\
&\qquad \leq \|f\|_\infty \mathcal{P}(B \triangle B_{n_k}) + \left|E\left[\left(E[f \mid \mathcal{F}_{n_k}] - h\right) 1_B\right]\right|,
\end{aligned}$$

the right-hand side of which tends to zero as $k \to \infty$. Thus we get $E[f 1_B] = \lim_{k\to\infty} E[f 1_{B_{n_k}}] = E[h 1_B]$, and hence $E[f \mid \mathcal{G}] = h$; in other words,

$$E[f \mid \mathcal{G}_{n_k}] \to E[f \mid \mathcal{G}] \quad \text{weakly in } L^2.$$

Now we prove that $E[f \mid \mathcal{G}_n] \to E[f \mid \mathcal{G}]$ weakly in L^2; suppose conversely that this is false. Then there is a $g \in L^2$ and an $\varepsilon > 0$ such that

$$\varlimsup_{n\to\infty} \left|E\left[\left(E[f \mid \mathcal{G}_n] - E[f \mid \mathcal{G}]\right) g\right]\right| > \varepsilon.$$

Select a subsequence $\{n_j\}$ so that

$$\left|E\left[\left(E[f \mid \mathcal{G}_{n_j}] - E[f \mid \mathcal{G}]\right) g\right]\right| > \varepsilon$$

for all $j \geq 1$. Since $\mathcal{G}_{n_j} \xrightarrow{P} \mathcal{G}$ and $\mathcal{G}_{n_j} \xrightarrow{\perp} \mathcal{G}$, there exists a subsequence $\{m_k\}$ of $\{n_j\}$ such that $E[f \mid \mathcal{G}_{m_k}] \to E[f \mid \mathcal{G}]$ weakly in L^2 by what we have proved above. This contradicts the choice of $\{n_j\}$, and the assertion of Step 2 is proved.

Step 3. *If $E[f \mid \mathcal{G}_n] \to E[f \mid \mathcal{G}]$ weakly in L^2 for every $f \in L^\infty$, then the convergence holds in L^2.*

Suppose $f \in L^\infty$. Since $E[f \mid \mathcal{G}_n] \to E[f \mid \mathcal{G}]$ and $E\left[E[f \mid \mathcal{G}] \mid \mathcal{G}_n\right] \to E[f \mid \mathcal{G}]$ weakly in L^2, we have $E\left[E[f \mid \mathcal{G}] \mid \mathcal{G}_n\right] - E[f \mid \mathcal{G}_n] \to 0$ weakly in L^2. Hence we have

$$\begin{aligned}
&\left\|E[f \mid \mathcal{G}_n] - E[f \mid \mathcal{G}]\right\|_2^2 \\
&\qquad = \left\langle f, E[f \mid \mathcal{G}_n] - E\left[E[f \mid \mathcal{G}] \mid \mathcal{G}_n\right]\right\rangle \\
&\qquad\qquad + \langle E[f \mid \mathcal{G}], E[f \mid \mathcal{G}] - E[f \mid \mathcal{G}_n]\rangle \to 0,
\end{aligned}$$

where $\langle \cdot\, , \cdot \rangle$ denotes the inner product of L^2.

Step 4. *If* $E[\,f\,|\,\mathcal{G}_n\,] \to E[\,f\,|\,\mathcal{G}\,]$ *in* L^2 *for every* $f \in L^\infty$*, then* $E[\,g\,|\,\mathcal{G}_n\,] \to E[\,g\,|\,\mathcal{G}\,]$ *in* L^p *for every* $g \in L^p$*, where* $1 \le p < \infty$*.*

