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論文目次

論 文 内 容 要 旨

The topic of this dissertation is global and local noncomputability for effectively closed sets (often called Π^0_1 classes) in certain topological spaces. A subset F of a topological space X is called Π^0_1 if $F=\{x\in X \mid f(x)=0\}$ for some computable function $f:X\to R$. From 1940's to 50's, Kleene, Kreisel, Lacombe, and Shoenfield investigated Basis Theorems and Nonbasis Theorems for Π^0_1 classes in Cantor space and in Baire space. The Non-basis Theorem states the existence of a Π^0_1 class in Cantor space which contains no computable points. While Turing's halting problem implies the existence of Π^0_1 set which is not computable globally, the Nonbasis Theorem suggests that there is a Π^0_1 class which is even locally noncomputable. In 1955, Medvedev introduced degrees of noncomputability for subsets of Baire space. For subsets X,Y of Baire space, X is Medvedev reducible to Y if there is a computable function from Y to X. The main theorem in Chapter 2 is following.

Theorem 1. There is a mechanical procedure that determines whether a given $\forall\exists$-sentence is true or not in the Π^0_1 Medvedev degrees in Cantor space.

When exploring computability in Analysis and Topology, one naturally encounters connected Π^0_1 classes. For instance, Penrose, an astrophysicist, was interested in computability of the Mandelbrot set, which is a simply connected planar Π^0_1 class, as shown by Penrose himself. Le Roux and Ziegler asked whether every simply connected planar Π^0_1 class contains a computable point. In Chapter 3, we solve this problem.

Theorem 2. There is a simply connected planar Π^0_1 set which contains no computable points.

In Chapter 4, we also investigate the global computability of planar continua, specifically, dendrites, and

dendroids. Let d_H (A, B) denote the Hausdorff distance between nonempty closed sets A and B. Let P be a class of continua. We say that a continuum A closely-includes a member of P if $\inf\{d\_H(A,B) : A \supset B \in P\} = 0$.

Theorem 3. (1) There is a planar Π^0_1 set such that it is simply connected and locally simply connected, but does not closely-include a connected computable closed set.

(2) There is a simply connected computable closed planar set which does not closely-include a connected locally connected Π^0_1 set.

A set is K-trivial if it has the trivial Kolmogorov complexity. In Chapter 5, we show the following.

Theorem 4. (1) The Medvedev degrees of K-trivial Π^0_1 classes are dense.

(2) The Medvedev degrees of K-trivial Π^0_1 classes are bounded by a Π^0_1 class generated by a c.e. incomplete Boolean algebra.

As it happens, K-triviality has been studied as an opposite notion of randomness (i.e., being incompressible). Moreover, it is well-known that the set of all incompressible strings (in the sense of Kolmogorov complexity) is immune, i.e., highly noncomputable. In Chapter 6, we investigate some notions of immunity for closed sets, in particular the central notion of tree-immunity. A Π^0_1 class is tree-immune if its corresponding Π^0_1 tree contains no computable tree. One of the main theorem in Chapter 5 is following:

Theorem 5. The Medvedev degrees of all tree-immune-free Π^0_1 classes form a principal prime ideal in the Π^0_1 Medvedev degrees.

Finally, when considering tree-immunity in this context, we encounter connections to the subject Learning Theory and Intuitionistic Mathematics. In particular, our method for proving theorems concerning tree-immunity gives rise to a "disjunction" operator under the limit-BHK interpretation of Limit Computable Mathematics. This allows us to define, in Chapter 7, the limit-BHK disjunction and the Popperian disjunction $\bigtriangledown$ as operations on the power set of Baire space. This representation enables us to compare degrees of difficulty of disjunctive notions. One of the important theorems in Chapter 7 is the following:

Theorem 6. If an undecidable proposition P can be represented as a Π^0_1 subset of Cantor space, then P is not intuitionistically derivable from the conjunction $(P \bigtriangledown P) \wedge R$ for any proposition R which does not derive P intuitionistically.

Conversely, these disjunctive notions turn out to be useful for analyzing the Medvedev lattice of Π^0_1 subsets of Cantor space. By iterating the Popperian disjunction along the first non-computable ordinal, we have the following:

Theorem 7. A transfinite non-cupping sequence of Π^0_1 Medvedev degrees exists in any one degree-spectrum.

論文審査の結果の要旨

本博士論文は，ユークリッド空間上のコンパクトな実効的閉集合の計算不可能性構造を様々な視点から研究したものである．その内容は，大きく3つに分類される．

第1は実効的閉集合全体がつくる非可解性の次数による束構造の研究である．計算枚挙可能集合の次数構造の研究は計算可能性理論の中心テーマの1つであり，その発展としての実効的閉集合の次数の束構造は2000年代以降盛んに研究されてきた．木原は，コンパクトな実効的閉集合のMedvedev束において，本質的に2つの量化記号しか含まない文に関して，その真偽が決定可能であることを示した．Shaferによる未発表の結果と合わせると，これによってMedvedev束に関する問題は完全に解けたことになる．また，K-trivialな閉集合の稠密性や上界に関するなどMedvedev束の部分構造に関する幾つかの結果も得ている．特に，木免疫不在コンパクト実効的集合はMedvedev束の単項素イデアルを形成することを示し，Cenzer-Weber-Wuの問題を解決した．

第2は実効的閉集合の位相的性質と計算不可能性についてである．実効的閉集合が計算可能な点を持つかどうかということは，ある種の数学的問題に対して，その解を得るアルゴリズムがあるかということに大きく関わっている．そのため実効的閉集合の点の複雑さに関する研究は古くから行われてきたが，近年になると，点の計算可能性に及ぼす位相的性質に注目しはじめた．その中で，Le Roux-ZieglerはMandelbrot 集合のような平面上のコンパクト単連結実効的閉集合が，常に計算可能な点を持つかと問うた．木原は計算可能な点を持たないデンドロイド集合を構成することにより，これを否定的に解いた．また一方で，局所単連結な平面上のコンパクト単連結実効的閉集合で，計算可能閉部分集合で近似できないものが存在することも示した．

第3は構成的証明や学習理論の観点を実効的閉集合のクラスに導入したことである．特に，実効的閉集合における limit-BHK 選言やポパー的選言に対応するものを定義し，それぞれの選言の複雑さについて調べた．これは証明構造・論理構造に関する新しい分析であるといえよう．

以上のことは，自立して研究活動を行うに必要な高度の研究能力と学識を有することを示している．したがって，木原貴行提出の博士論文は，博士（理学）の学位論文として合格と認める．