For simplicity, we write $f_n = E[\,f\,|\,\mathcal{G}_n\,]$ and $f_\infty = E[\,f\,|\,\mathcal{G}\,]$ for $f \in L^\infty$. Since $f_n \to f_\infty$ in L^2, we have $f_n \to f_\infty$ in probability. On the other hand, for any $\varepsilon > 0$, we have

$$\begin{aligned}\|f_n - f_\infty\|_p &= \big\|(f_n - f_\infty)\,1_{\{|f_n - f_\infty| > \varepsilon\}}\big\|_p + \big\|(f_n - f_\infty)\,1_{\{|f_n - f_\infty| \le \varepsilon\}}\big\|_p \\ &\le 2\,\|f\|_\infty\,\big\|1_{\{|f_n - f_\infty| > \varepsilon\}}\big\|_p + \varepsilon.\end{aligned}$$

Letting $n \to \infty$ and then $\varepsilon \to 0$, we see that $f_n \to f_\infty$ in L^p.

Now let $g \in L^p$ and choose $g_k \in L^\infty$ such that $0 \le |g_k| \le |g|$ and $g_k \to g$ in L^p. We have

$$\begin{aligned}&\big\|E[\,g\,|\,\mathcal{G}_n\,] - E[\,g\,|\,\mathcal{G}\,]\,\big\|_p \\ &\quad\le \big\|E[\,g - g_k\,|\,\mathcal{G}_n\,]\,\big\|_p + \big\|E[\,g_k\,|\,\mathcal{G}_n\,] - E[\,g_k\,|\,\mathcal{G}\,]\,\big\|_p + \big\|E[\,g_k - g\,|\,\mathcal{G}\,]\big\|_p \\ &\quad\le 2\,\|g - g_k\|_p + \big\|E[\,g_k\,|\,\mathcal{G}_n\,] - E[\,g_k\,|\,\mathcal{G}\,]\,\big\|_p.\end{aligned}$$

By what has been proved above, the last term on the right-hand side tends to zero as $n \to \infty$ for each $k \ge 1$. Hence, letting $n \to \infty$ and then $k \to \infty$, we obtain the assertion.

Starting from Step 1, we have proved that (ii) of Theorem 4.11 implies (i) of Theorem 4.11. Now we prove the converse.

Step 5. *If* $E[\,f\,|\,\mathcal{G}_n\,] \to E[\,f\,|\,\mathcal{G}\,]$ *in* L^1 *for every* $f \in L^1$*, then* $\mathcal{G}_n \xrightarrow{P} \mathcal{G}$ *and* $\mathcal{G}_n \xrightarrow{\perp} \mathcal{G}$*.*

Let $A \in \mathcal{G}$ be arbitrary and set $A_n = \{E[\,1_A\,|\,\mathcal{G}_n\,] > 1/2\}$; then $A_n \in \mathcal{G}_n$ for each $n \ge 1$. Since $E[\,1_A\,|\,\mathcal{G}_n\,] \to 1_A$ in probability, we have

$$P(A_n \triangle A\,) \le P\big\{|E[\,1_A\,|\,\mathcal{G}_n\,] - 1_A| \ge 1/2\big\} \to 0$$

as $n \to \infty$, and therefore $\mathcal{G}_n \xrightarrow{P} \mathcal{G}$.

Now suppose that $A_n \in \mathcal{G}_n$ for all $n \ge 1$. If $f \in L^2$, then

$$\begin{aligned}\big|E\big[\big(1_{A_n} - E[\,1_{A_n}\,|\,\mathcal{G}\,]\big)f\,\big]\big| &= \big|E\big[\big(E[\,f\,|\,\mathcal{G}_n\,] - E[\,f\,|\,\mathcal{G}\,]\big)\,1_{A_n}\,\big]\big| \\ &\le \big\|E[\,f\,|\,\mathcal{G}_n\,] - E[\,f\,|\,\mathcal{G}\,]\,\big\|_1 \to 0,\end{aligned}$$

which means that $\mathcal{G}_n \xrightarrow{\perp} \mathcal{G}$.

Thus the equivalence between (i) and (ii) of Theorem 4.11 is established.

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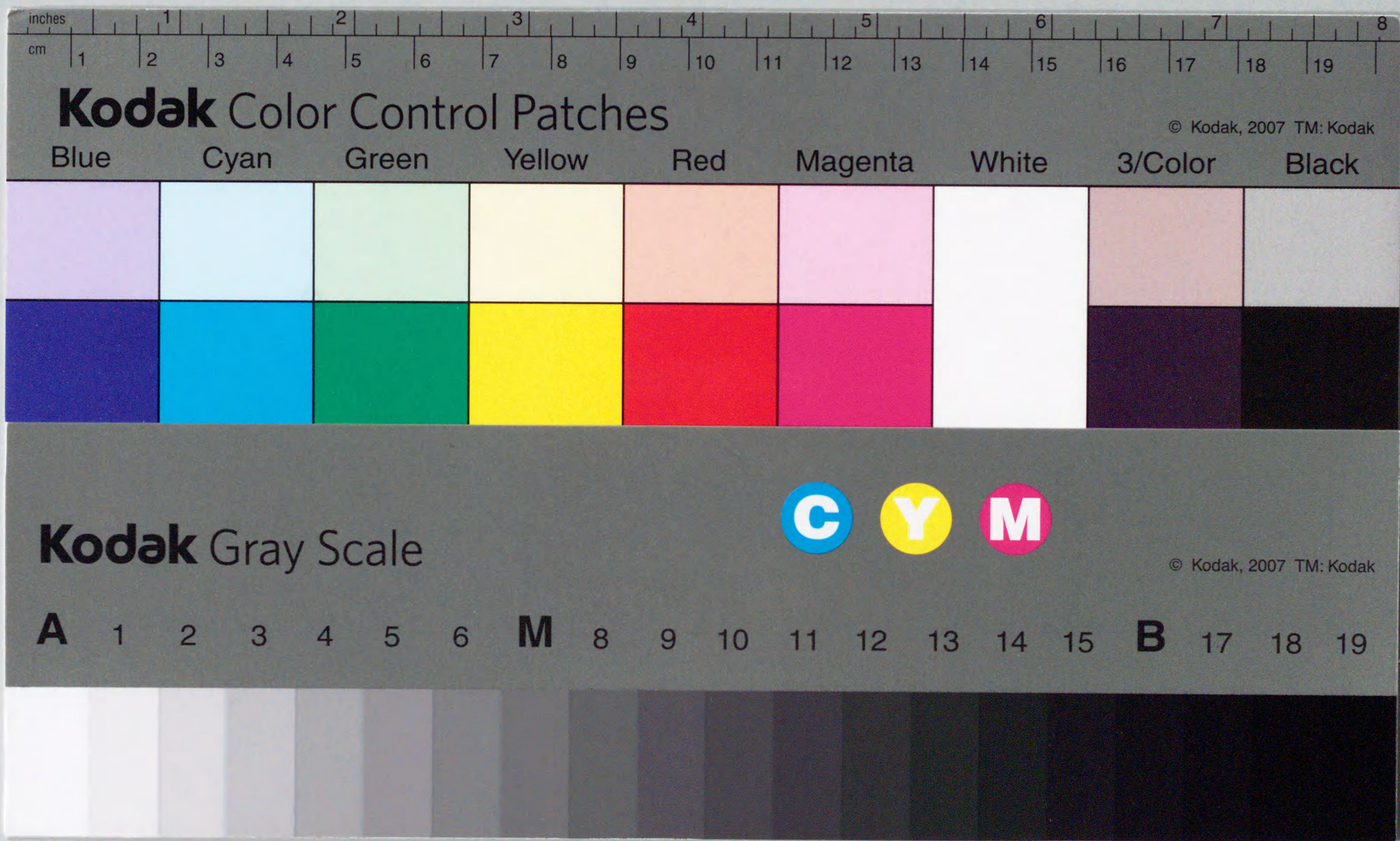
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